Convergence of Harder-Narasimhan polygons

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Abstract

We establish in this article convergence results of normalized Harder-Narasimhan polygons both in geometric and in arithmetic frameworks by introducing the Harder-Narasimhan filtration indexed by $\mathbb{R}$ and the associated Borel probability measure.

1 Introduction

Let X be a projective variety of dimension ≥ 1 over a field k and L be an ample line bundle on X. The Hilbert-Samuel theorem describing the asymptotic behaviour of $\operatorname{rk} H^0(X, L^{\otimes D})$ $(D \to \infty)$ is an important result in commutative algebra and in algebraic geometry, which is largely studied since Hilbert's article [Hil90]. Although numerous variants and generalizations of this theorem have been developed, many proofs have a common feature — the technic of unscrewing ("dévissage" in French). Let us recall a variant of Hilbert-Samuel theorem in relative geometric framework. Suppose that k is a field and C is a non-singular projective curve over $\operatorname{Spec} k$. We denote by $K = k(C)$ the field of rational functions on C. Let $\pi : X \to C$ be a projective and flat k-morphism and L be an invertible $\mathcal{O}_X$-module which is ample relatively to π. We denote by d the relative dimension of X over C. The Riemann-Roch theorem implies that

$$\deg(\pi_*(L^{\otimes D})) = \frac{c_1(L)^{d+1}}{(d+1)!} D^{d+1} + O(D^d) \qquad (D \to \infty).$$

Combining with the classical Hilbert-Samuel theorem

$$\operatorname{rk}(\pi_*(L^{\otimes D})) = \operatorname{rk} H^0(X_K, L_K^{\otimes D}) = \frac{c_1(L_K)^d}{d!} D^d + O(D^{d-1}),$$

we obtain the asymptotic formula

$$\lim_{D\to\infty} \frac{\mu(\pi_*(L^{\otimes D}))}{D} = \frac{c_1(L)^{d+1}}{(d+1)c_1(L_K)^d}, \tag{1}$$

where the *slope* μ of a non-zero locally free $\mathcal{O}_C$-module of finite type (in other words, non-zero vector bundle on C) is by definition the quotient of its degree by its rank. For a non-zero vector bundle E on C, there exists invariant which is much shaper than the slope. Namely, Harder and Narasimhan have proved in [HN75] that there exists a non-zero subbundle E_{des} whose slope is maximal among the slopes of non-zero subbundles of E and which contains all non-zero subbundles of E having the maximal slope. The slope of E_{des} is denoted by $\mu_{\max}(E)$, called

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the *maximal slope* of E. We say that E is *semistable* if and only if $E = E_{\mathrm{des}}$, or equivalently $\mu(E) = \mu_{\max}(E)$. By induction we obtain a sequence

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$$

of saturated subbundles of E such that $(E_i/E_{i-1}) = (E/E_{i-1})_{\mathrm{des}}$ for any $1 \leq i \leq n$. This sequence is called the *Harder-Narasimhan flag* of E. Clearly each sub-quotient E_i/E_{i-1} is semistable and we have $\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$. The last slope $\mu(E_n/E_{n-1})$ is called the *minimal slope* of E, denoted by $\mu_{\min}(E)$. Note that the Harder-Narasimhan flag of E_i is just $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_i$ and therefore $\mu_{\min}(E_i) = \mu(E_i/E_{i-1})$.

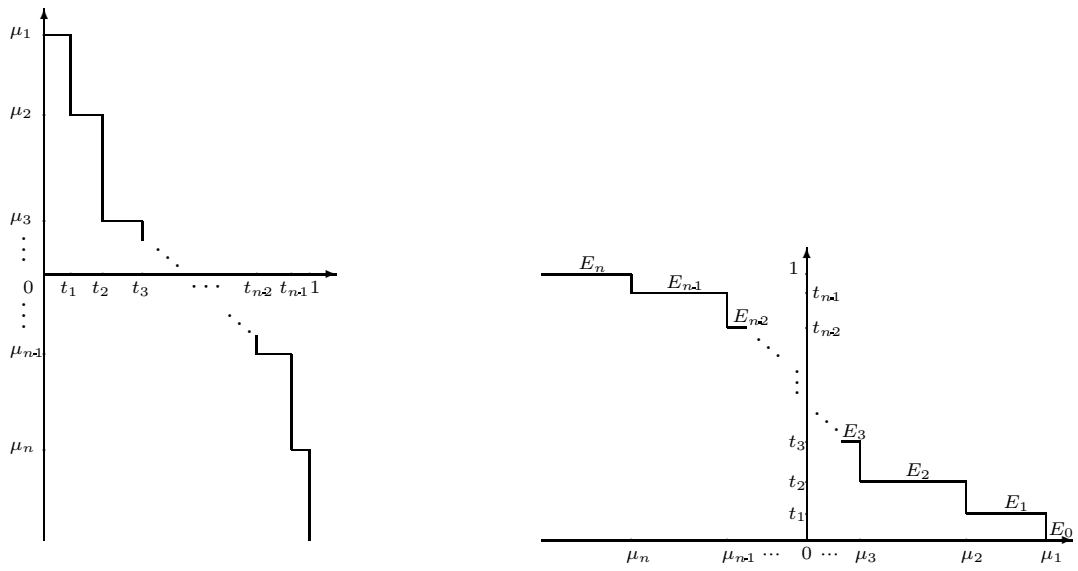
Recall that the *Harder-Narasimhan polygon* of E is by definition the concave function $\widetilde{P}_E$ on the interval $[0, \operatorname{rk} E]$ whose graph is the convex hull of points $(\operatorname{rk} F, \deg(F))$, where F runs over all subbundles of E. Therefore, the function $\widetilde{P}_E$ takes zero value at origin; it is piecewise linear and its slope on the interval $[\operatorname{rk} E_{i-1}, \operatorname{rk} E_i]$ is $\mu(E_i/E_{i-1})$. Let P_E be the function defined on $[0,1]$ whose graph is similar to that of $\widetilde{P}_E$, namely $P_E(t) = \widetilde{P}_E(t \operatorname{rk} E)/\operatorname{rk} E$, called the *normalized Harder-Narasimhan polygon of* E. Notice that $P_E(1) = \mu(E)$. Therefore (1) can be reformulated as

$$\lim_{D\to\infty} \frac{P_{\pi_*(L^{\otimes D})}(1)}{D} = \frac{c_1(L)^{d+1}}{(d+1)c_1(L_K)^d}. \tag{2}$$

It is then quite natural to study the convergence at other points in $[0,1]$. Here the major difficulty is that, unlike the degree function $\widetilde{P}_E(1) \operatorname{rk} E$, for other points $r \in]0, \operatorname{rk} E[$, the function $E \to \widetilde{P}_E(r) \operatorname{rk} E$ need not be additive with respect to short exact sequences. Therefore the unscrewing technic doesn't work.

The original idea of this article is to use Borel probability measures on $\mathbb{R}$ to study Harder-Narasimhan polygons. In Figure 1, the left graph presents the first order derivative of the

Figure 1: Derivative of the polygon and the corresponding distribution function



normalized Harder-Narasimhan polygon of E, where $\mu_i = \mu(E_i/E_{i-1})$ for $1 \leq i \leq n$ and

$t_i = \operatorname{rk} E_i / \operatorname{rk} E$ for $0 \leq i \leq n$. It is a step function on $[0,1]$. The right graph presents a decreasing step function on $\mathbb{R}$ valued in $[0,1]$ whose *quasi-inverse* corresponds to the left graph. Furthermore, this function is the difference between the constant function 1 and a probability distribution function and therefore corresponds to a Borel probability measure $\nu_E = \sum_{i=1}^{n}(t_i - t_{i-1})\delta_{\mu_i}$, where δ_x is the Dirac measure at the point x. If we place suitably the subbundles in the Harder-Narasimhan flag of E on the right graph, we obtain a decreasing $\mathbb{R}$-filtration of the vector bundle E, which induces naturally by restricting to the generic fiber a decreasing $\mathbb{R}$-filtration $\mathcal{F}^{\mathrm{HN}}$ of the vector space E_K, called the *Harder-Narasimhan filtration* of E_K. As we shall show later, the filtration $\mathcal{F}^{\mathrm{HN}}$ can be calculated explicitly from the vector bundle E, namely

$$\mathcal{F}^{\mathrm{HN}}_r E_K = \sum_{\substack{0 \neq F \subset E \\ \mu_{\min}(F) \geq r}} F_K.$$

From this filtration, one can recover easily the probability measure

$$\nu_E = \frac{1}{\operatorname{rk} E_K} \sum_{r \in \mathbb{R}} \Big(\operatorname{rk}(\mathcal{F}^{\mathrm{HN}}_r E_K) - \lim_{\varepsilon \to 0+} \operatorname{rk}(\mathcal{F}^{\mathrm{HN}}_{r+\varepsilon} E_K) \Big) \delta_r.$$

Furthermore, the function presented in the right graph is just $r \mapsto \operatorname{rk}(\mathcal{F}^{\mathrm{HN}}_r E_K)$. By passing to quasi-inverse (turning over the graph), we retrieve the first order derivative of the normalized Harder-Narasimhan polygon. This procedure is quite general and it works for an arbitrary (suitably) filtered finite dimensional vector space, where the word "suitably" means that the filtration is separated, exhaustive and left continuous, which we shall explain later in this article. Actually, we have natural mappings

$$\begin{array}{ccccc} \left\{\begin{array}{l}\text{(suitably) filtered finite}\\ \text{dimensional vector spaces}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{Borel probability measures}\\ \text{on } \mathbb{R} \text{ which are linear com-}\\ \text{binations of Dirac measures}\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{l}\text{polygones}\\ \text{on } [0,1]\end{array}\right\}, \\ V & \longmapsto & \nu_V & \longmapsto & P_V \end{array}$$

the last mapping being a bijection. If a probability measure ν corresponds to the polygon P, then we can verify that, for any real number $\varepsilon > 0$, the probability measure corresponding to εP is the direct image $T_\varepsilon \nu$ of ν by the dilation mapping $x \mapsto \varepsilon x$.

Let us go back to the convergence of polygons. To verify that a sequence of polygons converges uniformly, it suffice to prove that the corresponding sequence of measures converges vaguely to a probability measure. We state the main theorem of this article.

Theorem 1.1 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $\lim_{n \to +\infty} f(n)/n = 0$ and $B = \bigoplus_{n \geq 0} B_n$ be an integral graded K-algebra of finite type over K. Suppose that*

i) for sufficiently large integer n, the vector space $B_n \neq 0$,

ii) for any positive integer n, B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ which is separated, exhaustive and left continuous, such that B is an f-quasi-filtered graded K-algebra,

iii) $\sup\Big(\operatorname{supp} \nu_{B_n}\Big) = O(n)$.

For any integer $n > 0$, denote by $\nu_n = T_{\frac{1}{n}} \nu_{B_n}$. Then the supports of ν_n are uniformly bounded and the sequence of measures $(\nu_n)_{n \geq 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$. Therefore, the sequence of polygons $(\frac{1}{n} P_{B_n})$ converges uniformly to a concave function on $[0,1]$.

To apply the above theorem to the convergence of $\Big(\frac{1}{D}P_{\pi_*(L^{\otimes D})}\Big)_{D\geq 1}$, we point out that the graded algebra $\bigoplus_{D\geq 0} H^0(X_K, L_K^{\otimes D})$, equipped with Harder-Narasimhan filtrations, verifies the conditions in Theorem 1.1 for a suitable constant function f. The verification of this fact is easy. However, the proof of the theorem requires quite subtle technical arguments on almost super-additive sequences and on combinatorics of monomials, which will be presented in Section 3 and in Section 5 respectively. The idea is to prove that the sequence of measures $(\nu_n)_{n\geq 1}$ is "vaguely super-additive", and then apply a variant of Fekete's lemma to conclude the vague convergence.

We are now able to state our geometric convergence theorem.

Theorem 1.2 *With the notations above, the sequence of polygons* $\Big(\frac{1}{D}P_{\pi_*(L^{\otimes D})}\Big)_{D\geq 1}$ *converges uniformly to a concave function on* $[0,1]$.

The analogue of the formula (1) in Arakelov geometry was firstly proved by Gillet and Soulé [GS92], using their arithmetic Riemann-Roch theorem. Since then, this subject has been rediscussed by many authors such as Abbes and Bouche [AB95], Zhang [Zha95], Rumely, Lau and Varley [RLV00] and Randriambololona [Ran06].

Let K be a number field and $\mathcal{O}_K$ be its algebraic integer ring. We denote by Σ_∞ the set of all embedding of K in $\mathbb{C}$. If $\mathscr{X}$ is a projective arithmetic variety (i.e. scheme of finite type, projective and flat) over $\operatorname{Spec}\mathcal{O}_K$, we call *Hermitian vector bundle* on $\mathscr{X}$ any pair $\overline{\mathscr{E}} = (\mathscr{E}, (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty})$, where $\mathscr{E}$ is a locally free $\mathcal{O}_{\mathscr{X}}$-module of finit type, and $\|\cdot\|_\sigma$ is a continuous Hermitian metric on $\mathscr{E}_\sigma(\mathbb{C})$ such that the collection $(\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$ is invariant under the action of the complex conjugation. We call *Hermitian line bundle* on $\mathscr{X}$ any Hermitian vector bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$ such that $\mathscr{L}$ is of rank one. Notice that a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$ is nothing but the pair $\overline{E} = (E, (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty})$, where E is a projective $\mathcal{O}_K$-module of finite type, and $\|\cdot\|_\sigma$ is a Hermitian metric on $E_\sigma := E\otimes_{\mathcal{O}_K,\sigma}\mathbb{C}$ such that the collection $(\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$ is invariant by the complex conjugation. If $\overline{E}$ is a Hermitian vector bundle of rank r on $\operatorname{Spec}\mathcal{O}_K$, we define its (normalized) *Arakelov degree* to be

$$\widehat{\deg}_n(\overline{E}) := \frac{1}{[K:\mathbb{Q}]}\Big(\log\#\big(E/(\mathcal{O}_K s_1+\cdots+\mathcal{O}_K s_r)\big) - \frac{1}{2}\sum_{\sigma\in\Sigma_\infty}\log\det(\langle s_i,s_j\rangle_\sigma)\Big),$$

where $(s_1,\cdots,s_r)\in E^r$ is an arbitrary element in E^r which defines a base of E_K over K (see [Bos96], [Bos01], [CL02], [Bos04] and [BK07] for details). This is an analogue in Arakelov geometry of the classical notion of degree of a vector bundle on a smooth projective curve. Recall that the *slope* of a non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$ is by definition the quotient $\widehat{\mu}(\overline{E}) := \widehat{\deg}_n(\overline{E})/\operatorname{rk}(E)$. We denote by $\widehat{\mu}_{\max}(\overline{E})$ the maximal value of slopes of non-zero Hermitian subbundle of $\overline{E}$, and by $\widehat{\mu}_{\min}(\overline{E})$ the minimal value of slopes of non-zero Hermitian quotient bundle of $\overline{E}$. Stuhler [Stu76] and Grayson [Gra76] have proved that there exists a non-zero Hermitian subbundle $\overline{E}_{\mathrm{des}}$ of $\overline{E}$ such that $\widehat{\mu}(\overline{E}_{\mathrm{des}}) = \widehat{\mu}_{\max}(\overline{E})$ and that E_{des} contains all Hermitian subbundle of $\overline{E}$ having the maximal slope. We obtain therefore a flag

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$$

of E such that $\overline{E_i}/\overline{E}_{i-1} = (\overline{E}/\overline{E}_{i-1})_{\mathrm{des}}$ for any integer $1\leq i\leq n$, and that

$$\widehat{\mu}_{\max}(\overline{E}) = \widehat{\mu}(\overline{E}_1/\overline{E}_0) > \widehat{\mu}(\overline{E}_2/\overline{E}_1) > \cdots > \widehat{\mu}(\overline{E}_n/\overline{E}_{n-1}) = \widehat{\mu}_{\min}(\overline{E}).$$

The *Harder-Narasimhan polygon* of $\overline{E}$ is by definition the concave function $\widetilde{P}_{\overline{E}}$ defined on $[0,\operatorname{rk}E]$ whose graph is the convex hull of points of the form $(\operatorname{rk}F, \widehat{\deg}_n(\overline{F}))$, where $\overline{F}$ runs over

all non-zero sub-$\mathcal{O}_K$-modules of E equipped with induced metrics. The *normalized Harder-Narasimhan polygon* of $\overline{E}$ is the concave function $P_{\overline{E}}$ defined on $[0,1]$ such that $P_{\overline{E}}(t) = \widetilde{P}_{\overline{E}}(t \operatorname{rk} E)/\operatorname{rk} E$. Notice that we have $P_{\overline{E}}(1) = \widehat{\mu}(\overline{E})$. The measure theory approach in geometric case works without any modification in arithmetic case. Namely, to any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec} \mathcal{O}_K$, we associate a decreasing filtration $\mathcal{F}^{\mathrm{HN}}$ of E_K, called the *Harder-Narasimhan filtration*, such that

$$\mathcal{F}^{\mathrm{HN}}_r E_K = \sum_{\substack{0 \neq F \subset E \\ \widehat{\mu}_{\min}(\overline{F}) \geq r}} F_K.$$

This filtration induces a Borel probability measure $\nu_{\overline{E}}$ on $\mathbb{R}$ such that $\nu_{\overline{E}}([r, +\infty[) = \operatorname{rk}(\mathcal{F}^{\mathrm{HN}}_r E_K)/\operatorname{rk} E$. Finally the normalized Harder-Narasimhan polygon $P_{\overline{E}}$ is uniquely determined by $\nu_{\overline{E}}$.

Using Theorem 1.1, we obtain the following arithmetic convergence theorem.

Theorem 1.3 *Let $\pi : \mathscr{X} \to \operatorname{Spec} \mathcal{O}_K$ be a projective arithmetic variety and $\overline{\mathscr{L}}$ be a Hermitian line bundle on $\mathscr{X}$ such that the graded algebra $\bigoplus_{D \geq 0} H^0(\mathscr{X}_K, \mathscr{L}_K^{\otimes D})$ is of finite type over K, and that $H^0(\mathscr{X}_K, \mathscr{L}_K^{\otimes D}) \neq 0$ for $D \gg 0$. Then the sequence of polygons $(\frac{1}{D} P_{\pi_*(\overline{\mathscr{L}}^{\otimes D})})_{D \geq 1}$ converges uniformly to a concave function on $[0,1]$.*

Contrary to the geometric case, the verification of the fact that the algebra $\bigoplus_{D \geq 0} H^0(\mathscr{X}_K, \mathscr{L}_K^{\otimes D})$ equipped with Harder-Narasimhan filtrations is an f-quasi-filtered graded algebra for a function f of logarithmic increasing speed at infinity is subtle, which depends on the author's recent work [Che07b] on an upper bound of the maximal slope of the tensor product of several Hermitian vector bundles.

The article is organized as follows. In the second section, we introduce the notion of $\mathbb{R}$-filtrations of a vector space over a field and its various properties. We also explain how to associate to each filtered vector space of finite rank a Borel measure on $\mathbb{R}$, which is a probability measure if the vector space is non-zero. The third section is devoted to a generalization of Fekete's lemma on sub-additive sequences, which is useful in sequel. We present the main object of this article — quasi-filtered graded algebras in the fourth section. Then in the fifth section we establish the vague convergence of measures associated to a quasi-filtered symmetric algebra. In the sixth section we explain how to construct the polygon associated to a Borel probability measure which is a linear combination of Dirac measures. We show that the vague convergence of probability measures implies the uniform convergence of associated polygons. Combining the results obtained in previous sections, we establish in the seventh section the uniform convergence of polygons associated to a general quasi-filtered graded algebra. In the eighth and the ninth sections we apply the general result in the seventh section to relative geometric framework and to Arakelov geometric framework respectively to obtain the corresponding convergence of Harder-Narasimhan polygons. Finally in the tenth section, we propose another approach, inspired by Faltings and Wüstholz [FW94], to calculate explicitly the limit of the polygons. We conclude by providing an explicit example where the limit of the polygons is a non-trivial quadratic curve on $[0,1]$. In the appendix, we develop a variant of f-quasi-filtered graded algebra — f-pseudo-filtered graded algebra, where we require less algebraic conditions. With a stronger condition on the increment of f, we also obtain the convergence of polygons. Although this approach has not been used in this article, it may have applications elsewhere and therefore we include it as well.

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author to study the convergence of Harder-Narasimhan polygons, for his encouragement and for discussions.

2 Filtrations of vector spaces

We present some basic definition and properties of filtrations of vector spaces. Although the notion of filtrations in a general category has been discussed in [Che07a], we would like to introduce it in an explicit way for the particular case of vector spaces.

We fix in this section a field K. A (decreasing) $\mathbb{R}$-*filtration* of a vector space V is by definition a collection $\mathcal{F} = (\mathcal{F}(r))_{r\in\mathbb{R}}$ of K-vector subspaces of V such that $\mathcal{F}(r) \supset \mathcal{F}(r')$ if $r \leq r'$. We shall use the expression $\mathcal{F}_r V$ to denote $\mathcal{F}(r)$, or simply V_r if there is no ambiguity on the filtration $\mathcal{F}$. An $\mathbb{R}$-filtration $\mathcal{F}$ is said to be *separated* if $V_{+\infty} := \bigcap_{r\in\mathbb{R}} V_r = \{0\}$, and to be *exhaustive* if $V_{-\infty} := \bigcup_{r\in\mathbb{R}} V_r = V$.

Let V be a vector space over K, $\mathcal{F}$ be an $\mathbb{R}$-filtration of V. For any element $x \in V$, we call *index* of x *relatively to* $\mathcal{F}$ the element $\sup\{r \in \mathbb{R} \mid x \in \mathcal{F}_r V\}$ in $\mathbb{R} \cup \{\pm\infty\}$ (by convention $\sup \emptyset = -\infty$), denoted by $\lambda_{\mathcal{F}}(x)$, or simply $\lambda(x)$ if there is no ambiguity on $\mathcal{F}$. The mapping $\lambda_{\mathcal{F}} : V \to \mathbb{R} \cup \{\pm\infty\}$ is called the *index function* of $\mathcal{F}$.

Let x be an element in V. The set $\{r \in \mathbb{R} \mid x \in \mathcal{F}_r V\}$ is non-empty if and only if $\lambda(x) > -\infty$. In this case, it is either of the form $]-\infty, \lambda(x)[$ or of the form $]-\infty, \lambda(x)]$. The following properties of the function λ are easy to verify:

1) $\lambda(x) = +\infty$ if and only if $x \in V_{+\infty}$,

2) $\lambda(x) = -\infty$ if and only if $x \in V \setminus V_{-\infty}$,

3) $\lambda(x) > r$ if and only if $x \in \bigcup_{s>r} V_s$,

4) $\lambda(x) \geq r$ if and only if $x \in \bigcap_{s<r} V_s$.

We say that an $\mathbb{R}$-filtration $\mathcal{F}$ of V is *left continuous* if and only if for any $r \in \mathbb{R}$, $\mathcal{F}_r V = \bigcap_{s<r} \mathcal{F}_s V$. If $\mathcal{F}$ is an arbitrary filtration of V, then the filtration $\mathcal{F}^l = (\bigcap_{s<r} \mathcal{F}_s V)_{r\in\mathbb{R}}$ is a left continuous filtration of V.

For any element $x \in V$ and any $r \in \mathbb{R} \cup \{+\infty\}$, the fact that $x \in \mathcal{F}_r V$ implies $\lambda(x) \geq r$. The converse is true when $\mathcal{F}$ is left continuous.

Proposition 2.1 *Let V be a vector space over K and $\mathcal{F}$ be a filtration of V. The following assertions hold:*

1) if $a \in K^\times$ and if $x \in V$, then $\lambda(ax) = \lambda(x)$,

2) if x and y are two elements of V, then $\lambda(x+y) \geq \min(\lambda(x), \lambda(y))$,

3) if x and y are two elements of V such that $\lambda(x) \neq \lambda(y)$, then $x + y \neq 0$, and $\lambda(x+y) = \min(\lambda(x), \lambda(y))$,

4) if the rank of V is finite, then the image of λ is a finite subset of $\mathbb{R} \cup \{\pm\infty\}$ whose cardinal is bounded from above by $\mathrm{rk}_K V + 1$.

Proof. 1) For any $a \in K^\times$, $x \in \mathcal{F}_r V$ if and only if $ax \in \mathcal{F}_r V$. So $\{r \in \mathbb{R} \mid x \in \mathcal{F}_r V\} = \{r \in \mathbb{R} \mid ax \in \mathcal{F}_r V\}$, which implies that $\lambda(x) = \lambda(ax)$.

2) In fact, $\{t \mid x+y \in \mathcal{F}_t V\} \supset \{r \mid x \in \mathcal{F}_r V\} \cap \{s \mid y \in \mathcal{F}_s V\}$. Therefore $\sup\{t \mid x+y \in \mathcal{F}_t V\} \geq \min(\sup\{r \mid x \in \mathcal{F}_r V\}, \sup\{s \mid y \in \mathcal{F}_s V\})$.

3) If $x+y=0$, then $x=-y$. So $\lambda(x)=\lambda(y)$ by 1), which leads to a contradiction. Hence $x+y \neq 0$. We may suppose that $\lambda(x) < \lambda(y)$. For any $r \in]\lambda(x), \lambda(y)[$, we have $y \in \mathcal{F}_r V$ but $x \notin \mathcal{F}_r V$. Therefore $x+y \notin \mathcal{F}_r V$, in other words, $\lambda(x+y) \leq r$. Since r is arbitrary, we obtain $\lambda(x+y) \leq \lambda(x)$. Combining with 2), we get the equality.

4) Suppose that $x_1, \cdots, x_n$ are non-zero elements in V such that $\lambda(x_1) < \lambda(x_2) < \cdots < \lambda(x_n) < +\infty$. After 1) and 3), for any $(a_i)_{1 \leq i \leq n} \in K^n \setminus \{0\}$,

$$\lambda(a_1 x_1 + \cdots + a_n x_n) = \min\{\lambda(x_i) \mid a_i \neq 0\} < +\infty,$$

which implies that $a_1 x_1 + \cdots + a_n x_n \neq 0$. Therefore, $x_1, \cdots, x_n$ are linearly independent. So $n \leq \mathrm{rk}_K V$. □

Using the index function λ, we give some numerical characterizations for filtrations of vector spaces.

Proposition 2.2 *Let V be a vector space over K equipped with an $\mathbb{R}$-filtration $\mathcal{F}$. Then*

1) the filtration $\mathcal{F}$ is separated if and only if for any $x \in V \setminus \{0\}$, $\lambda(x) < +\infty$,

2) the filtration $\mathcal{F}$ is exhaustive if and only if for any $x \in V$, $\lambda(x) > -\infty$.

Proof. 1) If the filtration is separated, then for any non-zero element x of V, there exists $r \in \mathbb{R}$ such that $x \notin \mathcal{F}_r V$, so $\lambda(x) \leq r$. Conversely for any non-zero element $x \in V$ such that $\lambda(x) < +\infty$, if $\lambda(x) \in \mathbb{R}$, then $x \notin \mathcal{F}_{\lambda(x)+1} V$, otherwise $\lambda(x) = -\infty$ and by definition $x \notin \mathcal{F}_r V$ for every $r \in \mathbb{R}$.

2) If the filtration is exhaustive, then for any element x of V, there exists $r \in \mathbb{R}$ such that $x \in \mathcal{F}_r V$. Hence $\lambda(x) \geq r$. Conversely for any element $x \in V$ such that $\lambda(x) > -\infty$, either we have $\lambda(x) \in \mathbb{R}$, and therefore $x \in \mathcal{F}_{\lambda(x)-1} V$, or we have $\lambda(x) = +\infty$ and $x \in \mathcal{F}_r V$ for every $r \in \mathbb{R}$. □

Proposition 2.3 *Let V be a vector space over K and $\mathcal{F}$ be a filtration of V.*

1) For any element x of V, we have $\lambda_{\mathcal{F}}(x) = \lambda_{\mathcal{F}^l}(x)$.

2) If $\mathcal{F}$ is separated (resp. exhaustive), then also is $\mathcal{F}^l$.

Proof. 1) Since $\mathcal{F}_r V \subset \mathcal{F}^l_r V$, we have $\lambda_{\mathcal{F}}(x) \leq \lambda_{\mathcal{F}^l}(x)$. On the other hand, if $x \in \mathcal{F}^l_r V$, then for any $s < r$, we have $x \in \mathcal{F}_s V$, so $\lambda_{\mathcal{F}}(x) \geq r$. Hence $\lambda_{\mathcal{F}}(x) \geq \lambda_{\mathcal{F}^l}(x)$.

2) It's an easy consequence of 1) and Proposition 2.2. □

Consider now two vector spaces V and W over K. Let $\mathcal{F}$ be an $\mathbb{R}$-filtration of V and $\mathcal{G}$ be an $\mathbb{R}$-filtration of W. We say that a linear mapping $f : V \to W$ is *compatible* with the filtrations $(\mathcal{F}, \mathcal{G})$ if for any $r \in \mathbb{R}$, $f(\mathcal{F}_r V) \subset \mathcal{G}_r W$.

We introduce some functorial construction of filtrations. Let $f : V \to W$ be a K-linear mapping of vector spaces over K. If $\mathcal{G}$ is an $\mathbb{R}$-filtration of W, then the *inverse image* of $\mathcal{G}$ by f is by definition the filtration $f^*\mathcal{G}$ of V such that $(f^*\mathcal{G})_r V = f^{-1}(\mathcal{G}_r W)$. Clearly, if $\mathcal{G}$ is left continuous, then also is $f^*\mathcal{G}$. If $\mathcal{F}$ is an $\mathbb{R}$-filtration of V, the *weak direct image* of $\mathcal{F}$ by

f is by definition the filtration $f_\flat\mathcal{F}$ of W such that $(f_\flat\mathcal{F})_rW = f(\mathcal{F}_rV)$, and the *strong direct image* of $\mathcal{F}$ by f is by definition the filtration $f_*\mathcal{F} = (f_\flat\mathcal{F})^l$. Clearly the homomorphism f is compatible to filtrations $(f^*\mathcal{G}, \mathcal{G})$, $(\mathcal{F}, f_\flat\mathcal{F})$ and $(\mathcal{F}, f_*\mathcal{F})$.

Proposition 2.4 *If a K-linear mapping $f : V \to W$ is compatible with the filtrations $(\mathcal{F}, \mathcal{G})$, then for any $x \in V$, one has $\lambda(f(x)) \geq \lambda(x)$. The converse is true if $\mathcal{G}$ is left continuous.*

Proof. "$\Longrightarrow$": By definition we know that $\{r \in \mathbb{R} \mid x \in \mathcal{F}_rV\} \subset \{r \in \mathbb{R} \mid f(x) \in \mathcal{G}_rW\}$ for any $x \in V$, therefore $\lambda(x) \leq \lambda(f(x))$.

"$\Longleftarrow$": For any $r \in \mathbb{R}$ and any $x \in \mathcal{F}_rV$, we have $\lambda(x) \geq r$, and hence $\lambda(f(x)) \geq r$. Therefore, $f(x) \in \mathcal{G}_rW$ since the filtration $\mathcal{G}$ is left continuous. □

Proposition 2.5 *Let $f : V' \to V$ be an injective homomorphism and $\pi : V \to V''$ be a surjective homomorphism of vector spaces over K. Suppose that $\mathcal{F}$ is an $\mathbb{R}$-filtration of V. Then:*

1) if $\mathcal{F}$ is separated, also is $f^*\mathcal{F}$;

2) if $\mathcal{F}$ is separated and if the rank of V is finite, the filtration $\pi_\flat\mathcal{F}$ is also separated;

3) if $\mathcal{F}$ is exhaustive, the filtrations $f^*\mathcal{F}$, $\pi_\flat\mathcal{F}$ and $\pi_*\mathcal{F}$ are all exhaustive.

Proof. 1) As $\mathcal{F}$ is separated, $\bigcap\limits_{r\in\mathbb{R}} \mathcal{F}_rV = \{0\}$. Since f is injectif, we have

$$\bigcap_{r\in\mathbb{R}} (f^*\mathcal{F})_rV' = \bigcap_{r\in\mathbb{R}} f^{-1}(\mathcal{F}_rV) = f^{-1}\Big(\bigcap_{r\in\mathbb{R}} \mathcal{F}_rV\Big) = f^{-1}(\{0\}) = \{0\}.$$

Therefore $f^*\mathcal{F}$ is also separated.

2) If $\operatorname{rk} E < +\infty$, then $\lambda_\mathcal{F}$ takes only a finite number of values. Let $r_0 = \sup\Big(\lambda_\mathcal{F}(E) \setminus \{\pm\infty\}\Big) < +\infty$. For any real number $r > r_0$ and any $x \in \mathcal{F}_rV$ we have $\lambda_\mathcal{F}(x) \geq r > r_0$, so $\lambda_\mathcal{F}(x) = +\infty$, i.e., $x = 0$ since the filtration $\mathcal{F}$ is separated. Therefore, $\mathcal{F}_rV = 0$ and $(\pi_\flat\mathcal{F})_rV'' = \pi(\mathcal{F}_rV) = 0$.

3) Since the filtration $\mathcal{F}$ is exhaustive, we have $\bigcup\limits_{r\in\mathbb{R}} \mathcal{F}_rV = V$. Therefore,

$$\bigcup_{r\in\mathbb{R}} (f^*\mathcal{F})_rV' = \bigcup_{r\in\mathbb{R}} (V' \cap \mathcal{F}_rV) = V' \cap \Big(\bigcup_{r\in\mathbb{R}} \mathcal{F}_rV\Big) = V' \cap V = V',$$

$$\bigcup_{r\in\mathbb{R}} (\pi_\flat\mathcal{F})_rV'' = \bigcup_{r\in\mathbb{R}} \pi(\mathcal{F}_rV) = \pi\Big(\bigcup_{r\in\mathbb{R}} \mathcal{F}_rV\Big) = \pi(V) = V''.$$

So the filtrations $f^*\mathcal{F}$ and $\pi_\flat\mathcal{F}$ are exhaustive. Finally, after Proposition 2.3 2), $\pi_*\mathcal{F} = (\pi_\flat\mathcal{F})^l$ is exhaustive. □

The following proposition gives index description of functorial constructions of filtrations.

Proposition 2.6 *Let V and W be two finite dimensional vector spaces over K, $\mathcal{F}$ be an $\mathbb{R}$-filtration of V, $\mathcal{G}$ be an $\mathbb{R}$-filtration of W and $\varphi : V \to W$ be a K-linear mapping.*

1) Suppose that φ is injective. If $\mathcal{F} = \varphi^*\mathcal{G}$, then for any $x \in V$, one has $\lambda_\mathcal{F}(x) = \lambda_\mathcal{G}(\varphi(x))$. The converse is true if both filtrations $\mathcal{F}$ and $\mathcal{G}$ are left continuous.

2) Suppose that φ is surjective. If $\mathcal{G} = \varphi_*\mathcal{F}$, then for any $y \in W$, $\lambda_{\mathcal{G}}(y) = \sup_{x\in\varphi^{-1}(y)} \lambda_{\mathcal{F}}(x)$.

The converse is true if both filtrations $\mathcal{F}$ and $\mathcal{G}$ are left continuous.

Proof. 1) "$\Longrightarrow$": Since $\mathcal{F} = \varphi^*\mathcal{G}$, a non-zero element $x \in V$ lies in V_λ if and only if $\varphi(x) \in W_\lambda$, hence $\lambda(x) = \sup\{r \in \mathbb{R} \mid x \in W_r\} = \sup\{r \in \mathbb{R} \mid \varphi(x) \in V_r\} = \lambda(\varphi(x))$.

"$\Longleftarrow$": If $x \in V_r$, then $\lambda(\varphi(x)) \geq \lambda(x) \geq r$. So $\varphi(x) \in W_r$ since the filtration $\mathcal{G}$ is left continuous. On the other hand, if $0 \neq x \in \varphi^{-1}(W_r)$, then $\lambda(x) = \lambda(\varphi(x)) \geq r$, so $x \in V_r$ since the filtration $\mathcal{F}$ of V is left continuous. Therefore $V_r = \varphi^{-1}(W_r)$.

2) "$\Longrightarrow$": If $x \in V_r$, then $\varphi(x) \in W_r$, so $\lambda(\varphi(x)) \geq \lambda(x)$. Hence for any $y \in W \setminus \{0\}$, $\lambda(y) \geq \sup_{x\in\varphi^{-1}(y)} \lambda(x)$. On the other hand, $y \in W_r$ implies that $V_s \cap \varphi^{-1}(y) \neq \emptyset$ for any $s < r$. Therefore $r \leq \sup_{x\in\varphi^{-1}(y)} \lambda(x)$, and hence $\lambda(y) = \sup\{r \in \mathbb{R} \mid y \in W_r\} \leq \sup_{x\in\varphi^{-1}(y)} \lambda(x)$.

"$\Longleftarrow$": For any non-zero element y of W, if $y \in W_r$, then $\lambda(y) \geq r$, so $\sup_{x\in\varphi^{-1}(y)} \lambda(x) \geq r$. Therefore, for any $s < r$, there exists $x \in \varphi^{-1}(y)$ such that $\lambda(x) \geq s$. Since the filtration $\mathcal{F}$ is left continuous, we have $x \in V_s$. This implies $y \in \bigcap_{s<r} \varphi(V_s)$.

On the other hand, if y is a non-zero element in $\varphi(V_s)$, then there exists $x \in V_s$ such that $y = \varphi(x)$. So $\lambda(y) \geq \lambda(x) \geq s$. This implies that $y \in W_s$ since the filtration $\mathcal{G}$ is left continuous. Therefore, $\bigcap_{s<r} \varphi(V_s) \subset \bigcap_{s<r} W_s = W_r$. □

In the following, we use Borel measures on $\mathbb{R}$ to study $\mathbb{R}$-filtrations of vector spaces. For any finite dimensional vector space over K, equipped with a separated and exhaustive filtration, we shall associate a Borel probability measure on $\mathbb{R}$ to each base of the vector space, which is a linear combination of Dirac measures. Furthermore, there exists a "maximal base" whose associated measure captures full "numerical" information of the filtration. This technic will play an import role in the sequel.

If ν_1 and ν_2 are two bounded Borel measures on $\mathbb{R}$, we say that ν_1 is *on the right* of ν_2 and we write $\nu_1 \succ \nu_2$ or $\nu_2 \prec \nu_1$ if for any increasing and bounded function f, we have

$$\int_{\mathbb{R}} f \mathrm{d}\nu_1 \geq \int_{\mathbb{R}} f \mathrm{d}\nu_2,$$

which is also equivalent to say that for any $r \in \mathbb{R}$, $\int_{\mathbb{R}} \mathbb{1}_{[r,+\infty[} \mathrm{d}\nu_1 \geq \int_{\mathbb{R}} \mathbb{1}_{[r,+\infty[} \mathrm{d}\nu_2$. We say that ν_1 is *strictly on the right* of ν_2 if $\nu_1 \succ \nu_2$ but $\nu_2 \neq \nu_1$.

Definition 2.7 Let V be a vector space of rank $0 < n < +\infty$ over K, equipped with a separated and exhaustive filtration $\mathcal{F}$. If $\mathbf{e} = (e_i)_{1\leq i\leq n}$ is a base of V, we define a Borel probability measure on $\mathbb{R}$

$$\nu_{\mathcal{F},\mathbf{e}} := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda(e_i)},$$

called the *probability associated to $\mathcal{F}$ relatively to* $\mathbf{e}$. If there is no ambiguity on the filtration, we write also $\nu_{\mathbf{e}}$ instead of $\nu_{\mathcal{F},\mathbf{e}}$. Notice that Proposition 2.3 implies that $\nu_{\mathcal{F},\mathbf{e}} = \nu_{\mathcal{F}^l,\mathbf{e}}$. We say that a base $\mathbf{e}$ of V is *maximal* if for any base $\mathbf{e}'$ of V, we have $\nu_{\mathbf{e}} \succ \nu_{\mathbf{e}'}$. Clearly a base $\mathbf{e}$ is maximal for the filtration $\mathcal{F}$ if and only if it is maximal for the filtration $\mathcal{F}^l$.

Proposition 2.8 *Suppose that the filtration $\mathcal{F}$ of V is left continuous. Then a base $\mathbf{e} = (e_i)_{1\leq i\leq n}$ of V is maximal if and only if* $\mathrm{card}(\mathbf{e} \cap V_r) = \mathrm{rk}\, V_r$ *for any real number r.*

Proof. "$\Longrightarrow$": Since $\mathbf{e}$ is a base of V, we have $\mathrm{card}(\mathbf{e}\cap V_r)\leq \mathrm{rk}\, V_r$ for any $r\in\mathbb{R}$. Suppose that there exists a real number r such that $\mathrm{card}(\mathbf{e}\cap V_r)<\mathrm{rk}\, V_r$. Let V_r' be the sub-vector space of V_r generated by $\mathbf{e}\cap V_r$. We have clearly $\mathrm{rk}\, V_r'<\mathrm{rk}\, V_r$. Hence there exists $e'\in V_r\setminus V_r'$. Since $\mathbf{e}$ is a base of V, there exists $(a_i)_{1\leq i\leq n}\in K^n$ such that $e'=a_1e_1+\cdots+a_ne_n$. As $e'\notin V_r'$, there exists an integer $1\leq i\leq n$ such that $e_i\notin V_r$ and that $a_i\neq 0$. Therefore

$$\mathbf{e}'=(e_1,\cdots,e_{i-1},e',e_{i+1},\cdots,e_n)$$

is a base of V. Furthermore, as $e_i\notin V_r$, we have $\lambda(e_i)<r$ since the filtration is left continuous. On the other hand, since $e'\in V_r$, we have $\lambda(e')\geq r$. Let g be an increasing function such that $g(\lambda(e_i))<g(\lambda(e'))$. Then we have

$$\int_{\mathbb{R}}g\mathrm{d}\nu_{\mathbf{e}'}-\int_{\mathbb{R}}g\mathrm{d}\nu_{\mathbf{e}}=\frac{1}{n}\Big(g(\lambda(e'))-g(\lambda(e_i))\Big)>0,$$

which is absurd since $\mathbf{e}$ is maximal.

"$\Longleftarrow$": For any real number r and any base $\mathbf{e}'$ of V, we have $\mathrm{card}(\mathbf{e}'\cap V_r)=n\displaystyle\int_{\mathbb{R}}\mathbb{1}_{[r,+\infty[}\,\mathrm{d}\nu_{\mathbf{e}'}$. Hence for any real number r, we have $\displaystyle\int_{\mathbb{R}}\mathbb{1}_{[r,+\infty[}\,\mathrm{d}\nu_{\mathbf{e}'}\leq\int_{\mathbb{R}}\mathbb{1}_{[r,+\infty[}\,\mathrm{d}\nu_{\mathbf{e}}$. Therefore, $\nu_{\mathbf{e}'}\prec\nu_{\mathbf{e}}$. □

Proposition 2.9 *For any base $\mathbf{e}=(e_1,\cdots,e_n)^T$ of V, there exists an upper triangulated matrix $A\in M_{n\times n}(K)$ with $\mathrm{diag}(A)=(1,\cdots,1)$ such that $A\mathbf{e}$ is a maximal base of V.*

Proof. We may suppose that the filtration $\mathcal{F}$ is left continuous: it suffices to replace it by $\mathcal{F}^l$. We shall prove the proposition by induction on the rank n of V. If $n=1$, then

$$V_r=\begin{cases}V, & r\leq\lambda(e_1),\\ 0, & r>\lambda(e_1).\end{cases}$$

Hence $\mathrm{card}(V_r\cap\{e_1\})=\mathrm{rk}\, V_r$. In other words, $\mathbf{e}$ is already a maximal base.

Suppose that $n>1$. Let W be the quotient space V/Ke_n, equipped with the strong direct image filtration. Then $\widetilde{\mathbf{e}}=([e_1],\cdots,[e_{n-1}])^T$ is a base of W, where $[e_i]$ is the canonical image of e_i in W $(1\leq i\leq n-1)$. By the hypothesis of induction, there exists $\widetilde{A}\in M_{(n-1)\times(n-1)}(K)$ with $\mathrm{diag}(\widetilde{A})=(1,\cdots,1)$ such that $\overrightarrow{\alpha}=(\alpha_1,\cdots,\alpha_{n-1}):=\widetilde{A}\widetilde{\mathbf{e}}$ is a maximal base.

Let $\pi:V\to W$ be the canonical projection. For any $1\leq i\leq n-1$, choose $e_i'\in\pi^{-1}(\alpha_i)$ such that $\lambda(e_i')=\max\limits_{x\in\pi^{-1}(\alpha_i)}\lambda(x)$. This is always possible since the function λ takes only a finite number of values. Let $\mathbf{e}'=(e_1',\cdots,e_{n-1}',e_n)^T$. Notice that $\mathbf{e}'$ can be written as $A\mathbf{e}$, where

$$A=\begin{pmatrix}\widetilde{A} & *\\ 0 & 1\end{pmatrix}$$

is an upper triangulated matrix with diagonal $\mathrm{diag}(A)=(1,\cdots,1)$. Since $\overrightarrow{\alpha}$ is a maximal base, $\mathrm{card}(W_r\cap\overrightarrow{\alpha})=\mathrm{rk}\, W_r$ for any $r\in\mathbb{R}$. In addition, $e_i'\in V_r$ implies that $\alpha_i=\pi(e_i')\in W_r$. Hence

$$\mathrm{card}(\mathbf{e}'\cap V_r)\geq\begin{cases}\mathrm{rk}\, W_r\geq\mathrm{rk}\,\pi(V_r)=\mathrm{rk}\, V_r, & e_n\notin V_r,\\ \mathrm{rk}\, W_r+1\geq\mathrm{rk}\,\pi(V_r)+1=\mathrm{rk}\, V_r, & e_n\in V_r.\end{cases}$$

So we always have $\mathrm{card}(V_r\cap\mathbf{e}')=\mathrm{rk}\, V_r$, and hence $\mathbf{e}'$ is a maximal base. □

Proposition 2.9 can also be proved in the following way: the set X of complete flags of V is equipped with a transitive action of $\mathrm{GL}_n(K)$ and identifies with the homogeneous space $\mathrm{GL}_n(K)/B$, where B is the subgroup of upper triangulated matrices. The proposition is then a consequence of Bruhat's decomposition for invertible matrices.

Remark 2.10 Proposition 2.9 implies actually that there always exists a maximal base of V.

Definition 2.11 If $\mathbf{e}$ is a maximal base of V, the measure $\nu_{\mathcal{F},\mathbf{e}}$ is called the (probability) *measure* associated to $\mathcal{F}$.

It is clear that the measure associated to $\mathcal{F}$ doesn't depend on the choice of the maximal base $\mathbf{e}$, we shall denote it by $\nu_{\mathcal{F},V}$ (or simply ν_V if there is no ambiguity on $\mathcal{F}$). If V is the zero space, then ν_V is by convention the zero measure.

Let V be a finite dimensional vector space over K. A left continuous $\mathbb{R}$-filtration $\mathcal{F}$ of V is equivalent to the data of a flag $V^{(0)} \subsetneq V^{(1)} \subsetneq \cdots \subsetneq V^{(n)}$ together with a strictly decreasing real number sequence $(a_i)_{1\leq i\leq n}$, which describes the jumping points. We have

$$\mathcal{F}_r V = \begin{cases} V^{(0)} & \text{if } r \in]a_1, +\infty[, \\ V^{(i)} & \text{if } r \in]a_{i+1}, a_i], \quad 1 \leq i < n, \\ V^{(n)} & \text{if } r \in]-\infty, a_n]. \end{cases}$$

The filtration $\mathcal{F}$ is separated (resp. exhaustive) if and only if $V^{(0)} = \{0\}$ (resp. $V^{(n)} = V$). When $\mathcal{F}$ is separated and exhaustive, the measure associated to $\mathcal{F}$ equals to

$$\sum_{i=1}^{n} \frac{\operatorname{rk} V^{(i)} - \operatorname{rk} V^{(i-1)}}{\operatorname{rk} V} \delta_{a_i}.$$

Therefore, if V is non-zero, then for any $x \in \mathbb{R}$, we have the equality

$$1 - \frac{\operatorname{rk} V_x}{\operatorname{rk} V} = \nu_V\big(]-\infty, x[\big).$$

The probability distribution function of ν_V is therefore

$$F(x) = 1 - \lim_{y \to x+} \frac{\operatorname{rk} V_y}{\operatorname{rk} V}.$$

Proposition 2.12 *Let* $0 \longrightarrow V' \xrightarrow{\varphi} V \xrightarrow{\psi} V'' \longrightarrow 0$ *be a short exact sequence of finite dimensional vector spaces over* K *equipped with left continuous* $\mathbb{R}$*-filtrations. Suppose that the following conditions are verified:*

i) the space V *is non-zero and the filtration* $\mathcal{F}$ *of* V *is separated and exhaustive;*

ii) the filtration of V' *is the inverse image* $\varphi^*\mathcal{F}$*;*

iii) the filtration of V'' *is the strong direct image* $\psi_*\mathcal{F}$*.*

Then $\nu_V = \dfrac{\operatorname{rk} V'}{\operatorname{rk} V}\nu_{V'} + \dfrac{\operatorname{rk} V''}{\operatorname{rk} V}\nu_{V''}$.

Proof. If W is a finite dimensional vector space over K equipped with an $\mathbb{R}$-filtration, the filtration of W is separated and exhaustive if and only if the function $\lambda : W \setminus \{0\} \to \mathbb{R} \cup \{\pm\infty\}$ takes values in a bounded interval in $\mathbb{R}$ (see Proposition 2.1 4) and Proposition 2.2). Therefore, after Proposition 2.5, if $\mathcal{F}$ is separated and exhaustive, then also are $\varphi^*\mathcal{F}$ and $\psi_*\mathcal{F}$. So the measures $\nu_{V'}$ and $\nu_{V''}$ are well defined.

Let $\mathbf{e}' = (e'_i)_{1\leq i\leq n}$ (resp. $\mathbf{e}'' = (e''_j)_{1\leq j\leq m}$) be a maximal base of V' (resp. V''). Let

$$\mathbf{e} = (\varphi(e'_1), \cdots, \varphi(e'_n), e_{n+1}, \cdots, e_{n+m})$$

be a base of V such that, for any integer $1 \leq j \leq m$, $\psi(e_{n+j}) = e''_j$ and $\lambda(e_{n+j}) = \lambda(e''_j)$ (this is always possible after Proposition 2.1 4) and Proposition 2.6 2)). By definition we know that

$$\nu_{\mathbf{e}} = \frac{\operatorname{rk} V'}{\operatorname{rk} V}\nu_{\mathbf{e}'} + \frac{\operatorname{rk} V''}{\operatorname{rk} V}\nu_{\mathbf{e}''}.$$

It suffices then to verify that $\mathbf{e}$ is a maximal base.

Let r be a real number. First we have

$$\operatorname{card}(\{\varphi(e'_1), \cdots \varphi(e'_n)\} \cap V_r) = \operatorname{card}(\mathbf{e}' \cap V'_r) = \operatorname{rk} V'_r. \tag{3}$$

On the other hand, since $\lambda(e''_j) = \lambda(e_{n+j})$, $e''_j \in V''_r$ if and only if $e_{n+j} \in V_r$. Therefore

$$\operatorname{card}(\{e_{n+1}, \cdots, e_{n+m}\} \cap V_r) = \operatorname{card}(\mathbf{e}'' \cap V''_r) = \operatorname{rk} V''_r. \tag{4}$$

The sum of the inequalities (3) and (4) gives $\operatorname{card}(\mathbf{e} \cap V_r) = \operatorname{rk}(V'_r) + \operatorname{rk}(V''_r) = \operatorname{rk}(V_r)$, so $\mathbf{e}$ is a maximal base. □

3 Almost super-additive sequence

In this section we discuss a generalization of Fekete's lemma (see [Fek23] page 233 for a particular case) asserting that, for any sub-additive sequence $(a_n)_{n\geq 1}$ of real numbers (that's to say, $a_{n+m} \leq a_n + a_m$ for any $(m,n) \in \mathbb{Z}^2_{>0}$), the limit $\lim_{n\to+\infty} a_n/n$ exists in $\mathbb{R} \cup \{-\infty\}$. We shall show that the convergence of the sequence $(a_n/n)_{n\geq 1}$ is still valid if the sequence $(a_n)_{n\geq 1}$ is sub-additive up to a small error term. These technical results are crucial to prove the convergence theorems stated in the section of introduction.

Proposition 3.1 *Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}_{\geq 0}$ and $f : \mathbb{Z}_{>0} \to \mathbb{R}$ be a function such that $\lim_{n\to\infty} f(n)/n = 0$. If there exists an integer $n_0 > 0$ such that, for any integer $l \geq 2$ and any $(n_i)_{1\leq i\leq l} \in \mathbb{Z}^l_{\geq n_0}$, we have $a_{n_1+\cdots+n_l} \leq a_{n_1} + \cdots + a_{n_l} + f(n_1) + \cdots + f(n_l)$, then the sequence $(a_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}_{\geq 0}$.*

Proof. If n, p and $n \leq l < 2n$ are three integers $\geq n_0$, we have

$$\begin{aligned}\frac{a_{pn+l}}{pn+l} &\leq \frac{pa_n + a_l}{pn+l} + \frac{pf(n)+f(l)}{pn+l} \leq \frac{a_n}{n} + \frac{a_l}{pn} + \frac{pf(n)+f(l)}{pn+l} \\ &\leq \frac{a_n}{n} + \frac{a_l}{pn} + \frac{|f(n)|}{n} + \frac{|f(l)|}{pn}.\end{aligned}$$

Since $\lim_{p\to\infty} \dfrac{\max_{n\leq i<2n} a_i}{pn} + \dfrac{\max_{n\leq i<2n} |f(i)|}{pn} = 0$, we obtain, for any integer $n > 0$, that

$$\limsup_{m\to\infty} \frac{a_m}{m} \leq \frac{a_n}{n} + \frac{|f(n)|}{n}, \tag{5}$$

hence

$$\limsup_{m\to\infty}\frac{a_m}{m}\leq\liminf_{n\to\infty}\left(\frac{a_n}{n}+\frac{|f(n)|}{n}\right)\leq\liminf_{n\to\infty}\frac{a_n}{n}+\limsup_{n\to\infty}\frac{|f(n)|}{n}=\liminf_{n\to\infty}\frac{a_n}{n}.$$

Therefore, the sequence $(a_n/n)_{n\geq 1}$ has a limit, which is clearly ≥ 0, and is finite after (5). □

Corollary 3.2 *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $f:\mathbb{Z}_{>0}\to\mathbb{R}$ be a function such that $\lim_{n\to\infty} f(n)/n=0$. If the following two conditions are verified:*

1) there exists an integer $n_0>0$ such that, for any integer $l\geq 2$ and any $(n_i)_{1\leq i\leq l}\in\mathbb{Z}^l_{\geq n_0}$, we have $a_{n_1+\cdots+n_l}\geq a_{n_1}+\cdots+a_{n_l}-f(n_1)-\cdots-f(n_l)$,

2) there exists a constant $\alpha>0$ such that $a_n\leq\alpha n$ for any integer $n\geq 1$,

then the sequence $(a_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$.

Proof. Consider the sequence $(b_n=\alpha n-a_n)_{n\geq 1}$ of positive real numbers. If $n_1,\cdots,n_l$ are integers $\geq n_0$ and $n=n_1+\cdots+n_l$, then

$$b_n=\alpha n-a_n=\alpha\sum_{i=1}^{l}n_i-a_n\leq\alpha\sum_{i=1}^{l}n_i-\sum_{i=1}^{l}\Big(a_{n_i}-f(n_i)\Big)$$

$$=\sum_{i=1}^{l}\Big(\alpha n_i-a_{n_i}+f(n_i)\Big)=b_{n_1}+\cdots+b_{n_l}+f(n_1)+\cdots+f(n_l).$$

After Proposition 3.1, the sequence $(b_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$. As $b_n/n=\alpha-a_n/n$, the sequence $(a_n/n)_{n\geq 1}$ also has a limit in $\mathbb{R}$. □

Corollary 3.3 *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and c_1, c_2 be two positive constants such that*

1) $a_{m+n}\geq a_m+a_n-c_1$ for any pair (m,n) of sufficiently large integers,

2) $a_n\leq c_2 n$ for any integer $n\geq 1$,

then the sequence $(a_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$.

Proof. Let f be the constant function taking value c_1. By induction we obtain the following inequality for any finite sequence $(n_i)_{1\leq i\leq l}$ of sufficiently large integers:

$$a_{n_1+\cdots+n_l}\geq a_{n_1}+\cdots+a_{n_l}-f(n_1)-\cdots-f(n_l),$$

After Corollary 3.2, the sequence $(a_n/n)_{n\geq 1}$ converges in $\mathbb{R}$. □

4 Quasi-filtered graded algebras

In this section we introduce the notion of *quasi-filtered graded algebras*. Such algebras are fundamental objects in this article. We are particularly interested in the convergence of measures associated to a quasi-filtered graded algebra (Sections 5 to 7). Later we shall show that the graded algebras that we have mentioned in the section of introduction, equipped with Harder-Narasimhan filtrations, are quasi-filtered graded algebras. The results presented in this section is therefore a formalism which is useful to study the Harder-Narasimhan filtrations of graded algebras.

Let K be a filed. Recall that a *$\mathbb{Z}_{\geq 0}$-graded K-algebra* is a direct sum $B = \bigoplus_{n\geq 0} B_n$ of vector spaces over K indexed by $\mathbb{Z}_{\geq 0}$ equipped with a commutative unitary K-algebra structure such that $B_n B_m \subset B_{n+m}$ for any $(m,n) \in \mathbb{Z}_{\geq 0}^2$. We call *homogeneous element of degree n* any element in B_n. Clearly the unit element of B is homogeneous of degree 0. In the following, we shall use the expression "graded K-algebra" to denote a $\mathbb{Z}_{\geq 0}$-graded K-algebra. If B is a graded K-algebra, we call *graded B-module* any B-module M equipped with a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ into direct sum of vector subspaces over K such that $B_n M_m \subset M_{n+m}$ for any $(n,m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$. The elements in M_m are called *homogeneous element of degree m* of M. If x is a non-zero homogeneous element of M, we use $\mathrm{d}^\circ_M(x)$ or $\mathrm{d}^\circ(x)$ to denote the homogeneous degree of x. For reference on graded algebras and graded modules, one can consult [Bou85].

Definition 4.1 Let $B = \bigoplus_{n\geq 0} B_n$ be a graded K-algebra and $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. We say that the K-algebra B is *f-quasi-filtered* if each vector space B_n is equipped with an $\mathbb{R}$-filtration $(B_{n,s})_{s\in\mathbb{R}}$ satisfying the following condition:

> *there exists an integer $n_0 \geq 0$ such that, for any integer $r > 0$, any $(n_i)_{1\leq i\leq r} \in \mathbb{Z}^r_{\geq n_0}$ and any $(s_i)_{1\leq i\leq r} \in \mathbb{R}^r$, we have*
>
> $$\prod_{i=1}^r B_{n_i,s_i} \subset B_{N,S} \qquad \textit{where} \qquad N = \sum_{i=1}^r n_i, \quad S = \sum_{i=1}^r \big(s_i - f(n_i)\big).$$

If B is an f-quasi-filtered graded K-algebra. We say that a graded B-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is *f-quasi-filtered* if for any integer n, M_n is equipped with an $\mathbb{R}$-filtration $(M_{n,s})_{s\in\mathbb{R}}$ satisfying the following condition:

> *there exists an integer $n_0 \geq 0$ such that, for any integer $r > 0$, any $(n_i)_{1\leq i\leq r+1} \in \mathbb{Z}^{r+1}_{\geq n_0}$ and any $(s_i)_{1\leq i\leq r+1} \in \mathbb{R}^{r+1}$, we have*
>
> $$\Big(\prod_{i=1}^r B_{n_i,s_i}\Big) M_{n_{r+1},s_{r+1}} \subset M_{N,S} \qquad \textit{where} \qquad N = \sum_{i=1}^{r+1} n_i, \quad S = \sum_{i=1}^{r+1} \big(s_i - f(n_i)\big).$$

In particular, if $f \equiv 0$, we say that B is a *filtered graded* K-algebra, and M is a *filtered graded* B-module.

We now give some numerical criteria for a graded algebra (or graded module) equipped with $\mathbb{R}$-filtrations to be quasi-filtered.

Proposition 4.2 *Let B be a graded K-algebra and $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. Suppose that for each $n \in \mathbb{Z}_{\geq 0}$, B_n is equipped with an exhaustive and left continuous $\mathbb{R}$-filtration. Then the following conditions are equivalent:*

1) the graded algebra B is f-quasi-filtered,

2) there exists an integer $n_0 > 0$ such that, for any integer $r \geq 2$ and any non-zero homogeneous elements $a_1, \cdots, a_r$ of degree $\geq n_0$ of B, if we write $a = \prod_{i=1}^{r} a_i$, then

$$\lambda(a) \geq \sum_{i=1}^{r} \Big(\lambda(a_i) - f(\mathrm{d}^\circ(a_i))\Big). \tag{6}$$

Proof. The filtrations being exhaustive, the sum on the right side of (6) is well defined and takes value in $\mathbb{R} \cup \{+\infty\}$.

"1)$\Longrightarrow$2)": Since the filtrations are left continuous, we have $a_i \in \mathcal{F}_{\lambda(a_i)} B_{\mathrm{d}^\circ(a_i)}$. Let

$$d = \sum_{i=1}^{r} \mathrm{d}^\circ(a_i) \qquad \text{and} \qquad \eta = \sum_{i=1}^{r} \Big(\lambda(a_i) - f(\mathrm{d}^\circ(a_i))\Big).$$

Since B is f-quasi-filtered, we obtain $a \in \mathcal{F}_\eta B_d$, so $\lambda(a) \geq \eta$.

"2)$\Longrightarrow$1)": Suppose that $a_1, \cdots, a_r$ are homogeneous elements of degrees $\geq n_0$ of B. For any integer $1 \leq i \leq r$ let $d_i = \mathrm{d}^\circ(a_i)$. Let $a = \prod_{i=1}^{r} a_i$. If for any integer $1 \leq i \leq r$, we have $a_i \in \mathcal{F}_{t_i} B_{d_i}$, then we have $\lambda(a_i) \geq t_i$. Therefore, $\lambda(a) \geq \sum_{i=1}^{r} \Big(t_i - f(d_i)\Big)$. Hence $a \in \mathcal{F}_{t_1+\cdots+t_r-f(d_1)-\cdots-f(d_r)} B_{d_1+\cdots+d_r}$. □

Using the numerical criterion established above, we obtain the following corollary.

Corollary 4.3 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function and B be an f-quasi-filtered graded K-algebra. Suppose that for any integer $n \geq 0$, the filtration of B_n is exhaustive and left continuous.*

1) Let A be a sub-K-algebra of B generated by homogeneous elements, equipped with induced graduation. If for each $n \in \mathbb{Z}_{\geq 0}$, the vector space A_n is equipped with the inverse image filtration, then A is an f-quasi-filtered graded K-algebra.

2) Let I be a homogeneous ideal of B and let $C = B/I$, equipped with the quotient graduation. If for each $n \in \mathbb{Z}_{\geq 0}$, the vector space C_n is equipped with the strong direct image filtration, then C is an f-quasi-filtered graded K-algebra.

Proof. 1) After Proposition 2.5, the filtrations of A_n are exhaustive. Furthermore, they are left continuous. If a is a homogeneous element of A, then $\mathrm{d}^\circ_A(a) = \mathrm{d}^\circ_B(a)$. On the other hand, since the filtrations of A_n ($n \geq 0$) are inverse images filtrations, we obtain $\lambda_A(a) = \lambda_B(a)$. So for any integer $r \geq 2$ and any family $(a_i)_{1 \leq i \leq r}$ of homogeneous elements of degree $\geq n_0$ in A with $a = \prod_{i=1}^{r} a_i$, we have

$$\lambda_A(a) = \lambda_B(a) \geq \sum_{i=1}^{r} \Big(\lambda_B(a_i) - f(\mathrm{d}^\circ_B(a_i))\Big) = \sum_{i=1}^{r} \Big(\lambda_A(a_i) - f(\mathrm{d}^\circ_A(a_i))\Big).$$

So the graded algebra A is f-quasi-filtered.

2) After Proposition 2.5, the filtrations of homogeneous components of C are exhaustive. Let $\pi : B \to C$ be the canonical homomorphism. Suppose that $(a_i)_{1 \leq i \leq r}$ is a family of homogeneous elements of degree $\geq n_0$ in C. For any $1 \leq i \leq r$, let $d_i = \mathrm{d}^\circ(a_i)$ and $t_i = \lambda_C(a_i)$.

After Proposition 2.6 2), for any $1 \leq i \leq r$, there exists a sequence $(\alpha_j^{(i)})_{j\geq 1}$ in B_{d_i} such that $\pi(\alpha_j^{(i)}) = a_i$ for any $j \geq 1$ and that the sequence $(\lambda_B(\alpha_j^{(i)}))_{j\geq 1}$ is increasing and converge to t_i. Let $a = \prod_{i=1}^{r} a_i$ and for any $j \geq 1$, let $\alpha_j = \prod_{i=1}^{r} \alpha_j^{(i)}$. Clearly we have $a = \pi(\alpha_j)$ for any $j \geq 1$. Therefore, $\lambda_C(a) \geq \lambda_B(\alpha_j)$. On the other hand, $\lambda_B(\alpha_j) \geq \sum_{i=1}^{r} \Big(\lambda_B(\alpha_j^{(i)}) - f(d_i)\Big)$. Hence $\lambda_C(a) \geq \sum_{i=1}^{r} \Big(\lambda_B(\alpha_j^{(i)}) - f(d_i)\Big)$. By passing to the limit when $j \to +\infty$ we obtain $\lambda_C(a) \geq \sum_{i=1}^{r}(t_i - f(d_i))$. □

The following assertions give numerical criteria for quasi-filtered graded modules, the proofs are similar.

Proposition 4.4 *Let $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be a function, B be an f-quasi-filtered graded K-algebra and M be a graded B-module. Suppose that for any integer n, M_n is equipped with an exhaustive and left continuous $\mathbb{R}$-filtration. Suppose in addition that for any integer $n \geq 0$, the filtration of B_n is exhaustive and left continuous. Then the following conditions are equivalent:*

1) the graded B-module M is f-quasi-filtered;

2) there exists an integer $n_0 \geq 0$ such that, for any integer $r \geq 1$, any family $(a_i)_{1\leq i\leq r}$ of non-zero homogeneous elements of degree $\geq n_0$ of B and any non-zero homogeneous element x of degree $\geq n_0$ of M, if we write $y = (a_1 \cdots a_r)x$, then

$$\lambda(y) \geq \sum_{i=1}^{r} \Big(\lambda(a_i) - f(\mathrm{d}^\circ(a_i))\Big) + \lambda(x) - f(\mathrm{d}^\circ(x)).$$

Corollary 4.5 *Let $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be a function, B be an f-quasi-filtered graded K-algebra and M be an f-quasi-filtered graded B-module. Suppose that for any integer $n \geq 0$, the filtrations of B_n and of M_n are exhaustive and left continuous.*

1) Let M' be a graded sub-B-module. If each M'_n is equipped with the inverse image filtration, then M' is an f-quasi-filtered graded B-module.

2) Let M' be a homogeneous sub-B-module of M and let $M'' = M/M'$. If each M''_n is equipped with the strong direct image filtration, then M'' is an f-quasi-filtered graded B-module.

Corollary 4.6 *Let $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be a function, B be an f-quasi-filtered graded K-algebra, and M be an f-quasi-filtered graded B-module. Suppose that for any positive integer (resp. any integer) n, the filtration of B_n (resp. M_n) is exhaustive and left continuous.*

1) Let A be a sub-K-algebra of B generated by homogeneous elements, equipped with the induced graduation. If each vector space A_n is equipped with the inverse image filtration, then M is an f-quasi-filtered graded A-module.

2) Let I be a homogeneous ideal of B contained in $\mathrm{ann}(M)$ and $C = B/I$ which is equipped with the quotient graduation. If each C_n is equipped with the strong direct image filtration, then M is an f-quasi-filtered graded C-module.

5 Convergence for symmetric algebras

We now consider the symmetric algebra of a finite dimensional non-zero vector space, which is equipped with certain suitable filtrations. Each homogeneous component of the symmetric algebra contains a special base which consists of monomials. By introducing a combinatoric equality on monomials (Theorem 5.1), we establish a convergence result (Corollary 5.3) for quasi-filtered symmetric algebras. We shall show later in Section 7 that the general convergence can be deduced from this result in the special case of quasi-filtered symmetric algebras.

For any pair of integers (n,d) such that $n \geq 0$ and $d \geq 1$, let $\Delta_n^{(d)}$ be the subset of $\mathbb{Z}_{\geq 0}^d$ consisting of all decompositions of n into sum of d positive integers. We introduce the lexicographic order on $\Delta_n^{(d)}$: $(a_1, \cdots, a_d) \geq (b_1, \cdots, b_d)$ if and only if there exists an integer $1 \leq i \leq d$ such that $a_j = b_j$ for any $1 \leq j \leq i$ and that $a_{i+1} > b_{i+1}$ if $i < d$. The set $\Delta_n^{(d)}$ is totally ordered. On the other hand, for any integer $r \geq 2$ and any $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{Z}_{\geq 0}^r$, we have a mapping from $\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}$ to $\Delta_{n_1+\cdots+n_r}^{(d)}$ which sends $(\alpha_i)_{1\leq i\leq r}$ to $\alpha_1 + \cdots + \alpha_r$ (the addition being that in $\mathbb{Z}^d$). This mapping is not injective in general but is always surjective. Moreover, if $(\alpha_i)_{1\leq i\leq r}$ and $(\beta_i)_{1\leq i\leq r}$ are two elements of $\Delta_{n_1}^{(d)} \times \cdots \Delta_{n_r}^{(d)}$ such that $\alpha_i \geq \beta_i$ for any $1 \leq i \leq r$, then $\alpha_1 + \cdots + \alpha_r \geq \beta_1 + \cdots + \beta_r$.

For any $n \in \mathbb{Z}_{\geq 0}$, we denote by $\Gamma_n^{(d)}$ the subset of $\mathbb{Z}_{\geq 0}^{d-1}$ consisting of elements $(a_i)_{1\leq i\leq d-1}$ such that $0 \leq a_1 + \cdots + a_{d-1} \leq n$. We have a natural mapping $p_n^{(d)} : \Delta_n^{(d)} \to \Gamma_n^{(d)}$ defined by the projection on the first $d-1$ factors. The mapping $p_n^{(d)}$ is in fact a bijection and its inverse is the mapping which sends $(a_i)_{1\leq i\leq d-1}$ to $(a_1, \cdots, a_{d-1}, n - a_1 - \cdots - a_{d-1})$. For any $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{Z}_{\geq 0}^r$, we have the following commutative diagram

$$\begin{array}{ccc} \Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)} & \xrightarrow{+} & \Delta_{|\mathbf{n}|}^{(d)} \\ \downarrow{\scriptstyle p_{n_1}^{(d)} \times \cdots \times p_{n_r}^{(d)}} & & \downarrow{\scriptstyle p_N^{(d)}} \\ \Gamma_{n_1}^{(d)} \times \cdots \times \Gamma_{n_r}^{(d)} & \xrightarrow[+]{} & \Gamma_{|\mathbf{n}|}^{(d)} \end{array} \tag{7}$$

where $|\mathbf{n}| = n_1 + \cdots + n_r$ and the operators "+" are defined by the addition structures in the monoids $\mathbb{Z}_{\geq 0}^d$ and $\mathbb{Z}_{\geq 0}^{d-1}$ respectively.

Theorem 5.1 *Let $r \geq 2$ and $d \geq 1$ be two integers. For any $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{Z}_{\geq 0}^r$, there exists a probability measure $\rho_{\mathbf{n}}$ on $\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}$ such that the direct image of $\rho_{\mathbf{n}}$ by each of the r projections on $\Delta_{n_1}^{(d)}, \cdots, \Delta_{n_r}^{(d)}$ is equidistributed, and also is its direct image on $\Delta_{|\mathbf{n}|}^{(d)}$ by the operator "+".*

Proof. The theorem is trivial when $d = 1$ because in this case, for any $k \in \mathbb{Z}_{\geq 0}$, $\Delta_k^{(1)}$ is the one point set $\{k\}$. In the following, we suppose $d \geq 2$. By (7), it suffices to construct a probability measure $\rho_{\mathbf{n}}$ on $\Gamma_{n_1}^{(d)} \times \cdots \times \Gamma_{n_r}^{(d)}$ such that the direct image of $\rho_{\mathbf{n}}$ by each of the r projections on $\Gamma_{n_1}^{(d)}, \cdots, \Gamma_{n_r}^{d}$ is an equidistributed measure, and also is the direct image on $\Gamma_{|\mathbf{n}|}^{(d)}$ by the operator "+".

For any $\alpha = (a_i)_{1\leq i\leq d-1} \in \mathbb{Z}_{\geq 0}^{d-1}$, we define $|\alpha| = a_1 + \cdots + a_{d-1}$. The set $\Gamma_n^{(d)}$ can be written in the form $\Gamma_n^{(d)} = \{\alpha \in \mathbb{Z}_{\geq 0}^{d-1} \;\big|\; |\alpha| \leq n\}$. If $\alpha = (a_i)_{1\leq i\leq d-1}$ is an element of $\mathbb{Z}_{\geq 0}^{d-1}$, we write $\alpha! = a_1! \times \cdots \times a_{d-1}!$.

Consider the algebra of formal series in rd variables $R = \mathbb{Z}[\![\mathbf{t}, \mathbf{X}]\!]$, where $\mathbf{t} = (t_1, \cdots, t_r)$, $\mathbf{X} = (X_{i,j})_{\substack{1\leq i\leq r,\\ 1\leq j\leq d-1}}$. If $\alpha = (a_1, \cdots, a_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$ and if $1 \leq i \leq r$, we denote by X_i^α the

product $X_{i,1}^{a_1} \times \cdots \times X_{i,d-1}^{a_{d-1}}$. If $\mathbf{n} = (n_i)_{1 \leq i \leq r}$ is an element in $\mathbb{Z}_{\geq 0}^r$, we denote by $\mathbf{t}^{\mathbf{n}}$ the product $t_1^{n_1} \times \cdots \times t_r^{n_r}$. Let $H(\mathbf{t}, \mathbf{X})$ be the formal series

$$\sum_{\mathbf{n}=(n_i) \in \mathbb{Z}_{\geq 0}^r} \mathbf{t}^{\mathbf{n}} \sum_{\substack{(\alpha_i) \in (\mathbb{Z}_{\geq 0}^{d-1})^r \\ |\alpha_i| \leq n_i}} \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha_1! \cdots \alpha_r!} \frac{(n_1 + \cdots + n_r - |\alpha_1 + \cdots + \alpha_r|)!}{(n_1 - |\alpha_1|)! \cdots (n_r - |\alpha_r|)!} \prod_{j=1}^r X_j^{\alpha_j},$$

the coefficients of which are positive integers. If we perform the change of indices $m_i = n_i - |\alpha_i|$ and permute the summations by defining $(\beta_1, \cdots, \beta_{d-1}) = \alpha_1 + \cdots + \alpha_r$ and $m = m_1 + \cdots + m_r$, we obtain the following equality in $\mathbb{Z}[\![\mathbf{t}, \mathbf{X}]\!]$:

$$\begin{aligned} H(\mathbf{t}, \mathbf{X}) &= \sum_{(\alpha_i) \in (\mathbb{Z}_{\geq 0}^{d-1})^r} \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha_1! \cdots \alpha_r!} \prod_{j=1}^r t_j^{|\alpha_j|} X_j^{\alpha_j} \sum_{\mathbf{m}=(m_i) \in \mathbb{Z}_{\geq 0}^r} \frac{(m_1 + \cdots + m_r)!}{m_1! \cdots m_r!} \mathbf{t}^{\mathbf{m}} \\ &= \sum_{(\beta_i) \in \mathbb{Z}_{\geq 0}^{d-1}} \prod_{i=1}^{d-1} (t_1 X_{1,i} + \cdots + t_r X_{r,i})^{\beta_i} \sum_{m \in \mathbb{Z}_{\geq 0}} (t_1 + \cdots + t_r)^m \\ &= (1 - (t_1 + \cdots + t_r))^{-1} \prod_{i=1}^{d-1} (1 - (t_1 X_{1,i} + \cdots + t_r X_{r,i}))^{-1}. \end{aligned}$$

This calculation also implies (cf. [Hör90] chap. II §2.4) that the Reinhardt's absolute convergence domain of $H(\mathbf{t}, \mathbf{X})$ in $\mathbb{C}^{rd}$ is defined by the condition

$$\sum_{j=1}^r |t_j| < 1 \text{ and } \sum_{j=1}^r |t_j||X_{j,i}| < 1.$$

This observation enables us to substitute certain variables X_i by the vector $\mathbb{1} = (\underbrace{1, \cdots, 1}_{d-1 \text{ copies}})$ without examining convergence problems. By the change of variables $m_i = n_i - |\alpha_i|$ for $2 \leq i \leq r$, we obtain

$$\begin{aligned} &H(\mathbf{t}, \mathbf{X})|_{X_2 = \cdots = X_r = \mathbb{1}} \\ &= \sum_{n_1 \geq 0} t_1^{n_1} \sum_{|\alpha_1| \leq n_1} X_1^{\alpha_1} \sum_{\substack{(\alpha_i)_{i=2}^r \in (\mathbb{Z}_{\geq 0}^{d-1})^{r-1} \\ (m_i)_{i=2}^r \in \mathbb{Z}_{\geq 0}^{r-1}}} \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha_1! \cdots \alpha_r!} \frac{(n_1 + m_2 + \cdots + m_r - |\alpha_1|)!}{(n_1 - |\alpha_1|)!\, m_2! \cdots m_r!} \prod_{j=2}^r t_j^{m_j + |\alpha_j|} \\ &= \sum_{n_1 \geq 0} t_1^{n_1} \sum_{|\alpha_1| \leq n_1} X_1^{\alpha_1} \sum_{(\alpha_i)_{i=2}^r \in (\mathbb{Z}_{\geq 0}^{d-1})^{r-1}} \frac{(\alpha_1 + \cdots + \alpha_r)!}{\alpha_1! \cdots \alpha_r!} \prod_{j=2}^r t_j^{|\alpha_j|} \\ &\qquad \sum_{(m_i)_{i=2}^r \in \mathbb{Z}_{\geq 0}^{r-1}} \frac{(n_1 + m_2 + \cdots + m_r - |\alpha_1|)!}{(n_1 - |\alpha_1|)! m_2! \cdots m_r!} \prod_{j=2}^r t_j^{m_j}. \end{aligned}$$

For any $a \in \mathbb{Z}_{\geq 0}$, we have

$$\sum_{b \geq 0} \frac{(a+b)!}{a!b!} t^b = (1-t)^{-a-1},$$

hence we get

$$\begin{aligned}
&\sum_{(\alpha_i)_{i=2}^r\in(\mathbb{Z}_{\geq 0}^{d-1})^{r-1}} \frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha_1!\cdots\alpha_r!}\prod_{j=2}^r t_j^{|\alpha_j|}\\
&=\sum_{(\alpha_i)_{i=2}^r\in(\mathbb{Z}_{\geq 0}^{d-1})^{r-1}} \frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha_1!(\alpha_2+\cdots+\alpha_r)!}\frac{(\alpha_2+\cdots+\alpha_r)!}{\alpha_2!\cdots\alpha_r!}\prod_{j=2}^r t_j^{|\alpha_j|}\\
&=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{d-1}} \frac{(\alpha_1+\alpha)!}{\alpha_1!\alpha!}\sum_{\substack{(\alpha_i)_{i=2}^r\in(\mathbb{Z}_{\geq 0}^{d-1})^{r-1}\\ \alpha_2+\cdots+\alpha_r=\alpha}} \frac{(\alpha_2+\cdots+\alpha_r)!}{\alpha_2!\cdots\alpha_r!}\prod_{j=2}^r t_j^{|\alpha_j|}\\
&=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{d-1}} \frac{(\alpha_1+\alpha)!}{\alpha_1!\alpha!}(t_2+\cdots+t_r)^{|\alpha|}=(1-(t_2+\cdots+t_r))^{-|\alpha_1|-d+1},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{(m_i)_{i=2}^r\in\mathbb{Z}_{\geq 0}^{r-1}} \frac{(n_1+m_2+\cdots+m_r-|\alpha_1|)!}{(n_1-|\alpha_1|)!m_2!\cdots m_r!}\prod_{j=2}^r t_j^{m_j}\\
&=\sum_{(m_i)_{i=2}^r\in\mathbb{Z}_{\geq 0}^{r-1}} \frac{(n_1+m_2+\cdots+m_r-|\alpha_1|)!}{(n_1-|\alpha_1|)!(m_2+\cdots+m_r)!}\frac{(m_2+\cdots+m_r)!}{m_2!\cdots m_r!}\prod_{j=2}^r t_j^{m_j}\\
&=\sum_{M\geq 0} \frac{(n_1-|\alpha_1|+M)!}{(n_1-|\alpha_1|)!M!}\sum_{\substack{(m_i)_{i=2}^r\in\mathbb{Z}_{\geq 0}^{r-1}\\ m_2+\cdots+m_r=M}} \frac{(m_2+\cdots+m_r)!}{m_2!\cdots m_r!}\prod_{j=2}^r t_j^{m_j}\\
&=\sum_{M\geq 0} \frac{(n_1-|\alpha_1|+M)!}{(n_1-|\alpha_1|)!M!}(t_2+\cdots+t_r)^M=(1-(t_2+\cdots+t_r))^{-n_1+|\alpha_1|-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
H(\mathbf{t},\mathbf{X})|_{X_2=\cdots=X_r=\mathbb{1}} &= \sum_{n_1\geq 0} t_1^{n_1}(1-(t_2+\cdots+t_r))^{-n_1-d}\sum_{|\alpha_1|\leq n_1} X_1^{\alpha_1}\\
&= \sum_{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r} \mathbf{t}^{\mathbf{n}}\frac{(n_1+\cdots+n_r+d-1)!}{(n_1+d-1)!n_2!\cdots n_r!}\sum_{|\alpha_1|\leq n_1} X_1^{\alpha_1}.
\end{aligned}\tag{8}$$

Similarly, for any $1\leq j\leq r$, we have

$$\begin{aligned}
&H(\mathbf{t},\mathbf{X})|_{X_1=\cdots=X_{j-1}=X_{j+1}=\cdots=X_r=\mathbb{1}}\\
&=\sum_{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r} \mathbf{t}^{\mathbf{n}}\frac{(n_1+\cdots+n_r+d-1)!}{n_1!\cdots n_{j-1}!(n_j+d-1)!n_{j+1}!\cdots n_r!}\sum_{|\alpha_j|\leq n_j} X_j^{\alpha_j}.
\end{aligned}\tag{9}$$

On the other hand,

$$\begin{aligned}
&H(\mathbf{t},\mathbf{X})|_{X_1=\cdots=X_r=Y}\\
&=\sum_{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r} \mathbf{t}^{\mathbf{n}}\sum_{\substack{(\alpha_i)\in(\mathbb{Z}_{\geq 0}^{d-1})^r\\ |\alpha_i|\leq n_i}} \frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha_1!\cdots\alpha_r!}\frac{(n_1+\cdots+n_r-|\alpha_1+\cdots+\alpha_r|)!}{(n_1-|\alpha_1|)!\cdots(n_r-|\alpha_r|)!}Y^{\alpha_1+\cdots+\alpha_r}.
\end{aligned}$$

By the change of variables $m_i = n_i - |\alpha_i|$ for any $1 \leq i \leq r$, we obtain

$$\begin{aligned}
&H(\mathbf{t}, \mathbf{X})|_{X_1=\cdots=X_r=Y} \\
&= \sum_{(\alpha_i)\in(\mathbb{Z}_{\geq 0}^{d-1})^r} \left(\frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha_1!\cdots\alpha_r!} \prod_{j=1}^{r} t_j^{|\alpha_j|} \right) Y^{\alpha_1+\cdots+\alpha_r} \sum_{\mathbf{m}=(m_i)\in\mathbb{Z}_{\geq 0}^r} \frac{(m_1+\cdots+m_r)!}{m_1!\cdots m_r!} \mathbf{t}^{\mathbf{m}} \\
&= \sum_{N\geq 0} (t_1+\cdots+t_r)^N \sum_{\gamma\in\mathbb{Z}_{\geq 0}^{d-1}} Y^\gamma (t_1+\cdots+t_r)^{|\gamma|} = \sum_{M\geq 0} (t_1+\cdots+t_r)^M \sum_{|\gamma|\leq M} Y^\gamma,
\end{aligned}$$

where we have performed the change of variables $\gamma = \alpha_1 + \cdots + \alpha_r$ and $M = N + |\gamma|$ in the last equality. Therefore, we have

$$H(\mathbf{t}, \mathbf{X})|_{X_1=\cdots=X_r=Y} = \sum_{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r} \frac{(n_1+\cdots+n_r)!}{n_1!\cdots n_r!} t_1^{n_1}\cdots t_r^{n_r} \sum_{|\gamma|\leq n_1+\cdots+n_r} Y^\gamma. \tag{10}$$

Finally,

$$\begin{aligned}
&H(\mathbf{t}, (\mathbb{1}, \cdots \mathbb{1})) = (1-(t_1+\cdots+t_r))^{-d} \\
&= \sum_{N\geq 0} \frac{(N+d-1)!}{N!(d-1)!} \sum_{\substack{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r \\ n_1+\cdots+n_r=N}} \frac{N!}{n_1!\cdots n_r!} \mathbf{t}^{\mathbf{n}} \\
&= \sum_{\mathbf{n}=(n_i)\in\mathbb{Z}_{\geq 0}^r} \frac{(n_1+\cdots+n_r+d-1)!}{n_1!\cdots n_r!(d-1)!} \mathbf{t}^{\mathbf{n}}.
\end{aligned} \tag{11}$$

For any $\mathbf{n} = (n_i) \in \mathbb{Z}_{\geq 0}^r$, let

$$\rho_{\mathbf{n}} = \frac{(d-1)!n_1!\cdots n_r!}{(n_1+\cdots+n_r+d-1)!} \sum_{\substack{(\alpha_i)\in(\mathbb{Z}_{\geq 0}^{d-1})^r \\ |\alpha_i|\leq n_i}} \left(\frac{(\alpha_1+\cdots+\alpha_r)!}{\alpha_1!\cdots\alpha_r!} \frac{(n_1+\cdots+n_r-|\alpha_1+\cdots+\alpha_r|)!}{(n_1-|\alpha_1|)!\cdots(n_r-|\alpha_r|)!} \right) \delta_{(\alpha_1,\cdots,\alpha_r)}.$$

The definition of $H(\mathbf{t}, \mathbf{X})$ and the equalities (9), (10) and (11) implies that $\rho_{\mathbf{n}}$ verifies the required conditions. □

We introduce some operators on the space of Borel measures on $\mathbb{R}$ which we shall use later. We denote by $C_c(\mathbb{R})$ the space of continuous functions with compact support on $\mathbb{R}$. Recall that a *Radon measure* on $\mathbb{R}$ is nothing but a positive linear form on $C_c(\mathbb{R})$. Note that all bounded Borel measures on $\mathbb{R}$ are Radon measures. We denote by $\mathscr{M}_+$ the convex cone of Radon measures on $\mathbb{R}$ (in the space of all linear forms on $C_c(\mathbb{R})$) and by $\mathscr{M}_1$ the sub-space of Borel probability measures on $\mathbb{R}$. Note that $\mathscr{M}_1$ is a convex subset of $\mathscr{M}_+$.

If c is a real number, we denote by $\varphi_c : \mathbb{R} \to \mathbb{R}$ the mapping which sends x to $x + c$. It induces an automorphism of convex cone $\tau_c : \mathscr{M}_+ \to \mathscr{M}_+$ which sends $\nu \in \mathscr{M}_+$ to the direct image of ν by φ_c. Thus we define an action of $\mathbb{R}$ on $\mathscr{M}_+$ which keeps $\mathscr{M}_1$ invariant, and which preserves the order $\succ$ between Borel measures.

If ε is a strictly positive real number, we denote by $\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}$ the dilation mapping which sends $x \in \mathbb{R}$ to εx. This mapping induces by direct image an automorphism of the convex cone $T_\varepsilon : \mathscr{M}_+ \to \mathscr{M}_+$ which keeps $\mathscr{M}_1$ invariant and also preserves the order $\succ$.

We now consider a vector space V of finite dimension d over a field K. For any integer $n \geq 0$, let $B_n = S^n V$ be the n^{th} symmetric power of V, equipped with a separated, exhaustive

and left continuous $\mathbb{R}$-filtration. We shall use Theorem 5.1 to establish the almost super-additivity of the measures associated to B_n ($n \geq 1$) under the condition that the graded algebra $B = \bigoplus_{n\geq 0} B_n$ is quasi-filtered.

Choose a base $\mathbf{e} = (e_i)_{1\leq i\leq d}$ of V. We then have a mapping $\varphi_n : \Delta_n^{(d)} \to B_n$ which sends $\alpha = (\alpha_1, \cdots, \alpha_d)$ to $e^\alpha := e_1^{\alpha_1} \cdots e_d^{\alpha_d}$. The image of $\Delta_n^{(d)}$ by φ_n is a base of B_n. There exists, for each $n \in \mathbb{N}$, a maximal base $\mathbf{u}^{(n)} = (u_\alpha)_{\alpha\in\Delta_n^{(d)}}$ of B_n such that (see Proposition 2.9 *infra*)

$$u_\alpha \in e^\alpha + \sum_{\beta<\alpha} Ke^\beta. \tag{12}$$

If $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{Z}_{\geq 0}^r$ and $N = n_1 + \cdots + n_r$, for any $\gamma \in \Delta_N^{(d)}$, let $u_\gamma^{(\mathbf{n})}$ be an element in

$$\left\{ \prod_{i=1}^r u_{\alpha_i} \;\middle|\; \alpha_i \in \Delta_{n_i}^{(d)},\ \sum_{i=1}^r \alpha_i = \gamma \right\}.$$

such that

$$\lambda(u_\gamma^{(\mathbf{n})}) = \max_{\substack{\alpha_i\in\Delta_{n_i}^{(d)}\\ \alpha_1+\cdots+\alpha_r=\gamma}} \lambda(u_{\alpha_1}\cdots u_{\alpha_r}).$$

From (12), we deduce

$$u_\gamma^{(\mathbf{n})} \in e^\gamma + \sum_{\delta<\gamma} Ke^\delta.$$

Hence $\mathbf{u}^{(\mathbf{n})} := (u_\gamma^{(\mathbf{n})})_{\gamma\in\Delta_N^{(d)}}$ is a base of B_N.

Proposition 5.2 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function, c be a positive real number and $g : \mathbb{R} \to \mathbb{R}$ be a concave increasing c-Lipschitz function. Suppose that the graded algebra $B = \bigoplus_{n\geq 0} B_n$ is f-quasi-filtered. If for any integer $n \geq 0$, denote by*

$$I_n = \int_{\mathbb{R}} g \,\mathrm{d}\Big(T_{\frac{1}{n}}\nu_{B_n}\Big),$$

then for any integer $r \geq 2$ and any $\mathbf{n} = (n_i) \in \mathbb{Z}_{\geq n_0}^r$, by writing $N = n_1 + \cdots + n_r$, we have

$$NI_N \geq \sum_{i=1}^r \Big(n_i I_{n_i} - cf(n_i)\Big).$$

Proof. For any integer $n \geq 0$, denote by ξ_n the equidistributed measure on $\Delta_n^{(d)}$, by $\rho_{\mathbf{n}}$ a measure on $\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}$ satisfying the conditions of Theorem 5.1, and by $\mathbf{u}^{(\mathbf{n})}$ the base of B_N constructed as above. Then by Definition 2.11,

$$\begin{aligned} I_N &\geq \int_{\mathbb{R}} g \,\mathrm{d}\Big(T_{\frac{1}{N}}\nu_{\mathbf{u}^{(\mathbf{n})}}\Big) = \int_{\Delta_N^{(d)}} g\left(\frac{1}{N}\lambda(u_\gamma^{(\mathbf{n})})\right) \mathrm{d}\xi_N(\gamma) \\ &= \int_{\Delta_{n_1}^{(d)}\times\cdots\times\Delta_{n_r}^{(d)}} g\left(\frac{1}{N}\lambda(u_{\alpha_1+\cdots+\alpha_r}^{(\mathbf{n})})\right) \mathrm{d}\rho_{\mathbf{n}}(\alpha_1, \cdots, \alpha_r). \end{aligned}$$

Since g is an increasing function, by definition of $u_\gamma^{(\mathbf{n})}$, we have

$$I_N \geq \int_{\Delta_{n_1}^{(d)}\times\cdots\times\Delta_{n_r}^{(d)}} g\left(\frac{1}{N}\lambda(u_{\alpha_1}\cdots u_{\alpha_r})\right) \mathrm{d}\rho_{\mathbf{n}}(\alpha_1, \cdots, \alpha_n).$$

Since B is an f-quasi-filtered graded algebra and since g is increasing, we obtain

$$I_N \geq \int_{\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}} g\left(\frac{1}{N}\sum_{i=1}^{r}\Big(\lambda(u_{\alpha_i}) - f(n_i)\Big)\right) \mathrm{d}\rho_{\mathbf{n}}(\alpha_1, \cdots, \alpha_r).$$

Since the function g is c-Lipschitz, then

$$I_N \geq \int_{\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}} \left[g\left(\frac{1}{N}\sum_{i=1}^{r}\lambda(u_{\alpha_i})\right) - \frac{c}{N}\sum_{i=1}^{r} f(n_i)\right] \mathrm{d}\rho_{\mathbf{n}}(\alpha_1, \cdots, \alpha_r).$$

Then the concavity of g implies that

$$I_N \geq \int_{\Delta_{n_1}^{(d)} \times \cdots \times \Delta_{n_r}^{(d)}} \left[\sum_{i=1}^{r}\frac{n_i}{N} g\left(\frac{\lambda(u_{\alpha_i})}{n_i}\right)\right] \mathrm{d}\rho_{\mathbf{n}}(\alpha_1, \cdots, \alpha_r) - \frac{c}{N}\sum_{i=1}^{r} f(n_i)$$

Finally, since the direct image of $\rho_{\mathbf{n}}$ by the r projections are equidistributed, we obtain that

$$I_N \geq \sum_{i=1}^{r}\frac{n_i}{N} I_{n_i} - \frac{c}{N}\sum_{i=1}^{r} f(n_i).$$

□

Corollary 5.3 *With the notations of Proposition 5.2, if the sequence $(I_n)_{n\geq 0}$ is bounded from above (for example if g is bounded from above, or if there exists $a \in \mathbb{R}$ such that $\mathrm{supp}(\nu_{B_n}) \subset]-\infty, na]$ for any sufficiently large integer n) and if $\lim_{n\to+\infty} f(n)/n = 0$, then the sequence $(I_n)_{n\geq 0}$ has a limit when $n \to +\infty$.*

Proof. It is a consequence of Proposition 5.2 and Corollary 3.2. □

6 Polygon associated to a Borel measure

We explain in this section how to associate to a Borel probability measure on $\mathbb{R}$ a concave function on $[0,1]$ which takes zero value at the origin. Furthermore, if the measure is a linear combination of Dirac measures, then the associated concave function is piecewise linear, therefore is a *polygon* on $[0,1]$.

If $f : \mathbb{R} \to [0,1]$ is a right continuous decreasing function such that

$$\lim_{x\to-\infty} f(x) = 1, \quad \text{and} \quad \lim_{x\to+\infty} f(x) = 0,$$

we define the *quasi-inverse* of f the function $f^* :]0,1[\to \mathbb{R}$ which sends any $t \in]0,1[$ to $\sup\{x \mid f(x) > t\}$. The following properties of f^* are easy to verify.

Proposition 6.1 *Let $f : \mathbb{R} \to [0,1]$ be a right continuous decreasing function such that $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to+\infty} f(x) = 0$. Then*

1) for any $t \in]0,1[$ and any $y \in \mathbb{R}$, $f(y) > t$ if and only if $y < f^*(t)$;

2) f^* is a right continuous decreasing function;

3) $\sup_{t\in]0,1[} f^*(t) = \inf\{x \in \mathbb{R} \mid f(x) = 0\}$ *and* $\inf_{t\in]0,1[} f^*(t) = \sup\{x \in \mathbb{R} \mid f(x) = 1\}$ *(by convention* $\inf\emptyset = +\infty$ *and* $\sup\emptyset = -\infty$*).*

Proposition 6.2 *Let ν be a Borel probability measure on $\mathbb{R}$ which is a linear combination of Dirac measures. If we denote by $f : \mathbb{R} \to [0,1]$ the function $f(x) = \nu\big(]x, +\infty[\big)$, then the function on $[0,1]$ defined by $P(\nu)(t) := \int_0^t f^*(s)\mathrm{d}s$ is a polygon on $[0,1]$.*

Proof. Since ν is a linear combination of Dirac measures, the function f is decreasing, right continuous, and piecewise constant. Furthermore, $f(x) = 0$ (resp. $f(x)=1$) when x is sufficiently positive (resp. negative). Therefore, f^* is decreasing, right continuous, piecewise constant and bounded. As $P(\nu)$ is the primitive function of f^*, which takes zero value at the origin, we obtain that $P(\nu)$ is a polygon. □

Actually, $P(\nu)$ is just the *Legendre transformation* of the concave function $x \mapsto \int_0^x f(y)\mathrm{d}y$ (see [Hör94] II.2.2), which is called the *polygon* associated to the Borel probability measure ν.

We can calculate explicitly $P(\nu)$. Suppose that ν is of the form $\sum_{i=1}^{n}(t_i - t_{i-1})\delta_{a_i}$, where $a_1 > \cdots > a_n$, and $0 = t_0 < \cdots < t_n = 1$. Then $f(x) = \mathbb{1}_{]-\infty,a_n[}(x) + \sum_{i=1}^{n-1} t_i\, \mathbb{1}_{[a_{i+1},a_i[}(x)$, and hence $f^*(t) = a_0\, \mathbb{1}_{]0,t_1[}(t) + \sum_{i=2}^{n} a_i\, \mathbb{1}_{[t_{i-1},t_i[}(t)$. Therefore,

$$P(\nu)(t) = \sum_{i=1}^{j-1} a_i(t_i - t_{i-1}) + a_j(t - t_{j-1}), \quad t \in [t_{j-1}, t_j], \quad 1 \leq j \leq n.$$

If V is a non-zero vector space of finite rank over K and $\mathcal{F}$ is a separated and exhaustive filtration of V. We call *polygon* associated to the filtration $\mathcal{F}$ the polygon $P(\nu_{\mathcal{F},V})$ on $[0,1]$, denoted by $P_{\mathcal{F},V}$ (or simply P_V if there is no ambiguity on the filtration).

Suppose in addition that the filtration $\mathcal{F}$ is left continuous. Then $\mathcal{F}$ corresponds to a flag

$$0 = V^{(0)} \subsetneq V^{(1)} \subsetneq \cdots \subsetneq V^{(n)} = V$$

and a strictly decreasing sequence $(a_i)_{1\leq i\leq n}$. We have shown that its associated probability measure is $\nu_{\mathcal{F},V} = \sum_{i=1}^{n}\Big(\frac{\mathrm{rk}\, V^{(i)}}{\mathrm{rk}\, V} - \frac{\mathrm{rk}\, V^{(i-1)}}{\mathrm{rk}\, V}\Big)\delta_{a_i}$. Therefore we have, for any integer $1 \leq j \leq n$ and any $t \in \Big[\frac{\mathrm{rk}\, V^{(j-1)}}{\mathrm{rk}\, V}, \frac{\mathrm{rk}\, V^{(j)}}{\mathrm{rk}\, V}\Big]$,

$$P_{\mathcal{F},V}(t) = \sum_{i=1}^{j-1} a_i\Big(\frac{\mathrm{rk}\, V^{(i)}}{\mathrm{rk}\, V} - \frac{\mathrm{rk}\, V^{(i-1)}}{\mathrm{rk}\, V}\Big) + a_j\Big(t - \frac{\mathrm{rk}\, V^{(j-1)}}{\mathrm{rk}\, V}\Big).$$

For a general Borel probability measure ν, similarly to Proposition 6.2, we can also define a concave function $P(\nu)$. If ν_1 and ν_2 are two Borel probability measures on $\mathbb{R}$ such that $\nu_1 \succ \nu_2$, then $P(\nu_1) \geq P(\nu_2)$. Furthermore, for any real number a, $P(\tau_a\nu)(t) = P(\nu)(t) + at$ and for any strictly positive number ε, $P(T_\varepsilon\nu) = \varepsilon P(\nu)$.

In the following, we explain that the vague convergence of Borel probability measures implies the uniform convergence of associated polygons. With this observation, to prove the

convergence of polygons, it suffice to establish the vague convergence of corresponding probability measures. We begin by presenting some properties of Borel probability measures.

Lemma 6.3 *For any function $f \in C_c(\mathbb{R})$, we have*

$$\lim_{\varepsilon\to 0} \|f\circ\gamma_{1+\varepsilon} - f\|_{\sup} = 0, \quad \lim_{\varepsilon\to 0} \|f\circ\varphi_{\varepsilon} - f\|_{\sup} = 0.$$

Proof. Suppose that $\mathrm{supp}(f) \subset [-K, K]$ $(K > 0)$. For any number $-1/2 < \varepsilon < 1/2$,

$$\|f\circ\gamma_{1+\varepsilon} - f\|_{\sup} = \sup_{-2K\le x\le 2K} |f(x+\varepsilon x) - f(x)|.$$

Since f is uniformly continuous, $\lim_{\varepsilon\to 0} \sup_{-2K\le x\le 2K} |f(x+\varepsilon x) - f(x)| = 0$, so we have $\lim_{\varepsilon\to 0} \|f\circ\gamma_{1+\varepsilon} - f\|_{\sup} = 0$. The other assertion is just the definition of uniform continuity of f. □

Definition 6.4 If $(\nu_n)_{n\ge 1}$ is a sequence of Radon measures on $\mathbb{R}$ and if ν is a Radon measure on $\mathbb{R}$, we say that $(\nu_n)_{n\ge 1}$ *converges vaguely* to ν if for any function $f \in C_c(\mathbb{R})$, the sequence $(\int_{\mathbb{R}} f\mathrm{d}\nu_n)_{n\ge 1}$ converges to $\int_{\mathbb{R}} f\mathrm{d}\nu$.

Proposition 6.5 *Let $(\nu_n)_{n\ge 1}$ be a sequence of Radon measures on $\mathbb{R}$, ν be a Radon measure on $\mathbb{R}$, and $(a_n)_{n\ge 1}$ be a sequence of real numbers in $]-1, +\infty[$ which converges to 0. Suppose that the total masses of $(\nu_n)_{n\ge 1}$ are uniformly bounded. Then the following conditions are equivalents:*

1) the sequence $(\nu_n)_{n\ge 1}$ converges vaguely to ν;

2) the sequence $(T_{1+a_n}\nu_n)_{n\ge 1}$ converges vaguely to ν;

3) the sequence $(\tau_{a_n}\nu_n)_{n\ge 1}$ converges vaguely to ν.

Proof. Since $\tau_{a_n}^{-1} = \tau_{-a_n}$ and $T_{1+a_n}^{-1} = T_{(1+a_n)^{-1}} = T_{1-\frac{a_n}{1+a_n}}$, it suffices to verify "1)$\Longrightarrow$ 2)" and "1)$\Longrightarrow$3)", which are immediate consequences of Lemma 6.3. □

Lemma 6.6 *Let $(\nu_n)_{n\ge 1}$ be a sequence of Borel probability measures on $\mathbb{R}$ which converges vaguely to a measure ν. If the supports of $(\nu_n)_{n\ge 1}$ are uniformly bounded, then ν is also a probability measure.*

Proof. Suppose $\bigcup_{n\ge 1} \mathrm{supp}(\nu_n) \subset [m, M]$. If φ is a continuous function with compact support which takes values in $[0, 1]$ and such that $\varphi|_{[m,M]} = 1$. We have $\int_{\mathbb{R}} \varphi\mathrm{d}\nu = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi\mathrm{d}\nu_n = 1$. Since φ is arbitrary, we obtain $\nu(\mathbb{R}) = 1$. □

Proposition 6.7 *Let $(\nu_n)_{n\ge 1}$ be a sequence of Borel probability measures on $\mathbb{R}$ which converges vaguely to a measure ν. Suppose that the supports of ν_n are uniformly bounded. Let F_n (resp. F) be the distribution function of ν_n (resp. ν). Then there exists a numerable subset Z of $\mathbb{R}$ such that, for any point $x \in \mathbb{R} \setminus Z$, the sequence $(F_n(x))_{n\ge 1}$ converges to $F(x)$.*

Proof. After Lemma 6.6, ν is a probability measure. Let Z be the set of $x \in \mathbb{R}$ such that $\nu(\{x\}) \neq 0$. Since ν is of total mass 1, the set Z is numerable. If r is a point in $\mathbb{R} \setminus Z$, the set of discontinuous points of the function $\mathbb{1}_{]-\infty,r]}(x)$ is $\{r\}$, which is μ-negligible. After [Bou65] IV.5 Proposition 22, the sequence $(F_n(x))_{n\geq 1}$ converges to $F(x)$. □

Proposition 6.8 *Let $(f_n)_{n\geq 1}$ be a sequence of right continuous decreasing functions valued in $[0,1]$ such that*

i) $\sup_{n\geq 1} \inf\{x \in \mathbb{R} \mid f_n(x) = 0\} < +\infty$, $\inf_{n\geq 1} \sup\{x \in \mathbb{R} \mid f_n(x) = 1\} > -\infty$;

ii) there exists a numerable subset Z of $\mathbb{R}$ such that, for any $x \in \mathbb{R}\setminus Z$, the sequence $(f_n(x))_{n\geq 1}$ converges.

Let $f : \mathbb{R} \to [0,1]$ be a right continuous function such that $f(x) = \lim_{n\to+\infty} f_n(x)$ for any $x \in \mathbb{R}\setminus Z$. Then

1) the function f is decreasing;

2) if we write $A := \liminf_{n\to+\infty} \inf\{x \in \mathbb{R} \mid f_n(x) = 0\}$, $B := \limsup_{n\to+\infty} \sup\{x \in \mathbb{R} \mid f_n(x) = 1\}$, then $f|_{]A,+\infty[} \equiv 0$, $f|_{]-\infty,B[} \equiv 1$;

3) there exists a numerable subset Z' of $]0,1[$ such that $(f_n^*(t))_{n\geq 1}$ converges to $f^*(t)$ for any $t \in]0,1[\setminus Z'$;

4) the function sequence $(\int_0^t f_n^*(s)\mathrm{d}s)_{n\geq 0}$ converges uniformly to $\int_0^t f^*(s)\mathrm{d}s$.

Proof. 1) and 2) are easy to verify.

3) After the condition i), the function f_n^* is well defined for any $n \geq 1$. After 2), the function f^* is well defined. If t is a number in $]0,1[$ and if $y = f^*(t)$, then there exists a a strictly increasing sequence $(x_m)_{m\geq 1} \subset \mathbb{R} \setminus Z$ which converges to y. Since $x_m < y$, we have $f(x_m) > t$. Since $x_m \in \mathbb{R} \setminus Z$, there exists $N(m) \in \mathbb{Z}_{\geq 0}$ such that $f_n(x_m) > t$ (i.e., $x_m < f_n^*(t)$) for any $n > N(m)$, which implies that $\liminf_{n\to+\infty} f_n^*(t) \geq f^*(t)$.

For any integer $n \geq 1$, let Z_n' be the set of all $t \in]0,1[$ such that $f_n^{-1}(\{t\})$ has an interior point. Clearly Z_n' is a numerable set. Let Z'' be the set of $t \in]0,1[$ such that $f^{-1}(\{t\})$ has an interior point. Let Z' be the union of all Z_n' and Z''. It is also a numerable subset of $]0,1[$. Let t be a point in $]0,1[\setminus Z'$ and $y = f^*(t)$. We take a strictly decreasing sequence $(x_m)_{m\geq 1} \subset \mathbb{R} \setminus Z$ which converges to y. Since $y \notin Z''$, we have $f(x_m) < t$. Therefore, there exists $N(m) \in \mathbb{Z}_{>0}$ such that, for any $n > N(m)$, $f_n(x_m) < t$ and *a fortiori* $x_m \geq f_n^*(t)$. We then have $\limsup_{n\to+\infty} f_n^*(t) \leq f^*(t)$.

4) After Proposition 6.1 3), the condition i) implies that the functions f_n^* are uniformly bounded. On the other hand, $f_n^* - f$ converges almost everywhere to the zero function. After the Lebesgue's dominate convergence theorem, we obtain that

$$\left| \int_0^t f_n^*(s)\mathrm{d}s - \int_0^t f^*(s)\mathrm{d}s \right| \leq \int_0^1 |f_n^*(s) - f^*(s)|\mathrm{d}s$$

converges to 0 when $n \to +\infty$. □

7 Convergence of polygons of a quasi-filtered graded algebra

We establish in this section the convergence of polygons of a quasi-filtered graded algebra. By using the results obtained in Section 4, we show that the measures associated to a quasi-filtered graded algebra converge vaguely to a Borel probability measure on $\mathbb{R}$, and therefore, the associated polygons converge uniformly to a concave function on $[0,1]$.

We first recall some facts about Poincaré series of a graded module, which we shall use later. Let A be an Artinian ring, B be a graded A-algebra of finite type and genenrated by B_1, and M be a non-zero graded B-module of finite type. For any $n \in \mathbb{Z}$, M_n is an A-module of finite type, therefore of finite length. We denote by H_M the Poincaré series associated to M, i.e., $H_M(X) = \sum_{n\in\mathbb{Z}} \mathrm{len}_A(M_n)X^n \in \mathbb{Z}[\![X]\!]$. The theory of Poincaré series affirms (cf. [Bou83]) that there exists an integer $r \geq 0$ such that $H_M(X)$ can be written in the form

$$H_M(X) = a_r(X)(1-X)^{-r} + a_{r-1}(X)(1-X)^{-r+1} + \cdots + a_0(X), \tag{13}$$

where $a_0, \cdots, a_r$ are elements in $\mathbb{Z}[X, X^{-1}]$, a_r being non-zero and having positive coefficients if M is non-zero. Moreover, the values r and $a_r(1)$ don't depend on the choice of $(a_0, \cdots, a_r)$. In fact, r identifies with the dimension of M. We write $c(M) = a_r(1)$. Clearly we have

$$\mathrm{len}_A(M_n) = \frac{c(M)}{(r-1)!}n^{r-1} + o(n^{r-1})$$

when $n \to +\infty$, and there exists a polynom Q_M with coefficients in $\mathbb{Q}$ such that $Q_M(n) = \mathrm{len}_A(M_n)$ for sufficiently large integer n. If M is the zero B-module, by convention we define $\dim(M) = -\infty$ and $c(M) = 0$.

If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of graded B-modules of finite type, we have $H_M = H_{M'} + H_{M''}$. Therefore, $\dim M = \max(\dim M', \dim M'')$ and

$$c(M) = c(M')\,\mathbb{1}_{\{\dim M' \geq \dim M''\}} + c(M'')\,\mathbb{1}_{\{\dim M'' \geq \dim M'\}}\,. \tag{14}$$

Definition 7.1 Let K be a field, B be a graded K-algebra of finite type which is generated by B_1 and M be a graded B-module of finite type and of dimension $d > 0$. Suppose that for each integer $n \geq 0$, M_n is equipped with a separated, exhaustive and left continuous $\mathbb{R}$-filtration. We say that M satisfies the *vague convergence condition* and we write $\mathbf{CV}(M)$ if the sequence of Radon measures $(T_{\frac{1}{n}}\nu_{M_n})_{n\geq 1}$ converges vaguely. Finally, if N is a graded B-module which is of dimension 0 or is zero, then by convention N satisfies the vague convergence condition (in fact, for any sufficiently large integer n, we have $N_n = 0$, so $T_{\frac{1}{n}}\nu_{N_n}$ is the zero measure).

Although not explicitly stated, in Section 4, we have essentially proved the vague convergence of measures associated to a quasi-filtered symmetric algebra. We now state this result as follows.

Proposition 7.2 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $\lim_{n\to+\infty} f(n)/n = 0$ and V be a vector space of dimension $0 < d < +\infty$ over K. For any integer $n \geq 0$ let $B_n = S^nV$. Suppose that each vector space B_n is equipped with a separated, exhaustive and left continuous $\mathbb{R}$-filtration such that the graded algebra $B = \bigoplus_{n\geq 0} B_n$ is f-quasi-filtered. Then B satisfies the vague convergence condition.*

Proof. For any integer $n \geq 1$, denote by $\nu_n = T_{\frac{1}{n}}\nu_{B_n}$. Let G be the set of Borel functions g on $\mathbb{R}$ such that, for any $n \in \mathbb{Z}_{\geq 0}$, g is integrable with respect to ν_n and such that $(\int g\mathrm{d}\nu_n)_{n\geq 0}$

converges. Corollary 5.3 implies that G contains all increasing, bounded from above, concave and Lipschitz functions. Cleary G is a vector space over $\mathbb{R}$. Suppose that f is a function in $C_0^\infty(\mathbb{R})$. Let $I = [a, b]$ be an interval containing the support of f. Notice that f' and f'' are also smooth functions and the supports of f' and of f'' are contained in I. Therefore, f' and f'' are bounded functions. Let $C = \|f'\|_{\sup}$ and $C' = \|f''\|_{\sup}/2$. Let h be the function

$$h(x) = \begin{cases} C'(b-a)(2x-a-b) + C(x-b), & x \leq a, \\ -C'(b-x)^2 + C(x-b), & a < x \leq b, \\ 0, & x > b. \end{cases}$$

It's a concave, increasing and $(2C'(b-a)+C)$-Lipschitz function which is bounded from above by 0. Hence $h \in G$. On the other hand, $h + f$ is also a concave function since $h'' = -2C'$ on I. It is also increasing because $h'(x) \geq 0$ on $\mathbb{R}$ and $h'(x) \geq C$ on I. Furthermore, it is $(2C'(b-a)+2C)$-Lipschitz and bounded from above by $\|f\|_{\sup}$. Therefore, we have $h+f \in G$. We then deduce $f \in G$. Finally, since $C_0^\infty(\mathbb{R})$ is dense in the normed space $(C_c(\mathbb{R}), \|\cdot\|_{\sup})$, we obtain $C_c(\mathbb{R}) \subset G$.

Let $S : C_c(\mathbb{R}) \to \mathbb{R}$ be the opeartor which associates to each continous function g with compact support the limit of the sequence $(\int g \mathrm{d}\nu_n)_{n\geq 1}$. It's a linear operator. Furthermore, if g is a positive function in $C_c(\mathbb{R})$, then $\int g\mathrm{d}\nu_n \geq 0$ for any $n \in \mathbb{Z}_{\geq 0}$. Therefore, we have $S(g) \geq 0$. After Riesz's representation theorem, there exists a unique Radon measure ν on $\mathbb{R}$ such that $S(g) = \int g\mathrm{d}\nu$. By definition the sequence $(\nu_n)_{n\geq 1}$ converges vaguely to ν. □

In the following, we shall establish the vague convergence (Theorem 7.5) for a general quasi-filtered graded algebra of finite type over a field. We begin by introducing two technical lemmas (7.3 and 7.4), which are useful to prove Theorem 7.5.

Lemma 7.3 *Let B be a graded K-algebra of finite type which is generated by B_1,*

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\pi} M'' \longrightarrow 0$$

be a short exact sequence of graded B-modules of finite type. We denote by $d' = \dim M'$, $d = \dim M$ and $d'' = \dim M''$. Suppose that for any integer $n \geq 0$ (resp. n), B_n (resp. M_n) is equipped with a separated, exhaustive and left continuous $\mathbb{R}$-filtration. Suppose furthermore that for each integer $n \geq 0$, M'_n (resp. M''_n) is equipped with the inverse image filtration (resp. strong direct image filtration), then

1) if $d' > d''$, then $\mathbf{CV}(M') \Longleftrightarrow \mathbf{CV}(M)$;

2) if $d'' > d'$, then $\mathbf{CV}(M'') \Longleftrightarrow \mathbf{CV}(M)$;

3) if $d' = d''$, then $\mathbf{CV}(M')$ and $\mathbf{CV}(M'') \Longrightarrow \mathbf{CV}(M)$.

Proof. Let $\alpha' = c(M')$, $\alpha = c(M)$, and $\alpha'' = c(M'')$. If $\dim M' = 0$, then for sufficiently large n, we have $M_n = M''_n$, so $\mathbf{CV}(M'') \Longleftrightarrow \mathbf{CV}(M)$. Hence the proposition is true when $\dim M' = 0$. Similarly it is also true when $\dim M'' = 0$. In the following we suppose $\min(d', d'') \geq 1$. We then have $d = \max(d', d'')$. For any integer $n \geq 0$, let

$$\nu'_n = T_{\frac{1}{n}}\nu_{M'_n}, \quad \nu_n = T_{\frac{1}{n}}\nu_{M_n}, \quad \nu''_n = T_{\frac{1}{n}}\nu_{M''_n}$$

and

$$r'_n = \mathrm{rk}\, M'_n, \quad r_n = \mathrm{rk}\, M_n, \quad r''_n = \mathrm{rk}\, M''_n.$$

For sufficiently large integer n, r'_n, r_n and r''_n are strictly positive. Moreover, by Proposition 2.12, $\nu_n = \frac{r'_n}{r_n}\nu'_n + \frac{r''_n}{r_n}\nu''_n$. The measures ν'_n, ν_n and ν''_n are of bounded total masses, and we have the following estimations:

$$r'_n = \frac{\alpha'}{(d'-1)!}n^{d'-1} + o(n^{d'-1}), \quad r''_n = \frac{\alpha''}{(d''-1)!}n^{d''-1} + o(n^{d''-1}), \quad r_n = r'_n + r''_n.$$

1) If $d' > d''$, then $\lim_{n\to+\infty} r'_n/r_n = 1$, $\lim_{n\to+\infty} r''_n/r_n = 0$, so $(\nu_n)_{n\geq 1}$ converges vaguely if and only if $(\nu'_n)_{n\geq 1}$ converges vaguely, and if it is the case, they have the same limit.

2) It is similar to 1).

3) If $d'' = d'$, then $\alpha = \alpha' + \alpha''$, and

$$\lim_{n\to+\infty} \frac{r'_n}{r_n} = \frac{\alpha'}{\alpha'+\alpha''}, \quad \lim_{n\to+\infty} \frac{r''_n}{r_n} = \frac{\alpha''}{\alpha'+\alpha''}.$$

If $(\nu'_n)_{n\geq 1}$ converges vaguely to ν' and if $(\nu''_n)_{n\geq 1}$ converges vaguely to ν'', then $(\nu_n)_{n\geq 1}$ converges vaguely to $\frac{\alpha'}{\alpha'+\alpha''}\nu' + \frac{\alpha''}{\alpha'+\alpha''}\nu''$. □

Lemma 7.4 *Let V and V' be two vector spaces of finite rank over K, equipped with separated, exhaustive and left continuous $\mathbb{R}$-filtrations, $\varphi : V \to V'$ be an isomorphism of vector spaces over K and c be a real number. If $\lambda(x) \leq \lambda(\varphi(x)) + c$ for any element $x \in V$, then $\nu_V \prec \tau_c\nu_{V'}$.*

Proof. Let $\mathbf{e} = (e_i)_{1\leq i\leq n}$ be a maximal base of V. Then $\mathbf{e}' = (\varphi(e_i))_{1\leq i\leq n}$ is a base of V'. Hence $\tau_c\nu_{V'} \succ \tau_c\nu_{\mathbf{e}'} = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda(\varphi(e_i))+c} \succ \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda(e_i)} = \nu_E$. □

We now present the general convergence theorem. As we have already proved the special case of symmetric algebras, it is quite natural to expect that the general case follows by using the method of unscrewing. However, as we shall see in Remark 7.7, the theorem cannot be generalized to quasi-filtered graded modules. Even for modules generated by one homogeneous element, the convergence of associated probability measures fails in general. Therefore, the first step of unscrewing doesn't work. In fact, the major difference between filtration and grading is that, in a graded algebra, the homogeneous degree of the product of two homogeneous elements equals to the sum of homogeneous degrees, as for (quasi-)filtrated algebra, we only give a lower bound for the index of the product, which doesn't prevent the product going "far away" in the filtration. More precisely, the graded algebra associated to a filtered algebra of finite type over K need not be of finite type over K in general.

The proof of the theorem below uses the Noether's normalization theorem, which provides a subalgebra isomorphic to a symmetric algebra over which the algebra is finite. It is this finiteness which prevents the product of two element from going too "far away".

Theorem 7.5 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $\lim_{n\to+\infty} f(n)/n = 0$ and $B = \bigoplus_{n\geq 0} B_n$ be an integral graded K-algebra of finite type over K, which is generated by B_1 as K-algebra. Suppose that*

i) $d = \dim B$ is strictly positive,

ii) for any positive integer n, B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ which is separated, exhaustive and left continuous, such that B is an f-quasi-filtered graded K-algebra,

iii) $\limsup_{n\to+\infty} \sup_{0\neq a\in B_n} \dfrac{\lambda(a)}{n} < +\infty$.

For any integer $n>0$, we denote by $\nu_n = T_{\frac{1}{n}}\nu_{B_n}$. Then

1) $\lim_{n\to+\infty} \min_{a\in B_n} \dfrac{\lambda(a)}{n}$ *exists in* $\mathbb{R}$,

2) the supports of ν_n ($n\geq 1$) are uniformly bounded and the sequence of measures $(\nu_n)_{\geq 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$.

Proof. For any integer $n\geq 1$, let

$$\lambda_n^{\max} = \sup_{0\neq a\in B_n} \lambda(a) \text{ and } \lambda_n^{\min} = \min_{a\in B_n} \lambda(a).$$

The support of ν_n is contained in $[\lambda_n^{\min}/n, \lambda_n^{\max}/n]$. Since $0<d<+\infty$, for any integer $n>0$, B_n is a non-zero vector space of finite rank, so $\lambda_n^{\min}\in\mathbb{R}$ since the filtration of B_n is exhaustive. On the other hand, there exists an element a_n in B_n such that $\lambda_n^{\min}=\lambda(a_n)$. Let

$$W_n = \{b_1\cdots b_n \mid b_i\in B_1 \text{ for any } 1\leq i\leq n\}.$$

Since B is generated by B_1, B_n is generated as vector space over K (even as commutative group) by W_n. After Proposition 2.1 2), we may suppose $a_n\in W_n$. Clearly a_n is non-zero for any integer $n>1$. If $\mathbf{n}=(n_i)_{1\leq i\leq r}$ is a multi-index in $\mathbb{Z}_{>0}^r$ and if $N=n_1+\cdots+n_r$, we can write a_N as the product of r elements $c_1,\cdots,c_r$, where $c_i\in B_{n_i}\setminus\{0\}$. Therefore,

$$\lambda_N^{\min} = \lambda(a_N) \geq \sum_{i=1}^r \Big(\lambda(c_i)-f(n_i)\Big) \geq \sum_{i=1}^r \Big(\lambda_{n_i}^{\min} - f(n_i)\Big). \tag{15}$$

The condition iii) implies that $\limsup_{n\to+\infty} \lambda_n^{\min}/n < +\infty$, so the sequence $(\lambda_n^{\min}/n)_{n\geq 1}$ has a limit in $\mathbb{R}$ (by Corollary 3.2) and therefore is bounded from below. On the other hand, the condition iii) shows that the sequence $(\lambda_n^{\max}/n)_{n\geq 1}$ is bounded from above. Hence the supports of measures ν_n ($n\geq 1$) are uniformly bounded.

We now prove the second assertion of the theorem. After Lemma 6.6, it suffices to verify $\mathbf{CV}(B)$.

Step 1: some simplifications.

First, after possible extension of fields, we may suppose that K is infinite.

Let c be a real constant. We consider the filtration $\mathcal{F}^c$ of B such that $\mathcal{F}^c_t B_n = \mathcal{F}_{t-cn}B_n$. In other words, for any element $a\in B_n$, we have the equality $\lambda_{\mathcal{F}^c}(a)=\lambda_{\mathcal{F}}(a)+cn$. If $(n_i)_{1\leq i\leq r}\in\mathbb{Z}_{>0}^r$ is an multi-index and if for any i, a_i is an element in B_{n_i}, in writing $N=n_1+\cdots+n_r$, $a=a_1\cdots a_r$, we have

$$\lambda_{\mathcal{F}^c}(a) = \lambda_{\mathcal{F}}(a)+cN \geq \sum_{i=1}^r \Big(\lambda_{\mathcal{F}}(a_i)-f(n_i)\Big) + \sum_{i=1}^r cn_i = \sum_{i=1}^r \Big(\lambda_{\mathcal{F}^c}(a_i)-f(n_i)\Big),$$

in other words, B is f-quasi-filtered for the filtration $\mathcal{F}^c$. On the other hand, if we denote by $\nu^c_{B_n}$ the probability measure associated to B_n for the filtration $\mathcal{F}^c$, we have $\nu^c_{B_n}=\tau_{cn}\nu_{B_n}$. Therefore, $T_{\frac{1}{n}}\nu^c_{B_n} = T_{\frac{1}{n}}\tau_{cn}\nu_{B_n} = \tau_c T_{\frac{1}{n}}\nu_{B_n}$. Hence B satisfies the vague convergence condition for the filtration $\mathcal{F}$ if and only if it is the case for the filtration $\mathcal{F}^c$. After the proof of the first assertion we obtain $\lambda_n^{\min}=O(n)$. Since $f(n)=o(n)$, we have $\lambda_n^{\min}-f(n)=O(n)$. In replacing

the filtration $\mathcal{F}$ by $\mathcal{F}^c$, where $c \in \mathbb{R}_{>0}$ is sufficiently large, we reduce the problem to the case where $\lambda_n^{\min} - f(n) \geq 0$ for any $n \geq 1$. In particular, for any homogeneous element a of B of homogeneous degree n, we have $\lambda(a) - f(n) \geq 0$.

Step 2: Since K is an infinite field, by Noether's normalization (cf. [Eis95] Theorem 13.3), there exist d elements $x_1, \cdots, x_d$ in B_1 such that

1) the homomorphism from the polynomial algebra $K[T_1, \cdots, T_d]$ to B, which sends T_i to x_i, is an isomorphism of graded K-algebras from $K[T_1, \cdots, T_d]$ to its image,

2) if we denote by A this image, i.e., the sub-K-algebra of B generated by $x_1, \cdots, x_d$, then B is a graded A-module of finite type.

The sub-K-algebra A, equipped with the inverse image filtrations, is an f-quasi-filtered graded K-algebra. Moreover, B is an f-quasi-filtered graded A-module. Proposition 7.2 shows that we have $\mathbf{CV}(A)$.

Let a be a non-zero homogeneous element of A. We equip Aa with the inverse image filtration. Since $\dim A/Aa < \dim A$, we have $\mathbf{CV}(Aa)$ after Lemma 7.3. Furthermore, the sequences $(T_{\frac{1}{n}}\nu_{A_n})_{n\geq 1}$ and $(T_{\frac{1}{n}}\nu_{(Aa)_n})_{n\geq 1}$ of probability measures on $\mathbb{R}$ converge vaguely to the same probability measure on $\mathbb{R}$.

If x is a homogeneous element of degree $m > 0$ in B, then there exists a unitary polynomial $P \in A[X]$ of degree $p \geq 1$ such that $P(x) = 0$. We may suppose that P is minimal and is written in the form

$$P(X) = X^p + a_{p-1}X^{p-1} + \cdots + a_0.$$

Since P is minimal and since B is an integral ring, a_0 is non-zero. For any integer $0 \leq i < p$, let $\widetilde{a}_i$ be the component of degree $(p-i)m$ of a_i. If we write

$$\widetilde{P}(X) = X^p + \widetilde{a}_{p-1}X^{p-1} + \cdots + \widetilde{a}_0,$$

then we still have $\widetilde{P}(x) = 0$ since x is homogeneous of degree m. Therefore we can suppose that a_i is homogeneous of degree $(p-i)m$ for any $0 \leq i < p$. Let $y = x^{p-1} + a_{p-1}x^{p-2} + \cdots + a_1$, which is homogeneous of degree $(p-1)m$. Moreover we have $xy + a_0 = 0$. If u is a homogeneous element of degree n of A, then

$$\lambda(ua_0) = \lambda(uxy) \geq \lambda(ux) - f(n+m) + \lambda(y) - f((p-1)m) \geq \lambda(ux) - f(n+m). \tag{16}$$

We then deduce that $\lambda(ux) \leq \lambda(ua_0) + f(n+m)$. On the other hand,

$$\lambda(ux) \geq \lambda(u) + \lambda(x) - f(m) - f(n) \geq \lambda(u) - f(n). \tag{17}$$

Let $M = Aa_0$ and $M' = Ax$. The algebra B being integral, for any integer $n \geq 1$, the mapping $ux \mapsto ua_0$ ($u \in A_n$) from M'_{n+m} to M_{n+mp} is an isomorphism of vector spaces over K. After (16) and Lemma 7.4, we have $\nu_{M'_{n+m}} \prec \tau_{f(n+m)}\nu_{M_{n+mp}}$. On the other hand, the mapping $u \mapsto ux$ ($u \in A_n$) from A_n to M'_{n+m} is an isomorphism of vector spaces over K. After (17) and Lemma 7.4, we obtain $\nu_{A_n} \prec \tau_{f(n)}\nu_{M'_{n+m}}$, or equivalently $\tau_{-f(n)}\nu_{A_n} \prec \nu_{M'_{n+m}}$. So we have the estimation

$$\tau_{-f(n)}\nu_{A_n} \prec \nu_{M'_{n+m}} \prec \tau_{f(n+m)}\nu_{M_{n+mp}},$$

and hence

$$T_{\frac{1}{n+m}}\tau_{-f(n)}\nu_{A_n} \prec T_{\frac{1}{n+m}}\nu_{M'_{n+m}} \prec T_{\frac{1}{n+m}}\tau_{f(n+m)}\nu_{M_{n+mp}},$$

or equivalently

$$\tau_{-f(n)/(n+m)}T_{\frac{n}{n+m}}T_{\frac{1}{n}}\nu_{A_n} \prec T_{\frac{1}{n+m}}\nu_{M'_{n+m}} \prec \tau_{f(n+m)/(n+m)}T_{\frac{n+mp}{n+m}}T_{\frac{1}{n+mp}}\nu_{M_{n+mp}}. \tag{18}$$

As proved above, the sequences $(T_{\frac{1}{n}}\nu_{A_n})_{n\geq 1}$ and $(T_{\frac{1}{n}}\nu_{M_n})_{n\geq 1}$ converge vaguely to the same limit ν. After Proposition 6.5 and the estimation (18), we conclude that the sequence $(T_{\frac{1}{n}}\nu_{M'_n})_{n\geq 1}$ converges vaguely to ν.

Step 3: Since B is a finite algebra over A, the algebra $L\otimes_A B$ is of finite rank over L, where L is the quotient field of A. The A-module B is generated by homogeneous elements, hence there exist homogeneous elements $x_1,\cdots,x_s$ of B forming a base of $L\otimes_A B$ over L. If we write $H=Ax_1+\cdots+Ax_s$, then H is a free sub-A-module of base $(x_1,\cdots,x_s)$ of B. Let $H'=B/H$. We have an exact sequence:

$$0\longrightarrow H\xrightarrow{\psi} B\xrightarrow{\pi} H'\longrightarrow 0\ .$$

Since $1\otimes\psi : L\otimes_A H\to L\otimes_A B$ is an isomorphism, we have $L\otimes_A H'=0$, so H' is a torsion A-module. Then $\dim_A H'<\dim A=\dim_A H=\dim_A B$. After the step 2 we have $\mathbf{CV}(Ax_i)$ for any $1\leq i\leq s$. After Lemma 7.3, we obtain $\mathbf{CV}(H)$ and hence $\mathbf{CV}(B)$. □

Remark 7.6 In Theorem 7.5, if we suppose that the vector space B_n is non-zero for sufficiently large n (this condition is notably satisfied when $B_1\neq 0$), then the condition that B is generated by B_1 is not necessary. In fact, after [GD61] II, 2.1.6, there exists an integer $d>0$ such that $B^{(d)}=\bigoplus_{n\geq 0}B_{nd}$ is a B_0-algebra generated by $B_1^{(d)}=B_d$. Moreover, if we denote by $g:\mathbb{Z}_{\geq 0}\to\mathbb{R}_{\geq 0}$ the mapping such that $g(n)=f(nd)$, then $B^{(d)}$ is a g-quasi-filtered K-algebra. After Theorem 7.5, the algebra $B^{(d)}$ satisfies the vague convergence condition. Hence by an argument similar to the second step of the proof of Theorem 7.5, for any non-zero homogeneous element x of B, $B^{(d)}x$ satisfies the vague convergence condition, and the sequence of probability measures associated to $B^{(d)}x$ converges to the limit of that associated to $B^{(d)}$. We suppose that $B_n\neq 0$ for any $n\geq m_0$. Then for any integer $m_0\leq k<m_0+d$, the $B^{(d)}$-module $B^{(d,k)}=\bigoplus_{n\geq 0}B_{nd+k}$ is non-zero. By an argument similar to the third step of the proof of Theorem 7.5 using the fact that $B^{(d)}$ is an integral ring, we conclude that $B^{(d,k)}$ satisfies the vague convergence condition, and that the limit of the sequence of probability measures associated to $B^{(d,k)}$ coincides with that of probability measures associated to $B^{(d)}$. Finally, combining all these measure sequences, Proposition 6.5 shows that the sequence of probability measures associated to B converges vaguely.

Remark 7.7 1) Theorem 7.5 is not true in general for a quasi-filtered graded module. In fact, let B be the algebra $K[X]$ of polynomials in one variable, equipped with the usual graduation and with the filtration $\mathcal{F}$ such that

$$\mathcal{F}_t B_n=\begin{cases}B_n, & t\leq 0,\\ 0, & t>0.\end{cases}$$

Clearly B is a quasi-filtered graded K-algebra. Let M be a free graded B-module generated by one homogeneous element of degree 0. If $\varphi:\mathbb{Z}_{\geq 0}\to\mathbb{R}$ is an increasing function, we can define a filtration $\mathcal{F}^\varphi$ of M such that

$$\mathcal{F}^\varphi_t M_n=\begin{cases}M_n, & t\leq\varphi(n),\\ 0, & t>\varphi(n).\end{cases}$$

Then M is a quasi-filtered graded B-module, and for any integer $n\geq 0$, $\nu_{M_n}=\delta_{\varphi(n)}$. Notice that the condition $\mathbf{CV}(M)$ is equivalent to the assertion that $\lim_{n\to+\infty}\varphi(n)/n$ exists

in $\mathbb{R}\cup\{+\infty\}$. If $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is an increasing function such that the sequence $(\varphi(n)/n)_{n\geq 1}$ has more than one accumulation point — for example, if $\varphi(n) = 2^{\lfloor \log_2 n \rfloor}$, then $\mathbf{CV}(M)$ is no longer satisfied. This counter-example shows that it is not possible to prove Theorem 7.5 by using the classical version of unscrewing.

2) Theorem 7.5 is not true in general for a quasi-filtered graded algebra which is not integral. In fact, if B is a quasi-filtered algebra over K and if M is a quasi-filtered graded B-module. We suppose that $\mathbf{CV}(B)$ is satisfied, but the condition $\mathbf{CV}(M)$ is **not** satisfied (after 1), this is always possible). If we denote by C the nilpotent extension of B by M (see [Mat89] chap. 9 §25), then C is a filtered graded algebra over K, which is of finite type. But the condition $\mathbf{CV}(C)$ is not satisfied.

Corollary 7.8 *With the notations of Theorem 7.5, if for any $n \in \mathbb{N}$, we denote by P_n the polygon associated to the probability measure ν_n, then the sequence of polygons $(P_n)_{n\geq 1}$ converges uniformly to a concave function on $[0,1]$. If $B_n \neq 0$ for sufficiently large n, the same result remains true if we remove the condition that B is generated as K-algebra by B_1.*

8 Convergence of Harder-Narasimhan polygons: relative geometric case

Using the results established in the previous section, notably Theorem 7.5 and Remark 7.6, we obtain in Theorem 8.7 the convergence of normalized Harder-Narasimhan polygons for an algebra in vector bundles on a non-singular projective curve.

Let k be a field, C be a non-singular projective curve of genus g over k, η be the generic point of C and K be the field of rational functions on C. As explained in the introduction, we shall associate to each non-zero vector bundle E on C an $\mathbb{R}$-filtration of E_K which is separated, exhaustive and left continuous. Let

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$$

be the Harder-Narasimhan flag of E, which induces a flag

$$0 = E_{0,K} \subsetneq E_{1,K} \subsetneq E_{2,K} \subsetneq \cdots \subsetneq E_{n,K} = E_K$$

of the vector space E_K. Furthermore, if we write $\mu_i = \mu(E_i/E_{i-1})$ for $1 \leq i \leq n$, then the sequence of rational numbers $(\mu_i)_{1\leq i\leq n}$ is strictly decreasing.

Therefore, we obtain a filtration $\mathcal{F}^{\mathrm{HN}}$ of E_K such that

$$\mathcal{F}^{\mathrm{HN}}_s E_K = \begin{cases} 0, & \text{if } s > \mu_1, \\ E_{i,K}, & \text{if } \mu_{i+1} < s \leq \mu_i, \quad 1 \leq i \leq n, \\ E_K & \text{if } s \leq \mu_n, \end{cases}$$

called the *Harder-Narasimhan filtration* of E_K. Note that the normalized Harder-Narasimhan polygon of E identifies with the polygon associated to the Harder-Narasimhan filtration of E_K.

We recall that if $\varphi : F \to G$ is a non-zero homomorphism of vector bundles on C, then the inequality $\mu_{\min}(F) \leq \mu(\varphi(F)) \leq \mu_{\max}(G)$ holds. We obtain therefore the following proposition.

Proposition 8.1 *Let $\varphi : F \to E$ be a homomorphism of vector bundles on C. For any real number s, the image $\varphi_K(F_K)$ is contained in $\mathcal{F}^{\mathrm{HN}}_s E_K$ if $\mu_{\min}(F) \geq s$.*

Proof. The case where $\varphi = 0$ is trivial. We assume hence $\varphi \neq 0$. First, for any real number $s \in \mathbb{R}$, $\mathcal{F}_s^{\mathrm{HN}} E_K \in \{E_{0,K}, \cdots, E_{n,K}\}$. Since the vector bundles E_i are saturated in E, $\varphi_K(F_K) \subset E_{i,K}$ if and only if $\varphi(F) \subset E_i$. Therefore, if i is the smallest index such that $\varphi_K(F_K) \subset E_{i,K}$, which is always ≥ 1 because $\varphi \neq 0$, then $\mu_{\min}(F) \leq \mu_{\max}(E_i/E_{i-1}) = \mu_i$ since the composed homomorphism $F \xrightarrow{\varphi} E_i \longrightarrow E_i/E_{i-1}$ is non-zero. Therefore we have $s \leq \mu_i$, so $\varphi_K(F_K) \subset E_{i,K} \subset \mathcal{F}_s^{\mathrm{HN}} E_K$. □

Proposition 8.1 implies in particular that, for any subbundle $F \subset E$ such that $\mu_{\min}(F) \geq s$, F_K is contained in $\mathcal{F}_s^{\mathrm{HN}} E_K$. Therefore we have

$$\mathcal{F}_s^{\mathrm{HN}} E_K = \sum_{\substack{0 \neq F \subset E \\ \mu_{\min}(F) \geq s}} F_K.$$

Corollary 8.2 *Let $\varphi : F \to E$ be a homomorphism of vector bundles on C. For any real number s, the K-linear mapping φ_K sends $\mathcal{F}_s^{\mathrm{HN}} F_K$ into $\mathcal{F}_s^{\mathrm{HN}} E_K$. In other words, the homomorphism φ_K is compatible with Harder-Narasimhan filtrations.*

Proof. Let F_s be the saturated subbundle of F such that $F_{s,K} = \mathcal{F}_s^{\mathrm{HN}} F_K$. By the definition of Harder-Narasimhan filtrations, we know that $\mu_{\min}(F_s) \geq s$ once $\mathcal{F}_s^{\mathrm{HN}} F_K$ is non-zero. Therefore, the canonical mapping from $\mathcal{F}_s^{\mathrm{HN}} F_K$ to E_K factorizes through $\mathcal{F}_s^{\mathrm{HN}} E_K$. □

In the following, we shall introduce some easy estimations for the maximal and the minimal slope of the tensor product of vector bundles on C, which will be useful in Proposition 8.6.

Lemma 8.3 *Let E be a non-zero vector bundle on C. If $H^0(C, E)$ reduces to zero, then $\mu_{\max}(E) \leq g - 1$.*

Proof. As $H^0(C, E) = 0$, for any subbundle F of E, we have $H^0(C, F) = 0$. After Riemann-Roch theorem, we have $\mathrm{rk}_k H^0(C, F) - \mathrm{rk}_k H^1(C, F) = \deg(F) + \mathrm{rk}(F)(g-1)$. If $H^0(C, F) = 0$, then $\deg(F) + \mathrm{rk}(F)(1 - g) \leq 0$, i.e. $\mu(F) \leq g - 1$. □

Let $b(C) = \min\{\deg(H) \mid H \in \mathrm{Pic}(C),\ H \text{ is ample}\}$. It is a strictly positif integer, and the set of values $\{\deg(H) \mid H \in \mathrm{Pic}(C)\}$ is exactly $b(C)\mathbb{Z}$. We define $a(C) = b(C) + g$.

Proposition 8.4 *For any non-zero vector bundle E on C, there exists a line subbundle L of E such that $\deg(L) \geq \mu_{\max}(E) - a(C)$.*

Proof. Let M be a line bundle of degree $b(C)$ on C. We write $r = \lceil (g - \mu_{\max}(E))/b(C) \rceil$. Thus

$$\frac{g - \mu_{\max}(E)}{b(C)} \leq r < \frac{g - \mu_{\max}(E) + b(C)}{b(C)}.$$

Therefore $\mu_{\max}(E \otimes M^{\otimes r}) = \mu_{\max}(E) + rb(C) \geq g$. After Lemma 8.3, we obtain $H^0(C, E \otimes M^{\otimes r}) \neq 0$. So there exists an injective homomorphism from $\mathcal{O}_C$ to $E \otimes M^{\otimes r}$. Let $L = M^{\vee \otimes r}$. Then L is a subbundle of E. On the other hand, we have $\deg(L) = -r \deg(M) = -rb(C) > \mu_{\max}(E) - g - b(C)$. Since $a(C) = b(C) + g$, we obtain $\deg(L) \geq \mu_{\max}(E) - a(C)$. □

Proposition 8.5 *If E_1 and E_2 are two non-zero vector bundles on C, then*

1) $\mu_{\max}(E_1 \otimes E_2) < \mu_{\max}(E_1) + \mu_{\max}(E_2) + a(C)$;

2) $\mu_{\min}(E_1 \otimes E_2) > \mu_{\min}(E_1) + \mu_{\min}(E_2) - a(C)$.

Proof. 1) First we prove that if $\mu_{\max}(E_1) + \mu_{\max}(E_2) < 0$, then $\mu_{\max}(E_1 \otimes E_2) < g$. In fact, if $\mu_{\max}(E_1 \otimes E_2) \geq g$, then $H^0(C, E_1 \otimes E_2) \neq 0$ (see Lemma 8.3). Therefore, there exists a non-zero homomorphism from $E_1^\vee$ to E_2, which implies that

$$\mu_{\max}(E_2) \geq \mu_{\min}(E_1^\vee) = -\mu_{\max}(E_1),$$

i.e., $\mu_{\max}(E_1) + \mu_{\max}(E_2) \geq 0$. To prove 1), we take a line bundle L on C such that $-b(C) \leq \mu_{\max}(E_1) + \mu_{\max}(E_2) + \deg(L) < 0$. We then have $\mu_{\max}(E_1 \otimes L) + \mu_{\max}(E_2) < 0$ and hence, after the result established above, $\mu_{\max}(E_1 \otimes L \otimes E_2) < g$. Therefore,

$$\mu_{\max}(E_1 \otimes E_2) < g - \deg(L) \leq \mu_{\max}(E_1) + \mu_{\max}(E_2) + g + b(C) = \mu_{\max}(E_1) + \mu_{\max}(E_2) + a(C).$$

2) In fact,

$$\begin{aligned} &\mu_{\min}(E_1 \otimes E_2) = -\mu_{\max}((E_1 \otimes E_2)^\vee) = -\mu_{\max}(E_1^\vee \otimes E_2^\vee) \\ &> -\Big(\mu_{\max}(E_1^\vee) + \mu_{\max}(E_2^\vee) + a(C)\Big) = \mu_{\min}(E_1) + \mu_{\min}(E_2) - a(C). \end{aligned}$$

□

From Proposition 8.5, we obtain by induction that if $(E_i)_{1 \leq i \leq r}$ is a family of non-zero vector bundles on C, we have the estimation

$$\mu_{\min}(E_1 \otimes \cdots \otimes E_r) \geq \sum_{i=1}^{r} \mu_{\min}(E_i) - a(C)(r-1) \geq \sum_{i=1}^{r} \mu_{\min}(E_i) - a(C)r.$$

Actually, if the field k is of characteristic 0, then we have even the equality $\mu_{\min}(E_1 \otimes \cdots \otimes E_r) = \mu_{\min}(E_1) + \cdots + \mu_{\min}(E_r)$. This is a consequence of Ramanan and Ramanathan's result [RR84] asserting that the tensor product of two semistable vector bundles on C is semistable.

Proposition 8.6 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the constant function which sends any $n \in \mathbb{Z}_{\geq 0}$ to $a(C)$. Let $\mathscr{B} = \bigoplus_{n \geq 0} \mathscr{B}_n$ be a graded quasi-coherent $\mathcal{O}_C$-algebra. Suppose that for any integer $n \geq 0$, $\mathscr{B}_n$ is a vector bundle over C, and we denote by $B_n = \mathscr{B}_{n,K}$. Then $B = \bigoplus_{n \geq 0} B_n$, equipped with Harder-Narasimhan filtrations, is an f-quasi-filtered graded K-algebra.*

Proof. For any integer $n \geq 0$ and any real number s, let $\mathscr{B}_{n,s}$ be the saturated subbundle of $\mathscr{B}_n$ such that $\mathscr{B}_{n,s,K} = \mathcal{F}_s^{\mathrm{HN}} B_n$. Since $\mathscr{B}$ is an $\mathcal{O}_C$-algebra, for any integer $r \geq 2$ and any element $(n_i)_{1 \leq i \leq r} \in \mathbb{Z}_{\geq 0}^r$, we have a natural homomorphism φ from $\mathscr{B}_{n_1} \otimes \cdots \otimes \mathscr{B}_{n_r}$ to $\mathscr{B}_N$, where $N = n_1 + \cdots + n_r$. If $(t_i)_{1 \leq i \leq r}$ is a family of real numbers, the homomorphism φ induces by restriction a homomorphism ψ from $\mathscr{B}_{n_1,t_1} \otimes \cdots \otimes \mathscr{B}_{n_r,t_r}$ to $\mathscr{B}_N$. By the definition of Harder-Narasimhan filtration we obtain that if $\mathscr{B}_{n_i,t_i}$ is non-zero, then $\mu_{\min}(\mathscr{B}_{n_i,t_i}) \geq t_i$. Therefore, by using the convention $\mu_{\min}(0) = +\infty$, we have $\mu_{\min}(\mathscr{B}_{n_1,t_1} \otimes \cdots \otimes \mathscr{B}_{n_r,t_r}) \geq t_1 + \cdots + t_r - a(C)r$. After Corollary 8.2, ψ_K is compatible with Harder-Narasimhan filtrations, so ψ_K factorizes through $\mathcal{F}_{t_1 + \cdots + t_r - a(C)r} B_N$. Therefore, B is a quasi-filtered graded K-algebra. □

Theorem 8.7 *Let $\mathscr{B} = \bigoplus_{n \geq 0} \mathscr{B}_n$ be a quasi-coherent graded $\mathcal{O}_C$-algebra. Suppose that the following conditions are verified:*

i) $\mathscr{B}_n$ is a vector bundle on C for any integer $n \geq 0$;

ii) there exists a constant $a > 0$ such that $\mu_{\max}(\mathscr{B}_n) \leq an$ for any integer $n \geq 1$;

iii) $\mathscr{B}_K$ is an integral ring which is of finite type over K and $\mathscr{B}_n$ is non-zero for sufficiently large integer n.

For any integer $n \geq 1$, we denote by P_n the Harder-Narasimhan polygon of $\mathscr{B}_n$. Then

1) the sequence of numbers $(\frac{1}{n}\mu_{\min}(\mathscr{B}_n))_{n\geq 1}$ has a limit in $\mathbb{R}$.

2) the sequence $(\frac{1}{n}P_n)_{n\geq 1}$ converges uniformly on $[0,1]$.

Proof. Let f be the constant function $\mathbb{Z}_{\geq 0}$ with value $a(C)$. After Proposition 8.6, we obtain that $\mathscr{B}_K$ equipped with Harder-Narasimhan filtrations is an f-quasi-filtered graded K-algebra. The theorem is then proved by using Theorem 7.5 (see also Remark 7.6) and Corollary 7.8. □

Let $\pi : X \to C$ be a projective and flat morphism from an algebraic variety X to C and L be a line bundle on X. We shall apply Theorem 8.7 to the special case where $\mathscr{B}$ is the direct sum of the direct images by the morphism π of tensor powers of L.

Lemma 8.8 *There exists a constant ε such that, for any integer $n > 0$,*

$$\mu_{\max}(\pi_*(L^{\otimes n})) \leq \varepsilon n.$$

Proof. The variety X is projective over $\operatorname{Spec} k$. We can hence choose an ample line bundle $\mathscr{L}$ on X.

Let $d = \dim X$. Observe that $\pi_*(c_1(\mathscr{L})^{d-1}) = (\deg_{\mathscr{L}_K} X_K)[C]$ in the Chow group $\mathrm{CH}_1(C)$. Suppose that M is a line bundle on C and that $\varphi : M \to \pi_*(\mathscr{L}^{\otimes n})$ is an injective homomorphism. We denote by $\widetilde{\varphi} : \pi^*M \to \mathscr{L}^{\otimes n}$ the homomorphism of $\mathcal{O}_X$-modules corresponding to φ by adjunction, which identifies with a non-identically zero section of $\pi^*M^\vee \otimes \mathscr{L}^{\otimes n}$, whose divisor $\operatorname{div}\widetilde{\varphi}$ is effective. Then $\deg_X\left(c_1(\mathscr{L})^{d-1}[\operatorname{div}(\widetilde{\varphi})]\right) \geq 0$. On the other hand, $[\operatorname{div}\widetilde{\varphi}] = -\pi^*c_1(M) + nc_1(L)$ in $\mathrm{CH}^1(X)$. Hence

$$\begin{aligned}&\deg_X\left(c_1(\mathscr{L})^{d-1}[\operatorname{div}(\widetilde{\varphi})]\right) = \deg_X\left((-\pi^*c_1(M) + nc_1(L))c_1(\mathscr{L})^{d-1}\right)\\ &= -\deg_C\left(c_1(M)\pi_*(c_1(\mathscr{L})^{d-1})\right) + n\deg_X(c_1(L)c_1(\mathscr{L})^{d-1}).\end{aligned}$$

Therefore, $\deg_C(M) \leq n\dfrac{\deg_X(c_1(L)c_1(\mathscr{L})^{d-1})}{\deg_{\mathscr{L}_K} X_K}$. Finally, using the comparison established in Proposition 8.4, we deduce the upper bound of $\mu_{\max}(\pi_*(\mathscr{L}^{\otimes n}))$ by a linear function on n. □

Theorem 8.9 *Suppose that $H^0(X_K, L_K^{\otimes n}) \neq 0$ for sufficiently large integer n and that the graded algebra $\bigoplus_{n\geq 0} H^0(X_K, L_K^{\otimes n})$ is of finite type over K (this condition is satisfied notably when L_K is ample). For any integer $n \geq 1$, let P_n be the Harder-Narasimhan polygon of $\pi_*(L^{\otimes n})$. Then the sequence of numbers $(\frac{1}{n}\mu_{\min}(\pi_*(L^{\otimes n})))_{n\geq 1}$ has a limit in $\mathbb{R}$ and the sequence of polygons $(\frac{1}{n}P_n)_{n\geq 1}$ converges uniformly on $[0,1]$.*

Proof. After Lemma 8.8, $\mu_{\max}(\pi_*(L^{\otimes n})) = O(n)$ $(n \to +\infty)$. Therefore, the algebra $\mathscr{B} := \bigoplus_{n\geq 0}\pi_*(L^{\otimes n})$ verifies the conditions in Theorem 8.7. □

The convergence of polygons $(\frac{1}{n}P_n)$ suggests that the sequence of (normalized) maximal slopes $(\frac{1}{n}\mu_{\max}(\pi_*(L^{\otimes n})))_{n\geq 1}$ converges. However, this is not a formal consequence of Theorem 8.9. In Proposition 8.11, we shall justify the convergence of this sequence by using the same generalization of Fekete's lemma for almost super-additive sequences.

Lemma 8.10 *Let E_1 and E_2 be two vector bundles on X. If $\pi_*(E_1)$ and $\pi_*(E_2)$ are non-zero, then*

$$\mu_{\max}(\pi_*(E_1 \otimes E_2)) \geq \mu_{\max}(\pi_* E_1) + \mu_{\max}(\pi_* E_2) - 2a(C),$$

where $a(C)$ is the constant in Proposition 8.4.

Proof. Since $\pi_*(E_1)$ and $\pi_*(E_2)$ are non-zero, also is $\pi_*(E_1 \otimes E_2)$. After Proposition 8.4, there exist two line bundles M_1 and M_2 on C and two injective homomorphisms $M_1 \to \pi_*(E_1)$ and $M_2 \to \pi_*(E_2)$ such that $\deg M_1 \geq \mu_{\max}(\pi_* E_1) - a(C)$ and $\deg M_2 \geq \mu_{\max}(\pi_* E_2) - a(C)$. Since both $M_1^\vee \otimes \pi_*(E_1)$ and $M_2^\vee \otimes \pi_*(E_2)$ have global sections which do not vanish everywhere on C, then both $\pi^*(M_1)^\vee \otimes E_1$ and $\pi^*(M_2)^\vee \otimes E_2$ have global sections which do not vanish everywhere on X. Therefore, $H^0(X, \pi^*(M_1 \otimes M_2)^\vee \otimes (E_1 \otimes E_2)) = H^0(C, (M_1 \otimes M_2)^\vee \otimes \pi_*(E_1 \otimes E_2)) \neq 0$. So we have $0 \leq \mu_{\max}((M_1 \otimes M_2)^\vee \otimes \pi_*(E_1 \otimes E_2))$, and hence $\mu_{\max}(\pi_*(E_1 \otimes E_2)) \geq \deg M_1 + \deg M_2 \geq \mu_{\max}(\pi_*(E_1)) + \mu_{\max}(\pi_*(E_2)) - 2a(C)$. □

Proposition 8.11 *Let $\pi : X \to C$ be a projective and flat morphism from an algebraic variety X to C and L be a line bundle on X verifying the conditions of Theorem 8.9. Then the sequence $(\frac{1}{n}\mu_{\max}(\pi_*(L^{\otimes n})))_{n\geq 1}$ has a limit in $\mathbb{R}$.*

Proof. Denote by $a_n = \mu_{\max}(\pi_*(L^{\otimes n}))$ for any integer $n \geq 1$. After Lemma 8.8, there exists a constant $\varepsilon > 0$ such that $a_n \leq \varepsilon n$ for sufficiently large n. On the other hand, Lemma 8.10 shows that $a_{m+n} \geq a_m + a_n - 2a(C)$ for all integers m and n. After Corollary 3.3, the sequence $(a_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$. □

9 Convergence of Harder-Narasimhan polygons: arithmetic case

In this section, we establish the analogue of the results in the previous section in Arakelov geometry. Let K be a number field and $\mathcal{O}_K$ be its integer ring. We denote by Σ_f the set of all finite places of K, which coincides with the set of all closed points in $\operatorname{Spec}\mathcal{O}_K$. Let Σ_∞ be the set of all embeddings of K into $\mathbb{C}$. Suppose that E is a projective $\mathcal{O}_K$-module of finite type, then for any finite place $\mathfrak{p}$ of K, the structure of $\mathcal{O}_K$-module on E induces an ultranorm on $E_{K_\mathfrak{p}} := E \otimes_{\mathcal{O}_K} K_\mathfrak{p}$.

We have explained that to any non-zero Hermitian vector bundle $\overline{E}$ on $\operatorname{Spec}\mathcal{O}_K$, we can associated a flag $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E$ of E such that $\overline{E}_i/\overline{E}_{i-1}$ is semistable for any integer $1 \leq i \leq n$ and that

$$\widehat{\mu}(\overline{E}_1/\overline{E}_0) > \widehat{\mu}(\overline{E}_2/\overline{E}_1) > \cdots > \widehat{\mu}(\overline{E}_n/\overline{E}_{n-1}).$$

If we write $\mu_i = \widehat{\mu}(\overline{E}_i/\overline{E}_{i-1})$ for $1 \leq i \leq n$, then the sequence of real numbers $(\mu_i)_{1\leq i\leq n}$ is strictly decreasing. Furthermore, the flag of E above induces a flag $0 = E_{0,K} \subsetneq E_{1,K} \subsetneq \cdots \subsetneq E_{n,K} = E_K$ of the vector space E_K. We obtain therefore a filtration $\mathcal{F}^{\mathrm{HN}}$ of E_K such that

$$\mathcal{F}^{\mathrm{HN}}_s E_K = \begin{cases} 0, & \text{if } s > \mu_1, \\ E_{i,K}, & \text{if } \mu_{i+1} < s \leq \mu_i, \quad 1 \leq i \leq n, \\ E_K, & \text{if } s \leq \mu_n, \end{cases}$$

called the *Harder-Narasimhan filtration* of E_K. Notice that the normalized Harder-Narasimhan polygon of $\overline{E}$ coincides with the polygon associated to the Harder-Narasimhan filtration of E_K.

Let $\overline{F}$ and $\overline{G}$ be two non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$ and $\varphi : F_K \to G_K$ be a linear mapping. For any place $\mathfrak{p} \in \Sigma_f$, φ induces a linear mapping $\varphi_{\mathfrak{p}} : F_{K_{\mathfrak{p}}} \to G_{K_{\mathfrak{p}}}$. For any embedding $\sigma \in \Sigma_\infty$, φ induces a linear mapping $\varphi_\sigma : F_\sigma \to G_\sigma$. We define the *height* of φ to be

$$h(\varphi) = \frac{1}{[K:\mathbb{Q}]}\Big(\sum_{\mathfrak{p}\in\Sigma_f} \log\|\varphi_{\mathfrak{p}}\| + \sum_{\sigma\in\Sigma_\infty} \log\|\varphi_\sigma\|\Big).$$

Notice that if φ comes from an $\mathcal{O}_K$-linear homomorphism ϕ, i.e. $\varphi = \phi_K$, then for any $\mathfrak{p} \in \Sigma_f$, $\log\|\varphi_{\mathfrak{p}}\| \leq 0$. We recall the slope inequalities:

1) if φ is injective, then $\widehat{\mu}_{\max}(\overline{F}) \leq \widehat{\mu}_{\max}(\overline{G}) + h(\varphi)$;

2) if φ is surjective, then $\widehat{\mu}_{\min}(\overline{F}) \leq \widehat{\mu}_{\min}(\overline{G}) + h(\varphi)$;

3) if φ is non-zero, then $\widehat{\mu}_{\min}(\overline{F}) \leq \widehat{\mu}_{\max}(\overline{G}) + h(\varphi)$.

For the proof of the first inequality, one can consult [Bos01]. The second inequality is obtained by applying the first one on $\varphi^\vee : G_K^\vee \to F_K^\vee$. Finally if we apply the first two inequalities on the two homomorphisms in the decomposition $F_K \twoheadrightarrow \varphi(F_K) \hookrightarrow G_K$ respectively, we obtain the third inequality. Using the seconde slope inequality, we obtain the following proposition.

Proposition 9.1 *Let $\overline{F}$ and $\overline{E}$ be two Hermitian vector bundles. If $\varphi : \overline{F}_K \to \overline{E}_K$ is a K-linear homomorphism, then for any real number $s \leq \widehat{\mu}_{\min}(\overline{F}) - h(\varphi)$, the image $\varphi(F_K)$ is contained in $\mathcal{F}^{\mathrm{HN}}_s E_K$.*

Proof. The case where $\varphi = 0$ is trivial. Suppose that $\varphi \neq 0$. Let i be the smallest index such that $\varphi(F_K) \subset E_{i,K}$, which is always ≥ 1 since $\varphi \neq 0$. Consider the composed homomorphism $\psi : F_K \xrightarrow{\varphi} E_{i,K} \longrightarrow (E_i/E_{i-1})_K$, which is non-zero. By slope inequality, $s \leq \widehat{\mu}_{\min}(\overline{F}) \leq \widehat{\mu}_{\max}(\overline{E}_i/\overline{E}_{i-1}) + h(\psi) \leq \mu_i + h(\varphi)$, or equivalently $s - h(\varphi) \leq \mu_i$. Therefore $\varphi_K(F_K) \subset E_{i,K} \subset \mathcal{F}^{\mathrm{HN}}_{s-h(\varphi)} E_K$. □

Proposition 9.1 implies that, for any Hermitian subbundle $\overline{F}$ of $\overline{E}$ such that $\widehat{\mu}_{\min}(\overline{F}) \geq s$, F_K is contained in $\mathcal{F}^{\mathrm{HN}}_s E_K$ (the height of the inclusion mapping $F_K \to E_K$ is bounded from above by 0). Therefore we obtain the relation

$$\mathcal{F}^{\mathrm{HN}}_s E_K = \sum_{\substack{0\neq F\subset E\\ \widehat{\mu}_{\min}(\overline{F})\geq s}} F_K.$$

Corollary 9.2 *Let $\overline{F}$ and $\overline{E}$ be two Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$ and $\varphi : F_K \to E_K$ be a K-linear mapping. Then for any real number s, φ sends $\mathcal{F}^{\mathrm{HN}}_s F_K$ into $\mathcal{F}^{\mathrm{HN}}_{s-h(\varphi)} E_K$.*

Proof. Let F_s be the saturated subbundle of F such that $F_{s,K} = \mathcal{F}^{\mathrm{HN}}_s F_K$. By the definition of Harder-Narasimhan filtrations we know that $\widehat{\mu}_{\min}(F_s) \geq s$ if $\mathcal{F}^{\mathrm{HN}}_s F_K$ is non-zero. Therefore, the canonical mapping from $\mathcal{F}^{\mathrm{HN}}_s F_K$ to E_K factorizes through $\mathcal{F}^{\mathrm{HN}}_{s-h(\varphi)} E_K$. □

In Corollary 9.2, if the homomorphism φ is an isomorphism, then $\tau_{h(\varphi)}\nu_{\mathcal{F}^{\mathrm{HN}},E_K} \succ \nu_{\mathcal{F}^{\mathrm{HN}},F_K}$. Therefore, for any $t \in [0,1]$, $P_{\overline{F}}(t) \leq P_{\overline{E}}(t) + h(\varphi)t$. In particular, if E is a non-zero vector bundle on $\operatorname{Spec}\mathcal{O}_K$ and if $h = (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$ and $h' = (\|\cdot\|'_\sigma)_{\sigma\in\Sigma_\infty}$ are two Hermitian structures on E, then for any $t \in [0,1]$,

$$\big|P_{(E,h)}(t) - P_{(E,h')}(t)\big| \leq \frac{t}{[K:\mathbb{Q}]} \sum_{\sigma\in\Sigma_\infty} \sup_{0\neq x\in E_{\sigma,\mathbb{C}}} \Big|\log\|x\|_\sigma - \log\|x\|'_\sigma\Big| \tag{19}$$

Let $(\overline{\mathscr{B}}_n)_{n\geq 0}$ be a collection of non-zero Hermitian vector bundles on $\operatorname{Spec}\mathcal{O}_K$. For any integer $n \geq 0$ and any $s \in \mathbb{R}$, we denote by $\mathscr{B}_{n,s}$ the saturated subbundle of $\mathscr{B}_n$ such that $B_{n,s} := \mathscr{B}_{n,s,K} = \mathscr{F}_s^{\mathrm{HN}}\mathscr{B}_{n,K}$. Suppose that $B = \bigoplus_{n\geq 0}\mathscr{B}_{n,K}$ is equipped with a structure of commutative $\mathbb{Z}_{\geq 0}$-graded algebra over K. For any integer $r \geq 2$ and any element $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{N}^r$, we have a homomorphism $\varphi_{\mathbf{n}}$ from $B_{n_1}\otimes\cdots\otimes B_{n_r}$ to $B_{|\mathbf{n}|}$ defined by the structure of algebra. After [Che07b], if $(s_i)_{1\leq i\leq r}$ is an element in $\mathbb{R}^r$, we obtain, by using the convention $\widehat{\mu}_{\min}(0) = +\infty$,

$$\widehat{\mu}_{\min}(\overline{\mathscr{B}}_{n_1,s_1}\otimes\cdots\otimes\overline{\mathscr{B}}_{n_r,s_r}) \geq \sum_{i=1}^r \widehat{\mu}_{\min}(\overline{\mathscr{B}}_{n_i,s_i}) - \sum_{i=1}^r \log(\mathrm{rk}(B_{n_i,s_i})) \geq \sum_{i=1}^r \Big(s_i - \log(\mathrm{rk}(B_{n_i}))\Big).$$

If $\overline{E}$ is a Hermitian vector subbundle of $\overline{\mathscr{B}}_{|\mathbf{n}|}$ such that E_K coincides with the image of $B_{n_1,s_1}\otimes\cdots\otimes B_{n_r,s_r}$ in $B_{|\mathbf{n}|}$, after the slope inequality, we have

$$\widehat{\mu}_{\min}(\overline{E}) \geq \widehat{\mu}_{\min}(\overline{\mathscr{B}}_{n_1,s_1}\otimes\cdots\otimes\overline{\mathscr{B}}_{n_r,s_r}) - h(\varphi_{\mathbf{n}}) \geq \sum_{i=1}^r \Big(s_i - \log(\mathrm{rk}(B_{n_i}))\Big) - h(\varphi_{\mathbf{n}}).$$

Suppose that $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a function such that $h(\varphi_{\mathbf{n}}) \leq g(n_1)+\cdots+g(n_r)$ for n_i sufficiently large. For any integer $n \geq 1$, let $f(n) = g(n) + \log(\mathrm{rk}(B_n))$. Then B is an f-quasi-filtered graded K-algebra.

Theorem 9.3 *For any integer $n \geq 0$, denote by P_n the Harder-Narasimhan polygon of $\overline{\mathscr{B}}_n$. Suppose that $\lim_{n\to+\infty} f(n)/n = 0$ and that the sequence $(\widehat{\mu}_{\max}(\overline{\mathscr{B}}_n)/n)_{n\geq 1}$ is bounded. If B is an integral K-algebra of finite type and if $B_n \neq 0$ for sufficiently large n, then the sequence $(\widehat{\mu}_{\min}(\overline{\mathscr{B}}_n)/n)_{n\geq 1}$ has a limit in $\mathbb{R}$ and the function sequence $(P_n/n)_{n\geq 1}$ converges uniformly on $[0,1]$.*

Proof. In fact, P_n coincides with the polygon associated to the filtered space B_n. The theorem results therefore from Theorem 7.5 (see also Remark 7.6) and Corollary 7.8. □

In the following, we shall establish the analogue of Theorem 8.9 in Arakelov geometry. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathcal{O}_K$ be a scheme of finite type and flat over $\operatorname{Spec}\mathcal{O}_K$ such that $\mathscr{X}_K$ is proper. Let $\overline{\mathscr{L}}$ be a Hermitian line bundle on $\mathscr{X}$. For any integer $D \geq 0$, let E_D be the projective $\mathcal{O}_K$-module $\pi_*(\mathscr{L}^{\otimes D})$. Suppose that $E_D \neq 0$ for sufficiently large D and that the algebra $B := \bigoplus_{D\geq 0} E_{D,K}$ is of finite type over K. Clearly B is integral. We denote by $\|\cdot\|_{\sigma,\sup}$ the norm on $E_{D,\sigma}$ such that $\|s\|_{\sigma,\sup} = \sup_{x\in\mathscr{X}_\sigma(\mathbb{C})}\|s_x\|_\sigma$ for any $s \in E_{D,\sigma} = H^0(\mathscr{X}_{\sigma,\mathbb{C}}, \mathscr{L}_{\sigma,\mathbb{C}}^{\otimes D})$. In general this is not a Hermitian norm. For any integer $D \geq 0$ and any $\sigma \in \Sigma_\infty$, we choose a Hermitian norm $\|\cdot\|_\sigma$ on $E_{D,\sigma}$ such that

$$\sup_{0\neq s\in E_{D,\sigma}} \Big|\log\|s\|_\sigma - \log\|s\|_{\sigma,\sup}\Big| = O(\log D) \qquad (D \to +\infty). \tag{20}$$

This is always possible by Gromov's inequality in smooth metric case (see [GS92] Lemma 30), or by John's or Löwner's ellipsoid argument in general case (see [Gau07], [Tho96]). Suppose in addition that the collection $h_D = (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty}$ is invariant by the complex conjugation. Then $\overline{E}_D = (E_D, h_D)$ becomes a Hermitian vector bundle on $\operatorname{Spec}\mathcal{O}_K$. For any integer $r \geq 2$ and any element $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{N}^r$, let $\varphi_{\mathbf{n}}$ be the canonical homomorphism from $E_{n_1,K}\otimes\cdots\otimes E_{n_r,K}$ to $E_{|\mathbf{n}|,K}$. For any integer $D \geq 1$ and any $\sigma \in \Sigma_D$, we denote by

$$A_{D,\sigma} = \sup_{0\neq s\in E_{D,\sigma}} \Big|\log\|s\|_\sigma - \log\|s\|_{\sigma,\sup}\Big|.$$

From (20) we know that there exists an integer $n_0 \geq 2$ and a real number $\varepsilon > 0$ such that $A_{D,\sigma} \leq \varepsilon \log D$ for any $D \geq n_0$ and any $\sigma \in \Sigma_\infty$.

Lemma 9.4 *We have the following inequality*

$$h(\varphi_{\mathbf{n}}) \leq \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in \Sigma_\infty} A_{|\mathbf{n}|,\sigma} + \sum_{i=1}^{r} \Big(A_{n_i,\sigma} + \frac{1}{2} \log(\mathrm{rk}(E_{n_i})) \Big). \tag{21}$$

Proof. Since $\varphi_{\mathbf{n}}$ comes from a homomorphism of $\mathcal{O}_K$-modules, $\|\varphi_{\mathbf{n}}\|_{\mathfrak{p}} \leq 1$ for any finite place $\mathfrak{p}$ of K. Consider now an embedding $\sigma \in \Sigma_\infty$. If $(s_i)_{1 \leq i \leq r} \in E_{n_1,\sigma} \times \cdots \times E_{n_r,\sigma}$, then

$$\log \|s_1 \cdots s_r\|_\sigma \leq \log \|s_1 \cdots s_r\|_{\sigma,\sup} + A_{|\mathbf{n}|,\sigma} \leq \sum_{i=1}^{r} \log \|s_i\|_{\sigma,\sup} + A_{|\mathbf{n}|,\sigma}$$
$$\leq \sum_{i=1}^{r} \Big(\log \|s_i\|_\sigma + A_{n_i,\sigma} \Big) + A_{|\mathbf{n}|,\sigma} = \log \|s_1 \otimes \cdots \otimes s_r\|_\sigma + A_{|\mathbf{n}|,\sigma} + \sum_{i=1}^{r} A_{n_i,\sigma}.$$

Since $E_{n_1,\sigma} \otimes \cdots \otimes E_{n_r,\sigma}$ contains an orthogonal base which consists of $\mathrm{rk}(E_{n_1}) \cdots \mathrm{rk}(E_{n_r})$ elements of the forme $s_1 \otimes \cdots \otimes s_r$, using Cauchy-Schwarz inequality, we obtain

$$\log \|\varphi_{\mathbf{n}}\|_\sigma \leq A_{|\mathbf{n}|,\sigma} + \sum_{i=1}^{r} \Big(A_{n_i,\sigma} + \frac{1}{2} \log(\mathrm{rk}(E_{n_i})) \Big).$$

Therefore, (21) holds. □

Remark 9.5 Lemma 9.4 implies that

$$h(\varphi_{\mathbf{n}}) \leq \sum_{i=1}^{r} \Big(2\varepsilon \log n_i + \frac{1}{2} \log(\mathrm{rk}(E_{n_i})) \Big), \quad \text{for any } \mathbf{n} \in \mathbb{N}^r_{\geq n_0}.$$

Therefore, if we define $f(n) = 2\varepsilon \log n + \dfrac{3}{2} \log(\mathrm{rk}(E_n))$, then the graded algebra B equipped with Harder-Narasimhan filtrations is f-quasi-filtered. Notice that the function f satisfies $\lim\limits_{n \to \infty} f(n)/n = 0$.

We recall a result in [BK07], which is a reformulation of Minkowski's first theorem in Arakelov geometry.

Proposition 9.6 ([BK07]) *Let $\overline{E} = (E, (\|\cdot\|_\sigma)_{\sigma \in \Sigma_\infty})$ be a non-zero Hermitian vector bundle on* $\operatorname{Spec} \mathcal{O}_K$. *The following inequality holds:*

$$\widehat{\mu}_{\max}(\overline{E}) - \frac{1}{2} \log([K:\mathbb{Q}] \operatorname{rk} E) - \frac{\log |\Delta_K|}{2[K:\mathbb{Q}]} \leq -\frac{1}{2} \log \sup_{0 \neq s \in E} \Big(\sum_{\sigma \in \Sigma_\infty} \|s\|_\sigma^2 \Big) \leq \widehat{\mu}_{\max}(\overline{E}) - \frac{1}{2} \log [K:\mathbb{Q}]. \tag{22}$$

Lemma 9.7 *There exists a constant C such that $\widehat{\mu}_{\max}(\overline{E}_D) \leq CD$ for any sufficiently large integer D.*

Proof. Let $\overline{L}$ be a Hermitian line bundle on $\mathscr{X}$ which is arithmetically ample and such that $c_1(\overline{L}) > 0$. Suppose that s is a section of $\mathscr{L}^{\otimes D}$ on $\mathscr{X}$, then $\operatorname{div} s$ is an effective divisor of $\mathscr{X}$. Therefore, we have

$$h_{\overline{L}}(\operatorname{div} s) = \widehat{c}_1(\overline{L})^d \cdot \widehat{c}_1(\overline{\mathscr{L}}^{\otimes D}) + \int_{\mathscr{X}(\mathbb{C})} \log \|s\| c_1(\overline{L})^d \geq 0.$$

On the other hand, since $c_1(\overline{L}) > 0$, we obtain

$$\int_{\mathscr{X}(\mathbb{C})} \log \|s\| c_1(\overline{L})^d \leq \max_{\sigma \in \Sigma_\infty} \log \|s\|_{\sigma,\sup} \int_{\mathscr{X}(\mathbb{C})} c_1(\overline{L})^d.$$

Therefore, by defining $C_1 = \widehat{c}_1(\overline{L})^d \cdot \widehat{c}_1(\overline{\mathscr{L}}) \left(\int_{\mathscr{X}(\mathbb{C})} c_1(\overline{L})^d \right)^{-1}$, we have $-\max_\sigma \log \|s\|_{\sigma,\sup} \leq C_1 D$, which implies $-\log \|s\|_\sigma \leq -\log \|s\|_{\sigma,\sup} + A_{D,\sigma} \leq C_1 D + \varepsilon \log D$ for any $\sigma \in \Sigma_\infty$. We then obtain after Proposition 9.6 that

$$\begin{aligned} \widehat{\mu}_{\max}(\overline{E}_D) &\leq - \sup_{0 \neq s \in E_D} \frac{1}{2} \log \Big(\sum_{\sigma \in \Sigma_\infty} \|s\|_\sigma^2 \Big) + \frac{1}{2} \log([K : \mathbb{Q}] \operatorname{rk} E_D) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} \\ &\leq C_1 D + \varepsilon \log D + \frac{1}{2} \log(\operatorname{rk} E_D) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} = O(D). \end{aligned}$$

□

Theorem 9.8 *For any sufficiently large integer D, we denote by P_D the normalized Harder-Narasimhan polygon of $\overline{E}_D$. Then the sequence $(\widehat{\mu}_{\min}(\overline{E}_D)/D)_{D \geq 1}$ has a limit in $\mathbb{R}$ and the sequence of polygons $(P_D/D)_{D \geq 1}$ converges uniformly to a concave function on $[0, 1]$.*

Proof. Notice that P_D coincides with the polygon associated to the Harder-Narasimhan filtration of $E_{D,K}$. Therefore, the theorem follows from Theorem 9.3. □

Remark 9.9 The limit of polygons in Theorem 9.8 does not depend on the choice of Hermitian metrics $\| \cdot \|_\sigma$. Suppose that for any integer $D \geq 0$ and any $\sigma \in \Sigma_\infty$, we choose another Hermitian metric $\| \cdot \|_\sigma^*$ on $E_{D,\sigma}$ such that the collection $h_D^* := (\| \cdot \|_\sigma^*)_{\sigma \in \Sigma_\infty}$ is invariant under complex conjugation and such that

$$A_{D,\sigma}^* := \sup_{0 \neq s \in E_{D,\sigma}} \Big| \log \|s\|_\sigma^* - \log \|s\|_{\sigma,\sup} \Big| = O(\log D).$$

We denote by P_D^* the normalized Harder-Narasimhan polygon of (E_D, h_D^*). After (19), we have

$$|P_D^*(t) - P_D(t)| \leq \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma \in \Sigma_\infty} \Big(A_{D,\sigma} + A_{D,\sigma}^* \Big).$$

Since $\lim_{D \to +\infty} A_{D,\sigma}/D = \lim_{D \to +\infty} A_{D,\sigma}^*/D = 0$, we know that the two sequences $(P_D^*/D)_{D \geq 1}$ and $(P_D/D)_{D \geq 1}$ converge to the same limit. Similarly the slope inequality implies that

$$\lim_{D \to +\infty} \frac{1}{D} \Big| \widehat{\mu}_{\min}(E_D, h_D) - \widehat{\mu}_{\min}(E_D, h_D^*) \Big| = 0.$$

We establish now (Proposition 9.11) the analogue of Proposition 8.11 in Arakelov geometry.

Lemma 9.10 *For any integer $r \geq 2$ and any element $\mathbf{n} = (n_i)_{1\leq i\leq r} \in \mathbb{N}^r_{\geq n_0}$, we have*

$$\widehat{\mu}_{\max}(\overline{E}_{|\mathbf{n}|}) \geq \sum_{i=1}^{r} \widehat{\mu}_{\max}(\overline{E}_{n_i}) - 2\varepsilon \sum_{i=1}^{r} \log n_i - \sum_{i=1}^{r} \Big(\frac{1}{2}\log([K:\mathbb{Q}]\operatorname{rk} E_{n_i}) + \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}\Big). \quad (23)$$

Proof. Suppose that for any integer $1 \leq i \leq r$, s_i is a non-zero element in E_{n_i}. Then for any $\sigma \in \Sigma_\infty$,

$$\begin{aligned}\|s_1 \cdots s_r\|_\sigma &\leq \|s_1 \cdots s_r\|_{\sigma,\sup} \exp(A_{|\mathbf{n}|,\sigma}) \leq \prod_{i=1}^{r} \|s_i\|_{\sigma,\sup} \exp(A_{|\mathbf{n}|,\sigma}) \\ &\leq \exp(A_{|\mathbf{n}|,\sigma}) \prod_{i=1}^{r} \Big(\|s_i\|_\sigma \exp(A_{n_i,\sigma})\Big) \leq |\mathbf{n}|^\varepsilon n_1^\varepsilon \cdots n_r^\varepsilon \prod_{i=1}^{r} \|s_i\|_\sigma.\end{aligned}$$

Therefore,

$$\sum_{\sigma\in\Sigma_\infty} \|s_1 \cdots s_r\|_\sigma^2 \leq |\mathbf{n}|^{2\varepsilon}\Big(\prod_{i=1}^{r} n_i^{2\varepsilon}\Big) \sum_{\sigma\in\Sigma_\infty}\sum_{j=1}^{r} \|s_j\|_\sigma^2 \leq \Big(\prod_{i=1}^{r} n_i^{4\varepsilon}\Big)\Big(\prod_{j=1}^{r}\sum_{\sigma\in\Sigma_\infty} \|s_j\|_\sigma^2\Big),$$

$$\text{and} \quad -\frac{1}{2}\log\Big(\sum_{\sigma\in\Sigma_\infty} \|s_1 \cdots s_r\|_\sigma^2\Big) \geq -2\varepsilon\sum_{i=1}^{r}\log n_i - \sum_{i=1}^{r}\frac{1}{2}\log\Big(\sum_{\sigma\in\Sigma_\infty}\|s_i\|_\sigma^2\Big).$$

After Proposition 9.6, we obtain

$$\begin{aligned}\widehat{\mu}_{\max}(\overline{E}_{|\mathbf{n}|}) &\geq -\frac{1}{2}\log\Big(\sum_{\sigma\in\Sigma_\infty} \|s_1 \cdots s_r\|_\sigma^2\Big) \geq -2\varepsilon\sum_{i=1}^{r}\log n_i - \sum_{i=1}^{r}\frac{1}{2}\log\Big(\sum_{\sigma\in\Sigma_\infty}\|s_i\|_\sigma^2\Big) \\ &\geq \sum_{i=1}^{r}\widehat{\mu}_{\max}(\overline{E}_{n_i}) - 2\varepsilon\sum_{i=1}^{r}\log n_i - \sum_{i=1}^{r}\Big(\frac{1}{2}\log([K:\mathbb{Q}]\operatorname{rk} E_{n_i}) + \frac{\log|\Delta_K|}{2[K:\mathbb{Q}]}\Big).\end{aligned}$$

□

Proposition 9.11 *The sequence $(\frac{1}{D}\widehat{\mu}_{\max}(\overline{E}_D))_{D\geq 1}$ has a limit in $\mathbb{R}$.*

Proof. This is a direct consequence of Lemma 9.10 and Corollary 3.2. □

From the slope inequality we know immediately that the limits in Proposition 9.11 do not depend on the choice of Hermitian metrics $\|\cdot\|_\sigma$.

10 Calculation of the limit of polygons for a bigraded algebra

We present in this section an explicit calculation of the limit of polygons in the case where the bigraded algebra associated to the quasi-filtered graded algebra is of finite type. The method used in this section is inspired by an article of Faltings and Wüstholz [FW94], which applies the theory of Poincaré series in two variables.

Definition 10.1 Let A be a commutative ring. We call *bigraded* A-algebra any $\mathbb{N}^2$-graded commutative A-algebra. If B is a bigraded A-algebra, we call *bigraded* B-module any B-module M equipped with a $\mathbb{Z}^2$-graduation in A-modules such that, for any $(n,d) \in \mathbb{N}^2$ and any $(n',d') \in \mathbb{Z}^2$, we have $B_{n,d}M_{n',d'} \subset M_{n+n',d+d'}$. We call *homogeneous sub-B-module* of M any sub-B-module M' of M such that $M' = \bigoplus_{(n,d)\in\mathbb{Z}^2} M' \cap M_{n,d}$. M' is therefore canonically equipped with a structure of graded B-module. In particular, if B is a bigraded A-algebra, then B is canonically equipped with a structure of bigraded B-module. The homogeneous sub-B-modules of B are called *homogeneous ideals* of B.

If B is a bigraded A-algebra and if M is a bigraded B-module, for any $(n,d) \in \mathbb{Z}^2$, we denote by $M(n,d)$ the graded B-module such that $M(n,d)_{n',d'} = M_{n+n',d+d'}$ for any $(n',d') \in \mathbb{Z}^2$.

Let $\mathbf{f}$ be a mapping from $\{1,\cdots,n\}$ to $\mathbb{N}^2$. The ring $A[T_1,\cdots,T_n]$ of polynomials is canonically equipped with an $\mathbb{N}^2$-graduation such that T_i is homogeneous of bidegree $\mathbf{f}(i)$. We obtain hence a bigraded A-algebra, denote by $A[\mathbf{f}]$.

If B is a bigraded A-algebra of finite type, then B is generated by a finite number of homogeneous elements $x_1,\cdots,x_m$. We suppose that x_i is of bidegree (n_i,d_i). Let $\mathbf{f} : \{1,\cdots,m\} \to \mathbb{N}^2$ be the function which sends i to (n_i,d_i). Then the surjective homomorphism of A-algebras from $A[\mathbf{f}] \cong A[T_1,\cdots,T_m]$ to B which sends T_i to x_i is compatible with $\mathbb{N}^2$-graduations. It is therefore a homomorphism of bigraded algebras. In this case, any bigraded B-module M can be considered as a bigraded $A[\mathbf{f}]$-module, which is of finite type if M is a B-module of finite type.

Definition 10.2 Let $\mathbf{f} = (f_1,f_2)$ be a mapping from $\{1,\cdots,m\}$ to $\mathbb{N}^2$ and M be a bigraded $A[\mathbf{f}]$-module of finite type whose homogeneous component are all A-modules of finite length. We call *Poincaré series* of M the element $P_M \in Z\llbracket X,Y\rrbracket[X^{-1},Y^{-1}]$ defined by the formula $P_M = \sum_{(n,d)\in\mathbb{Z}^2} \mathrm{len}_A(M_{n,d})X^nY^d$. We write $Q_M = P_M\prod_{i=1}^{m}(1-X^{f_1(i)}Y^{f_2(i)})$.

Proposition 10.3 *We have $Q_M \in \mathbb{Z}[X,Y,X^{-1},Y^{-1}]$.*

Proof. By replacing A with $A/\operatorname{ann}_A(M)$, we reduce the problem to the case where $\operatorname{ann}_A(M) = 0$. Since M is an $A[\mathbf{f}]$-module of finite type, there exist integers $a < b$ such that M is generated as $A[\mathbf{f}]$-module by $M' = \bigoplus_{(n,d)\in[a,b]^2\cap\mathbb{Z}^2} M_{n,d}$. Since M' is an A-module of finite length, and since $\operatorname{ann}_A(M') = \operatorname{ann}_A(M) = 0$, the ring A is Artinian, so is Notherian.

We deduce by induction on m. If $m = 0$, then $A[\mathbf{f}] = A$. Since M is an A-module of finite type, we have $P_M \in \mathbb{Z}[X,Y,X^{-1},Y^{-1}]$. Suppose that the proposition has been proved for $1,\cdots,m-1$. Let $\mathbf{f}'$ be the restriction of $\mathbf{f}$ on $\{1,\cdots,m-1\}$. We write $(n_m,d_m) = \mathbf{f}(m)$. The mapping $T_m : M(-n_m,-d_m) \longrightarrow M$ is a homomorphism of bigraded $A[\mathbf{f}]$-modules. Let N be its kernel (considered as homogeneous sub-$A[\mathbf{f}]$-module of M). We have an exact sequence

$$0 \longrightarrow N(-n_m,-d_m) \longrightarrow M(-n_m,-d_m) \longrightarrow M \longrightarrow M/T_mM \longrightarrow 0\ .$$

Therefore, $P_M - X^{n_m}Y^{d_m}P_M = P_{M/T_mM} - X^{n_m}Y^{d_m}P_N$. Since M/T_mM and N are $A[f'] = A[f]/(T_m)$-modules of finite type, by induction hypothesis, we obtain

$$Q_M = Q_{M/T_mM} - X^{n_m}Y^{d_m}Q_N \in \mathbb{Z}[X,Y,X^{-1},Y^{-1}].$$

□

Remark 10.4 Let $\mathbf{f} = (f_1, f_2) : \{1, \cdots, m\} \to \mathbb{N}^2$ be a mapping such that $f_1 \equiv 1$ and M be a bigraded $A[\mathbf{f}]$-module of finite type, whose homogeneous components are A-modules of finite length. The algebra $A[\mathbf{f}]$, equipped with the first graduation, is the usually graded algebra of polynomials in m variables. We can also consider the first graduation of M for which the n^{th} homogeneous component of M is $\bigoplus_{d\in\mathbb{Z}} M_{n,d}$. This homogeneous component is an A-module of finite length since there exist only a finite number of integers d such that $M_{n,d} \neq 0$. With this graduation, M is a graded module of finite type over the polynomial algebra $A[T_1, \cdots, T_m]$ (with the usual grading). If we denote by H_M the Poncaré series associated to M (for the first grading), we have $H_M(X) = P_M(X, 1)$. The notions $\dim M$ and $c(M)$ are hence defined, as in Section 7.

The following theorem is an analogue in the two variables case of the formula (13) for Poincaré series.

Theorem 10.5 *With the notations of Remark 10.4, the series P_M is written in the forme*

$$P_M(X, Y) = \sum_{r=0}^{h} \sum_{\substack{\alpha\subset\{1,\cdots,m\}\\ \#\alpha=r}} I_\alpha(X, Y) \prod_{i\in\alpha} (1 - XY^{f_2(i)})^{-1},$$

where

1) $I_\alpha \in \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$,

2) if $\#\alpha = h$, the coefficients of I_α are positive,

3) if $M \neq 0$, there exists at least an subset $\alpha \subset \{1, \cdots, m\}$ of cardinal h such that $I_\alpha \neq 0$.

Remark 10.6 With the notations of Theorem 10.5, we have

$$H_M(X) = \sum_{r=0}^{h} \left(\sum_{\substack{\alpha\subset\{1,\cdots,m\}\\ \#\alpha=r}} I_\alpha(X, 1) \right) (1 - X)^{-r}.$$

Therefore, if M is non-zero, then $\dim M = h$ and $c(M) = \displaystyle\sum_{\substack{\alpha\subset\{1,\cdots,m\}\\ \#\alpha=h}} I_\alpha(1, 1)$.

To simplify the proof of Theorem 10.5, we introduce the following notation. If M is a bigraded $A[\mathbf{f}]$-module satisfying the assertion of Theorem 10.5, we say that M verifies the the condition $\mathbb{P}$, noted by $\mathbb{P}(M)$. The assertion of Theorem 10.5 then becomes:

For any $A[\mathbf{f}]$-module M, we have $\mathbb{P}(M)$.

For any integer $m > 0$, let Θ_m be the set $\{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i \leq m,\ j > 0\} \cup \{(-\infty, 0)\}$. We equip it with the lexicographic relation "$\leq$" as follows:

$$(i, j) \leq (i', j') \text{ if and only if } i < i' \text{ or if } i = i',\ j \leq j'.$$

We verify easily that it is an order relation on Θ_m and that the set Θ_m is totally ordered for this relation. We use the expression $(i, j) < (i', j')$ to present the condition $(i, j) \leq (i', j')$ but $(i, j) \neq (i', j')$.

Lemma 10.7 *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be a short exact sequence of bigraded* $A[\mathbf{f}]$*-module. Suppose that* $M'_{n,d}$ *and* $M''_{n,d}$ *are* A*-modules of finite length for any* $(n,d) \in \mathbb{Z}^2$*. Then* $M_{n,d}$ *are* A*-modules of finite length, and*

1) $\dim M = \max(\dim M', \dim M'')$*,*

2)

$$c(M) = \begin{cases} c(M') + c(M''), & \dim M' = \dim M'', \\ c(M'), & \dim M' > \dim M'', \\ c(M''), & \dim M'' > \dim M'. \end{cases}$$

3) $\mathbb{P}(M')$ *and* $\mathbb{P}(M'') \Longrightarrow \mathbb{P}(M)$*.*

Proof. In fact, we have $P_M = P_{M'} + P_{M''}$ and $H_M = H_{M'} + H_{M''}$. By definition we know that 1) and 2) are true. Finally, 3) is a consequence of 1) and of the fact that $P_M = P_{M'} + P_{M''}$. □

Proof of Theorem 10.5. By the same argument as that for the proof of Proposition 10.3, we can suppose that A is an Artinian ring. We shall prove the theorem by induction on m. First we prove that the theorem is true in the case where $\dim M \leq 0$. If M is of dimension ≤ 0, then the Poincaré series $H_M(X) = P_M(X,1)$ of M is an element of $\mathbb{Z}[X, X^{-1}]$, and $P_M \in \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$. Hence we have $\mathbb{P}(M)$. Since $\dim M \leq m$, the theorem is true when $m = 0$. Suppose that the theorem is true for bigraded modules of an A-algebra of polynomials in j variables $(0 \leq j < m)$. Let $\mathbf{f} = (f_1, f_2) : \{1, \cdots, m\} \to \mathbb{N}^2$ be a mapping such that $f_1 \equiv 1$ and let M be a bigraded $A[\mathbf{f}] = A[T_1, \cdots, T_m]$-module of finite type such that $M_{(n,d)}$ is of finite length over A for any $(n,d) \in \mathbb{Z}^2$. Suppose that $f_2(m) = d$.

We begin another procedure of induction on $(\dim M, c(M))$. We have already proved $\mathbb{P}(M)$ for $\dim M \leq 0$. Suppose that we have proved $\mathbb{P}(M)$ for $(\dim M, c(M)) < (r, s)$, where $0 < r \leq m$, $s > 0$. In the following, we shall prove $\mathbb{P}(M)$ in the case where $(\dim M, c(M)) = (r, s)$. Consider the homothetic transformation $T_m : M(-1, -d) \longrightarrow M$, which is a homomorphism of bigraded $A[\mathbf{f}]$-modules. We denote by $\mathbf{f}'$ the restriction of $\mathbf{f}$ on $\{1, \cdots, m-1\}$. Let N_1 be the kernel of T_m (considered as homogeneous sub-$A[\mathbf{f}]$-module). It is a bigraded $A[\mathbf{f}']$-module of finite type. After the induction hypothesis, we have $\mathbb{P}(N_1)$. Let $M_1 = M/N_1$. After Lemma 10.7 3), to prove $\mathbb{P}(M)$, it suffices to prove $\mathbb{P}(M_1)$. If $\dim N_1 = \dim M$, then either $\dim M_1 < \dim M$, or $\dim M_1 = \dim M$ and $c(M_1) = c(M) - c(N_1) < c(M)$. So we always have $(\dim M_1, c(M_1)) < (\dim M, c(M))$. After the induction hypothesis, we have $\mathbb{P}(M_1)$. Otherwise we have $\dim N_1 < \dim M$ and $(\dim M_1, c(M_1)) = (\dim M, c(M))$. If $\mathbb{P}(M)$ is not true, by iterating the procedure above, we obtain an increasing sequence of homogeneous submodules

$$N_1 \subset N_2 \subset \cdots N_j \subset N_{j+1} \subset \cdots \tag{24}$$

of M such that (we define $M_0 = M$)

i) $N_j = \operatorname{Ker} T_m^j$,

ii) $\dim N_j < \dim M$,

iii) $M_j := M/N_j$ don't satisfy the condition $\mathbb{P}$, and $(\dim M_j, c(M_j)) = (\dim M, c(M))$.

Since $A[\mathbf{f}]$ is a Noetherian ring, the sequence (24) is stationary. In other words, there exists $j \in \mathbb{N}$ such that $M_j = M_{j+1}$. Since M_{j+1} identifies canonically with the image of M_j by the homothetic transformation T_m,we have the exact sequence

$$0 \longrightarrow M_j(-1, -d) \xrightarrow{T_m} M_j \longrightarrow M_j/T_m M_j \longrightarrow 0\ .$$

We write $N' = M_j/T_m M_j$. It is actually an $A[\mathbf{f}']$-module of finite type. After induction hypothesis, we have $\mathbb{P}(N')$. Finally, since $(1-XY^d)P_{M_j}(X,Y) = P_{N'}(X,Y)$, we have $\mathbb{P}(M_j)$, which is absurd. Hence we have $\mathbb{P}(M)$.

□

Let P be the formal series in $\mathbb{Z}[\![X,Y]\!][X^{-1},Y^{-1}]$ with positive coefficients. Then P is written in the forme $P(X,Y) = \sum_{(n,d)\in\mathbb{Z}^2} a_{n,d}(P)X^nY^d$. For any $n \in \mathbb{N}$, we write $S_n(P) = \sum_{d\in\mathbb{Z}} a_{n,d}(P)$ and denote by $\nu_{n,P}$ the Borel measure on $\mathbb{R}$ defined by

$$\nu_{n,P} = \sum_{d\in\mathbb{Z}} \frac{a_{n,d}(P)}{S_n(P)}\delta_{d/n}.$$

If $S_n(P) = 0$, then $\nu_{n,P}$ is by convention the zero measure.

Remark 10.8 We keep the notations of Theorem 10.5 in supposing that A is a field. If for any integer n, we equip the space $M_{n,\bullet} := \bigoplus_{d\in\mathbb{Z}} M_{n,d}$ with the $\mathbb{R}$-filtration $\mathcal{F}$ defined by $\mathcal{F}_\lambda M_{n,\bullet} = \bigoplus_{d\geq\lambda} M_{n,d}$, then the measure $\nu_{n,P}$ identifies with $T_{\frac{1}{n}}\nu_{M_{n,\bullet}}$. This observation is crucial because it enables us to use the Poincaré series to study measures of a bigraded algebra over a field.

Proposition 10.9 *If P is a series in $\mathbb{Z}[\![X,Y]\!]$ of the forme $P(X,Y) = \prod_{i=1}^{m}(1-XY^{d_i})^{-1}$, then*

1) the Borel measures $\nu_{n,P}$ converge vaguely to a Borel measure ν_P when $n \to +\infty$;

2) the sequence of functions $\left(F_{n,P} : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]}\,\mathrm{d}\nu_{n,P}\right)_{n\geq 1}$ converges simply to $F_P : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]}\,\mathrm{d}\nu_P$.

Proof. 1) We have

$$P(X,Y) = \prod_{i=1}^{m}\Big(\sum_{n\geq 0} X^nY^{nd_i}\Big) = \sum_{(n,d)\in\mathbb{N}\times\mathbb{Z}}\left(\sum_{\substack{(u_1,\cdots,u_m)\in\mathbb{N}^m\\ u_1+\cdots+u_m=n,\\ u_1d_1+\cdots+u_md_m=d}} 1\right)X^nY^d$$

$$= 1 + \sum_{\substack{(n,d)\in\mathbb{Z}^2\\ n>0}}\left(\sum_{\substack{(\mu_1,\cdots,\mu_m)\in\frac{1}{n}\mathbb{N}^m\\ \mu_1+\cdots+\mu_m=1,\\ \mu_1d_1+\cdots+\mu_md_m=d/n}} 1\right)X^nY^d.$$

On the other hand, $S_n(P) = \sum_{\substack{(\mu_1,\cdots,\mu_m)\in\frac{1}{n}\mathbb{N}^m\\ \mu_1+\cdots+\mu_m=1}} 1$. Let Δ_m be the simplex $\{(\mu_1,\cdots,\mu_m) \in \mathbb{R}_+^m \mid \mu_1+\cdots+\mu_m = 1\}$, $\varphi : \Delta_m \to \mathbb{R}$ be the mapping which sends $(\mu_1,\cdots,\mu_m)$ to $\mu_1d_1+\cdots+\mu_md_m$. For any integer $n > 0$, let $\eta_{n,P}$ be the measure on Δ_m defined by $\eta_{n,P} = \sum_{\mu\in\frac{1}{n}\mathbb{N}^m\cap\Delta_m} \frac{1}{S_n(P)}\delta_\mu$. We observe that $\nu_{n,P}$ is the direct image of $\eta_{n,P}$ by φ. Therefore,

$\nu_{n,P}$ is supported by $\varphi(\Delta_m)$. Hence for any continuous function $f : \mathbb{R} \to \mathbb{R}$, f is integrable with respect to the measure $\nu_{n,P}$. Furthermore, we have $\displaystyle\int_{\mathbb{R}} f \mathrm{d}\nu_{n,P} = \int_{\Delta_m} (f \circ \varphi) \mathrm{d}\eta_{n,P}$, which is the n^{th} Riemann sum of the function $f \circ \varphi : \Delta_m \to \mathbb{R}$. So the sequence $\Big(\displaystyle\int_{\mathbb{R}} f \mathrm{d}\nu_{n,P} \Big)_{n\geq 1}$ converges to $\displaystyle\int_{\Delta_m} f \circ \varphi \mathrm{d}\eta = \int_{\mathbb{R}} f \mathrm{d}\varphi_* \eta$ where η is the Lebesgue measure on Δ_m. We then obtain that the measures $\nu_{n,P}$ converge vaguely to the measure $\nu_P = \varphi_* \eta$.

2) The mapping φ can be extended to an affine mapping Φ from $\{(\mu_1, \cdots, \mu_m) \in \mathbb{R}^m \mid \mu_1 + \cdots + \mu_m = 1\}$ to $\mathbb{R}$ by simply defining $\Phi(\mu_1, \cdots, \mu_m) = \mu_1 d_1 + \cdots \mu_m d_m$. If $d_1 = d_2 = \cdots = d_m = d$, then $P(X, Y) = (1 - XY^d)^{-m}$. Therefore, for any $n \geq 1$, $\nu_{n,P} = \nu_P = \delta_d$. The assertion is then evident. Otherwise the image of Φ is the whole set $\mathbb{R}$ and for any point $x \in \operatorname{Im} \varphi$, $\varphi^{-1}(x)$ is a negligible subset of Δ_m for the Lebesgue measure. Therefore, the one point set $\{x\}$ is negligible for the measure λ_P. After [Bou65] IV.5 Proposition 22, since x is the only discontinuous point of the function $\mathbb{1}_{]-\infty,x]}$, we obtain that the sequence $\Big(\displaystyle\int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_{n,P} \Big)_{n\geq 1}$ converges to $\displaystyle\int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_P$. □

Proposition 10.10 *Suppose that Q is a non-zero series in $\mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$ with positive coefficients, and $P \in \mathbb{Z}[\![X, Y]\!][X^{-1}, Y^{-1}]$ is of the form $P(X, Y) = Q(X, Y) \displaystyle\prod_{i=1}^{m} (1 - XY^{d_i})^{-1}$.*

1) The Borel measures $\nu_{n,P}$ converge vaguely to a Borel measure ν_P when $n \to +\infty$.

2) Define the functions

$$\Big(F_{n,P} : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_{n,P} \Big)_{n\geq 1} \qquad \text{and} \qquad F_P : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_P.$$

i) If $d_1 = \cdots = d_m = d$, then for any $x \neq d$, the sequence $(F_{n,P}(x))_{n\geq 1}$ converges to $F_P(x)$.

ii) If d_i's are not identical, then the sequence of functions $(F_{n,P})_{n\geq 1}$ converges simply to F_P.

Furthermore, if we denote by P' the series $P'(X, Y) = \displaystyle\prod_{i=1}^{m} (1 - XY^{d_i})^{-1}$, then we have $\nu_P = \nu_{P'}$, and hence $F_P = F_{P'}$.

Proof. 1) Suppose that Q is of the form $Q(X, Y) = \displaystyle\sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} X^{n'} Y^{d'}$ where $c_{n',d'} \geq 0$. Since $P = P'Q$, we obtain $a_{n,d}(P) = \displaystyle\sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} \, a_{n-n',d-d'}(P')$ and

$$\begin{aligned} S_n(P) &= \sum_{d\in\mathbb{Z}} a_{n,d}(P) = \sum_{d\in\mathbb{Z}} \sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} \, a_{n-n',d-d'}(P') \\ &= \sum_{|n'|\leq e} \sum_{|d'|\leq r} \sum_{d\in\mathbb{R}} c_{n',d'} \, a_{n-n',d-d'}(P') = \sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} \, S_{n-n'}(P'). \end{aligned}$$

Denote by $C_{n'} = \sum_{|d'|\leq r} c_{n',d'}$, then we have $S_n(P) = \sum_{|n'|\leq e} C_{n'} S_{n-n'}(P')$. If $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support, then

$$\int_{\mathbb{R}} g \, \mathrm{d}\nu_{n,P} = \sum_{d\in\mathbb{Z}} \frac{a_{n,d}(P)}{S_n(P)} \, g(d/n) = \frac{1}{S_n(P)} \sum_{d\in\mathbb{Z}} \sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} \, a_{n-n',d-d'}(P') \, g(d/n).$$

Notice that

$$\begin{aligned}
&\frac{1}{S_n(P)} \sum_{|n'|\leq e} \sum_{|d'|\leq r} \sum_{d\in\mathbb{Z}} c_{n',d'} \, a_{n-n',d-d'}(P') \, g\left(\frac{d-d'}{n-n'}\right) \\
&= \frac{1}{S_n(P)} \sum_{|n'|\leq e} \sum_{|d'|\leq r} c_{n',d'} S_{n-n'}(P') \int_{\mathbb{R}} g \, \mathrm{d}\nu_{n-n',P'} = \frac{1}{S_n(P)} \sum_{|n'|\leq e} C_{n'} S_{n-n'}(P') \int_{\mathbb{R}} g \, \mathrm{d}\nu_{n-n',P'}
\end{aligned}$$

converges to $\int_{\mathbb{R}} g \, \mathrm{d}\nu_{P'}$ since $\nu_{n,P'}$ converges vaguely to $\nu_{P'}$ when $n \to \infty$. Finally, the function g is uniformly continuous on $\mathbb{R}$. For any number $\delta > 0$, there exists a number $\varepsilon > 0$ such that, for all $x, y \in \mathbb{R}$ such that $|x - y| < \varepsilon$, we have $|g(x) - g(y)| < \delta$. On the other hand, since $P' = \prod_{i=1}^{m} (1 - XY^{d_i})^{-1}$, if $|d| > |n| \max_{1\leq i\leq m} |d_i|$, we have $a_{n,d}(P') = 0$. Hence for all integers d, n such that $|d| > \max_{1\leq i\leq m} |d_i|(|n| + e) + r$, we have $a_{n-n',d-d'}(P') = 0$ for any $|n'| \leq e$ and any $|d'| \leq r$. Therefore, for all integers $n > e$, $d \in \mathbb{Z}$, $|n'| \leq e$ and $|d'| \leq r$ such that $a_{n-n',d-d'}(P') \neq 0$, we always have

$$\left|\frac{d}{n} - \frac{d-d'}{n-n'}\right| = \left|\frac{d'n - dn'}{n(n-n')}\right| \leq \frac{r}{n-n'} + \max_{1\leq i\leq m} |d_i| \frac{e(n+e+r)}{n(n-n')}.$$

Therefore, there exists an integer $N > 0$ such that, for all integers $n > N$, $d \in \mathbb{Z}$, $|n'| \leq e$ and $|d'| \in r$, we have either $a_{n-n',d-d'}(P') = 0$, or $\left|\frac{d}{n} - \frac{d-d'}{n-n'}\right| < \varepsilon$. Hence we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} g \, \mathrm{d}\nu_{n,P} - \frac{1}{S_n(P)} \sum_{|n'|\leq e} \sum_{|d'|\leq r} \sum_{d\in\mathbb{Z}} c_{n',d'} \, a_{n-n',d-d'}(P') \, g\left(\frac{d-d'}{n-n'}\right) \right| \\
&\leq \frac{1}{S_n(P)} \sum_{|n'|\leq e} \sum_{|d'|\leq r} \sum_{d\in\mathbb{Z}} c_{n',d'} \, a_{n-n',d-d'}(P') \left| g\left(\frac{d}{n}\right) - g\left(\frac{d-d'}{n-n'}\right) \right| \\
&\leq \frac{\delta}{S_n(P)} \sum_{|n'|\leq e} \sum_{|d'|\leq r} \sum_{d\in\mathbb{Z}} c_{n',d'} \, a_{n-n',d-d'}(P') = \delta.
\end{aligned}$$

We then deduce the vague convergence of $\nu_{n,P}$ to $\nu_{P'}$.

2) If $d_1 = \cdots = d_m = d$, then $\nu_P = \delta_d$. So for any $x \neq d$, the set of discontinuous points of $\mathbb{1}_{]-\infty,x]}$, i.e., $\{x\}$, is negligible for the measure ν_P. Hence $\int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_{n,P}$ converges to $\int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_P$. If d_i's are not identical, then any discrete subset of $\mathbb{R}$ is negligible for the measure ν_P, so the sequence of functions $(F_{n,P})_{n\geq 1}$ converges simply to the function F_P. □

Remark 10.11 With the notations of Proposition 10.10, the limit measure ν_P depends only on the vector $(d_1, \cdots, d_m) \in \mathbb{N}^m$ (or simply the equivalence class of $(d_1, \cdots, d_m)$ in $\mathbb{N}^m/\mathfrak{S}_m$, the quotient of $\mathbb{N}^m$ by the symmetric group $\mathfrak{S}_m$). In the following, we denote by $\nu_{(d_1,\cdots,d_m)}$ this measure. Actually, when $m > 0$, it is a probability measure. When $m = 0$, $\nu_\emptyset$ is the zero measure.

The following theorem is an immediate consequence of Proposition 10.10.

Theorem 10.12 *Let* $(d_1, \cdots, d_m) \in \mathbb{Z}_+^m$ *and* $P(X,Y) = \sum_{r=0}^{h} \sum_{\substack{\alpha \subset \{1,\cdots,m\} \\ \operatorname{card} \alpha = r}} I_\alpha(X,Y) \prod_{i \in \alpha} (1 - XY^{d_i})^{-1}$ *be a series in* $\mathbb{Z}[\![X,Y]\!][X^{-1},Y^{-1}]$ *where*

a) the coefficients of P *are positive,*

b) $I_\alpha \in \mathbb{Z}[X,Y,X^{-1},Y^{-1}]$*,*

c) for any $\alpha \subset \{1,\cdots,m\}$ *of cardinal* h*, the coefficients of* I_α *are positive,*

d) there exists at least one $\alpha \subset \{1,\cdots,m\}$ *of cardinal* h *such that* $I_\alpha \neq 0$*.*

Then

1) the Borel measures $\nu_{n,P}$ *converge vaguely to a Borel measure* ν_P *when* $n \to +\infty$*,*

2) there exists a finite subset Ω *of* $\mathbb{R}$ *such that the sequence of functions*

$$\Big(F_{n,P} : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_{n,P}\Big)_{n \geq 1}$$

converges pointwise on $\mathbb{R} \setminus \Omega$ *to the function* $F_P : x \longmapsto 1 - \int_{\mathbb{R}} \mathbb{1}_{]-\infty,x]} \, \mathrm{d}\nu_P$*.*

Furthermore, if for any $\alpha = \{i_1 < \cdots < i_h\}$*, we write* $d_\alpha = (d_{i_1}, \cdots, d_{i_h})$*, then the limit measure* ν_P *equals to* $\sum_{\substack{\alpha \subset \{1,\cdots,m\} \\ \operatorname{card} \alpha = h}} \frac{I_\alpha(1,1)}{S} \nu_{d_\alpha}$ *where* $S = \sum_{\substack{\alpha \subset \{1,\cdots,m\} \\ \operatorname{card} \alpha = h}} I_\alpha(1,1)$*. So* ν_P *is a probability measure when* $h > 0$*. If* $h = 0$*, then* ν_P *is the zero measure.*

The results obtained above, notably Theorem 10.5 and Theorem 10.12, imply immediately the following theorem.

Theorem 10.13 *Let* K *be a field,* $\mathbf{f} = (f_1, f_2) : \{1,\cdots,m\} \to \mathbb{N}^2$ *be a mapping such that* $f_1 \equiv 1$ *and* M *be a finite generated bigraded* $K[\mathbf{f}]$*-module. If for any integer* $n \geq 1$*, we denote by* ν_n *the Borel measure associated to the vector space* $M_{n,\bullet} := \bigoplus_{d \in \mathbb{Z}} M_{n,d}$ *which is equipped with the filtration induced by the second grading, then the sequence of Borel measures* $T_{\frac{1}{n}} \nu_n$ *converges vaguely to a Borel measure* ν *on* $\mathbb{R}$*. If furthermore* $M_{n,\bullet}$ *is non-zero for sufficiently large* n*, then the limit measure* ν *is a probability measure, and the polygons associated to* $T_{\frac{1}{n}} \nu_n$ *converge uniformly to a concave curve defined on* $[0,1]$*.*

Remark 10.14 Let K be a field and B be an $\mathbb{N}$-filtered graded K-algebra (that is to say, the jumping set is contained in $\mathbb{N}$) which is of finite type over K and is generated as K-algebra by B_1. We can introduce a bigraded K-algebra $\widetilde{B}$ by defining $\widetilde{B}_{n,d} = \mathcal{F}_d B_n / \mathcal{F}_{d+1} B_n$. Notice that the filtered vector spaces $\widetilde{B}_{n,\bullet}$ (whose filtration is induced by the second grading) and B_n have the same associated measure. Therefore, if $\widetilde{B}$ is an algebra of finite type over K which is generated by $\widetilde{B}_{1,\bullet}$, then the previous theorem shows that the normalized polygons of B_n converge uniformly. However, this condition is not satisfied in general. We can for example consider the algebra $B = K[X]$ of polynomials, equipped with the usual graduation and the filtration such that $\lambda(X^n) = n - 1$ for any $n \geq 1$. Then B is a filtered graded algebra since

$\lambda(X^{n+m}) = n+m-1 > n-1+m-1 = \lambda(X^n) + \lambda(X^m)$. On the other hand, the bigraded algebra $\widetilde{B}$ identifies with the algebra $K[T_1, \cdots, T_n, \cdots]$, where the bidegree of T_n is $(n, n-1)$, modulo the homogeneous ideal generated by all elements of the form $T_n T_m$. This is not an algebra of finite type over K.

Finally, we shall give an example of the limit of normalized Harder-Narasimhan polygons in relative geometric framework. Let k be a field, and C be a smooth projective curve over k. We denote by K the field of rational functions on C. Let $(E_i)_{1\leq i\leq m}$ be a finite family of locally free $\mathcal{O}_C$-modules of finite type which are semistable. We suppose in addition that for any family $(n_i)_{1\leq i\leq m}$ of positive integers, the $\mathcal{O}_C$-module $S^{n_1}E_1 \otimes \cdots \otimes S^{n_m}E_m$ is semistable. This condition is satisfied notably when one of the following conditions is satisfied:

1) the $\mathcal{O}_C$-modules $E_1, \cdots, E_m$ are all of rank 1;

2) C is the projective space $\mathbb{P}^1$;

3) C is an elliptic curve over k (see [Ati57]);

4) k is of characteristic 0.

Let E be the direct sum $E = E_1 \oplus \cdots \oplus E_m$. Let $\mathscr{B}$ be the symmetric algebra of E, which is a graded $\mathcal{O}_C$-algebra. For any integer $n \geq 1$, we have

$$\mathscr{B}_n = S^n E = \bigoplus_{\substack{(d_i)_{1\leq i\leq m}\in\mathbb{N}^m \\ d_1+\cdots+d_m=n}} \left(S^{d_1}E_1 \otimes \cdots \otimes S^{d_m}E_m\right)^{\oplus \frac{n!}{d_1!\cdots d_m!}}.$$

Denote by B the graded algebra over K such that $B_n = \mathscr{B}_{n,K}$. For any integer $1 \leq i \leq m$, we denote by r_i the rank of E_i and by μ_i the slope of E_i, and we choose a base $\mathbf{u}_i = (u_{i,j})_{1\leq i\leq r_j}$ of $E_{i,K}$. We write $\mathbf{u} = \mathbf{u}_1 \amalg \cdots \amalg \mathbf{u}_m$ and $r = r_1 + \cdots + r_m$ the rank of E. The algebra B identifies hence with the algebra of polynomials $K[\mathbf{u}]$. If $\alpha : \mathbf{u} \to \mathbb{R}$ is a mapping, denote by $|\alpha|$ the sum $\sum_{i=1}^m \sum_{j=1}^{r_i} \alpha(u_{ij})$. For any integer $n \geq 1$, we denote by $\nu_n = T_{\frac{1}{n}} \nu_{\mathscr{B}_n}$ and we have

$$\begin{aligned}\nu_{\mathscr{B}_n} &= \sum_{\substack{(d_i)_{1\leq i\leq m}\in\mathbb{N}^m \\ d_1+\cdots+d_m=n}} \frac{n!}{d_1!\cdots d_m!} \frac{\mathrm{rk}(S^{d_1}E_1 \otimes \cdots \otimes S^{d_m}E_m)}{\mathrm{rk}(S^n E)}\, \delta_{d_1\mu_1+\cdots+d_m\mu_m} \\ &= \sum_{\substack{\alpha:\mathbf{u}\to\mathbb{N} \\ |\alpha|=n}} \frac{1}{\mathrm{rk}(S^n E)}\, \delta_{\sum_{i=1}^m \mu_i \sum_{j=1}^{r_i} \alpha(u_{ij})}.\end{aligned}$$

Therefore, $\nu_n = \sum_{\substack{\beta:\mathbf{u}\to\frac{1}{n}\mathbb{N} \\ |\beta|=1}} \frac{1}{\mathrm{rk}(S^n E)}\, \delta_{\sum_{i=1}^m \mu_i \sum_{j=1}^{r_i} \beta(u_{ij})}$. Denote by Δ the simplex of dimension $r-1$ in $\mathbb{R}^r$ (considered as the function space of $\mathbf{u}$ in $\mathbb{R}$) defined by the relation

$$\Delta := \{x : \mathbf{u} \to \mathbb{R}_{\geq 0} \mid |x| = 1\}.$$

and by $\Phi : \Delta \to \mathbb{R}$ the mapping which sends $(x : \mathbf{u} \to \mathbb{R}) \in \mathbb{R}^r$ to $\sum_{i=1}^m \mu_i \sum_{j=1}^{r_i} x(u_{ij})$. This is a continuous function. For any integer $n \geq 1$, let $\Delta^{(n)}$ be the subset of Δ of functions valued in $n^{-1}\mathbb{N}$. Then ν_n is the direct image by $\Phi|_{\Delta^{(n)}}$ of the equidistributed probability measure w_n on $\Delta^{(n)}$. By abuse of language, we still use the expression w_n to denote the direct image of w_n by the inclusion mapping from $\Delta^{(n)}$ in Δ. Then $\nu_n = \Phi_*(w_n)$. Since the sequence of measures

$(w_n)_{n\geq 1}$ converges vaguely to the uniform measure on Δ, the limit ν of the measure sequence $(\nu_n)_{n\geq 1}$ exists and equals to the direct image of the uniform measure on Δ by the mapping Φ. Therefore the uniform limit of polygons associated to ν_n exists and equals to the "polygon" (it is in fact a concave function) associated to the limit measure ν.

Example 10.15 Let E be the direct sum of two invertible modules L_1 and L_2. We write $\mu_1 = \deg(L_1)$ and $\mu_2 = \deg(L_2)$, and we suppose that $\mu_1 < \mu_2$. In this case, $\Delta = \{(x, 1-x) \mid 0 \leq x \leq 1\} \subset \mathbb{R}^2$ is parametered by $[0,1]$. The mapping $\Phi : \Delta \to \mathbb{R}$ sends $(x, 1-x)$ to $\mu_1 x + \mu_2(1-x)$. Therefore, the limit measure ν is the equidistributed probability measure on $[\mu_1, \mu_2]$. Let f be the function defined by $f(t) = \mathbb{E}^{\nu}[\mathbb{1}_{\{x>t\}}]$. Then we have $f(x) = \frac{1}{\mu_2 - \mu_1}\Big((\mu_2 - x)_+ - (\mu_1 - x)_+\Big)$. The quasi-inverse of f is therefore $f^*(t) = \mu_1 t + \mu_2(1-t)$. Finally, the limit of normalized Harder-Narasimhan polygons of $S^n E$ is given by the quadratic curve

$$\mu_2 x - \frac{\mu_2 - \mu_1}{2} x^2,$$

which is non-trivial in general.

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A Pseudo-filtered graded algebra

In this section, we propose another generalization of filtered graded algebras which is weaker than the notion of f-quasi-filtered graded algebras. By imposing a condition on f which is stronger than $\lim_{n\to+\infty} f(n)/n = 0$, we also obtain the vague convergence of measures associated to filtrations and hence the uniform convergence of polygons.

Definition A.1 Let $B = \bigoplus_{n\geq 0} B_n$ be a graded K-algebra and $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be a function. We say that B is an *f-pseudo-filtered graded* K-algebra if each B_n is equipped with a decreasing $\mathbb{R}$-filtration such that, for all sufficiently large integers n, m, we have

$$B_{n,s}B_{m,t} \subset B_{n+m,s+t-f(n)-f(m)}.$$

If B is an f-pseudo-filtered graded K-algebra, we say that a graded B-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is *f-pseudo-filtered* if for any integer n, M_n is equipped with a decreasing $\mathbb{R}$-filtration such that, for all sufficiently large integers n, m, we have

$$B_{n,s}M_{m,t} \subset M_{n+m,s+t-f(n)-f(m)}.$$

Remark A.2 Note that B is an f-pseudo-filtered graded B-module. If $f \equiv 0$, then B is a filtered graded K-algebra and M is a filtered graded B-module. If g is another function dominating f, then B is a g-pseudo-filtered graded K-algebra and M is a g-pseudo-filtered graded B-module.

Some results which are analogues to those in Section 4 can be stated and verified without difficulty for pseudo-filtered graded algebras and for pseudo-filtered graded modules, notably the corollaries 4.3, 4.5 and 4.6 where we only need to replace "quasi-filtered" by "pseudo-filtered" in the statement of the results.

Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing function, V be a vector space of rank $0 < d < +\infty$ over K, and B be the symmetric algebra generated by V, equipped with the usual graduation. We suppose that for any positive integer n, B_n is equipped with an $\mathbb{R}$-filtration which is separated, exhaustive and left continuous such that B is an f-pseudo-filtered graded K-algebra. Let

g be an increasing function which is concave and c-Lipschitz. For any integer $n \geq 0$, let $I_n = \int_{\mathbb{R}} g \, \mathrm{d}T_{\frac{1}{n}}\nu_{B_n}$. Then for all sufficiently large integers m and n, we have

$$\begin{aligned}
I_{m+n} &\geq \int_{\mathbb{R}} g \, \mathrm{d}\Big(T_{\frac{1}{m+n}}\nu_{\mathbf{u}^{(m,n)}}\Big) = \int_{\Delta_{m+n}^{(d)}} g\left(\frac{1}{m+n}\lambda(u_\gamma^{(m,n)})\right)\mathrm{d}\xi_{m+n}(\gamma) \\
&= \int_{\Delta_m^{(d)}\times\Delta_n^{(d)}} g\left(\frac{1}{m+n}\lambda(u_{\alpha+\beta}^{(m,n)})\right)\mathrm{d}\rho_{(m,n)}(\alpha,\beta) \\
&\geq \int_{\Delta_m^{(d)}\times\Delta_n^{(d)}} g\left(\frac{1}{m+n}\lambda(u_\alpha u_\beta)\right)\mathrm{d}\rho_{(m,n)}(\alpha,\beta) \\
&\geq \int_{\Delta_m^{(d)}\times\Delta_n^{(d)}} g\left(\frac{\lambda(u_\alpha)+\lambda(u_\beta)-f(n)-f(m)}{m+n}\right)\mathrm{d}\rho_{(m,n)}(\alpha,\beta) \\
&\geq \int_{\Delta_m^{(d)}\times\Delta_n^{(d)}} \left[g\left(\frac{\lambda(u_\alpha)+\lambda(u_\beta)}{m+n}\right) - \frac{c(f(n)+f(m))}{m+n}\right]\mathrm{d}\rho_{(m,n)}(\alpha,\beta) \\
&\geq \int_{\Delta_m^{(d)}\times\Delta_n^{(d)}} \left[\frac{n}{m+n}g\left(\frac{\lambda(u_\alpha)}{n}\right) + \frac{m}{n+m}g\left(\frac{\lambda(u_\beta)}{m}\right)\right]\mathrm{d}\rho_{(m,n)}(\alpha,\beta) - \frac{c(f(n)+f(m))}{m+n} \\
&= \frac{n}{n+m}I_n + \frac{m}{n+m}I_m - c\,\frac{f(n)+f(m)}{m+n}.
\end{aligned} \tag{25}$$

If the sequence $(I_n)_{n\geq 0}$ is bounded from above and if $\sum_{\alpha\geq 0} f(2^\alpha)/2^\alpha < +\infty$, then $(I_n)_{n\geq 0}$ converges, which implies that the sequence of measures $(T_{\frac{1}{n}}\nu_{B_n})_{n\geq 1}$ converges vaguely. In other words, B satisfies the vague convergence condition. The convergence of $(I_n)_{n\geq 0}$ is based on Corollary A.5, which we shall present as below.

Lemma A.3 *If $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ is an increasing function such that $\sum_{\alpha\geq 1} f(2^\alpha)/2^\alpha < +\infty$, then*

$$\lim_{\alpha\to+\infty} 2^{-\alpha}\sum_{i=0}^{\alpha} f(2^i) = 0.$$

Proof. For any integer $\alpha \geq 0$, let $S_\alpha = \sum_{i\geq\alpha} f(2^i)/2^i$. By Abel's summation formula,

$$\sum_{i=0}^{\alpha} f(2^i) = \sum_{i=0}^{\alpha}(S_i - S_{i+1})2^i = S_0 - S_{\alpha+1}2^\alpha + \sum_{i=1}^{\alpha} S_i 2^{i-1}.$$

Since $\lim_{\alpha\to+\infty} S_\alpha = 0$, we have $2^{-\alpha}\sum_{i=1}^{\alpha} S_i 2^{i-1}$ converges to 0 when $\alpha \to +\infty$, which implies the lemma. □

Proposition A.4 *Let $(b_n)_{n\geq 1}$ be a sequence of positive real numbers and $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be an increasing function such that $\sum_{\alpha\geq 1} f(2^\alpha)/2^\alpha < +\infty$. If there exists an integer $n_0 > 0$ such that, for any pair (m,n) of integers $\geq n_0$, we have $b_{n+m} \leq b_m + b_n + f(m) + f(n)$, then the sequence $(b_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}_{\geq 0}$.*

Proof. First let us treat the case where $n_0 = 1$. Since f is an increasing function we obtain that for any $(m,n) \in \mathbb{Z}_{>0}^2$,

$$b_{m+n} \leq b_n + b_m + 2f(m+n). \tag{26}$$

For any integer $\alpha \geq 0$, let $S_\alpha = \sum_{i \geq \alpha} f(2^i)/2^i$. then $\lim_{\alpha \to +\infty} S_\alpha = 0$. If $2^\beta \leq n < 2^{\beta+1}$ is an integer, we have, for any $\alpha \in \mathbb{N}$,

$$b_{2^\alpha n} \leq 2^\alpha b_n + \sum_{i=1}^{\alpha} 2^{\alpha+1-i} f(2^{i-1}n) \leq 2^\alpha b_n + \sum_{i=1}^{\alpha} 2^{\alpha+1-i} f(2^{\beta+i}). \tag{27}$$

Suppose that $p = \sum_{i=0}^{k} \epsilon_i 2^i$, where $\epsilon_i \in \{0,1\}$ for any $0 \leq i < k$ and $\epsilon_k = 1$. If $0 \leq r < n$ is another integer, we have after (26) the following inequality:

$$b_{np+r} \leq b_{np} + b_r + 2f(np+r) \leq \sum_{i=0}^{k} \epsilon_i b_{2^i n} + b_r + 2\sum_{i=0}^{k} \epsilon_i f\Big(\sum_{j=0}^{i} \epsilon_j 2^j n\Big) + 2f(np+r) \tag{28}$$

After (27), we have

$$b_{np+r} \leq \sum_{i=0}^{k} \epsilon_i 2^i b_n + b_r + \sum_{i=1}^{k} \epsilon_i \sum_{j=1}^{i} 2^{i+1-j} f(2^{\beta+j}) + 2\sum_{i=0}^{k} \epsilon_i f(2^{i+\beta+2}) + 2f(2^{k+\beta+2}). \tag{29}$$

Therefore,

$$\begin{aligned} \frac{b_{np+r}}{np+r} \leq \frac{pb_n}{np+r} + \frac{b_r}{np+r} &+ 2^{-k-\beta} \sum_{i=1}^{k} \sum_{j=1}^{i} 2^{i+1-j} f(2^{\beta+j}) \\ &+ 2^{-k-\beta+1} \sum_{i=0}^{k} f(2^{i+\beta+2}) + 2^{-k-\beta+1} f(2^{k+\beta+2}). \end{aligned}$$

Since

$$2^{-k-\beta} \sum_{i=1}^{k} \sum_{j=1}^{i} 2^{i+1-j} f(2^{\beta+j}) = 2^{-k-\beta} \sum_{j=1}^{k} \sum_{i=j}^{k} f(2^{\beta+j}) 2^{i+1-j} \leq \sum_{j=1}^{k} f(2^{\beta+j}) 2^{2-j-\beta} = 4S_{\beta+1},$$

we obtain that

$$\frac{b_{np+r}}{np+r} \leq \frac{pb_n}{np+r} + \frac{b_r}{np+r} + 4S_{\beta+1} + 2^{-k-\beta+1} \sum_{i=0}^{k+\beta+3} f(2^i).$$

After Lemma A.3, we have

$$\limsup_{m \to +\infty} \frac{b_m}{m} \leq \liminf_{n \to +\infty} \Big(\frac{b_n}{n} + 4S_{\lfloor \log_2 n \rfloor + 1}\Big) = \liminf_{n \to +\infty} \frac{b_n}{n}.$$

Therefore, the sequence $(b_n/n)_{n \geq 1}$ converges.

For the general case, by applying the above result on the subsequence $(b_{n_0 k})_{k \geq 1}$ and the function $g(k) = f(n_0 k)$, we obtain that the sequence $(b_{n_0 k}/k)_{k \geq 1}$ has a limit in $\mathbb{R}_{\geq 0}$. On the other hand, if $n_0 \leq l < 2n_0$ is an integer, then for any integer $k \geq 1$, we have the inequality

$$b_{n_0(k+2)} - b_{2n_0 - l} - f(n_0 k + l) - f(2n_0 - l) \leq b_{n_0 k + l} \leq b_{n_0 k} + b_l + f(n_0 k) + f(l). \tag{30}$$

If we divide (30) by $n_0 k + l$, we obtain, by passing to the limit $k \to +\infty$,

$$\lim_{k \to +\infty} \frac{b_{n_0 k + l}}{n_0 k + l} = \lim_{k \to +\infty} \frac{b_{n_0 k}}{n_0 k}.$$

Since l is arbitrary, the proposition is proved. □

Corollary A.5 *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers, $f : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ be an increasing function and $c > 0$ be a constant. Suppose that*

1) for sufficiently large integers n, m, $a_{n+m} \geq a_n + a_m - f(n) - f(m)$,

2) $a_n \leq cn$ for any integer $n \geq 1$,

3) $\sum_{\alpha\geq 0} f(2^\alpha)/2^\alpha < +\infty$.

Then the sequence $(a_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$.

Proof. Consider the sequence $(b_n = cn - a_n)_{n\geq 1}$ of positive real numbers. If n and m are two sufficiently large integers, we have

$$b_{n+m} = c(n+m) - a_{n+m} \leq cn + cm - a_n - a_m + f(n) + f(m) = b_n + b_m + f(n) + f(m).$$

After Proposition A.4, the sequence $(b_n/n)_{n\geq 1}$ has a limit in $\mathbb{R}$. Since $a_n/n = c - b_n/n$, the sequence $(a_n/n)_{n\geq 1}$ also has a limit in $\mathbb{R}$. □

We establish finally the vague convergence for normalized measures associated to a pseudo-filtered graded algebra.

Theorem A.6 *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function such that $\sum_{\alpha\geq 0} f(2^\alpha)/2^\alpha < +\infty$, B be an integral graded K-algebra of finite type over K, which is generated by B_1. Suppose that*

i) $d = \dim B$ is strictly positive,

ii) for any positive integer n, B_n is equipped with an $\mathbb{R}$-filtration $\mathcal{F}$ which is separated, exhaustive and left continuous such that B is an f-pseudo-filtered graded K-algebra,

iii) $\limsup_{n\to+\infty} \sup_{0\neq a\in B_n} \dfrac{\lambda(a)}{n} < +\infty.$

If for any integer $n > 0$, we write $\nu_n = T_{\frac{1}{n}}\nu_{B_n}$, then the supports of ν_n ($n \geq 1$) are uniformly bounded and the sequence of measures $(\nu_n)_{\geq 1}$ converges vaguely to a Borel probability measure on $\mathbb{R}$.

Proof. We apply the proof of Theorem 7.5 in making some modifications. First we replace the inequality (15) by $\lambda^{\min}_{m+n} = \lambda^{\min}_n + \lambda^{\min}_m - f(n) - f(m)$ for all sufficiently large integers m, n. After Corollary A.5, the sequence $(\lambda^{\min}_n/n)_{n\geq 1}$ converges, so is bounded from below.

For the first step, since $\sum_{\alpha\geq 0} f(2^\alpha)/2^\alpha < +\infty$, we have $\lim_{\alpha\to+\infty} f(2^\alpha)/2^\alpha = 0$. As f is an increasing function, $\lim_{n\to+\infty} f(n)/n = 0$. Therefore, the first step of the proof of Theorem 7.5 remains valid. Moreover, the third step is a formal argument for the vague convergence condition, and therefore works without problem. It remains to verify that for any homogeneous element x of B, the graded A-module Ax, equipped with the inverse image filtration, satisfies the vague convergence condition. This corresponds to the second step of the proof of Theorem 7.5. Finally, no modification to the second step is necessary since in inequalities (16) and (17), involves only the product of **two** homogeneous elements in B. □

Corollary A.7 *With the notations of Theorem A.6, the polygons associated to probability measures ν_n converge uniformly to a concave function on $[0, 1]$.*

Remark A.8 Instead of supposing that B is generated by B_1, if we suppose that B_n is non-zero for sufficiently large n, Theorem A.6 remains true, we have also the uniform convergence of polygons.