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# THE INTRINSIC FUNDAMENTAL GROUP OF A LINEAR CATEGORY\*

Claude Cibils, María Julia Redondo and Andrea Solotar

## Abstract

We provide an intrinsic definition of the fundamental group of a linear category over a ring as the automorphism group of the fibre functor on Galois coverings. We prove that this group is isomorphic to the inverse limit of the Galois groups associated to Galois coverings. Moreover, the graduation deduced from a Galois covering enables us to describe the canonical monomorphism from its automorphism group to the first Hochschild-Mitchell cohomology vector space.

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## 1 Introduction

The purpose of this work is to provide a positive answer to the question of the existence of an intrinsic and canonical fundamental group  $\pi_1$  associated to a  $k$ -category  $\mathcal{B}$ , where  $k$  is a commutative ring. The fundamental group we introduce takes into account the linear structure of the category  $\mathcal{B}$ , it differs from the fundamental group of the underlying category obtained as the classifying space of its nerve ([21, 20, 22]).

The fundamental group that we define is intrinsic in the sense that it does not depend on the presentation of the  $k$ -category by generators and relations. In case a universal covering exists, we obtain that the fundamental groups constructed by R. Martínez-Villa and J.A. de la Peña (see [17], and [3, 11, 1]) depending on a presentation of the category by a quiver and relations are in fact quotients of the intrinsic  $\pi_1$  that we introduce. Note that those groups can vary according to different presentations of the same  $k$ -category (see for instance [1, 4, 15]) while the group that we introduce

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is intrinsic, since we define it as the automorphisms of the fibre functor of the Galois coverings over a fixed object.

In fact if a universal covering  $U : \mathcal{U} \rightarrow \mathcal{B}$  exists, the fundamental group that we define is isomorphic to the automorphism group  $\text{Aut}U$ . Otherwise we show that the fundamental group is isomorphic to the inverse limit of the automorphisms groups of the Galois coverings of  $\mathcal{B}$ . In case each connected component of the category of the Galois coverings admits an initial object – in other words if "locally" universal coverings exist – the intrinsic group  $\pi_1$  that we define is isomorphic to the direct product of the corresponding automorphism groups. Particular cases of this fact have been noticed before by several authors, see for instance [4]. We also prove that changing the base object provides isomorphic intrinsic fundamental groups.

The methods we use are inspired for instance in the topological case considered in R. Douady and A. Douady's book [7]. They are closely related to the way the fundamental group is considered in algebraic geometry after A. Grothendieck and C. Chevalley. The present paper is very much indebted to the pioneer work of P. Le Meur in his thesis [16], see also [14]. He has shown that under some hypotheses on the category, there exists an optimal fundamental group in the sense that all other "fundamental groups" deduced from different presentations are quotients of the optimal one. His method consists mainly in tracing all the possible presentations of a given category, and relating the diverse resulting "fundamental groups". As already quoted, we adopt a different point of view in this paper.

In Section 2 we provide the definition of a covering of a  $k$ -category and we prove properties about morphisms between coverings as initiated in [14, 16]. In Section 3 we define Galois coverings and next we study some properties of this kind of coverings. The main results are Theorem 3.7 which describes the structure of Galois coverings and Theorem 3.9 which concerns morphisms between Galois coverings and the relation between the associated groups of automorphisms. We provide the definition of the universal covering in the category of Galois coverings of a fixed  $k$ -category  $\mathcal{B}$ . In a forthcoming paper we will study the behaviour of Galois coverings through fibre products, as well as a criterion for a covering to be Galois or universal. Differences with the usual algebraic topology setting will appear, since the fibre product of coverings of  $k$ -categories do not provide in general a covering through the projection functor.

Finally we suppose that  $k$  is a field and that the endomorphism algebra of each object of the  $k$ -category is reduced to  $k$ . We recover in a simple way the canonical  $k$ -linear embedding (see [1, 19, 6]) of the abelian characters of the automorphism group of a Galois covering to the first Hochschild-Mitchell cohomology vector space of the category. We use our description of Galois coverings as well as the canonical graduation of the  $k$ -category deduced from a Galois covering, as obtained in [5]. This way Euler derivations are considered, see also [8, 9].

Note that it will be interesting to explore the behaviour of the intrinsic fundamental group with respect at least to Morita equivalences of  $k$ -categories. As expected the fundamental group is an invariant of the equivalence class of a  $k$ -category but not of its Morita class. Of course in case the category admits a unique basic representative in its Morita class, the fundamental group attached to this category can be considered as the canonical fundamental group of the Morita equivalence class.

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## 2 $k$ -categories, stars and coverings

Let  $k$  be a commutative ring. A  $k$ -category is a small category  $\mathcal{B}$  such that each morphism set  ${}_y\mathcal{B}_x$  from an object  $x \in \mathcal{B}_0$  to an object  $y \in \mathcal{B}_0$  is a  $k$ -module, the composition of morphisms is  $k$ -bilinear and  $k$ -multiples of the identity at each object are central in the endomorphism ring of the object. Note that such  $k$ -categories are also called *linear categories over  $k$* . In particular each endomorphism set of an object is a  $k$ -algebra, and  ${}_y\mathcal{B}_x$  is a  ${}_y\mathcal{B}_y - {}_x\mathcal{B}_x$ -bimodule.

Each  $k$ -algebra  $A$  provides a single object  $k$ -category  $\mathcal{B}_A$  with endomorphism ring  $A$ . The structure of  $A$  can be described more precisely by choosing a finite set  $E$  of orthogonal idempotents of  $A$ , such that  $\sum_{e \in E} e = 1$  in the following way: the  $k$ -category  $\mathcal{B}_{A,E}$  has set of objects  $E$  and morphisms from  $e$  to  $f$  the  $k$ -module  $fAe$ . Note that  $\mathcal{B}_{A,\{1\}} = \mathcal{B}_A$ . This approach is meaningful since clearly the category of left  $A$ -modules is isomorphic to the category of  $k$ -functors from  $\mathcal{B}_{A,E}$  to the category of  $k$ -modules, where a  $k$ -functor is a functor which is  $k$ -linear when restricted to morphisms.

**Definition 2.1** *The star  $\text{St}_{b_0}\mathcal{B}$  of a  $k$ -category  $\mathcal{B}$  at an object  $b_0$  is the direct sum of all the morphisms with source or target  $b_0$  :*

$$\text{St}_{b_0}\mathcal{B} = \left( \bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_{b_0} \right) \oplus \left( \bigoplus_{y \in \mathcal{B}_0} {}_{b_0}\mathcal{B}_y \right).$$

*Note that this  $k$ -module counts twice the endomorphism algebra at  $b_0$ .*

**Definition 2.2** *Let  $\mathcal{C}$  and  $\mathcal{B}$  be  $k$ -categories. A  $k$ -functor  $F : \mathcal{C} \rightarrow \mathcal{B}$  is a covering of  $\mathcal{B}$  if it is surjective on objects and if  $F$  induces  $k$ -isomorphisms between the corresponding stars. More precisely, for each  $b_o \in \mathcal{B}_0$  and each  $x$  in the non-empty fibre  $F^{-1}(b_o)$ , the map*

$$F_{b_o}^x : \text{St}_x\mathcal{C} \longrightarrow \text{St}_{b_o}\mathcal{B}.$$

*induced by  $F$  is a  $k$ -isomorphism.*

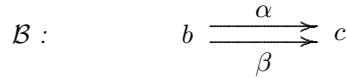
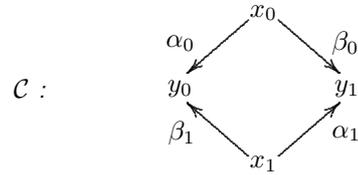
**Remark 2.3** *Each star is the direct sum of the source star  $\text{St}_{b_0}^- \mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_{b_0}$  and the target star  $\text{St}_{b_0}^+ \mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_{b_0}\mathcal{B}_y$ . Since  $\text{St}^-$  and  $\text{St}^+$  are preserved under any  $k$ -functor, the condition of the definition is equivalent to the requirement that the corresponding target and source stars are isomorphic through  $F$ .*

*Moreover this splitting goes further : for  $b_1 \in \mathcal{B}_0$ , the restriction of  $F$  to  $\bigoplus_{y \in F^{-1}(b_0)} {}_y\mathcal{C}_x$  is  $k$ -isomorphic to the corresponding  $k$ -module  ${}_{b_1}\mathcal{B}_{b_0}$ . The same holds with respect to the target star and morphisms starting at all objects in a single fibre.*

**Remark 2.4** *The previous facts show that Definition 2.2 coincides with the one given by K. Bongartz and P. Gabriel in [3].*

To each small category  $\mathcal{A}$  one may associate its linearization  $k\mathcal{A}$  in the following way: objects of  $k\mathcal{A}$  are the objects of  $\mathcal{A}$ , while morphisms are free  $k$ -modules on the sets of morphisms of  $\mathcal{A}$ . Such linearized  $k$ -categories are rather special, since they admit by construction a multiplicative basis of morphisms, in fact they are free  $k$ -categories. Any usual covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  of categories provides by linearization a covering  $kF : k\mathcal{C} \rightarrow k\mathcal{B}$ .

**Example 2.5** [14] *Not all the coverings of a linearized category are obtained in the preceding way. Consider the following  $k$ -categories  $\mathcal{C}$  and  $\mathcal{B}$  obtained by linearization of the categories given by the corresponding diagrams:*



and the covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  given by

$$\begin{aligned}
 F(x_0) = F(x_1) = b, \quad F(y_0) = F(y_1) = c \\
 F(\alpha_0) = \alpha + \beta \\
 F(\alpha_1) = \alpha, \quad F(\beta_0) = F(\beta_1) = \beta.
 \end{aligned}$$

**Definition 2.6** *Given  $k$ -categories  $\mathcal{B}, \mathcal{C}, \mathcal{D}$ , the set  $\text{Mor}(F, G)$  from a covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  to a covering  $G : \mathcal{D} \rightarrow \mathcal{B}$  is the set of  $k$ -linear functors  $g : \mathcal{C} \rightarrow \mathcal{D}$  such that  $Gg = F$ . The group of automorphisms  $\text{Aut}^F$  of a covering  $F$  is the group of invertible endomorphisms of  $F$ . The category of coverings of  $\mathcal{B}$  and morphisms of coverings will be denoted  $\text{Cov}(\mathcal{B})$ .*

Our purpose is to show that the automorphism group of a connected covering acts freely on each fibre.

**Definition 2.7** *A  $k$ -category  $\mathcal{B}$  is connected if any two objects  $b$  and  $c$  of  $\mathcal{B}$  can be connected by a finite walk made by non zero morphisms, more precisely there exist a finite sequence of objects  $x_1, \dots, x_n$  and non zero morphisms  $\varphi_1, \dots, \varphi_n$  such that  $x_1 = b$ ,  $x_n = c$ , where  $\varphi_i$  belongs either to  ${}_{x_{i+1}}\mathcal{B}_{x_i}$  or to  ${}_{x_i}\mathcal{B}_{x_{i+1}}$ .*

**Proposition 2.8** *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a covering of  $k$ -categories. If  $\mathcal{C}$  is connected, then  $\mathcal{B}$  is connected.*

**Proof.** Let  $b$  and  $c$  be objects in  $\mathcal{B}_0$ , and let  $x_0$  and  $y_0$  be two objects respectively chosen in their fibres. Consider a walk of non zero morphisms connecting  $x_0$  and  $y_0$  in  $\mathcal{C}$ . Since  $F$  induces  $k$ -isomorphisms at each star, the image by  $F$  of a non zero morphism is a non zero morphism in  $\mathcal{B}$ .  $\diamond$

**Proposition 2.9** [14] *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  and  $G : \mathcal{D} \longrightarrow \mathcal{B}$  be coverings of  $k$ -linear categories. Assume  $\mathcal{C}$  is connected. Two morphisms  $g$  and  $h$  from  $F$  to  $G$  which coincide on some object are equal.*

**Proof.** Let  $g$  be a morphism of coverings, let  $x_0$  be an object of  $\mathcal{C}$  and consider the map between stars induced by  $g$ :

$$g_{g(x_0)}^{x_0} : \text{St}_{x_0}\mathcal{C} \longrightarrow \text{St}_{g(x_0)}\mathcal{D}.$$

Observe that  $Gg(x_0) = F(x_0)$ . There is a commutative diagram

$$\begin{array}{ccc} \text{St}_{x_0}\mathcal{C} & \xrightarrow{g_{g(x_0)}^{x_0}} & \text{St}_{g(x_0)}\mathcal{D} \\ & \searrow F_{F(x_0)}^{x_0} & \swarrow G_{Gg(x_0)}^{g(x_0)} \\ & \text{St}_{F(x_0)}\mathcal{B} & \end{array}$$

where the morphisms  $F_{F(x_0)}^{x_0}$  and  $G_{Gg(x_0)}^{g(x_0)}$  are  $k$ -isomorphisms. Consequently  $g$  is an isomorphism at each star level, determined by  $F$  and  $G$ . In case  $g$  and  $h$  are morphisms verifying  $g(x_0) = h(x_0)$ , the  $k$ -linear maps  $\text{St}_{x_0}\mathcal{C} \longrightarrow \text{St}_{g(x_0)}\mathcal{D} = \text{St}_{h(x_0)}\mathcal{D}$  induced by  $g$  and  $h$  are equal, hence  $g$  and  $h$  coincide on morphisms starting or ending at  $x_0$ , in particular  $g$  and  $h$  coincide on objects related to  $x_0$  by a non-zero morphism. Since  $\mathcal{C}$  is connected, it follows that  $g$  and  $h$  coincide on every object and on every morphism of  $\mathcal{C}$ .  $\diamond$

**Corollary 2.10** *The group of automorphisms of a connected covering of  $k$ -linear categories acts freely on each fibre.*

### 3 Galois and universal coverings

We start this section with the definition of a Galois covering in order to study properties of this kind of coverings. The main results are the description of the structure of Galois coverings, and the relation between Galois coverings and the group of automorphisms. Finally we consider the definition of the universal covering in the category of Galois coverings of a fixed  $k$ -category  $\mathcal{B}$ .

**Definition 3.1** A covering  $F : \mathcal{C} \longrightarrow \mathcal{B}$  of  $k$ -categories is a Galois covering if  $\mathcal{C}$  is connected and if  $\text{Aut}F$  acts transitively on some fibre.

It is natural to expect that the automorphism group should act transitively at each fibre whenever it acts transitively on a particular one. In order to prove this fact, we shall use a construction introduced in [11, 3], see also [6].

**Definition 3.2** Let  $G$  be a group acting by  $k$ -autofunctors on a  $k$ -category  $\mathcal{C}$ , such that the action on the objects is free, meaning that if  $sx = x$  for some object, then  $s = 1$ . The set of objects of the categorical quotient  $\mathcal{C}/G$  is the set of  $G$ -orbits of  $\mathcal{C}_0$ . The  $k$ -module of morphisms from an orbit  $\alpha$  to an orbit  $\beta$  is

$${}_{\beta}(\mathcal{C}/G)_{\alpha} = \left( \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \right) / G$$

where for a  $kG$ -module  $X$  we denote  $X/G$  the  $k$ -module  $X/(\text{Ker } \epsilon)X$ , which is the quotient of  $X$  by the augmentation ideal, where  $\epsilon : kG \rightarrow k$  is given by  $\epsilon(s) = 1$  for all  $s \in G$ .

**Remark 3.3** One can easily check that the previous definition provides a  $k$ -category: the composition is well defined precisely because the action of  $G$  on the objects is free.

**Proposition 3.4** Let  $G$  be a group acting by  $k$ -autofunctors on a connected  $k$ -category  $\mathcal{C}$ , and assume that the action on the objects is free. Then the projection functor  $P : \mathcal{C} \longrightarrow \mathcal{C}/G$  is a Galois covering with automorphism group  $G$ .

**Proof.** The projection functor is a covering since it is surjective on objects. For each choice of an object  $x_0 \in \alpha$  and  $y_0 \in \beta$  we clearly have  $k$ -isomorphisms

$$\bigoplus_{y \in \beta} {}_y\mathcal{C}_{x_0} \rightarrow {}_{\beta}(\mathcal{C}/G)_{\alpha} \quad \text{and} \quad \bigoplus_{x \in \alpha} {}_{y_0}\mathcal{C}_x \rightarrow {}_{\beta}(\mathcal{C}/G)_{\alpha}$$

which can be assembled in order to provide the required isomorphism of stars. Observe that the fibres of  $P$  are the orbits sets by construction, therefore the action of  $\text{Aut}P = G$  is transitive on each fibre.

Consider now  $g \in \text{Aut}P$  and let  $x_0 \in \mathcal{C}_0$ . Since the action of  $G$  on  $\mathcal{C}_0$  is free, there exists a unique  $s \in G$  such that  $sx_0 = g(x_0)$ . By definition, the element  $s$  provides an autofunctor of  $\mathcal{C}$  which verifies  $Ps = P$ . Both automorphisms of the connected covering  $P$  coincide on an object, consequently they are equal as automorphisms of  $P$  by Proposition 2.9.  $\diamond$

**Lemma 3.5** Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a connected covering of  $k$ -categories and suppose there exists a singleton fibre. Then every fibre is a singleton and  $F$  is an isomorphism of  $k$ -categories.

**Proof.** Let  $b \in \mathcal{B}_0$  be an object such that  $F^{-1}(b) = \{x\}$ . Since  $\mathcal{C}$  is connected it is enough to show that for a non zero morphism in  $\mathcal{C}$  with target or source  $x$ , the other extreme object  $y$  is such that  $F^{-1}(F(y)) = \{y\}$ . We denote  $c = F(y)$ . Assume  $\varphi \in {}_y\mathcal{C}_x$  is non-zero and let  $y' \in F^{-1}(c)$ , then  $\varphi \in \text{St}_x\mathcal{C}$  and  $F(\varphi) \in \text{St}_b\mathcal{B}$ . Moreover,  $F(\varphi)$  belongs to  $\text{St}_c\mathcal{B}$ . Since  $F$  induces an isomorphism  $F_c^{y'} : \text{St}_{y'}\mathcal{C} \longrightarrow \text{St}_c\mathcal{B}$ , there is a unique  $k$ -linear combination  $\sum_z {}_{y'}h_z$  of morphisms from the fibre of  $b$  to  $y'$  such that  $F_c^{y'}(\sum_z {}_{y'}h_z) = F(\varphi)$ . Now the fibre of  $b$  is reduced to  $x$ , which means that there is a non-zero morphism  $\psi \in {}_{y'}\mathcal{C}_x$  such that  $F\psi = F\varphi$ . Note that  $\psi$  also belongs to  $\text{St}_x\mathcal{C}$ , and recall that  $F_b^x$  is an isomorphism between the corresponding stars. Hence  $\varphi = \psi$  and in particular their ending objects are the same, namely  $y = y'$ . Finally since all the fibres are singletons, the star property of a covering implies immediately that  $F$  is an isomorphism.  $\diamond$

We are now able to prove the following proposition.

**Proposition 3.6** *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a Galois covering. Then  $\text{Aut}F$  acts transitively on each fibre.*

**Proof.** First consider the categorical quotient  $P : \mathcal{C} \longrightarrow \mathcal{C}/\text{Aut}F$ . There is a unique functor  $F' : \mathcal{C}/\text{Aut}F \longrightarrow \mathcal{B}$  such that  $F'P = F$ , defined as follows: let  $\alpha$  be an object of  $\mathcal{C}/\text{Aut}F$ , that is, an orbit of  $\mathcal{C}_0$  under the action of  $\text{Aut}F$ . Choose an object  $x \in \alpha$  and define  $F'\alpha = Fx$ . Clearly  $F'$  is well defined on objects. In order to define  $F'$  on morphisms, let  $\alpha$  and  $\beta$  be objects in  $\mathcal{C}/\text{Aut}F$ , and recall that

$${}_{\beta}(\mathcal{C}/\text{Aut}F)_{\alpha} = \left( \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \right) / \text{Aut}F.$$

Next observe that the morphism

$$F : \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x \longrightarrow {}_{F'(\beta)}\mathcal{B}_{F'(\alpha)}$$

verifies  $F(s\varphi) = F(\varphi)$  for any  $\varphi \in \bigoplus_{x \in \alpha, y \in \beta} {}_y\mathcal{C}_x$  and any  $s \in \text{Aut}F$ . Finally the commutative triangle of morphisms between corresponding stars shows that  $F'$  is indeed a covering.

Since  $F$  is a Galois covering, there exists a fibre where the action of  $\text{Aut}F$  is transitive, which means that the corresponding fibre of  $F'$  is a singleton. Since  $F$  is a Galois covering,  $\mathcal{C}$  is connected as well as  $\mathcal{C}/\text{Aut}F$  by Proposition 2.8. The preceding Lemma asserts that all the fibres of  $F'$  are singletons, which exactly means that the action of  $\text{Aut}F$  is transitive on each fibre of  $F$ .  $\diamond$

As a consequence we obtain the following description of Galois coverings.

**Theorem 3.7** *Let  $F : \mathcal{C} \longrightarrow \mathcal{B}$  be a Galois covering. Then  $F$  is isomorphic to the categorical quotient of  $\mathcal{C}$  by  $\text{Aut}F$ , more precisely there exists a unique isomorphism of categories  $F' : \mathcal{C}/\text{Aut}F \longrightarrow \mathcal{B}$  such that  $F'P = F$ , where  $P : \mathcal{C} \longrightarrow \mathcal{C}/\text{Aut}F$  is the Galois covering given by the categorical quotient.*

**Proof.** The proof of the preceding results provides the covering  $F'$ , which has a singleton fibre. Then all the fibres of  $F'$  are singletons and  $F'$  is an isomorphism.  $\diamond$

The following example shows that coverings are not Galois coverings in general.

**Example 3.8** Consider  $F : \mathcal{C} \rightarrow \mathcal{B}$  as in Example 2.5. This covering has trivial automorphism group, as an easy computation shows. Consequently this covering is not Galois, since each fibre has two objects. This fact has been observed by Le Meur [14, 16].

Next we recall a result of P. Le Meur concerning factorizations of Galois coverings.

**Theorem 3.9** [14, 16] Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  and  $G : \mathcal{D} \rightarrow \mathcal{B}$  be Galois coverings, and let  $H$  be a morphism from  $F$  to  $G$ . Then there is an epimorphism of groups  $\lambda : \text{Aut}F \rightarrow \text{Aut}G$  and  $\text{Ker } \lambda = \text{Aut}H$ . Moreover  $H$  is a Galois covering.

**Proof.** First we assert that for each  $f \in \text{Aut}F$  there exists a unique element  $\lambda(f) \in \text{Aut}G$  such that  $\lambda(f)H = Hf$ . For this purpose, let  $c_0$  be an object of  $\mathcal{C}$ . Then  $GHfc_0 = Ffc_0 = Fc_0 = GHc_0$ , hence  $Hfc_0$  and  $Hc_0$  are in the same  $G$ -fibre. Then there exists a unique automorphism of  $G$ , which we denote  $\lambda(f)$ , such that  $Hfc_0 = \lambda(f)Hc_0$ . Moreover both functors  $\lambda(f)H$  and  $Hf$  are morphisms from  $F$  to  $G$  which coincide on  $c_0$ , hence they are equal. The uniqueness of  $\lambda(f)$  and the equalities

$$\lambda(f_1 f_2)H = Hf_1 f_2 = \lambda(f_2)Hf_1 = \lambda(f_1)\lambda(f_2)H$$

imply that  $\lambda$  is a group morphism. Moreover  $\lambda$  is an epimorphism. Let  $g \in \text{Aut}G$ . In order to find  $f \in \text{Aut}F$  such that  $gHc_0 = Hfc_0$ , let  $c$  be an element in the  $H$ -fibre of  $gHc_0$ . A simple computation shows that  $c$  and  $c_0$  are in the same  $F$ -fibre. Since  $F$  is a Galois covering, there exists  $f \in \text{Aut}F$  such that  $fc_0 = c$ , and  $Hfc_0 = Hc = gHc_0$ . Note that  $f \in \text{Ker } \lambda$  if and only if  $Hf = H$  which means precisely that  $f \in \text{Aut}H$ .

In order to prove that  $H$  is a Galois covering, let us first prove that  $H$  is surjective on objects. Let  $y \in \mathcal{D}_0$  and let  $x$  be an object in the  $F$ -fibre of  $Gy$ . Clearly  $Hx$  and  $y$  are in the same  $G$ -fibre, hence there exists  $g \in \text{Aut}G$  such that  $y = gHx$ . Let  $f \in \text{Aut}F$  such that  $\lambda(f) = g$ . We assert that  $Hfx = y$ . Indeed,  $Hf = \lambda(f)H$ , hence  $Hfx = \lambda(f)Hx = gHx = y$ .

The functor  $H$  induces isomorphisms between stars since  $GH = F$ , hence the same equality is valid at the stars level where  $F$  and  $G$  induce  $k$ -isomorphisms.

We have proven that  $H$  is a covering. In order to obtain that  $H$  is Galois, let  $x$  and  $x'$  be in the same  $H$ -fibre. They are also in the same  $F$ -fibre, hence there exists  $f \in \text{Aut}F$  such that  $fx = x'$ . We assert that in fact  $f \in \text{Aut}H$ : indeed,  $Hf$  and  $H$  are both morphisms from  $F$  to  $G$  with the same value on  $x$ , hence they are equal by Proposition 2.9.  $\diamond$

**Definition 3.10** Let  $\mathcal{B}$  be a  $k$ -category and let  $b_0$  be a fixed object. The category  $\text{Gal}(\mathcal{B}, b_0)$  of pointed Galois coverings of  $(\mathcal{B}, b_0)$  is defined as follows: objects are triples  $(\mathcal{C}, x_0, F)$ , where  $F : (\mathcal{C}, x_0) \rightarrow (\mathcal{B}, b_0)$  is a Galois covering of  $\mathcal{B}$  verifying  $F(x_0) = b_0$ . Morphisms from  $(\mathcal{C}, x_0, F)$  to  $(\mathcal{D}, y_0, G)$  in  $\text{Gal}(\mathcal{B}, b_0)$  are pointed morphisms, i.e.  $k$ -functors  $H : (\mathcal{C}, x_0) \rightarrow (\mathcal{D}, y_0)$  such that  $H(x_0) = y_0$  and  $GH = F$ .

We recall that a small category is called *thin* if there is at most one morphism from  $x$  to  $y$  for any couple of objects. Note that this definition concerns usual categories without  $k$ -structure as the category of Galois coverings  $\text{Gal}(\mathcal{B}, b_0)$ .

**Lemma 3.11** The category  $\text{Gal}(\mathcal{B}, b_0)$  is thin. Moreover morphisms are Galois coverings of  $k$ -categories.

**Proof.** Since Galois coverings are connected by definition and morphisms in  $\text{Gal}(\mathcal{B}, b_0)$  are pointed, the result follows from Proposition 2.9. The preceding Theorem shows that morphisms are Galois coverings.  $\diamond$

**Lemma 3.12** Given two isomorphic  $k$ -linear pointed categories  $(\mathcal{B}, b_0)$  and  $(\mathcal{B}', b'_0)$ , the categories  $\text{Gal}(\mathcal{B}, b_0)$  and  $\text{Gal}(\mathcal{B}', b'_0)$  are isomorphic.

**Proof.** Let  $\varphi : \mathcal{B} \rightarrow \mathcal{B}'$  be the isomorphism of categories, such that  $\varphi(b_0) = b'_0$ . The left composition with  $\varphi$  of each pointed covering of  $\mathcal{B}$  gives a pointed covering of  $\mathcal{B}'$ .  $\diamond$

**Definition 3.13** A universal covering  $U$  is an initial object in  $\text{Gal}(\mathcal{B}, b_0)$ . In other words  $U$  is a pointed Galois covering  $U : (\mathcal{U}, u_0) \rightarrow (\mathcal{B}, b_0)$  such that for any pointed Galois covering  $F$  there exists a unique morphism from  $U$  to  $F$ .

**Remark 3.14** In case of existence, a universal covering is unique up to a unique isomorphism of pointed Galois coverings.

The following example shows that there exist  $k$ -categories which do not admit universal covering. It has been obtained by Geiss and de la Peña in [12]:

**Example 3.15** Let  $k$  be a field and  $\text{char}(k) = 2$ . Consider the  $k$ -linear categories

$$\begin{array}{c}
 \mathcal{C}_1 : \\
 \begin{array}{ccccc}
 x_0 & \xrightarrow{\alpha_0} & y_0 & \xrightarrow{\gamma_0} & z_0 \\
 & \searrow \beta_0 & & \searrow \delta_0 & \\
 & & & & \\
 x_1 & \xrightarrow{\alpha_1} & y_1 & \xrightarrow{\gamma_1} & z_1 \\
 & \nearrow \beta_1 & & \nearrow \delta_1 & \\
 & & & & 
 \end{array} \\
 \\
 \mathcal{B} : \\
 \begin{array}{ccccc}
 x & \xrightarrow{\alpha} & y & \xrightarrow{\gamma} & z \\
 & \searrow \beta & & \searrow \delta & 
 \end{array}
 \end{array}$$

with  $\mathcal{C}_1$  satisfying all commutativity relations and  $\mathcal{B}$  satisfying the relations

$$\gamma\alpha = \delta\beta, \quad \gamma\beta = \delta\alpha.$$

It is clear that  $\mathcal{C}_1$  is a Galois covering of  $\mathcal{B}$ . Since  $\text{char}(k) = 2$ , if we set  $a = \alpha + \beta, b = \beta, c = \gamma + \delta, d = \delta$ , we get that  $\mathcal{B}$  satisfies the relations

$$ca = 0, \quad cb = da.$$

In this case,

$$\mathcal{C}_2 : \begin{array}{ccccc} & & \vdots & & \vdots \\ & & x_{-1} & \xrightarrow{b_{-1}} & y_{-1} & \xrightarrow{\quad} & z_{-1} \\ & & \downarrow a_{-1} & & \downarrow c_{-1} & & \\ & & x_0 & \xrightarrow{b_0} & y_0 & \xrightarrow{d_0} & z_0 \\ & & \downarrow a_0 & & \downarrow c_0 & & \\ x_1 & \xrightarrow{\quad} & y_1 & \xrightarrow{d_1} & z_1 & & \\ & & \vdots & & \vdots & & \end{array}$$

with all commutativity relations and  $c_i a_{i-1} = 0$ , is also a Galois covering of  $\mathcal{B}$ . Now  $\mathcal{C}_1$  and  $\mathcal{C}_2$  admit no proper Galois covering since they are simply connected, see [17, 2], and there is no morphism between them.

## 4 Fundamental group

As quoted in the Introduction, our main purpose is to provide an intrinsic definition of the fundamental group  $\pi_1$  of a  $k$ -category, where  $k$  is a commutative ring. Previous definitions, provided for instance by J.A. de la Peña and R. Martinez-Villa [17], see also K. Bongartz and P. Gabriel [3], depend on the presentation of the category as a quotient of a free  $k$ -category by an ideal generated by some set of minimal relations. Different presentations of the same  $k$ -category may provide different groups through this construction, see for instance [1, 4, 15].

We will prove the following facts concerning the group  $\pi_1$  that we will define:

- if the universal covering exists, its group of automorphisms is isomorphic to the group  $\pi_1$ ; in this case any group obtained through the presentation construction is a quotient of  $\pi_1$ ;
- $\pi_1$  is the direct limit of the automorphism groups of the class of all Galois coverings of a given  $k$ -category.

**Definition 4.1** Let  $(\mathcal{B}, b_0)$  be a pointed  $k$ -category and consider  $\text{Gal}(\mathcal{B}, b_0)$  the category of Galois coverings. Let  $\Phi : \text{Gal}(\mathcal{B}, b_0) \rightarrow \text{Sets}$  be the fibre functor which associates to each Galois covering  $F$  the  $F$ -fibre  $F^{-1}(b_0)$ . We define

$$\pi_1(\mathcal{B}, b_0) = \text{Aut}\Phi.$$

**Remark 4.2** The fundamental group  $\pi_1(\mathcal{B}, b_0)$  is the group of natural isomorphisms  $\sigma : \Phi \rightarrow \Phi$ . In other words an element of the fundamental group is a family of invertible set maps  $\sigma_F : F^{-1}(b_0) \rightarrow F^{-1}(b_0)$  for each Galois covering  $F$ , which are compatible with morphisms of Galois coverings; namely for each morphism  $H : F \rightarrow G$  in  $\text{Gal}(\mathcal{B}, b_0)$  the corresponding square

$$\begin{array}{ccc} F^{-1}(b_0) & \xrightarrow{\sigma_F} & F^{-1}(b_0) \\ H \downarrow & & \downarrow H \\ G^{-1}(b_0) & \xrightarrow{\sigma_G} & G^{-1}(b_0) \end{array}$$

is commutative.

We have already proven that the category  $\text{Gal}(\mathcal{B}, b_0)$  is thin. For the convenience of the reader we recall some easy facts concerning inverse limits over thin categories. Let  $L$  be a functor from a thin category  $\mathfrak{L}$  to the category of groups. The inverse limit  $\varprojlim L$  of  $L$  is given by

$$\varprojlim L = \left\{ (s_i) \in \prod_{i \in \mathfrak{L}_0} L(i) \mid s_j = L(j f_i)(s_i) \right\}$$

which is clearly a subgroup of  $\prod_{i \in \mathfrak{L}_0} L(i)$ .  $\diamond$

The following result is immediate.

**Proposition 4.3** Let  $\mathfrak{L}$  and  $L$  be as above and assume that  $\mathfrak{L}$  has an initial object  $u$ . Then  $\varprojlim L = L(u)$

The following theorem provides a characterization of the fundamental group  $\pi_1$ .

**Theorem 4.4** Let  $\mathcal{B}$  be a connected  $k$ -category, let  $b_0$  be an object of  $\mathcal{B}$  and let  $\mathbf{Aut} : \text{Gal}(\mathcal{B}, b_0) \rightarrow \text{Grp}$  be the functor which associates to each pointed Galois covering its group of automorphisms. Then

$$\pi_1(\mathcal{B}, b_0) \simeq \varprojlim \mathbf{Aut}$$

**Proof.** In order to simplify notation, we denote  $(F, x)$  a Galois covering  $F : \mathcal{C} \rightarrow \mathcal{B}$  such that  $F(x) = b_0$ . Note that  $\text{Aut}(F, x)$  denotes the group  $\text{Aut}F$  of non-pointed morphisms.

Let  $\sigma$  be an automorphism of the fibre functor  $\Phi$  and consider  $\sigma_{(F, x)} : F^{-1}(b_0) \rightarrow F^{-1}(b_0)$  for each Galois covering. There exists a unique  $g_{\sigma, (F, x)} \in \text{Aut}(F, x)$  such that

$$\sigma_{(F, x)} x = g_{\sigma, (F, x)} x.$$

We define  $\alpha : \pi_1(\mathcal{B}, b_0) \rightarrow \prod_{(F,x)} \mathbf{Aut}(F, x)$  by  $\alpha(\sigma) = (g_{\sigma, (F,x)})$  and we assert that  $\alpha(\sigma)$  belongs to the inverse limit  $\varprojlim \mathbf{Aut}$ .

Indeed let  $H \in \text{Mor}((F, x), (G, y))$  be such that  $Hx = y$ , and let  $\lambda_H$  be the morphism of groups obtained in Theorem 3.9 defined by the property  $Hg = \lambda_H(g)H$  for any  $g \in \text{Aut}F$ . We have to show that

$$\lambda_H(g_{\sigma, (F,x)}) = g_{\sigma, (G,y)},$$

in other words we have to verify that  $Hg_{\sigma, (F,x)} = g_{\sigma, (G,y)}H$ . This last equality is true since  $\sigma$  is an automorphism of the fibre functor, see Remark 4.2.

In fact  $\alpha$  is a group antimorphism: we have  $\alpha(\sigma\sigma') = \alpha(\sigma')\alpha(\sigma)$  since

$$\sigma'_{(F,x)}\sigma_{(F,x)}x = \sigma'_{(F,x)}g_{\sigma, (F,x)}x = g_{\sigma, (F,x)}\sigma'_{(F,x)}x = g_{\sigma, (F,x)}g_{\sigma', (F,x)}x = g_{\sigma\sigma', (F,x)}x,$$

where the second equality follows from Remark 4.2. Of course since  $\alpha$  is just an antimorphism, the right map to consider in order to have a group morphism is  $\alpha$  followed by the inverse group map. This morphism is injective: let  $\sigma$  be such that  $g_{\sigma, (F,x)} = 1$  for all  $(F, x)$ . Then  $\sigma_{(F,x)}x = x$ . For each  $y \in F^{-1}(b_0)$  there exists  $s \in \text{Aut}F$  such that  $y = sx$  because  $F$  is Galois. Using Remark 4.2 with  $H = s$  we get that  $\sigma_{(F,x)}y = s\sigma_{(F,x)}x = sx = y$ . Hence  $\sigma = \text{Id}$ .

Finally  $\alpha$  is surjective: let  $(l_F) \in \varprojlim \mathbf{Aut}$ , that is,  $(l_F)_{(F,x) \in \text{Gal}(\mathcal{B}, b_0)_0}$  verifies  $l_G = \lambda_H(l_F)$  for each morphism  $H$  from  $F$  to  $G$ . We wish to obtain  $\sigma \in \pi_1$  such that  $\alpha(\sigma) = (g_{\sigma, (F,x)}) = (l_F)$ . Let  $\sigma_F : F^{-1}(b_0) \rightarrow F^{-1}(b_0)$  be defined on  $x$  by  $\sigma_F(x) = l_F(x)$ , while for  $y \in F^{-1}(b_0)$  there exists a unique  $s$  such that  $y = sx$ . We define  $\sigma_F(y) = s[l_F(x)]$ . In order to prove that  $\sigma$  is natural with respect to any morphism  $H \in \text{Mor}((F, x), (G, y))$ , note that

$$l_G(y) = l_G(H(x)) = \lambda_H(l_F)(H(x)) = H(l_F(x)).$$

◇

The following result is now immediate using Proposition 4.3.

**Corollary 4.5** *Suppose that a pointed connected  $k$ -category  $(\mathcal{B}, b_0)$  admits a universal covering  $U$ . Then*

$$\pi_1(\mathcal{B}, b_0) \simeq \text{Aut}U.$$

**Corollary 4.6** *Let  $\mathcal{B}$  be a  $k$ -category admitting a universal covering, and consider a presentation of  $\mathcal{B}$  given by a quiver  $Q$  and an admissible two-sided ideal  $I$  provided with a minimal set of generators  $R$  given by parallel paths. Let  $\pi_1(Q, R, b_0)$  be the group of the presentation as defined in [17], with respect to a vertex  $b_0$ . Then there is a group surjection  $\pi_1(\mathcal{B}, b_0) \rightarrow \pi_1(Q, R, b_0)$ .*

**Proof.** In [17] it is proven that the group  $\pi_1(Q, R, b_0)$  can be realized as the automorphism group of a Galois covering. Then the universal covering  $U$  of  $\mathcal{B}$  provides an epimorphism from the automorphism group of  $U$  to  $\pi_1(Q, R, b_0)$ . ◇

Our next purpose is to show the invariance of the fundamental group with respect to the base point. This will be a consequence of the following easy result on inverse limits over thin categories.

**Proposition 4.7** *Let  $\mathcal{L}$  and  $\mathcal{L}'$  be thin categories, let  $L : \mathcal{L} \rightarrow \text{Grp}$  and  $L' : \mathcal{L}' \rightarrow \text{Grp}$  be two functors, and let  $E : \mathcal{L} \rightarrow \mathcal{L}'$  be an isomorphism of categories such that  $L'E = L$ . Then the inverse limits  $\varprojlim L$  and  $\varprojlim L'$  are isomorphic.*

**Lemma 4.8** *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  and  $G : \mathcal{D} \rightarrow \mathcal{B}$  be Galois coverings of a connected  $k$ -category  $\mathcal{B}$  and assume that there is a morphism  $H$  from  $F$  to  $G$ . Let  $b_0$  be an object of  $\mathcal{B}$ . Then for any couple of objects  $x \in F^{-1}b_0$  and  $y \in G^{-1}b_0$  there exists a unique morphism  $H'$  from  $F$  to  $G$  such that  $H'x = y$ .*

**Proof.** The uniqueness is provided by Proposition 2.9. Since  $G$  is a Galois covering and since  $y$  and  $Hx$  are in the same  $G$ -fibre, there exists a unique  $u \in \text{Aut}G$  such that  $y = uHx$ . Define  $H' = uH$ .  $\diamond$

**Theorem 4.9** *Let  $\mathcal{B}$  be a connected  $k$ -category and let  $b_0$  and  $b_1$  be two objects. Then  $\pi_1(\mathcal{B}, b_0)$  and  $\pi_1(\mathcal{B}, b_1)$  are isomorphic.*

**Proof.** From Proposition 4.7, it is enough to construct an isomorphism of categories  $E : \text{Gal}(\mathcal{B}, b_0) \rightarrow \text{Gal}(\mathcal{B}, b_1)$  such that  $\mathbf{Aut}_{b_1} E = \mathbf{Aut}_{b_0}$ . For each Galois covering  $F$  choose objects  $x_{0,F}$  and  $x_{1,F}$  in the fibre of  $b_0$  and  $b_1$  respectively. Let  $F : (\mathcal{C}, x) \rightarrow (\mathcal{B}, b_0)$  be a pointed Galois covering. Since the action of  $\text{Aut}F$  on the fibre of  $b_0$  is free and transitive, there exists a unique  $s \in \text{Aut}F$  such that  $x = sx_{0,F}$ . We define  $E(F) \in \text{Gal}(\mathcal{B}, b_1)$  as the Galois covering  $F : (\mathcal{C}, sx_{1,F}) \rightarrow (\mathcal{B}, b_1)$ . Moreover, let  $H$  be a morphism of pointed Galois coverings from  $F : (\mathcal{C}, x) \rightarrow (\mathcal{B}, b_0)$  to  $G : (\mathcal{D}, y) \rightarrow (\mathcal{B}, b_0)$ . Let  $s \in \text{Aut}F$  such that  $x = sx_{0,F}$  and let  $t \in \text{Aut}G$  such that  $y = tx_{0,G}$ . Let  $E(H)$  be the morphism from  $F : (\mathcal{C}, sx_{1,F}) \rightarrow (\mathcal{B}, b_1)$  to  $G : (\mathcal{D}, tx_{1,G}) \rightarrow (\mathcal{B}, b_1)$  verifying  $E(H)(sx_{1,F}) = tx_{1,G}$ . The existence and uniqueness of  $E(H)$  are insured by the lemma above. This argument also proves that  $E$  is a functor. Clearly  $E$  commutes with the  $\text{Aut}$  functor.  $\diamond$

We state now another result concerning inverse limits over thin categories. Recall that a usual category is called connected if two objects can be connected by a finite walk of morphisms. In other words the  $k$ -linearization of the category is connected in the sense of Definition 2.7.

**Proposition 4.10** *Let  $\mathcal{L}$  be a thin category and let  $(\mathcal{L}_i)_{i \in I}$  be its connected components. Let  $L$  be a covariant functor from  $\mathcal{L}$  to groups, and let  $L_i$  be the restriction of  $L$  to  $\mathcal{L}_i$ . Then*

$$\varprojlim L \simeq \prod_{i \in I} \varprojlim L_i.$$

**Corollary 4.11** *Let  $(\mathcal{B}, b_0)$  be a connected pointed linear  $k$ -category and consider  $\text{Gal}(\mathcal{B}, b_0)^*$ , the category of Galois coverings without the trivial Galois covering. Let  $(\text{Gal}(\mathcal{B}, b_0)_i)_{i \in I}^*$  be the connected components admitting an initial object  $U_i$ , and let  $\widetilde{\pi}_1$  be the inverse limit of the Galois groups on the union of components without initial objects. Then*

$$\pi_1(\mathcal{B}, b_0) \simeq \prod_{i \in I} \text{Aut}U_i \times \widetilde{\pi}_1.$$

## 5 First Hochschild cohomology and Galois groups

In this section our main purpose is to provide a canonical embedding from the additive characters of the intrinsic  $\pi_1$  that we have defined to the first Hochschild-Mitchell cohomology vector space of  $\mathcal{B}$ . This will be achieved in case  $k$  is a field and assuming that the endomorphism of each object of the category is reduced to  $k$ .

First we will provide an intrinsic and direct way of describing the injective morphism from the additive characters of the group of automorphisms of a Galois covering to the first Hochschild-Mitchell cohomology vector space of  $\mathcal{B}$ .

Assem and de la Peña have described this map in [1] when  $G$  is the fundamental group of a triangular finite dimensional algebra presented by a quiver with relations. In [19] de la Peña and Saorín noticed that the triangular hypothesis is superfluous. This map has also been obtained in a spectral sequence context in [6].

In order to provide the canonical morphism, we first recall (see [13, 5]) that a Galois covering of  $\mathcal{B}$  provides a graduation for each choice of objects in the fibres. As expected another choice of objects provides a conjugated grading. We will translate in this setting the connectivity hypothesis of the Galois covering. Finally the definition of Hochschild-Mitchell derivations as well as the inner ones will provide the context for a natural definition of the required map.

**Definition 5.1** *Let  $\mathcal{B}$  be a  $k$ -category, where  $k$  is a ring. A  $G$ -graduation  $Z$  of  $\mathcal{B}$  by a group  $G$  is a decomposition of each morphism space as a direct sum of vector spaces  $Z_s$  indexed by  $G$ . For each couple of objects  $b$  and  $c$  we have  ${}_c\mathcal{B}_b = \bigoplus_{s \in G} Z_s({}_c\mathcal{B}_b)$  and*

$$Z_t({}_d\mathcal{B}_c) Z_s({}_c\mathcal{B}_b) \subset Z_{ts}({}_d\mathcal{B}_b),$$

where elements of  $Z_s({}_c\mathcal{B}_b)$  are called homogeneous morphisms of degree  $s$  from  $b$  to  $c$ .

The following result is clear:

**Proposition 5.2** *Let  $\mathcal{B}$  be a  $k$ -category as above, equipped with a  $G$ -graduation  $Z$ . Let  $(t_b)_{b \in \mathcal{B}_0}$  be a family of elements of  $G$  associated to the objects of  $\mathcal{B}$ . Define  $Y_s({}_c\mathcal{B}_b) = Z_{t_c \cdot s \cdot t_b^{-1}}({}_c\mathcal{B}_b)$ . Then  $Y$  is also a  $G$ -graduation of  $\mathcal{B}$ .*

**Theorem 5.3** [5] *Let  $\mathcal{C} \rightarrow \mathcal{B}$  be a Galois covering of categories. Let  $(x_b)_{b \in \mathcal{B}_0}$  be a choice of objects of  $\mathcal{C}$ , where  $x_b$  belongs to the  $F$ -fibre of  $b$  for each  $b \in \mathcal{B}_0$ . Then there is an  $\text{Aut}F$ -graduation of  $\mathcal{B}$ . Another choice of fibre objects provides a graduation with the same homogeneous components as described in the preceding Proposition.*

**Proof.** Since  $F$  is a covering, each morphism space of  $\mathcal{B}$  is equipped with a direct sum decomposition  $\bigoplus_{y \in F^{-1}c} F(y\mathcal{C}_{x_b})$ . Moreover since  $F$  is Galois, for each  $y$  in the  $F$ -fibre of  $c$  there exists a unique automorphism  $s$  such that  $y = sx_c$ . This element  $s$  will provide the degree of the direct summand, more precisely

$$Z_s(c\mathcal{B}_b) = F(sx_c\mathcal{C}_{x_b}).$$

It is straightforward to verify that this is indeed an  $\text{Aut}F$ -graduation. Moreover, a different choice of objects in the fibres  $(t_b x_b)_{b \in \mathcal{B}_0}$  where  $(t_b)_{b \in \mathcal{B}_0}$  is a family of elements of  $\text{Aut}F$  provides another graduation, which is precisely the graduation described in the proposition above.  $\diamond$

**Remark 5.4** In [5] the converse is obtained: in case  $\mathcal{B}$  is  $G$ -graded where  $G$  is an arbitrary group, the smash group category construction provides a Galois covering with automorphism group  $G$ . As expected, one recovers the original graduation as the graduation induced by the smash Galois covering as defined in [5].

**Definition 5.5** An homogeneous walk  $w$  in a  $G$ -graded  $k$ -category from an object  $b$  to an object  $c$  is a sequence of non zero homogeneous paths. It consists of a sequence of objects  $x_1 = b, \dots, x_i, \dots, x_n = c$ , a sequence of signs  $\epsilon_1, \dots, \epsilon_n$  where  $\epsilon_i \in \{-1, +1\}$ , and non-zero homogeneous morphisms  $\varphi_1, \dots, \varphi_n$  such that if  $\epsilon_i = 1$  then  $\varphi_i \in {}_{x_{i+1}}\mathcal{B}_{x_i}$  while if  $\epsilon_i = -1$  then  $\varphi_i \in {}_{x_i}\mathcal{B}_{x_{i+1}}$ . The degree of  $w$  is the following ordered product of elements of  $G$ :

$$\deg w = (\deg \varphi_n)^{\epsilon_n} \dots (\deg \varphi_i)^{\epsilon_i} \dots (\deg \varphi_1)^{\epsilon_1}$$

**Remark 5.6** Note that an homogeneous non zero endomorphism involved in an homogeneous walk at position  $i$  intervenes with its degree, or the inverse of its degree, according to the value of  $\epsilon_i$ .

**Definition 5.7** Let  $\mathcal{B}$  be a  $G$ -graded  $k$ -category. The grading is called connected if for any couple of objects  $b$  and  $c$  of  $\mathcal{B}$  and for any element  $s \in G$ , there exists an homogeneous walk from  $b$  to  $c$  of degree  $s$ .

**Theorem 5.8** Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a Galois covering. Then the induced grading on  $\mathcal{B}$  is connected.

**Proof.** Let  $(x_b)$  be a choice of an object in each  $F$ -fibre, providing a graduation of  $\mathcal{B}$ . Note that a morphism  $\varphi$  in  $\mathcal{C}$  from  $sx_b$  to  $tx_c$  has homogeneous image of degree  $s^{-1}t$ , since  $F\varphi = Fs^{-1}\varphi$ . This observation shows that a walk in  $\mathcal{C}$  from  $x_b$  to some  $sx_d$  projects to an homogeneous walk from  $b$  to  $d$  of degree  $s$ . Since  $\mathcal{C}$  is connected, the theorem is proved.  $\diamond$

Let us briefly recall the definition of Hochschild-Mitchell cohomology in degree one (see for instance [18]). This cohomology coincides with usual Hochschild cohomology of algebras in case the  $k$ -category has a finite number of objects.

**Definition 5.9** Let  $\mathcal{B}$  be a  $k$ -category. A derivation of  $\mathcal{B}$  is a collection  $D = ({}_c D_b)_{c,b \in \mathcal{D}_0}$  of  $k$ -linear endomorphisms of each  $k$ -module of morphisms  ${}_c \mathcal{B}_b$ , such that  $D(gf) = gD(f) + D(g)f$ . More precisely if  ${}_d g_c$  and  ${}_c f_b$  are morphisms of  $\mathcal{B}$ , then

$${}_d D_b(gf) = g_c D_b(f) + {}_d D_b(g)f.$$

An inner derivation  $D_\alpha$  associated to a collection  $\alpha$  of endomorphisms at each object  $(\alpha_b)_{b \in \mathcal{B}_0}$  is obtained in the usual way, namely

$$D_\alpha({}_c f_b) = \alpha_c f - f \alpha_b.$$

The first Hochschild cohomology  $k$ -module  $H^1(\mathcal{B}, \mathcal{B})$  is the quotient of the  $k$ -module of derivations by the inner ones.

**Theorem 5.10** Let  $k$  be a field and let  $\mathcal{B}$  be a  $k$ -category such that the endomorphism ring of each object is reduced to  $k$ . Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  be a Galois covering. Then there exists a canonical injective morphism

$$\Delta : \text{Hom}(\text{Aut}F, k^+) \longrightarrow H^1(\mathcal{B}, \mathcal{B}).$$

**Proof.** Let  $\chi : \text{Aut}F \rightarrow k^+$  be an abelian character of  $\text{Aut}F$ . In order to define  $\Delta_\chi$  as a  $k$ -endomorphism of each  $k$ -module of morphisms of  $\mathcal{B}$ , we define  $\Delta_\chi$  on the homogeneous components of a graduation induced by the covering  $F$ . Let  $f$  be a morphism of degree  $s$ . By definition  $\Delta_\chi(f) = \chi(s)f$ .

A standard computation shows that  $\Delta_\chi$  is indeed a derivation, which corresponds to the well known construction of Euler derivations, see also [8, 9]. Moreover if the graduation is changed through a different choice of objects in the fibres according to Theorem 5.3, the derivation  $\Delta_\chi$  is modified by an inner derivation, hence the morphism  $\Delta$  is canonic.

Assume  $\Delta_\chi$  is an inner derivation. Let  $f$  be a non-zero homogeneous morphism of degree  $s$ , then  $\chi(s)f = \alpha_c f - f \alpha_b = (\alpha_c - \alpha_b) f$  since  $\alpha_b \in k$  for each object  $b$ . Since  $k$  is a field and  $f$  is a non zero element of a vector space,  $\chi(s) = \alpha_c - \alpha_b$ . Moreover if  $w$  is an homogeneous walk of degree  $s$  from  $b$  to  $c$ , we also have  $\chi(s) = \alpha_c - \alpha_b$  by an easy computation. Finally since  $F$  is Galois, we know by Theorem 5.8 that for each automorphism  $s$  there exist an homogeneous walk from an object  $b$  to itself of degree  $s$ . Consequently  $\chi(s) = 0$  for every  $s$  and  $\Delta$  is injective.  $\diamond$

The abelian characters of the automorphisms of the Galois coverings provide a contravariant functor from the thin category of Galois coverings to the category of vector spaces. Moreover the family of canonical linear maps that we have obtained commutes with the maps between abelian characters. Note that if  $\mathfrak{L}$  is a thin category and  $L$  is a contravariant functor from  $\mathfrak{L}$  to the category of vector spaces, the direct limit is

$$\varinjlim L = \bigoplus_{i \in \mathfrak{L}_0} L(i) / \{L({}_i f_j)(s_i) - L({}_i f_k)(s_i) \mid i, j, k \in \mathfrak{L}_0, s_i \in L(i)\}.$$

Finally we state the following result

**Theorem 5.11** *Let  $\mathcal{B}$  be a  $k$ -category where  $k$  is a field and such that the endomorphisms of each object is reduced to  $k$ . Then there is a canonical morphism*

$$\lim_{\rightarrow} \text{Hom}(\mathbf{Aut}, k^+) \longrightarrow H^1(\mathcal{B}, \mathcal{B}).$$

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