Stability and bifurcation with moving discontinuities

Claude Stolz, Rachel-Marie Pradeilles-Duval

STABILITY AND BIFURCATION WITH MOVING DISCONTINUITIES

Claude Stolz
CNRS UMR7649, Laboratoire de Mécanique des Solides
Ecole Polytechnique, 91128 Palaiseau cedex
stolz@l ms. polytech niq ue.

Rachel-Marie Pradeilles-Duval
CNRS UMR7649, Laboratoire de Mécanique des Solides
Ecole Polytechnique, 91128 Palaiseau cedex
rachel @l ms.polytech niq ue.

Abstract The propagation of moving surface inside a body is analysed in the framework of thermodynamics, when the moving surface is associated with an irreversible change of mechanical properties. The thermodynamical force associated to the propagation has the form of an energy release rate. Quasistatic rate boundary value problem is given when the propagation of the interface is governed by a normality rule. Extension to generalised media to study delamination is also investigated.

1. INTRODUCTION

This paper is concerned mostly with the description of damage involved on the evolution of a moving interface along which mechanical transformation occurs, (Pradeilles-Duval and Stolz, 1995). Some connection can be made with the notion of configurational forces, (Gurtin, 1995 ; Maugin, 1995; Truskinovsky, 1987).

A domain Ω is composed of two distinct volumes Ω_1, Ω_2 of two linear elastic materials with different characteristics. The bounding between the two phases is perfect and the interface is denoted by Γ, ($\Gamma = \partial\Omega_1 \cap \partial\Omega_2$). The external surface $\partial\Omega$ is decomposed in two parts $\partial\Omega_{\mathrm{u}}$ and $\partial\Omega_{\mathrm{T}}$ on which the displacement $\underline{u}^d$ and the loading $\underline{\mathrm{T}}^d$ are prescribed respectively. The material 1 changes into material 2 along the interface Γ by an irreversible process. Hence Γ moves with the normal velocity $\underline{c} = \phi \underline{\nu}$ in the reference state, where $\underline{\nu}$ is the outward Ω_2 normal vector and ϕ is positive. When the surface Γ is moving volume average of any mechanical quantity f has a rate defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(\Gamma)} f \,\mathrm{d}\Omega = \int_{\Omega(\Gamma)} \dot{f} \,\mathrm{d}\Omega - \int_{\Gamma} [\, f \,]_{\Gamma} \,\underline{c}.\underline{\nu} \,\mathrm{d}S, \tag{1}$$

where $[\, f \,]_{\Gamma} = f_1 - f_2$ represents the jump of f accross Γ. The state of the system is characterized by the displacement field $\underline{u}$, from which the strain field $\boldsymbol{\varepsilon}$ is derived, and by the spatial distribution of the two phases defined by the position of Γ. The two phases have the same mass density ρ. The free energy density w_i is a quadratic function of the strain $\boldsymbol{\varepsilon}$, $\rho w_i = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C}^i : \boldsymbol{\varepsilon}$ in Ω_i. The potential energy $\mathcal{E}$ of the structure Ω $(\Omega_1 \cup \Omega_2)$ has the following form

$$\mathcal{E}(\tilde{\underline{u}}, \Gamma, \tilde{\underline{T}}^d) = \sum_{i=1,2} \int_{\Omega_i(\Gamma)} \rho \, w_i(\boldsymbol{\varepsilon}(u)) \,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \underline{T}^d.\underline{u} \,\mathrm{d}S.$$

When Γ is known, the body is a heterogeneous medium with two elastic phases, and under prescribed loading an equilibrium state $\underline{u}^{sol}$ is given by the stationnarity of the potential energy written as

$$\frac{\partial \mathcal{E}}{\partial \underline{u}} \cdot \delta\underline{u} = \sum_{i=1,2} \int_{\Omega_i} \rho \frac{\partial w_i}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\delta\underline{u}) \,\mathrm{d}\Omega - \int_{\partial\Omega_{\mathrm{T}}} \underline{T}^d.\delta\underline{u} \,\mathrm{d}S = 0, \tag{2}$$

for all $\delta\underline{u}$ kinematically admissible field satisfying $\delta\underline{u} = 0$ over $\partial\Omega_{\mathrm{u}}$. Then the solution $\underline{u}^{sol}$ satisfies (PB1):

- the local constitutive relations: $\boldsymbol{\sigma} = \rho \frac{\partial w_i}{\partial \boldsymbol{\varepsilon}}$, on Ω_i,
- the momentum equations: div $\boldsymbol{\sigma} = 0$, on Ω , $\boldsymbol{\sigma}.\underline{n} = \underline{T}^d$ over $\partial\Omega_{\mathrm{T}}$,
- the compatibility relations: $2\boldsymbol{\varepsilon} = \nabla\underline{u} + \nabla^t\underline{u}$, in Ω, $\underline{u} = \underline{u}^d$ over $\partial\Omega_{\mathrm{u}}$,
- the perfect bounding over Γ: $[\,\boldsymbol{\sigma}\,]_{\Gamma}.\underline{\nu} = 0$ and $[\,\underline{u}\,]_{\Gamma} = 0$.

For a prescribed history of the loading, we must determine the rate of all mechanical fields and the normal propagation ϕ to characterize the position of the interface Γ at each time. Let us introduce the convected derivative $\mathcal{D}_\phi$ of any function $f(\underline{X}_\Gamma, t)$ defined by

$$\mathcal{D}_\phi f = \lim_{\tau \to 0} \frac{f(\underline{X}_\Gamma + \phi\underline{\nu}\tau, t + \tau) - f(\underline{X}_\Gamma, t)}{\tau}. \tag{3}$$

For example, the equation of a moving surface is $S(\underline{X}_{\Gamma}), t) = 0$, then $\mathcal{D}_\phi S = 0$. This defines the normal velocity ϕ as $\phi = -\frac{\partial S}{\partial t} / \parallel \frac{\partial S}{\partial \underline{X}} \parallel$ and $\underline{\nu} = \frac{\partial S}{\partial \underline{X}} / \parallel \frac{\partial S}{\partial \underline{X}} \parallel$.

Hadamard's relations. The bounding being perfect between the phases, the displacement and the stress vector are continuous along Γ. Their rates ($\underline{v}$, $\dot{\boldsymbol{\sigma}}$) have discontinuities according to the compatibility conditions of Hadamard, rewritten with the convected derivative: the continuity of displacement induces

$$[\,\underline{u}\,]_\Gamma = 0 \Rightarrow \mathcal{D}_\phi([\,\underline{u}\,]_\Gamma) = [\,\underline{v}\,]_\Gamma + \phi[\,\nabla \underline{u}\,]_\Gamma.\underline{\nu} = 0, \tag{4}$$

and continuity of stress vector implies:

$$[\,\boldsymbol{\sigma}\,]_\Gamma.\underline{\nu} = 0 \Rightarrow \mathcal{D}_\phi([\,\boldsymbol{\sigma}\,]_\Gamma.\underline{\nu}) = [\,\dot{\boldsymbol{\sigma}}\,]_\Gamma.\underline{\nu} - \mathrm{div}_\Gamma([\,\boldsymbol{\sigma}\,]_\Gamma\ \phi) = 0. \tag{5}$$

The last equation is obtained taking the equilibrium equation into account and the surface divergence defined by $\mathrm{div}_\Gamma F = \mathrm{div} F - \underline{\nu}.\nabla F.\underline{\nu}$.

Orthogonality property for discontinuities. Since the displacement and the stress vector are continuous along the interface,

$$[\,\underline{u}\,]_\Gamma = 0, \Rightarrow [\,\nabla \underline{u}\,]_\Gamma.\underline{e}_\alpha = 0, \quad [\,\boldsymbol{\sigma}\,]_\Gamma.\underline{\nu} = 0, \tag{6}$$

the discontinuities of the stress $\boldsymbol{\sigma}$ and of $\nabla \underline{u}$ have the property of orthogonality:

$$[\,\boldsymbol{\sigma}\,]_\Gamma : [\,\nabla \underline{u}\,]_\Gamma = 0. \tag{7}$$

Dissipation analysis. The total dissipation of the system is

$$\mathrm{D} = \int_{\Omega(\Gamma)} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\ \mathrm{d}\Omega - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(\Gamma)} \rho w\ \mathrm{d}\Omega = -\frac{\partial \mathcal{E}}{\partial \Gamma}.\dot{\Gamma} = \int_\Gamma \mathcal{G}\phi\ \mathrm{d}S \geq 0, \tag{8}$$

taking the momentum conservation and the boundary conditions into account. The quantity $\mathcal{G}(\underline{X}_\Gamma, t) = [\,w\,]_\Gamma - \boldsymbol{\sigma} : [\,\boldsymbol{\varepsilon}\,]_\Gamma$ has an analogous form to the driving traction force acting on a surface of strain discontinuity proposed by Abeyratne and Knowles, 1990.

2. EVOLUTION OF THE INTERFACE

Let denotes $\Gamma^+ = \{x \in \Gamma / \mathcal{G}(x) = G_c\}$ and let considers the rule defined as a generalization of Griffith's law: $\phi \geq 0$, on $\Gamma^+, \phi = 0$, otherwise. At the point $x_\Gamma(t) \in \Gamma^+$, we have $\mathcal{G}(x_\Gamma(t), t) = G_c$, this is an implicit equation for the position of the interface, and the derivative of $\mathcal{G}$ following the moving surface vanishes: $\mathcal{D}_\phi \mathcal{G} = 0$. This leads to the consistency condition written for all point on Γ^+

$$(\phi - \phi^*)\mathcal{D}_\phi \mathcal{G} \geq 0, \forall \phi^* \geq 0, \text{ over } \Gamma^+, \tag{9}$$

and ϕ and ϕ^* are in $\mathcal{K} = \{\beta/\beta \geq 0 \text{ on } \Gamma^+, \beta = 0 \text{ otherwise}\}$. The evaluation of $\mathcal{D}_\phi \mathcal{G}$ is obtained easily

$$\begin{aligned} \mathcal{D}_\phi \mathcal{G} &= \mathcal{D}_\phi [\,w\,]_\Gamma - \mathcal{D}_\phi \boldsymbol{\sigma}_2 : [\,\nabla \underline{u}\,]_\Gamma - \boldsymbol{\sigma}_2 : [\,\mathcal{D}_\phi \nabla \underline{u}\,]_\Gamma & (10) \\ &= [\,\boldsymbol{\sigma}\,]_\Gamma : \nabla \underline{v}_1 - \dot{\boldsymbol{\sigma}}_2 : [\,\nabla \underline{u}\,]_\Gamma - \phi G_n, & (11) \end{aligned}$$

where $G_n = -[\,\boldsymbol{\sigma}\,]_\Gamma : (\nabla\nabla \underline{u}_1 . \underline{\nu}) + \nabla \boldsymbol{\sigma}_2 . \underline{\nu} : [\,\nabla \underline{u}\,]_\Gamma$.

The rate boundary value problem. The solution is governed by the derivation of (PB1) relatively to time, taking account of Hadamard's relations (4,5) along Γ and of the propagation law (9). Defining the functional $\mathcal{F}$

$$\begin{aligned} \mathcal{F}(\tilde{\underline{v}}, \tilde{\phi}, \dot{\underline{\mathrm{T}}}^d) &= \int_\Omega \frac{1}{2} \boldsymbol{\varepsilon}(\underline{v}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\underline{v})\, \mathrm{d}\Omega - \int_{\partial\Omega_\mathrm{T}} \dot{\underline{\mathrm{T}}}^d . \underline{v}\, \mathrm{d}S & (12) \\ &- \int_\Gamma \phi [\,\boldsymbol{\sigma}\,]_\Gamma : \nabla \underline{v}_1\, \mathrm{d}S + \int_\Gamma \frac{1}{2} \phi^2 G_n\, \mathrm{d}S, & (13) \end{aligned}$$

the solution satisfies the inequality

$$0 \leq \frac{\partial \mathcal{F}}{\partial \underline{v}} (\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial \phi} (\beta - \phi), \tag{14}$$

amoung the set $\mathcal{K}.\mathcal{A}$ of admissible fields $(\underline{v}^*, \phi^*)$:

$$\mathcal{K}.\mathcal{A} = \Big\{ (\,\underline{v}, \phi) / \,\underline{v} = \underline{v}^d \text{ over } \partial\Omega_\mathrm{u}, \mathcal{D}_\phi \underline{u} = 0, \text{ on } \Gamma, \; \phi \in \mathcal{K} \Big\}. \tag{15}$$

Some typical examples are presented in Pradeilles-Duval and Stolz, 1995.

Stability and bifurcation. Consider the velocity $\underline{v}$ solution of the rate boundary value problem for any given velocity ϕ. This field $\underline{v}$ is solution of a classical problem of heterogeneous elasticity with non classical boundary conditions on Γ : $\mathcal{D}_\phi([\,\boldsymbol{\sigma}\,]_\Gamma . \underline{\nu}) = 0, \quad \mathcal{D}_\phi [\,\underline{u}\,]_\Gamma = 0$. Consider the value W of $\mathcal{F}$ for this solution $\underline{v}(\phi, \underline{v}^d, \dot{\underline{\mathrm{T}}}^d)$

$$W(\phi, \underline{v}^d, \dot{\underline{\mathrm{T}}}^d) = \mathcal{F}(\,\underline{v}(\phi, \underline{v}^d, \dot{\underline{\mathrm{T}}}^d), \phi, \dot{\underline{\mathrm{T}}}^d). \tag{16}$$

The stability of the actual state is determined by the condition of the existence of a solution

$$\delta\phi \frac{\partial^2 W}{\partial\phi\partial\phi} \delta\phi \geq 0, \; \delta\phi \in \mathcal{K} - \{0\}, \tag{17}$$

and the uniqueness and non bifurcation is characterized by

$$\delta\phi \frac{\partial^2 W}{\partial\phi\partial\phi}\delta\phi \geq 0, \ \delta\phi \in \{\beta/\beta = 0 \text{ over } \Gamma - \Gamma^+\} - \{0\}. \tag{18}$$

3. DELAMINATION OF LAMINATES

The change of mechanical characteristics along moving front is used for the study of the degradation of laminates. We consider that a sound laminate (domain 0) with known characteristics is transformed into two laminates (1 and 2), separated by the crack of delamination as shown is the figure 26.1. Each laminate is described by an homogeneous plate or by an homogeneous beam, whose middle surface is denoted by S. The sound part has suffix 0, the two others parts are designed by suffices $1, 2$.

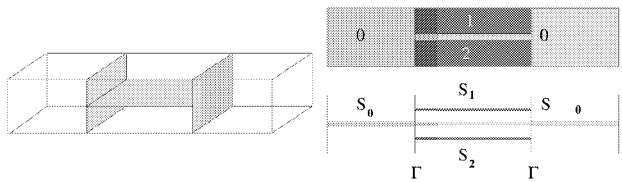

Figure 26.1. Modelization with beams.

The kinematic modeling of the beams or of the plates has a great influence on the behaviour of the delamination as well as the modeling of the continuity relations on the displacement along the delamination front which induces specific value for the energy release rate.

Kinematics of plates. A point of the middle surface S has curvilinear coordinates (x_1, x_2). The normal to this surface is denoted by $\underline{e}_3$. The normal coordinate is x_3. The displacement of the point (x_1, x_2, x_3) is defined by :

$$\underline{\xi} = \underline{u}(x_1, x_2) + w(x_1, x_2)\underline{e}_3 + \underline{\theta}(x_1, x_2)x_3, \quad (x_1, x_2) \in S. \tag{19}$$

where $\underline{u}$ is the plane displacement, w is the normal displacement and θ is the local rotation of the normal vector to the middle surface. With these fields, the strain inside the plate has the following form :

$$\boldsymbol{\varepsilon}(\underline{\xi}) = \boldsymbol{\varepsilon}(\underline{u}) + \frac{1}{2}(\underline{e}_3 \otimes \underline{\gamma} + \underline{\gamma} \otimes \underline{e}_3) - x_3\boldsymbol{\kappa}. \tag{20}$$

The distortion γ is defined by $\underline{\gamma} = \nabla w - \underline{\theta}$; the local rotation $\underline{\theta}$ gives $\boldsymbol{\kappa} = \frac{1}{2}(\nabla\underline{\theta} + \nabla^T\underline{\theta})$, and the membrane strain $\boldsymbol{\varepsilon}(\underline{u})$ satisfies $2\boldsymbol{\varepsilon}(\underline{u}) = (\nabla\underline{u} + \nabla^T\underline{u})$. The free energy of the plate is choosen naturally as a function of the generalized strains : $W = W(\boldsymbol{\varepsilon}(\underline{u}), \boldsymbol{\kappa}, \underline{\gamma})$.

The assemblage of plates. Consider the generalized parameters $\underline{q}_i = (\underline{u}_i, w_i, \underline{\theta}_i)$ and the generalized strains $\nabla \underline{q}_i = (\nabla \underline{u}_i, \nabla w_i, \nabla \underline{\theta}_i)$. We obtain the generalized stresses derived from the free energy W_i:

$$W_i(\boldsymbol{\varepsilon}(\underline{u}_i), \boldsymbol{\kappa}_i, \gamma_i) = F_i(\underline{q}_i, \nabla \underline{q}_i), \quad \boldsymbol{\sigma}_i = \frac{\partial F_i}{\partial \nabla \underline{q}_i}, \quad \underline{T}_i = \frac{\partial F_i}{\partial \underline{q}_i}. \tag{21}$$

Along the front, the section of the sound plate ($i = 0$) imposes its motion to the two others plates ($i = 1, 2$) :

$$\underline{u}_i = \underline{u}_o + h_i \underline{\theta}_o, \quad w_i = w_o, \quad \underline{\theta}_i = \underline{\theta}_o. \tag{22}$$

These continuity conditions are easily rewritten with the set of generalized parameters :

$$\underline{q}_i = \boldsymbol{l}_i . \underline{q}_o, \quad \underline{x} \in \Gamma. \tag{23}$$

Hence the corresponding Hadamard's compatiblity relations are

$$\mathcal{D}_\phi \underline{q}_i = \boldsymbol{l}_i : \mathcal{D}_\phi \underline{q}_o. \tag{24}$$

Equilibrium state. A state of equilibrium $\underline{q}$ is a stationnary value of the potential energy $\mathcal{E}$ of the system:

$$\mathcal{E}(\underline{\tilde{q}}, \underline{\mathrm{T}}^d) = \sum_{i=0}^{2} \int_{S_i} F_i(\underline{q}, \nabla \underline{q})\, \mathrm{d}S - \int_{\partial S_T} \underline{\mathrm{T}}^d . \underline{q}\, \mathrm{d}s. \tag{25}$$

The variation of $\mathcal{E}$ amoung the set of kinematically admissible fields

$$\mathcal{K}.\mathcal{A}. = \{ \underline{q} / \underline{q} = \underline{q}^d \text{ over } \partial S_q, \ \underline{q}_i = \boldsymbol{l}_i . \underline{q}_o . \text{ over } \Gamma \},$$

gives rize to the equilibrium equations :

$$\begin{aligned} 0 &= \frac{\partial F_i}{\partial \underline{q}_i} - \mathrm{div} \boldsymbol{\sigma}_i = \underline{T}_i - \mathrm{div} \boldsymbol{\sigma}_i, \text{ on } S_i, && (26) \\ 0 &= \underline{\nu}.(-\sum_{i=1}^{2} \boldsymbol{\sigma}_i . \boldsymbol{l}_i + \boldsymbol{\sigma}_o) = \underline{\nu}.[|\boldsymbol{\sigma}|]_\Gamma, \text{along } \Gamma, && (27) \\ \underline{\mathrm{T}}^d &= \boldsymbol{\sigma}_o . \underline{n}, \text{ along } \partial S_T. && (28) \end{aligned}$$

where the tractions $\underline{\mathrm{T}}^d$ are imposed on ∂S_T complementary part of ∂S_q. We have introduced the usefull notation $[|f|]_\Gamma = f_o - \sum_{i=1}^{2} f_i l_i$.

Analysis of the dissipation. The dissipation is given by the balance of the power of external loading and the reversible stored energy :

$$\mathrm{D} = \mathcal{P}_e - \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=0}^{2} \int_{S_i} F_i \, \mathrm{d}S = \int_\Gamma \mathcal{G} \phi \, \mathrm{d}S = -\frac{\partial \mathcal{E}}{\partial \Gamma} . \dot{\Gamma} \geq 0, \tag{29}$$

where the thermodynamical force $\mathcal{G}$ is the field :

$$\mathcal{G}(s) = -\frac{\partial \mathcal{E}}{\partial \Gamma}(s) = F_o - \underline{\nu}.\boldsymbol{\sigma}_o.\nabla \underline{q}_o.\underline{\nu} - \sum_{i=1}^{2}(F_i - \underline{\nu}.\boldsymbol{\sigma}_i.\nabla \underline{q}_i.\underline{\nu}). \tag{30}$$

Propagation law. The propagation law is defined as before :

$$\mathcal{G}(s,t) < G_c, \quad \phi = 0 \text{ and } \quad \mathcal{G}(s,t) = G_c, \quad \phi \geq 0. \tag{31}$$

The propagation is then possible when the critical value is reached. The velocity ϕ is included in the set of admissible propagation :

$$\mathcal{K} = \{\phi^*/\phi^*(s) \geq 0, \mathcal{G}(s) = G_c, \phi^* = 0, \text{otherwise}\}.$$

The rate boundary value problem. The rate boundary value problem is written in terms of rate of displacement. The solution $(\dot{\underline{q}}, \phi)$ of the rate boundary value satisfies the set of local equations obtained by the derivation of the constitutive law (21) and of the conservation of the momentum (26) with respect to time, the continuity relations along the front (24), the continuity relations on the stress vector :

$$\mathcal{D}_\phi(\underline{\nu}.[|\boldsymbol{\sigma}|]_\Gamma) = \underline{\nu}.[|\dot{\boldsymbol{\sigma}}|]_\Gamma - \mathrm{div}_\Gamma(\phi[|\boldsymbol{\sigma}|]_\Gamma) + \phi[|\underline{T}|]_\Gamma = 0, \tag{32}$$

the propagation law:

$$(\phi - \phi^*)\, \mathcal{D}_\phi \mathcal{G} \geq 0, \quad \phi \in \mathcal{K}, \quad \forall \phi^* \in \mathcal{K}, \tag{33}$$

and the boundary conditions: $\dot{\underline{q}} = \dot{\underline{q}}^d$ over ∂S_q and $\dot{\boldsymbol{\sigma}}.\underline{n} = \dot{\underline{\mathrm{T}}}^d$ over ∂S_T. The effective expression of the consistency condition gives a relation between the rate of the displacement and the velocity of propagation. The variation of $\mathcal{G}$ is given by :

$$\mathcal{D}_\phi \mathcal{G} = [|\boldsymbol{\sigma}|]_\Gamma : \nabla \dot{\underline{q}}_o + [|\underline{T}|]_\Gamma . \dot{\underline{q}}_o + \sum_{i=1}^{2} \dot{\boldsymbol{\sigma}}_i : (\nabla \underline{q}_i - \boldsymbol{l}_i.\nabla \underline{q}_o) - \phi G_n,$$

$$G_n = -[|\boldsymbol{\sigma}|]_\Gamma : \nabla\nabla \underline{q}_o.\underline{\nu} - [|\underline{T}|]_\Gamma.\nabla \underline{q}_o.\underline{\nu} + \sum_{i=1}^{2}(\nabla \boldsymbol{\sigma}_i.\underline{\nu}) : (\nabla \underline{q}_i - \boldsymbol{l}_i.\nabla \underline{q}_o).$$

Introducing now the potential $\mathcal{F}(\dot{\underline{q}}, \phi, \dot{\underline{\mathrm{T}}}^d)$:

$$\begin{aligned}
\mathcal{F} = \sum_{i=0}^{2} \int_{S_i} \frac{1}{2}(&\dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \underline{q}_i} . \dot{\underline{q}}_i + \dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \underline{q}_i \partial \nabla \underline{q}_i} . \nabla \dot{\underline{q}}_i \\
&+ \nabla \dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \underline{q}_i} . \dot{\underline{q}}_i + \nabla \dot{\underline{q}}_i . \frac{\partial^2 F_i}{\partial \nabla \underline{q}_i \partial \nabla \underline{q}_i} . \nabla \dot{\underline{q}}_i)\, \mathrm{d}S \\
&- \int_\Gamma \Big([|\boldsymbol{\sigma}|]_\Gamma : \nabla \dot{\underline{q}}_o + [|\underline{T}|]_\Gamma . \dot{\underline{q}}_o\Big)\phi\, \mathrm{d}S + \int_\Gamma \frac{\phi^2}{2} G_n\, \mathrm{d}S - \int_{\partial S_T} \dot{\underline{\mathrm{T}}}^d . \dot{\underline{q}}\, \mathrm{d}s,
\end{aligned}$$

the solution of the rate boundary value problem $(\tilde{\underline{q}}, \tilde{\phi}) \in \mathcal{K}.\mathcal{A}$ is a solution of the variational inequality:

$$\frac{\partial \mathcal{F}}{\partial \tilde{\dot{\underline{q}}}}.(\tilde{\dot{\underline{q}}} - \tilde{\dot{\underline{q}}}^*) + \frac{\partial \mathcal{F}}{\partial \tilde{\dot{\phi}}}.(\tilde{\dot{\phi}} - \tilde{\dot{\phi}}^*) \geq 0, \tag{34}$$

amongs the set $(\tilde{\dot{q}}^*, \tilde{\dot{\phi}}^*) \in \mathcal{K}.\mathcal{A}$.

$$\begin{aligned} \mathcal{K}.\mathcal{A}. \quad = \quad & \Big\{ (\tilde{\dot{q}}, \tilde{\dot{\phi}}) / \, \dot{\underline{q}}_i + \phi \nabla \, \underline{q}_i . \underline{\nu} = \boldsymbol{l}_i . (\dot{\underline{q}}_o + \phi \nabla \, \underline{q}_o . \underline{\nu}), \text{ over } \Gamma, \\ & \underline{q}_o = \underline{q}^d \text{ over } \partial S_{\mathbf{q}}, \quad \phi \in \mathcal{K} \Big\}. \end{aligned}$$

This framework can be extended to different models of beams or plates and to dynamics using of kinetic energy and hamiltonian formalism, (Stolz, 1995, 2000).

References

Abeyratne, R., Knowles, J., On the driving traction acting on a surface of strain discontinuity in a continuum, *J. Mech. Phys. Solids*, **38**, 3, pp. 345-360, 1990.

Gurtin, M. E., The nature of configurational forces, *Arch. Rat. Mech. Anal.*, **131**, pp. 67-100, 1995.

Maugin, G. A., Material forces : concept and applications, *ASME, Appl. Mech. Rev.*, **48**, pp. 213-245, 1995.

Pradeilles-Duval, R.M., Stolz, C., Mechanical transformations and discontinuities along a moving surface, *J. Mech. Phys. Solids*, **43**, 1, pp. 91-121, 1995.

Stolz, C., Functional approach in non linear dynamics, *Arch. Mech*, **47**, 3, 421-435, 1995.

Stolz, C., Thermodynamical description of running discontinuities, in *Continuum Thermodynamics*, pp 410-412, G.A. Maugin et al. (eds), Kluwer Academic Publishers, 2000.

Truskinovsky, L. M., Dynamics of non equilibrium phases boundaries in a heat conducting non linearly elastic medium, *P.M.M*, **51**, pp. 777-784, 1987.