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# INVARIANT DEFORMATIONS OF ORBIT CLOSURES IN $\mathfrak{sl}(n)$

Sébastien Jansou and Nicolas Ressayre

## Abstract

We study deformations of orbit closures for the action of a connected semisimple group  $G$  on its Lie algebra  $\mathfrak{g}$ , especially when  $G$  is the special linear group.

The tools we use are on the one hand the invariant Hilbert scheme and on the other hand the sheets of  $\mathfrak{g}$ . We show that when  $G$  is the special linear group, the connected components of the invariant Hilbert schemes we get are the geometric quotients of the sheets of  $\mathfrak{g}$ . These quotients were constructed by Katsylo for a general semisimple Lie algebra  $\mathfrak{g}$ ; in our case, they happen to be affine spaces.

## Introduction

Let  $G$  be a complex reductive group, and  $V$  be a finite dimensional  $G$ -module. A fundamental problem is to endow some sets of orbits of  $G$  in  $V$  with a structure of variety. The geometric invariant theory is the classical answer in this context: the set of closed orbits of  $G$  in  $V$  has a natural structure of affine variety. We denote by  $V//G$  this variety, equipped with a  $G$ -invariant quotient map  $\pi : V \rightarrow V//G$ .

Recently, Alexeev and Brion defined in [AB] a structure of quasiprojective scheme on some sets of  $G$ -stable closed affine subscheme of  $V$ . A natural question is to wonder what happens when one applies Alexeev-Brion's construction to the orbit closures of  $G$  in  $V$ . Here, we study this construction in the case of a well known  $G$ -module, namely the adjoint representation of a semisimple group  $G$ , especially when  $G$  is the special linear group  $\mathrm{SL}(n)$ .

From now on, we assume that  $G$  is semisimple, and denote by  $\mathfrak{g}$  its Lie algebra. Let us recall that a sheet of  $\mathfrak{g}$  is an irreducible component of the set of points in  $\mathfrak{g}$  whose  $G$ -orbit has a fixed dimension. Let us fix a sheet  $\mathcal{S}$ . We show that the  $G$ -module structure on the affine algebra  $\mathbb{C}[\overline{G \cdot x}]$  of the orbit closure  $\overline{G \cdot x}$  of  $x$  doesn't depend on  $x$  in  $\mathcal{S}$ . This allows us to define a set-theoretical application from  $\mathcal{S}$  to some Alexeev-Brion's invariant Hilbert scheme of  $V$ :

$$\begin{array}{ccc} \pi_{\mathcal{S}} : \mathcal{S} & \longrightarrow & \mathrm{Hilb}_{\mathcal{S}}^G(V) \\ x & \longmapsto & \overline{G \cdot x}. \end{array}$$

A unique sheet is open in  $\mathfrak{g}$ : we call it the *regular* one, and denote it by  $\mathfrak{g}_{\mathrm{reg}}$ .

Firstly, we prove that  $\mathrm{Hilb}_{\mathfrak{g}_{\mathrm{reg}}}^G(V)$  is canonically isomorphic to the categorical quotient  $V//G$ . Moreover, via this isomorphism, the application  $\pi_{\mathfrak{g}_{\mathrm{reg}}}$  identifies with the restriction of the quotient map  $\pi : V \rightarrow V//G$ ; in particular, it is a morphism.

In a second part, we study any sheet  $\mathcal{S}$  for  $G = \mathrm{SL}(n)$ . Let us denote by  $\pi : \mathcal{S} \rightarrow \mathcal{S}/\mathrm{SL}(n)$  the geometric quotient of  $\mathcal{S}$ , constructed by Katsylo in [Ka]. We show that there is a canonical morphism

$$\begin{aligned} \theta : \mathcal{S}/\mathrm{SL}(n) &\longrightarrow \mathrm{Hilb}_{\mathcal{S}}^{\mathrm{SL}(n)}(V) \\ \mathrm{SL}(n) \cdot x &\longmapsto \overline{\mathrm{SL}(n) \cdot x} \end{aligned}$$

which is actually an isomorphism onto a connected component of  $\mathrm{Hilb}_{\mathcal{S}}^{\mathrm{SL}(n)}(V)$ .

Another motivation for this work is to understand examples of invariant Hilbert schemes. Indeed, the construction of Alexeev and Brion is indirect and only few examples are known (see [J], [BC]). Here, the connected components of invariant Hilbert schemes we obtain happen to be affine spaces, as in [J] and [BC]. Note that this answers in the case of  $\mathrm{SL}(n)$  to a question of Katsylo who asked if the geometric quotient  $\mathcal{S}/G$  is normal.

## 1 Hilbert's sheets

We consider schemes and affine algebraic groups over  $\mathbb{C}$ . Let  $G$  be a connected semisimple group. We choose a Borel subgroup  $B$ , and a maximal torus  $T$  contained in  $B$ . We denote by  $U$  the unipotent radical of  $B$ ; we have  $B = TU$ .

We denote by  $\Lambda$  the character group of  $T$ . We denote by  $\Lambda^+$  the set of elements of  $\Lambda$  that are dominant weights with respect to  $B$ . The set  $\Lambda^+$  is in bijection with the set of isomorphism classes of simple rational  $G$ -modules. If  $\lambda$  is an element of  $\Lambda^+$ , we denote by  $V(\lambda)$  a simple  $G$ -module associated, that is of highest weight  $\lambda$ .

If  $V$  is a rational  $G$ -module, we denote by  $V_{(\lambda)}$  its isotypical component of type  $\lambda$ , that is the sum of its submodules isomorphic to  $V(\lambda)$ . We have the decomposition  $V = \bigoplus_{\lambda \in \Lambda^+} V_{(\lambda)}$ .

In any decomposition of  $V$  as a direct sum of simple modules, the multiplicity of the simple module  $V(\lambda)$  is the dimension of  $V_{(\lambda)}^U$ . We say that  $V$  has *finite multiplicities* if these multiplicities are finite (for any dominant weight  $\lambda$ ).

Let us recall some definitions from [AB, §1]. A *family of affine  $G$ -schemes* over some scheme  $S$  is a scheme  $\mathfrak{X}$  equipped with an action of  $G$  and with a morphism  $\pi : \mathfrak{X} \rightarrow S$  that is affine, of finite type and  $G$ -invariant. We have a  $G$ -equivariant morphism of  $\mathcal{O}_S$ -modules

$$\pi_* \mathcal{O}_{\mathfrak{X}} \simeq \bigoplus_{\lambda \in \Lambda^+} \mathcal{F}_{\lambda} \otimes_{\mathbb{C}} V(\lambda),$$

where each  $\mathcal{F}_{\lambda} := (\pi_* \mathcal{O}_{\mathfrak{X}})_{(\lambda)}^U$  is equipped with the trivial action of  $G$ . Let  $h : \Lambda^+ \rightarrow \mathbb{N}$  be a function. The family  $\mathfrak{X}$  is said to be of *Hilbert function*  $h$  if each  $\mathcal{F}_{\lambda}$  is an  $\mathcal{O}_S$ -module locally free of rank  $h(\lambda)$ . (Then the morphism  $\pi$  is flat.)

Let  $X$  be an affine  $G$ -scheme, and  $h : \Lambda^+ \rightarrow \mathbb{N}$  a function. A *family of  $G$ -stable closed subschemes of  $X$*  over some scheme  $S$  is a  $G$ -stable closed subscheme  $\mathfrak{X} \subseteq S \times X$ . The projection  $S \times X \rightarrow S$  induces a family of affine  $G$ -schemes  $\mathfrak{X} \rightarrow S$ . The contravariant functor:  $(\mathrm{Schemes})^{\circ} \rightarrow (\mathrm{Sets})$  that associates to every scheme  $S$  the set of families  $\mathfrak{X} \subseteq S \times X$  of Hilbert

function  $h$  is represented by a quasiprojective scheme denoted by  $\text{Hilb}_h^G(X)$  ([AB, §1.2].

The dimension of an affine  $G$ -scheme whose affine algebra has finite multiplicities can be read on its Hilbert function:

**Proposition 1.1.** *Let  $h : \Lambda^+ \rightarrow \mathbb{N}$  be a function. Let  $Y$  and  $Z$  be two affine schemes of Hilbert function  $h$ . Then  $\dim Y = \dim Z$ .*

*Proof.* Let us denote by  $A$  the affine ring of  $Y$ .

If  $Y$  is *horospherical*, that is ([AB, Lemma 2.4]) if for any dominant weights  $\lambda, \mu$ , we have  $A_{(\lambda)} \cdot A_{(\mu)} \subseteq A_{(\lambda+\mu)}$ , it is clear that the dimension of  $Y$  can be read on its Hilbert function. Indeed, let us denote by  $\theta_0$  the linear map from  $\Lambda \otimes \mathbb{Q}$  to  $\mathbb{Q}$  which associates to any fundamental weight the value 1. We denote by  $\theta$  the group homomorphism from  $\Lambda$  to  $\mathbb{Z}$  that is the restriction of  $\theta_0$ . We associate to  $\theta$  a graduation of the algebra  $A$  by  $\mathbb{N}$ : its homogeneous component of degree  $d$  is

$$A_d := \bigoplus_{\lambda \in \Lambda^+, \theta(\lambda)=d} A_{(\lambda)}.$$

The dimension of  $A_d$  is finite, and depends only on  $h$ :

$$\dim A_d = \sum_{\lambda \in \Lambda^+, \theta(\lambda)=d} h(\lambda) \dim V(\lambda).$$

So the Hilbert polynomial of the graded algebra  $A$  depends only on  $h$ , and so does the dimension of  $Y$ .

We can deduce the proposition. Indeed,  $Y$  admits a flat degeneration over a connected scheme to a horospherical  $G$ -scheme  $Y'$  that admits the same Hilbert function (by [AB, Theorem 2.7]). So  $\dim Y = \dim Y'$  depends only on  $h$ .  $\square$

We will use the method of “asymptotic cones” of Borho and Kraft ([PV, §5.2]): let  $V$  be a finite dimensional rational  $G$ -module and  $F$  the closure of an orbit in  $V$  (or, more generally, any  $G$ -stable closed subvariety contained in a fiber of the categorical quotient  $V \rightarrow V//G$ ). We embed  $V$  into the projective space  $\mathbb{P}(\mathbb{C} \oplus V)$  of vector lines of  $\mathbb{C} \oplus V$  by the inclusion  $v \mapsto [1 \oplus v]$ . The closure of  $F$  in  $\mathbb{P}(\mathbb{C} \oplus V)$  is denoted by  $\overline{F}$ . The affine cone in  $\mathbb{C} \oplus V$  over  $\overline{F}$  is the closed cone  $\mathfrak{X}$  generated by  $F$ .

The vector space  $\mathbb{C} \oplus V$ , equipped with its natural scheme structure, is denoted by  $\mathbb{A}^1 \times V$ . The cone  $\mathfrak{X} \subseteq \mathbb{A}^1 \times V$ , viewed as a reduced closed subscheme, is a flat family of affine  $G$ -schemes. Its fibers over non-zero elements are homothetic to  $F$ . Its fiber over 0 is a reduced cone, denoted by  $\hat{F}$ . It is contained in the null-cone of  $V$  (that is the fiber of the categorical quotient  $V \rightarrow V//G$  containing 0). Its dimension is the same as  $F$ .

We consider the adjoint action of  $G$  on its Lie algebra  $\mathfrak{g}$ . If  $x$  is an element of  $\mathfrak{g}$ , the affine algebra of the closure of its orbit, viewed as a reduced scheme, has finite multiplicities. Let us denote by  $h_x$  its Hilbert function; we call it the Hilbert function associated to  $x$ . In this paper, we are interested in the connected component denoted  $\text{Hilb}_x^G$  of the scheme  $\text{Hilb}_{h_x}^G(\mathfrak{g})$  that contains

$\overline{G \cdot x}$ . It gives the  $G$ -invariant deformations of  $\overline{G \cdot x}$  embedded in  $\mathfrak{g}$ . We determine it when  $x$  is in  $\mathfrak{g}_{\text{reg}}$  in §2, and for any  $x$  when  $G$  is the special linear group in §3.

Let us denote by  $G_x$  the stabilizer of  $x$  in  $G$ , and  $\mathfrak{g}_x$  its Lie algebra. The coadjoint action of  $G_x$  is its natural action on the dual vector space  $\mathfrak{g}_x^*$ .

**Proposition 1.2.** *Let us assume the orbit closure  $\overline{G \cdot x}$  to be normal. The tangent space  $T_{\overline{G \cdot x}} \text{Hilb}_x^G$  to  $\text{Hilb}_x^G$  at the point  $\overline{G \cdot x}$  is canonically isomorphic to the space of invariants of the coadjoint action of  $G_x$ .*

*Proof.* The tangent space to  $\overline{G \cdot x}$  at the point  $x$  is  $\mathfrak{g} \cdot x$ ; it is stable under the action of  $G_x$ . We denote by  $[\mathfrak{g}/\mathfrak{g} \cdot x]^{G_x}$  the space of invariants under the action of  $G_x$  on the quotient vector space  $\mathfrak{g}/\mathfrak{g} \cdot x$ . According to [AB, Proposition 1.15 (iii)], we have a canonical isomorphism

$$T_{\overline{G \cdot x}} \text{Hilb}_x^G \cong [\mathfrak{g}/\mathfrak{g} \cdot x]^{G_x}. \quad (1)$$

Indeed, the orbit closure  $\overline{G \cdot x}$  is assumed to be normal. Moreover, every orbit in  $\mathfrak{g}$  has even dimension, and has a finite number of orbits in its closure ([PV, Corollary 3 page 198]), so the codimension of the boundary of  $G \cdot x$  in  $\overline{G \cdot x}$  is at least 2, and the proposition of [AB] can be applied.

To transform (1) into the isomorphism of the proposition, we will use the Killing form on  $\mathfrak{g}$ , denoted by  $\kappa$ . As  $\mathfrak{g}$  is semisimple, its Killing form gives an isomorphism

$$\begin{aligned} \phi: \mathfrak{g} &\longrightarrow \mathfrak{g}^* \\ y &\longmapsto \kappa(y, \cdot). \end{aligned}$$

The isomorphism  $\phi$  is  $G$ -equivariant, thus  $G_x$ -equivariant. It sends  $\mathfrak{g} \cdot x$  onto the space  $\mathfrak{g}_x^\perp$  of linear forms on  $\mathfrak{g}$  that vanish on  $\mathfrak{g}_x$ . Indeed, the common zeros of the elements of  $\phi(\mathfrak{g} \cdot x)$  are the elements  $y$  in  $\mathfrak{g}$  such that

$$\forall z \in \mathfrak{g}, \kappa([z, x], y) = 0,$$

that is

$$\forall z \in \mathfrak{g}, \kappa(z, [x, y]) = 0,$$

and this last condition means that  $y$  belongs to  $\mathfrak{g}_x$  since  $\kappa$  is non-degenerate.

Thus the short exact sequence of  $G_x$ -modules

$$0 \longrightarrow \mathfrak{g} \cdot x \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g} \cdot x \longrightarrow 0$$

identifies (thanks to  $\phi$ ) with

$$0 \longrightarrow \mathfrak{g}_x^\perp \longrightarrow \mathfrak{g}^* \longrightarrow (\mathfrak{g}_x)^* \longrightarrow 0,$$

and the proposition follows from (1). □

A *sheet* of  $\mathfrak{g}$  is a maximal irreducible subset of  $\mathfrak{g}$  consisting of  $G$ -orbits of a fixed dimension. Every sheet of  $\mathfrak{g}$  contains a unique nilpotent orbit. A *regular element* of  $\mathfrak{g}$  is an element of  $\mathfrak{g}$  whose orbit has maximal dimension. The open subset of  $\mathfrak{g}$  whose elements are the regular elements is a sheet denoted by  $\mathfrak{g}_{\text{reg}}$ .

Let us call *Hilbert's sheet* a maximal irreducible subset of  $\mathfrak{g}$  consisting of elements admitting a fixed associated Hilbert function.

**Proposition 1.3.** *The Hilbert's sheets of  $\mathfrak{g}$  coincide with its sheets.*

*Proof.* According to Proposition 1.1, any Hilbert's sheet is contained in some sheet. It just remains to check that two points of some sheet  $\mathcal{S}$  have the same associated Hilbert function.

Let  $F$  be the closure of an orbit in  $\mathcal{S}$ . We recalled that its asymptotic cone  $\hat{F}$  is a degeneration of  $F$ . In particular, it is contained in the closure of  $\mathcal{S}$ . Moreover,  $\hat{F}$  is contained in the null-cone of  $\mathfrak{g}$ , and its dimension is the same as  $F$ . So  $\hat{F}$  is the closure of the nilpotent orbit of  $\mathcal{S}$ .

The affine algebra of  $\mathfrak{g}$  is the symmetric algebra of  $\mathfrak{g}^*$ . Its graduation induces a  $G$ -invariant filtration on the affine algebra  $A$  of  $F$ . The affine algebra of the asymptotic cone  $\hat{F}$  is isomorphic, as an algebra equipped with an action of  $G$ , to the graded algebra  $\hat{A}$  associated to the filtered algebra  $A$ . In particular,  $A$  and  $\hat{A}$  are isomorphic as  $G$ -modules, and their multiplicities are equal: the Hilbert function of  $F$  is equal to that of  $\hat{F}$ , and the proposition is proved.  $\square$

## 2 Regular case

Let us denote by  $h_{\text{reg}}$  the Hilbert function associated to the regular elements of  $\mathfrak{g}$  (Proposition 1.3). In this section, we prove that the invariant Hilbert scheme  $\mathbb{H}_{\text{reg}} := \text{Hilb}_{h_{\text{reg}}}^G(\mathfrak{g})$  is the categorical quotient  $\mathfrak{g} // G$ , that is an affine space whose dimension is the rank of  $G$ .

### 2.1 A morphism from $\mathfrak{g} // G$ to $\mathbb{H}_{\text{reg}}$

Let  $\mathfrak{X}_{\text{reg}}$  be the graph of the canonical projection  $\mathfrak{g} \rightarrow \mathfrak{g} // G$ . It is a family of  $G$ -stable closed subschemes of  $\mathfrak{g}$  over  $\mathfrak{g} // G$ .

**Proposition 2.1.** *The closed subscheme  $\mathfrak{X}_{\text{reg}}$  is a family of  $G$ -stable closed subschemes of  $\mathfrak{g}$  with Hilbert function  $h_{\text{reg}}$ .*

*Proof.* Let us denote by  $\pi : \mathfrak{X}_{\text{reg}} \rightarrow \mathfrak{g} // G$  the canonical projection, and by  $\mathcal{R} := \pi_* \mathcal{O}_{\mathfrak{X}_{\text{reg}}}$  the direct image by  $\pi$  of the structural sheaf of  $\mathfrak{X}_{\text{reg}}$ . We have to prove that for any dominant weight  $\lambda$ , we have that  $\mathcal{R}_{(\lambda)}^U$  is a locally free sheaf on  $\mathfrak{g} // G$  of rank  $h(\lambda)$ .

Let us first study the case where  $\lambda = 0$ . The morphism  $\pi // G : \mathfrak{X}_{\text{reg}} // G \rightarrow \mathfrak{g} // G$  induced by  $\pi$  is clearly an isomorphism. So  $\mathcal{R}^G = \mathcal{R}_{(0)}^U$  is a free module on  $\mathfrak{g} // G$  of rank  $1 = h_{\text{reg}}(0)$ .

Let  $\lambda$  be a dominant weight. It is known (see [AB, Lemma 1.2]) that  $\mathcal{R}_{(\lambda)}^U$  is a coherent  $\mathcal{R}^G$ -module. Thus it is a coherent module on  $\mathfrak{g} // G$ . To see that it is locally free, we just have to check that its rank is constant. The fibers of  $\pi$  are those of the canonical projection  $\mathfrak{g} \rightarrow \mathfrak{g} // G$ , so they are the orbit closures of the regular elements, and all of them admit  $h_{\text{reg}}$  as Hilbert function. So the rank of  $\mathcal{R}_{(\lambda)}^U$  at any closed point of  $\mathfrak{g} // G$  is  $h(\lambda)$ , and the proposition is proved.  $\square$

This gives us a canonical morphism

$$\phi_{\text{reg}} : \mathfrak{g} // G \longrightarrow \mathbb{H}_{\text{reg}}.$$

We will prove in the following of §2 that  $\phi_{\text{reg}}$  is an isomorphism.

**Lemma 2.2.** *The morphism  $\phi_{\text{reg}}$  realises a bijection from the set of closed points of  $\mathfrak{g}//G$  to the set of closed points of  $H_{\text{reg}}$ .*

*Proof.* We remark that  $\phi_{\text{reg}}$  is injective. Let us check it is surjective: in other words, that any  $G$ -invariant closed subscheme of  $\mathfrak{g}$  of Hilbert function  $h_{\text{reg}}$  is a fiber of  $\mathfrak{g} \rightarrow \mathfrak{g}//G$ .

Let  $Y$  be such a subscheme. As  $h_{\text{reg}}(0) = 1$ , it has to be contained in some fiber  $F$  of  $\mathfrak{g} \rightarrow \mathfrak{g}//G$  over a reduced closed point. But  $F$  already corresponds to a closed point of  $H_{\text{reg}}$  in the image of  $\phi_{\text{reg}}$ . Moreover,  $F$  admits no proper closed subscheme admitting the same Hilbert function, so  $F = Y$ , and the lemma is proved.  $\square$

Let us denote by  $r$  the rank of  $G$ . The quotient  $\mathfrak{g}//G$  is an affine space of dimension  $r$ . A consequence of Lemma 2.2 is:

**Corollary 2.3.** *The dimension of  $H_{\text{reg}}$  is  $r$ .*

## 2.2 Tangent space

In this section, we prove:

**Proposition 2.4.** *The scheme  $H_{\text{reg}}$  is smooth.*

*Proof.* Let  $Z$  be a closed point of  $H_{\text{reg}}$ . We have to prove that the dimension of the tangent space  $T_Z H_{\text{reg}}$  is  $r$ . We still denote by  $Z$  the closed subscheme of  $\mathfrak{g}$  corresponding to  $Z$ . By Lemma 2.2, we know that  $Z$  is a fiber of the morphism  $\mathfrak{g} \rightarrow \mathfrak{g}//G$ , thus the closure of some regular element  $x$ . It is a normal variety. By Proposition 1.2, we have to prove that the dimension of

$$(\mathfrak{g}_x^*)^{G_x}$$

is  $r$ , or simply that it is lower or equal to  $r$  (by Corollary 2.3).

Let us prove that the dimension of the bigger space

$$(\mathfrak{g}_x^*)^{\mathfrak{g}_x}$$

is  $r$ , and the proposition will be proved.

A linear form on  $\mathfrak{g}_x$  is  $\mathfrak{g}_x$ -invariant iff it vanishes on the derived algebra  $[\mathfrak{g}_x, \mathfrak{g}_x]$ , so we have to prove that

$$(\mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x])^*$$

is  $r$ -dimensional. We will prove that  $\mathfrak{g}_x$  is an  $r$ -dimensional abelian algebra, and the proposition will be proved. This is true if  $x$  is semisimple, because then  $\mathfrak{g}_x$  is a Cartan subalgebra of  $\mathfrak{g}$ . If the regular element  $x$  is not assumed to be semisimple, the dimension of  $\mathfrak{g}_x$  is still  $r$ , because this doesn't depend on the regular element  $x$ , by definition. Let us check that  $\mathfrak{g}_x$  is abelian.

Let us denote by  $\text{Grass}_r(\mathfrak{g})$  the grassmannian of  $r$ -dimensional subspaces of  $\mathfrak{g}$ , endowed with its projective variety structure. The subset of  $\mathfrak{g}_{\text{reg}} \times \text{Grass}_r(\mathfrak{g})$ :

$$\{(z, \mathfrak{h}) \in \mathfrak{g}_{\text{reg}} \times \text{Grass}_r(\mathfrak{g}) \mid \mathfrak{h} \cdot z = 0 \text{ and } [\mathfrak{h}, \mathfrak{h}] = 0\}$$

is closed, so its image by the natural projection into  $\mathfrak{g}_{\text{reg}}$  is closed too. As its image contains the semisimple elements of  $\mathfrak{g}_{\text{reg}}$ , it is equal to  $\mathfrak{g}_{\text{reg}}$ . Thus  $\mathfrak{g}_x$  is abelian for any regular  $x$ , and the proposition is proved.  $\square$

### 2.3 Conclusion

We can now conclude that the family  $\mathfrak{X}_{\text{reg}}$  of Proposition 2.1 is the universal family:

**Theorem 2.5.** *The morphism  $\phi_{\text{reg}}$  from  $\mathfrak{g} // G$  to  $\mathbf{H}_{\text{reg}}$  is an isomorphism.*

*Proof.* The morphism  $\phi_{\text{reg}}$  is bijective (Lemma 2.2) and  $\mathbf{H}_{\text{reg}}$  is normal. According to Zariski's main theorem,  $\phi_{\text{reg}}$  is an isomorphism.  $\square$

**Remark 2.6.** One knows there is a canonical morphism

$$\psi_{\text{reg}} : \mathbf{H}_{\text{reg}} \longrightarrow \mathfrak{g} // G$$

that associates to any closed point  $F$  of  $\mathbf{H}_{\text{reg}}$  (viewed as a closed subscheme of  $\mathfrak{g}$ ) its categorical quotient  $F // G$  (viewed as a closed point of  $\mathfrak{g} // G$ ). This morphism is a particular case of morphism

$$\eta : \text{Hilb}_h^G(V) \longrightarrow \text{Hilb}_{h(0)}(V // G)$$

defined in [AB, §1.2], because  $h_{\text{reg}}(0) = 1$  and thus the punctual Hilbert scheme that parameterizes closed subschemes of length 1 in  $\mathfrak{g} // G$  identifies with  $\mathfrak{g} // G$  itself. The morphism  $\psi_{\text{reg}}$  is clearly the inverse morphism of  $\phi_{\text{reg}}$ .

### 3 Case of $\mathfrak{sl}(n)$

We denote by  $t$  an indeterminate over  $\mathbb{C}$ , and  $I_n$  the identity matrix of size  $n \times n$ . If  $x$  is an element of  $\mathfrak{sl}(n)$  and  $i = 1 \cdots n$ , we denote by  $Q_i^x(t)$  the monic greatest common divisor (in the ring  $\mathbb{C}[t]$ ) of the  $(n+1-i) \times (n+1-i)$ -sized minors of  $x - tI_n$ , and  $Q_{n+1}^x(t) := 1$ .

Then we put

$$q_i^x(t) := Q_i^x(t) / Q_{i+1}^x(t).$$

The polynomials  $q_1^x(t), \dots, q_n^x(t)$  are the invariant factors of the matrix  $x - tI_n$  with coefficients in the euclidean ring  $\mathbb{C}[t]$ , ordered in such a way that  $q_{i+1}^x(t)$  divides  $q_i^x(t)$ .

If  $x, y$  are elements of  $\mathfrak{sl}(n)$ , then  $y$  is in the closure of the orbit  $\text{SL}(n) \cdot x$  of  $x$  if and only if for any  $i = 1 \dots n$ , the polynomial  $Q_i^x(t)$  divides  $Q_i^y(t)$ . In other words, iff for any  $i$ , the polynomial  $Q_i^x(t)$  divides the  $(n+1-i) \times (n+1-i)$ -sized minors of  $y - tI_n$ .

According to [W], when  $x$  is nilpotent, these conditions defines the closure of  $\text{SL}(n) \cdot x$  as a reduced scheme: to be more precise, when one divides a  $(n+1-i) \times (n+1-i)$ -sized minor of  $y - tI_n$  by  $Q_i^x(t)$  using Euclid algorithm, the remainder he gets is a regular function of  $y$ . All such functions generate the ideal of the closure of  $\text{SL}(n) \cdot x$ . We will deduce easily from this difficult result that the same remains true if  $x$  is no longer assumed to be nilpotent.

One sees easily that the set of sheets of  $\mathfrak{sl}(n)$  is in bijection with the set of partitions  $n$ , that is of sequences  $\sigma = (b_1 \geq b_2 \geq b_3 \geq \dots)$  of nonnegative integers such that  $b_1 + b_2 + b_3 + \dots = n$ . Namely, if  $\sigma$  is a partition of  $n$ , the elements of the correspondent sheet  $\mathcal{S}_\sigma$  are those elements  $x$



such that for any  $i$ , the polynomial  $q_i^x(t)$  is of degree  $b_i$ . We denote by  $\hat{\sigma} = (c_1 \geq c_2 \geq c_3 \geq \dots)$  the conjugate partition, where  $c_j$  is the number of  $i$  such that  $b_i \geq j$ . We denote by  $h_\sigma$  the Hilbert function associated to the points of  $\mathcal{S}_\sigma$  (Proposition 1.3). We denote by  $Z_\sigma$  the closure of the nilpotent orbit of  $\mathcal{S}_\sigma$ . The connected component of  $\text{Hilb}_{h_\sigma}^{\text{SL}(n)}(\mathfrak{sl}(n))$  that contains  $Z_\sigma$  as a closed point is denoted  $\text{H}_\sigma$ . We will prove in this section that  $\text{H}_\sigma$  is an affine space of dimension  $b_1 - 1$ . The proof is similar to §2.

We recall that the sheets of  $\mathfrak{sl}(n)$  are smooth ([Kr]).

### 3.1 A construction of the geometric quotient of $\mathcal{S}_\sigma$

Katslylo showed in [Ka] that any sheet of a semisimple Lie algebra admits a geometric quotient. Although his proof contains an explicit construction, it doesn't make clear the geometric properties of the quotient. Here we present a simple description of the quotient in the case of the Lie algebra  $\mathfrak{sl}(n)$ . It takes on the invariant factors theory. We get that the quotient is an affine space.

**Lemma 3.1.** *Given some  $i$ , the application  $\mathcal{S}_\sigma \rightarrow \mathbb{A}^{b_i}$  that associates to any  $x$  the coefficients of  $q_i^x(t) = t^{b_i} + \lambda_{b_i-1}^x t^{b_i-1} + \dots + \lambda_0^x t^0$  is regular.*

*Proof.* Up to scalar multiplication, the polynomial  $q_i^x(t)$  is the unique nonzero polynomial of degree less or equal to  $b_i$  such that

$$\dim \ker q_i^x(x) \geq N := \sum_{j=1}^{b_i} c_j. \quad (2)$$

Thus the closed subset of  $\mathcal{S}_\sigma \times \mathbb{P}^{b_i}$  consisting of elements  $(x, [\mu_0 : \dots : \mu_{b_i}])$  such that

$$\dim \ker \left( \sum_{j=0}^{b_i} \mu_j x^j \right) \geq N$$

is the graph of the application

$$\begin{aligned} \psi : \mathcal{S}_\sigma &\longrightarrow \mathbb{P}^{b_i} \\ x &\longmapsto [\lambda_0^x : \dots : \lambda_{b_i-1}^x : 1] \end{aligned}$$

According to [Hr, Exercise 7.8 p 76], this graph is also the graph of a rational map  $\phi$  from  $\mathcal{S}_\sigma$  to  $\mathbb{P}^{b_i}$ . On the open subset  $\Omega$  of  $\mathcal{S}_\sigma$  where  $\phi$  is regular,  $\phi$  coincides with  $\psi$ , so the functions  $x \mapsto \lambda_j^x$  are regular functions from  $\Omega$  to  $\mathbb{A}^1$ . As  $\mathcal{S}_\sigma$  is smooth, the complementary of  $\Omega$  in  $\mathcal{S}_\sigma$  has codimension at least 2 ([S, Thm 3 chap II.3.1]). We conclude that the functions extend to regular functions from  $\mathcal{S}_\sigma$  to  $\mathbb{A}^1$ . By continuity, these extensions satisfy (2), so they coincide with the functions  $x \mapsto \lambda_j^x$  on  $\mathcal{S}_\sigma$ .  $\square$

Let us define, for any  $x$  in  $\mathcal{S}_\sigma$ , the monic polynomial of degree  $b_i - b_{i+1}$ :

$$p_i^x(t) := q_i^x(t)/q_{i+1}^x(t)$$

(where  $q_{n+1}^x := 1$ ). It follows from the previous lemma that its coefficients, viewed as functions of  $x$ , are regular functions from  $\mathcal{S}_\sigma$  to  $\mathbb{A}^1$ .

Given an  $x$ , the family  $(p_1^x(t), \dots, p_n^x(t))$  can be any family of monic polynomials of degrees  $b_i - b_{i+1}$ , provided the following relation is satisfied, where  $S(p_i^x)$  denotes the sum of the roots of  $p_i^x$ , counted with multiplicities (given by its first nondominant coefficient):

$$\sum_{i=1}^n iS(p_i^x) = 0$$

(this relation simply means that the trace of  $x$  is zero).

Thus, associating to any  $x$  the coefficients of the family  $(p_1^x(t), \dots, p_n^x(t))$ , we get a regular map  $\pi$  from  $\mathcal{S}_\sigma$  to a linear hyperplane of  $\mathbb{C}^{b_1}$ , which we will denote by  $\mathbb{A}^{b_1-1}$ .

**Proposition 3.2.** *The map  $\pi : \mathcal{S}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$  is the geometric quotient of  $\mathcal{S}_\sigma$ .*

*Proof.* This map is surjective, and its fibers are exactly the orbits of  $\mathcal{S}_\sigma$  under the action of  $\mathrm{SL}(n)$ . Let us denote by  $\mathcal{S}_\sigma/\mathrm{SL}(n)$  the geometric quotient of  $\mathcal{S}_\sigma$  (whose existence is proved in [Ka]). The map  $\pi$  is the composite of the canonical projection from  $\mathcal{S}_\sigma$  to  $\mathcal{S}_\sigma/\mathrm{SL}(n)$  with a regular bijection

$$\mathcal{S}_\sigma/\mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}.$$

This last map is bijective (thus birational), and the space  $\mathbb{A}^{b_1-1}$  is normal. According to Zariski's main theorem, it is an isomorphism.  $\square$

### 3.2 A morphism from $\mathcal{S}_\sigma/\mathrm{SL}(n)$ to $H_\sigma$

If  $z = (p_1(t), \dots, p_n(t))$  is a closed point of  $\mathbb{A}^{b_1-1}$  corresponding to the orbit  $\mathrm{SL}(n) \cdot x$  in  $\mathcal{S}_\sigma$ , the polynomial

$$Q_i^z(t) = p_i(t) \cdot (p_{i+1}(t))^2 \cdot \dots \cdot (p_n(t))^{n-i+1}$$

only depends on  $z$ . Let us denote it by  $Q_i^z(t)$ . Its coefficients are regular functions from  $\mathbb{A}^{b_1-1}$  to  $\mathbb{A}^1$ .

Let us consider the closed subscheme  $\mathfrak{X}_\sigma$  of  $\{(z, y) \in \mathbb{A}^{b_1-1} \times \mathfrak{sl}(n)\}$  defined by the vanishing, for  $i = 1 \dots n$ , of the remainders we get when we divide the  $(n+1-i) \times (n+1-i)$ -minors of  $y - tI_n$  by  $Q_i^z(t)$ . We denote by  $I_\sigma$  the ideal generated by these remainders. The underlying set of  $\mathfrak{X}_\sigma$  consists of all the couples  $(z, y)$  such that  $y$  is in the closure of the orbit corresponding to  $z$ .

**Proposition 3.3.** *The closed subscheme  $\mathfrak{X}_\sigma$  is a family of  $\mathrm{SL}(n)$ -stable closed subschemes of  $\mathfrak{sl}(n)$  with Hilbert function  $h_\sigma$ .*

*Proof.* The proof is similar to that of Proposition 2.1. The subscheme  $\mathfrak{X}_\sigma$  is a family of  $\mathrm{SL}(n)$ -stable closed subschemes of  $\mathfrak{sl}(n)$  over  $\mathbb{A}^{b_1-1}$ . Let us denote by  $\pi$  the morphism  $\mathfrak{X}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$ .

As previously, let us first remark that the morphism

$$\pi//\mathrm{SL}(n) : \mathfrak{X}_\sigma//\mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}$$

induced by  $\pi$  is an isomorphism. To do this, let us verify that the comorphism

$$(\pi // \mathrm{SL}(n))^* : \mathbb{C}[\mathbb{A}^{b_1-1}] \longrightarrow \mathbb{C}[\mathbb{A}^{b_1-1}] \otimes \mathbb{C}[\mathfrak{sl}(n)]^{\mathrm{SL}(n)} / I_\sigma^{\mathrm{SL}(n)}$$

is an isomorphism. It is injective, as  $\pi$  is surjective. Its surjectivity comes from the relations that define  $\mathfrak{X}_\sigma$ : they give, for  $i = 1$ , that  $Q_1^z(t)$  divides the determinant of  $tI_n - y$ , that is the characteristic polynomial of  $y$ . As their degrees are equal,  $Q_1^z(t)$  and the characteristic polynomial of  $y$  are equal. This gives the surjectivity.

We go on as previously: let  $\lambda$  be a dominant weight. The  $R^{\mathrm{SL}(n)}$ -module  $R_{(\lambda)}^U$  is of finite type ([AB, Lemma 1.2]). Thus  $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$  is a coherent  $\mathcal{O}_{\mathbb{A}^{b_1-1}}$ -module. To see that it is locally free, we just have to check that its rank is constant. Let us assume that the origin  $0 \in \mathbb{A}^{b_1-1}$  corresponds to the nilpotent orbit in  $\mathcal{S}_\sigma$ . The fiber of  $\pi$  over 0 is the closure of this orbit, fitted with its structure of reduced scheme. Thus, the rank of  $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$  at 0 is  $h_\sigma(\lambda)$ . If  $z$  is any point of  $\mathbb{A}^{b_1-1}$ , the fiber of  $\pi$  over  $z$  is as a set the closure in  $\mathfrak{sl}(n)$  of the corresponding orbit. So, by Proposition 1.3 the rank of  $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$  at  $z$  is at least  $h_\sigma(\lambda)$ . To conclude, we use the action of the multiplicative group on  $\mathfrak{sl}(n)$  (by homotheties) and the induced action on  $\mathbb{A}^{b_1-1}$ , that makes  $\pi$  equivariant. The orbit of  $z$  goes arbitrary close to 0, and the rank of a coherent sheaf is upper semicontinuous, so the rank of  $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$  is  $h_\sigma(\lambda)$  at  $z$ .  $\square$

### 3.3 Tangent space

In this section, we compute the dimension of the tangent space to  $H_\sigma$  at the point  $Z_\sigma$ :

**Proposition 3.4.** *The dimension of  $T_{Z_\sigma} H_\sigma$  is  $b_1 - 1$ .*

*Proof.* Let  $x$  be an element in the open orbit in  $Z_\sigma$ . It is known that  $Z_\sigma$  is normal ([KP]). So by Proposition 1.2, we just have to prove that the dimension of

$$(\mathfrak{sl}(n)_x^*)^{\mathrm{SL}(n)_x}$$

is  $b_1 - 1$ . Let us consider  $\mathrm{SL}(n)$  as a closed subgroup of the general linear group  $\mathrm{GL}(n)$ , and  $\mathfrak{sl}(n)$  as a subalgebra of  $\mathfrak{gl}(n)$ . The stabilizer  $\mathrm{GL}(n)_x$  of  $x$  in  $\mathrm{GL}(n)$  is generated by  $\mathrm{SL}(n)_x$  and the center of  $\mathrm{GL}(n)$ . It is clearly equivalent to prove that the dimension of

$$(\mathfrak{gl}(n)_x^*)^{\mathrm{GL}(n)_x}$$

is  $b_1$ . The group  $\mathrm{GL}(n)_x$  is connected, so the last space is isomorphic to

$$(\mathfrak{gl}(n)_x^*)^{\mathfrak{gl}(n)_x}.$$

A linear form on  $\mathfrak{gl}(n)_x$  is  $\mathfrak{gl}(n)_x$ -invariant iff it vanishes on the derived algebra  $[\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x]$ , so we have to prove that

$$(\mathfrak{gl}(n)_x / [\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x])^*$$

is  $b_1$ -dimensional. This fact is the following elementary lemma.  $\square$

**Lemma 3.5.** *Let  $E = \bigoplus_{i=1}^{c_1} E_i$  be a graded vector space over  $\mathbb{C}$ , where each  $E_i$  is  $b_i$ -dimensional. We denote by  $\mathfrak{h} := \mathfrak{gl}(E)$  the Lie algebra of endomorphisms of  $E$ . Let  $x$  be a nilpotent element of  $\mathfrak{h}$  such that each subspace  $E_i$  is stabilized by  $x$ , and the restriction of  $x$  to each  $E_i$  is cyclic.*

*Let us denote by  $\mathfrak{h}_x$  the stabilizer of  $x$  in  $\mathfrak{h}$ . Then the codimension of the derived algebra  $[\mathfrak{h}_x, \mathfrak{h}_x]$  in  $\mathfrak{h}_x$  is  $b_1$ .*

*Proof.* The graduation of  $E$  induces a graduation on the vector space  $\mathfrak{h}$ :

$$\mathfrak{h} = \bigoplus_{i,j} \text{Hom}(E_i, E_j).$$

Let us denote by  $p_i : E \rightarrow E_i$  the natural projections. As they commute with  $x$ , the subspace  $\mathfrak{h}_x$  of  $\mathfrak{h}$  is homogeneous:

$$\mathfrak{h}_x = \bigoplus_{i,j} \text{Hom}_x(E_i, E_j),$$

where  $\text{Hom}_x(E_i, E_j)$  denotes the space of homomorphisms that commute with  $x$ . Let us choose, for any  $i$ , an element  $e_i$  of  $E_i$  such that  $x^{b_i-1}e_i \neq 0$ . We put  $n_{ij} := b_j - b_i$  if  $j < i$  and 0 otherwise. We denote by  $f_{ij} : E_i \rightarrow E_j$  the unique homomorphism that commutes with  $x$  and that sends  $e_i$  to  $x^{n_{ij}}e_j$ . Then any homomorphism from  $E_i$  to  $E_j$  that commutes with  $x$  is the composite of  $f_{ij}$  with a polynomial in  $x$ :

$$\text{Hom}_x(E_i, E_j) = \mathbb{C}[x] \cdot f_{ij}.$$

We notice that if  $i \neq j$ , then  $\text{Hom}_x(E_i, E_j)$  is contained in  $[\mathfrak{h}_x, \mathfrak{h}_x]$ . Indeed, for any  $u : E_i \rightarrow E_j$ , we have  $[u, p_i] = u$ .

So we have to prove that the codimension in  $\bigoplus_i \text{Hom}_x(E_i, E_i)$  of

$$[\mathfrak{h}_x, \mathfrak{h}_x] \cap \bigoplus_i \text{Hom}(E_i, E_i)$$

is  $b_1$ . The last vector space is generated by its elements of the form

$$P(x)[f_{ji}, f_{ij}] = P(x)x^{|b_i-b_j|}(\text{id}_{E_i} - \text{id}_{E_j}),$$

where  $P(x)$  is a polynomial in  $x$ .

One checks easily that a basis of a supplementary in  $\bigoplus_i \text{Hom}_x(E_i, E_i)$  of this space is given by the family of elements

$$x^k \text{id}_{E_i}$$

where  $0 \leq k < b_i - b_{i+1}$ , and the lemma is proved.  $\square$

### 3.4 Conclusion

In this section, we prove that the family  $\mathfrak{X}_\sigma$  of Proposition 3.3 is the universal family:

**Theorem 3.6.** *The morphism  $\phi_\sigma$  from  $\mathcal{S}_\sigma/\text{SL}(n)$  to  $\text{H}_\sigma$  obtained in §3.2 is an isomorphism.*

We denote by  $\overline{\mathcal{S}}_\sigma$  the closure of  $\mathcal{S}_\sigma$  in  $\mathfrak{sl}(n)$ , equipped with its reduced scheme structure. The invariant Hilbert scheme  $H'_\sigma := \text{Hilb}_{h_\sigma}^{\text{SL}(n)}(\overline{\mathcal{S}}_\sigma)$  parametrizing the closed subschemes of  $\overline{\mathcal{S}}_\sigma$  of Hilbert function  $h_\sigma$  is canonically identified with a closed subscheme of  $\text{Hilb}_{h_\sigma}^{\text{SL}(n)}(\mathfrak{sl}(n))$ . The morphism  $\phi_\sigma$  factorizes by a morphism  $\psi_\sigma : \mathcal{S}_\sigma/\text{SL}(n) \rightarrow H'_\sigma$ .

To prove the theorem, we will get that the morphism  $\psi_\sigma$  is an isomorphism from  $\mathcal{S}_\sigma/\text{SL}(n)$  to  $H'_\sigma$  and that  $H'_\sigma$  is a connected component of  $H_\sigma$  (Corollary 3.10).

**Lemma 3.7.** *The morphism  $\psi_\sigma$  induces a bijection from the set of closed points of  $\mathcal{S}_\sigma/\text{SL}(n)$  to the set of closed points of  $H'_\sigma$ .*

*Proof.* We know that  $\psi_\sigma$  is injective. Let us check it is surjective: in other words, that any  $\text{SL}(n)$ -invariant closed subscheme of  $\overline{\mathcal{S}}_\sigma$  with Hilbert function  $h_\sigma$  is the closure of some orbit in  $\mathcal{S}_\sigma$ .

Let  $X$  be such a subscheme. As  $h_\sigma(0) = 1$ , it has to be contained in some fiber  $F$  of the categorical quotient  $\overline{\mathcal{S}}_\sigma \rightarrow \overline{\mathcal{S}}_\sigma//\text{SL}(n)$  over a reduced closed point. But  $F$  already corresponds to a closed point of  $H'_\sigma$  in the image of  $\psi_\sigma$ . Moreover,  $F$  admits no proper closed subscheme admitting the same Hilbert function, so  $F = X$ , and the lemma is proved.  $\square$

**Corollary 3.8.** *The dimension of  $H'_\sigma$  is  $b_1 - 1$ .*

The action of the multiplicative group  $\mathbb{G}_m$  on  $\mathfrak{sl}(n)$  by homotheties induces canonically an action of  $\mathbb{G}_m$  on  $H_\sigma$ , and on  $H'_\sigma$  (because it stabilizes  $\overline{\mathcal{S}}_\sigma$ ). The cone  $Z_\sigma$  is a  $\mathbb{G}_m$ -fixed point of  $H'_\sigma$ . In fact, it is in the closure of the  $\mathbb{G}_m$ -orbit of any point of  $H'_\sigma$ :

**Proposition 3.9.** *Let  $F$  be a closed point of  $H'_\sigma$ . The morphism  $\eta : \mathbb{G}_m \rightarrow H'_\sigma$ ,  $t \mapsto t.X$  extends to a morphism  $\mathbb{A}^1 \rightarrow H'_\sigma$ ,  $0 \mapsto Z_\sigma$ .*

*Proof.* The point  $F$  corresponds to a  $\text{SL}(n)$ -invariant closed subscheme of  $\overline{\mathcal{S}}_\sigma$  admitting Hilbert function  $h_\sigma$ . We still denote it by  $F$ . As  $h_\sigma(0) = 1$ , it is contained in the fiber of the categorical quotient  $\mathfrak{sl}(n) \rightarrow \mathfrak{sl}(n)//\text{SL}(n)$  over some closed point. Thus we can apply to it the method of asymptotic cones: we obtain a flat family over  $\mathbb{A}^1$  whose fiber over 0 must be  $Z_\sigma$  (as in the proof of Proposition 1.3). It gives a morphism from  $\mathbb{A}^1$  to  $H'_\sigma$  whose restriction outside 0 is  $\eta$ .  $\square$

From the proposition, we deduce that the dimension of the tangent space to  $H_\sigma$  at any point of  $H'_\sigma$  is lower or equal to that at  $Z_\sigma$ , that is  $b_1 - 1$ . As the dimension of  $H'_\sigma$  is  $b_1 - 1$ , we get:

**Corollary 3.10.**

- *The scheme  $H'_\sigma$  is reduced and smooth.*
- *It is a connected component of  $H_\sigma$ .*

The morphism  $\psi_\sigma$  is bijective (Lemma 3.7) and  $H'_\sigma$  is normal. According to Zariski's main theorem,  $\psi_\sigma$  is an isomorphism. So Theorem 3.6 is proved, thanks to the second point of Corollary 3.10.

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