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## WEAKLY MAXIMAL DECIDABLE STRUCTURES

ALEXIS BÈS<sup>1</sup> AND PATRICK CÉGIELSKI<sup>1</sup>

**Abstract.** We prove that there exists a structure  $M$  whose monadic second order theory is decidable, and such that the elementary theory of every expansion of  $M$  by a constant is undecidable.

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### INTRODUCTION

In [3], Elgot and Rabin ask whether there exist maximal decidable structures, i.e. structures  $M$  whose elementary theory is decidable and such that the elementary theory of any expansion of  $M$  by a non-definable predicate is undecidable.

As far as we know, the only related results were obtained by Soprunov in [8], where he proves (by a forcing argument) that every structure in which a regular ordering is interpretable is not maximal. A partial ordering  $(B, <)$  is said to be regular if for every  $a \in B$  there exist distinct elements  $b_1, b_2 \in B$  such that  $b_1 < a$ ,  $b_2 < a$ , and no element  $c \in B$  satisfies both  $c < b_1$  and  $c < b_2$ . As a corollary he also proves that there is no maximal decidable structure if we replace “elementary theory” by “weak monadic second-order theory”.

In this paper we consider a weakening of Elgot-Rabin’ question, namely the question whether there exist maximal decidable theories with respect to non-definable constants. We answer positively this question for elementary theories as well as monadic second-order theories, by proving that there exists a structure  $M$  with a decidable monadic second-order theory, and such that every expansion of  $M$  by a constant has an undecidable elementary theory. The proof makes uses of Shelah’s composition theorem and Büchi’s result on the monadic second order theory of  $(\mathbb{N}, <)$ .

We also prove that  $M$  is not maximal in the sense of Elgot and Rabin, even if we replace “elementary theory” by “monadic second-order theory”.

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## 1. SOME USEFUL TOOLS

Given a structure  $M$  we denote by  $FO(M)$  (respectively  $MSO(M)$ ) the first-order (respectively monadic second-order) theory of  $M$ . The base set of  $M$  is denoted by  $|M|$ . We shall sometimes confuse logical symbols and their interpretation.

Given a relational language  $L$  and a countable  $L$ -structure  $M$ , we say that  $M$  is *computable* if its atomic diagram is computable, i.e. if there exists an injective map  $f : |M| \rightarrow \mathbb{N}$  such that the range of  $f$ , as well as all sets  $\{(f(a_1), \dots, f(a_n)) : M \models R(a_1, \dots, a_n)\}$  for every relation  $R$  of  $L$ , are recursive (see e.g. [6]).

### 1.1. EXTENSIONS OF THE MONADIC SECOND-ORDER THEORY OF $(\mathbb{N}, <)$

The following proposition is an easy corollary of Büchi's result [1] about the monadic second order theory of  $(\mathbb{N}, <)$ .

**Proposition 1.1.** *Let  $R$  be a unary predicate. It is decidable whether a MSO sentence  $\phi$  over the language  $\{<, R\}$  admits a model of the form  $(\mathbb{N}; <, R)$  where  $<$  is interpreted as the natural ordering over  $\mathbb{N}$ .*

*Moreover if it admits such a model then it admits a model where the interpretation of  $R$  is an ultimately periodic subset of  $\mathbb{N}$  (which can be computed effectively from  $\phi$ ).*

*Proof.* By [1] for every monadic second-order sentence  $\phi$  there exists an automaton  $\mathcal{A}$  over  $\omega$ -words such that  $(\mathbb{N}, <, R) \models \phi$  if and only if the word  $w_R \in \{0, 1\}^\omega$  which corresponds to the characteristic sequence of  $R$  is recognized by  $\mathcal{A}$ . Moreover  $\mathcal{A}$  can be computed effectively from  $\phi$ . Thus the question whether a MSO sentence  $\phi$  over the language  $\{<, R\}$  admits a model of the form  $(\mathbb{N}; <, R)$  is reducible to the emptiness problem for  $\mathcal{A}$ , which is decidable by [1].

The second part of the proposition comes from the fact that if the automaton  $\mathcal{A}$  accepts some  $\omega$ -word then it accepts some ultimately periodic  $\omega$ -word, which can be computed effectively from  $\mathcal{A}$ .  $\square$

We shall use the notion of *rich word* (see [2, 5]).

**Definition 1.2.** Given a finite alphabet  $A$ , a  $\mathbb{Z}$ -word over  $A$  is a sequence  $w : \mathbb{Z} \rightarrow A$ . We say that a  $\mathbb{Z}$ -word  $w$  is *rich* if every finite word over  $A$  appears infinitely many times in  $w$  in both directions; that is, for every non-empty word  $u = u_1 \dots u_n$  over  $A$  and every  $j \in \mathbb{Z}$  there exists  $k > j$  and  $k' < j$  such that  $w(k+1) \dots w(k+n) = u$  and  $w(k'+1) \dots w(k'+n) = u$ .

Compton proved that the MSO theory of  $(\mathbb{Z}, <)$  extended by any rich word is decidable [2] :

**Theorem 1.3.** *Let  $w : \mathbb{Z} \rightarrow A$  be a rich word. For every  $a \in A$  let  $X_a \subseteq \mathbb{Z}$  be defined as  $X_a = \{z \in \mathbb{Z} : w(z) = a\}$ . Then the MSO theory of  $(\mathbb{Z}; <, (X_a)_{a \in A})$  is decidable.*

## 1.2. THE COMPOSITION METHOD

We shall use Shelah's composition method [7] which allows to reduce the MSO theory of a sum of structures to the MSO theories of the components and the MSO theory of the index structure. This method is a generalization of the Feferman-Vaught technique [4]. For a recent and detailed presentation of Shelah's method we refer the reader to the paper [9], from which we borrow most of our definitions and notations.

We state definitions and results for the special case where the language  $L$  of the component structures equals  $L = \{<, R\}$  where  $R$  denotes a unary predicate (as in [9]).

**Definition 1.4.** We start from an index structure  $Ind = (I, <^I)$  where  $<^I$  is a linear ordering. Consider the relational language  $L = \{<, R\}$  where  $R$  denotes a unary predicate, and a family  $(M_i)_{i \in I}$  of  $L$ -structures  $M_i = (A_i; <^i, R^i)$  such that the interpretation  $<^i$  of  $<$  in each  $M_i$  is a linear ordering. We define the *ordinal sum* of the family  $(M_i)_{i \in I}$  as the  $L$ -structure  $M = (A; <^M, R^M)$  where

- $A$  equals the disjoint union of the  $A_i$ 's
- $x <^M y$  holds if and only if  $(x \in A_i \text{ and } y \in A_j \text{ for some } i <^I j)$ , or  $(x, y \in A_i \text{ and } x <^i y)$
- for every  $x \in A$ ,  $R^M(x)$  holds if and only if  $M_i \models R^i(x)$  where  $i$  is such that  $x \in A_i$ .

We shall use the notation  $M = \sum_{i \in I} M_i$ .

In view of Theorem 1.7 one has to consider a slight modification of the MSO formalism:

- we deal with MSO logic in which only monadic second-order variables occur; this is always possible by introducing the predicate "to be a singleton set" and replacing each relation which involves FO variables by relations which involve MSO variables restricted to singletons.
- for every  $n \geq 1$  we add to the language the predicates  $Cov_n(X_1, \dots, X_n)$  and  $Part_n(X_1, \dots, X_n)$ , which are respectively interpreted as " $X_1 \cup \dots \cup X_n$  equals the whole domain", and " $(X_1, \dots, X_n)$  is a partition of the domain". These predicates do not add any expressive power to the full MSO theory since they are already MSO-definable, but their use allows to extend the expressive power of quantifier-free formulas.

**Definition 1.5.** Given  $n \in \mathbb{N}$  and a sequence  $\bar{k} = (k_1, \dots, k_n)$  of natural numbers, we call  $\bar{k}$ -*formula* a MSO formula in prenex form which consists in a block of  $k_1$  quantifiers of the same kind, followed by a block of  $k_2$  quantifiers of the same kind, etc... and ends with a block of  $k_n$  quantifiers of the same kind followed by a quantifier-free formula. For example the formula  $\forall X_1 \forall X_2 \exists X_3 \phi(\bar{X})$ , where  $\phi$  is quantifier-free, is a  $(2, 1)$ -formula. A  $\bar{k}$ -*sentence* is a  $\bar{k}$ -formula which is closed.

**Definition 1.6.** The  $\bar{k}$ -theory of a structure  $M$ , which we denote by  $T^{\bar{k}}(M)$ , is the set of  $\bar{k}$ -sentences which are true in  $M$ . We denote  $M \equiv_{\bar{k}} M'$  if  $M$  and  $M'$  have the same  $\bar{k}$ -theory.

Observe that the equivalence classes of  $\equiv_{\bar{k}}$  can be defined in MSO logic: for every set  $T$  of  $\bar{k}$ -sentences there exists a  $\bar{k}$ -sentence  $\psi_T$  such that  $M \models \psi_T$  if and only if the  $\bar{k}$ -theory of  $M$  equals  $T$ . Moreover  $\psi_T$  can be computed effectively from  $T$ .

We state Shelah's composition theorem [7] as in [9]:

**Theorem 1.7.** *Let  $M = \sum_{i \in I} M_i$  be the ordinal sum of the family  $(M_i)_{i \in I}$ . Given a sequence  $\bar{k} = (k_1, \dots, k_n)$  one can compute a sequence  $\bar{r} = (r_1, \dots, r_n)$  such that the  $\bar{k}$ -theory of  $M$  is determined (in an effective way) by the  $\bar{r}$ -theory of the structure  $(I, <^I, Q_1, \dots, Q_p)$  where*

$$Q_j = \{i \in I : T^{\bar{k}}(M_i) = \tau_j\} \quad j = 1, \dots, p$$

and  $\tau_1, \dots, \tau_p$  denote the  $p$  distinct  $\bar{k}$ -theories of the  $M_i$ 's.

## 2. A WEAKLY MAXIMAL DECIDABLE STRUCTURE

We state now the main result of the paper.

**Theorem 2.1.** *There exist a relational language  $L$  and a countable  $L$ -structure  $M$  such that:*

- $M$  is computable;
- the monadic second-order theory of  $M$  is decidable;
- every  $(L \cup \{c\})$ -expansion of  $M$  by a constant symbol  $c \notin L$  has an undecidable elementary theory.

**Proof of Theorem 2.1.** Let us state first the main lines of the proof. We shall define  $M$  as an ordinal sum (indexed by  $\mathbb{Z}$  with its natural ordering) of structures of the form  $(\mathbb{N}; <, R)$  where  $R$  is some unary predicate whose interpretation varies from one component to another. The idea is to "hide" a copy of some structure  $(\mathbb{N}; <, B)$  among other components of the ordinal sum, in such a way that, on one hand, this copy cannot be distinguished easily from other components by a MSO sentence, and on the other hand the structure  $(\mathbb{N}; <, B)$  has an undecidable FO theory. The undecidability of  $FO(M)$  enriched by any constant will come from the fact that the constant allows to localize the component to which it belongs, from which we can reach the copy of  $(\mathbb{N}; <, B)$  and use the undecidability of  $FO(\mathbb{N}; <, B)$ . The decidability of  $MSO(M)$  will be obtained by using Theorem 1.7, which will allow to reduce the decision problem for a given MSO sentence  $\phi$  to the decision problem for an extension of  $MSO(\mathbb{Z}, <)$  by a rich word, which is decidable by Theorem 1.3.

Let  $W \subseteq \mathbb{N}$  be a recursively enumerable not recursive predicate. Let  $g_W : \mathbb{N} \rightarrow \mathbb{N}$  be an enumerator for  $W$ , i.e. a total recursive function whose range is  $W$ . Let

$B \subseteq \mathbb{N}$  be the set of elements of the sequence  $(b_n)_{n \in \mathbb{N}}$  defined by  $b_0 = g_W(0)$  and  $b_{n+1} = b_n + g_W(n) + 1$ .

**Lemma 2.2.** *FO( $\mathbb{N}; <, B$ ) is undecidable.*

*Proof.* One can associate effectively to every  $m \in \mathbb{N}$  some FO-sentence  $\phi_m$  such that  $m \in W$  if and only if  $(\mathbb{N}; <, B) \models \phi_m$ ; this sentence simply expresses the fact that there exist two consecutive elements of  $B$  at distance  $m + 1$ .  $\square$

Let us define the structure  $M$ . Consider a recursive function  $f$  which enumerates all finite non-empty sequences of ultimately periodic subsets of  $\mathbb{N}$  (such a function exists). Given  $n \in \mathbb{N}$  we shall use the notation  $f(n) = (E_{n,1}, E_{n,2}, \dots, E_{n,l(n)})$  where  $E_{n,i} \subseteq \mathbb{N}$  is ultimately periodic, and  $l(n)$  denotes the length of the sequence  $f(n)$ .

**Definition 2.3.** Let  $M$  be the  $\{<, R\}$ -structure defined as  $M = \sum_{i \in \mathbb{Z}} M_i$ , where for every  $i \in \mathbb{Z}$  we have  $M_i = (\mathbb{N}; <, R^i)$  where the interpretation of  $R^i$  is defined as follows:

- $R^0 = B$  (i.e.  $R$  coincides with  $B$  in the component with index 0).
- $R^i = E_{0,i}$  for every  $i \in [1, l(0)]$  (i.e.  $R$  coincides with  $E_{0,i}$  in the components whose index  $i$  belongs to  $[1, l(0)]$ ).
- $R^i = E_{1,-i}$  for every  $i \in [-l(1), -1]$  (i.e.  $R$  coincides with  $E_{1,-i}$  in the components whose index  $i$  belongs to  $[-l(1), -1]$ ).
- For every  $n \in \mathbb{N}$  we have:
  - For every  $i \in [1, l(2n+2)]$ , if  $z = \sum_{0 \leq j \leq n} l(2j) + i$  then  $R^z = E_{2n+2,i}$
  - For every  $i \in [1, l(2n+3)]$ , if  $z = -\sum_{0 \leq j \leq n} l(2j+3) + i - 1$  then  $R^z = E_{2n+3,i}$

Let us prove that  $M$  satisfies the conditions stated in Theorem 2.1. The proof of the following lemma is a straightforward consequence of the above definition and the fact that the set  $B$  is recursive.

**Lemma 2.4.** *The structure  $M$  is computable.*

Let us prove now that  $M$  is maximal with respect to non-definable constants.

**Lemma 2.5.** *Let  $c$  be a constant symbol. For any interpretation of  $c$ , the structure  $M'$  defined as the expansion of  $M$  with respect to  $c$  has an undecidable elementary theory.*

*Proof.* This comes from Lemma 2.2, and the fact that  $FO(\mathbb{N}; <, B)$  is interpretable into  $FO(M')$ . Indeed with the constant  $c$  one can easily define in  $M'$  the first element  $x$  of the  $M_i$  to which  $c$  belongs: this is the greatest element less than or equal to  $c$  and without predecessor. Then from  $x$  one can define the first element  $y$  of  $M_0$  (seen as an element of  $|M|$ ): in case  $i \geq 0$  then the formula expresses that  $y \leq x$ ,  $y$  has no predecessor, and there are  $i$  elements without predecessor between  $y$  and  $x$ ; the case  $i < 0$  is similar. Then from  $y$  one defines easily the subset  $|M_0|$  (seen as a subset of  $|M|$ ). Finally since  $M_0$  is a copy of

$(\mathbb{N}; <, B)$ , classical relativization techniques allow to translate every  $L$ -sentence  $\varphi$  in a  $L \cup \{c\}$ -sentence  $\varphi'$  such that  $(\mathbb{N}; <, B) \models \varphi$  if and only if  $M' \models \varphi'$ .  $\square$

Note that the previous arguments can be extended in an easy way to the case where one expands  $M$  by any unary predicate  $R$  whose interpretation is a finite or co-finite subset of  $|M|$ .

**Lemma 2.6.** *The MSO theory of  $M$  is decidable.*

*Proof.* Let us prove that the application which maps  $\bar{k}$  to the  $\bar{k}$ -theory of  $M$  is effective. By Theorem 1.7 it is sufficient to prove that for every  $\bar{k}$

- (1) one can compute effectively the set  $S(\bar{k}) = \{T^{\bar{k}}(M_i) \mid i \in \mathbb{Z}\}$
- (2) if we denote by  $\tau_1, \dots, \tau_{p_{\bar{k}}}$  the distinct elements of  $S(\bar{k})$ , then the MSO theory of the structure  $(\mathbb{Z}; <, Q_1, \dots, Q_{p_{\bar{k}}})$ , where

$$Q_j = \{i \in \mathbb{Z} : T^{\bar{k}}(M_i) = \tau_j\},$$

is decidable.

First of all, observe that for every set  $\tau$  of  $\bar{k}$ -sentences there exists a sentence  $\phi$  such that for every  $L$ -structure  $Z$ , we have  $Z \models \phi$  if and only if  $T^{\bar{k}}(Z) = \tau$ . Now by Proposition 1.1 we can check effectively whether  $\phi$  admits a model  $Z$  of the form  $(\mathbb{N}; <, R)$  where  $<$  is interpreted as the natural ordering over  $\mathbb{N}$ . If such a model  $Z$  exists, then Proposition 1.1 implies that there exists an ultimately periodic set  $X \subseteq \mathbb{N}$  such that  $(\mathbb{N}; <, R) \models \phi$  when one interprets  $R$  by  $X$ , and thus by definition of the  $M_i$ 's there exists  $i \in \mathbb{Z}$  such that  $M_i \models \phi$ , i.e.  $T^{\bar{k}}(M_i) = \tau$ . On the other hand if no such model  $Z$  exists then *a fortiori* there is no  $M_i$  such that  $T^{\bar{k}}(M_i) = \tau$ . This proves (1).

Let us prove (2). By Theorem 1.3 it suffices to prove that the word  $w : \mathbb{Z} \rightarrow \{1, \dots, p_{\bar{k}}\}$  defined by  $w(z) = i$  if and only if  $z \in Q_i$ , is a rich word. This amounts to show that for all  $n, j \in \mathbb{N}$  and every non-empty sequence  $(t_1, \dots, t_n)$  of elements of  $\{\tau_1, \dots, \tau_{p_{\bar{k}}}\}$ , there exist  $z > j$  (resp.  $z < j$ ) such that the  $\bar{k}$ -theories of the structures  $M_{z+1}, \dots, M_{z+n}$  are  $t_1, \dots, t_n$ , respectively. Now by Proposition 1.1 it suffices to prove that for every sequence  $(F_1, \dots, F_n)$  of ultimately periodic subsets of  $\mathbb{N}$  there exist  $z > j$  (resp.  $z < j$ ) such that  $R^{z+i} = F_i$  for every  $i \in \{1, \dots, n\}$ . This comes directly from the very construction of  $M$ .  $\square$

This completes the proof of Theorem 2.1. This theorem shows that the structure  $M$  is maximal with respect to non-definable constants. However  $M$  is not maximal in the sense of Elgot and Rabin, even if we consider monadic second-order logic.

**Proposition 2.7.** *There exists  $S \subseteq |M|$  which is not MSO-definable in  $M$ , and such that the  $(L \cup \{S\})$ -expansion of  $M$  induced by  $S$  has a decidable MSO-theory.*

*Proof.* The set  $S$  is defined by induction on  $n \in \mathbb{N}$ . The idea is to mark for each  $n$  finitely many components  $M_i$  of  $M$ , either in  $S$  or in the complement of  $S$  (i.e. either all elements, or no element, of  $M_i$  belong to  $S$ ).

For every  $i \in \mathbb{Z}$  we denote by  $S^i$  the interpretation of  $S$  in the structure  $M_i$ , and by  $M'_i$  the  $(L \cup \{S\})$ -expansion of  $M_i$  induced by  $S^i$ .

Let  $g$  be a function which enumerates all pairs  $(H, B)$  where  $H$  denotes a finite and non empty sequence of ultimately periodic subsets of  $\mathbb{N}$ , and  $B$  is a subset of  $\{1, \dots, m\}$  where  $m$  denotes the length of  $H$  (such a function exists). We shall use the notation  $g(n) = ((H_{n,1}, H_{n,2}, \dots, H_{n,\lambda(n)}), B_n)$ .

For every  $n \in \mathbb{N}$  we proceed as follows:

- (1) Consider the set  $\mathcal{F}_n$  of MSO formulas  $\varphi(X)$  in the language  $L$  with one free variable  $X$ , with quantifier rank less than or equal to  $n$ , and such that the subsets  $X \subseteq |M|$  for which  $M \models \varphi(X)$  holds are unions of sets  $|M_i|$ .

The language  $L$  is finite and relational, thus up to equivalence there exist finitely many formulas in  $\mathcal{F}_n$ . Therefore there exist  $a_n \geq 1$  and a partition  $(Y_{n,1}, \dots, Y_{n,a_n})$  of  $|M|$  such that every  $Y_{n,i}$  is definable by a formula of  $\mathcal{F}_n$ , and every set  $X \subseteq |M|$  which is definable by a formula in  $\mathcal{F}_n$  is a finite union of some of the  $Y_{n,i}$ 's. Since  $|M|$  is infinite there exists  $j$  such that  $Y_{n,j}$  is the union of infinitely many sets  $|M_i|$ . Let  $j'$  be the least positive integer such that  $|M_{j'}| \subseteq Y_{n,j}$  and  $M_{j'}$  is not marked yet. We mark all elements of  $|M_{j'}|$  in  $S$ , that is we set  $S^{j'} = |M_{j'}|$ . Now let  $j''$  be the least positive integer greater than  $j'$  and such that  $|M_{j''}| \subseteq Y_{n,j}$  and  $M_{j''}$  is not marked yet. We mark all elements of  $|M_{j''}|$  in the complement of  $S$ , that is we set  $S^{j''} = \emptyset$ .

This step of the marking ensures that both  $S$  and its complement intersect  $Y_{n,j}$ , which implies (by definition of the partition  $(Y_{n,1}, \dots, Y_{n,a_n})$ ) that  $S$  is not definable by any formula of  $\mathcal{F}_n$ .

- (2) Let  $k$  be the least positive integer such that no  $M_t$  with  $t \geq k$  is marked yet, and moreover  $R^{k+i} = H_{n,i}$  for every  $i \in \{1, \dots, \lambda(n)\}$ . By definition of  $M$  such a  $k$  exists. For every  $i \in \{1, \dots, \lambda(n)\}$  we set  $S^{k+i} = |M^{k+i}|$  if  $i \in B_n$ , and  $S^{k+i} = \emptyset$  otherwise. Now let  $k'$  be the greatest negative integer such that no  $M_t$  with  $t < k' + \lambda(n)$  is marked yet, and moreover  $R^{k'+i} = H_{n,i}$  for every  $i \in \{1, \dots, \lambda(n)\}$ . By definition of  $M$  such a  $k'$  exists. For every  $i \in \{1, \dots, \lambda(n)\}$  we set  $S^{k'+i} = |M^{k'+i}|$  if  $i \in B_n$ , and  $S^{k'+i} = \emptyset$  otherwise.
- (3) For every  $i \in [k', k]$  such that  $M_i$  is not marked yet, we mark  $M_i$  in the complement of  $S$ , i.e. we set  $S^i = \emptyset$ .

The proof that  $S$  is not MSO-definable in  $M$  easily follows from step (1) of the construction.

Let  $M'$  denote the  $(L \cup \{S\})$ -expansion of  $M$  induced by  $S$ . The proof that  $MSO(M')$  is decidable is similar to the one of Lemma 2.6. Let us state the main arguments. We have to prove the following:

- the function  $h : \bar{k} \rightarrow \{T^{\bar{k}}(M'_i) \mid i \in \mathbb{Z}\}$  is recursive;
- for all  $\bar{k}$ ,  $m \geq 1$ , every sequence  $(\tau_1, \dots, \tau_m)$  of elements of  $h(\bar{k})$ , and every  $j \in \mathbb{Z}$ , there exist  $z > j$  (resp.  $z < j$ ) such that the  $\bar{k}$ -theories of the structures  $M'_{z+1}, \dots, M'_{z+m}$  are  $\tau_1, \dots, \tau_m$ , respectively.



For every  $i \in \mathbb{Z}$ , let  $a_i = 1$  if  $S^i = |M_i|$ , and  $a_i = 0$  otherwise. It is easy to prove that for every  $i$ , the  $\bar{k}$ -theory of the  $(L \cup \{S\})$ -expansion  $M'_i$  of  $M_i$  induced by  $S$  can be computed effectively from  $a_i$  and the  $\bar{k}$ -theory of  $M_i$ . This fact together with Proposition 1.1 yields that  $h$  is recursive.

For the second item, by Proposition 1.1 it suffices to prove that for every  $m \geq 1$ , for every  $m$ -tuple  $(E_1, \dots, E_m)$  of ultimately periodic subsets of  $\mathbb{N}$  and every  $m$ -tuple  $(b_1, \dots, b_m)$  of elements of  $\{0, 1\}$ , there exist infinitely many positive integers and infinitely many negative integers  $i$  such that for every  $j \in \{1, \dots, m\}$ ,

- $R^{i+j} = E_j$
- $S^{i+j} = |M_{i+j}|$  if  $b_j = 1$ , and  $S^{i+j} = \emptyset$  if  $b_j = 0$ .

This fact is a direct consequence of the step (2) in the construction of  $S$ .

Combining the two items above with Theorem 1.3 and Theorem 1.7 (as in the proof of Lemma 2.6) yields the required result.  $\square$

Let us note that in the proof above, we show the existence of  $S$  but do not prove that one can define  $S$  effectively, i.e. that the structure  $M'$  is computable. We do not know whether such an effective definition is possible.

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