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# Bifurcation of equilibrium solutions and defects nucleation

Claude Stolz

**Abstract** The purpose of this article is to revise some concepts on defects nucleation based on bifurcation of equilibrium solutions. Equilibrium solutions are obtained on a homogeneous body and on a body with an infinitesimal defect such as cavity under the same prescribed dead load. First void formation and growth in non linear mechanics are examined. A branch of radial transformation bifurcates from the undeformed configuration in presence of a small cavity. Two cases of behaviour are examined. One case is the growth of the cavity by only the deformation of the shell. In another modelling the cavity evolves like a damaged zone, the transition between the sound part and the damaged one is governed by a local criterium. Each configuration leads to the definition of a nucleation criterion based on a presence of a bifurcation state, common state of the homogeneous body and a body with an infinitesimal defect.

**Keywords** Bifurcation · Nucleation of defects · Hyperelasticity · Local damage · Composite sphere

## 1 Introduction

We consider a composite sphere with external radius  $R_e$  composed by a cavity of radius  $R_i$  surrounded by

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a material. Under a uniform radial tensile loading  $p$ , the solution is assumed to be radial. We investigate the possible branches of radially symmetric configurations involving a traction free internal cavity bifurcation from the homogeneous sphere. At the bifurcation point  $p_c$ , the equilibrium solution for the porous sphere under the load  $p_c$  tends to the equilibrium solution of the homogeneous sphere when the volume of the cavity tends to zero.

Ball (1982) and Horgan and Pence (1989) have studied different classes of bifurcation problems for non linear incompressible elasticity, which simulate the appearance of a cavity inside a homogeneous sphere. Example in finite incompressible elastoplasticity have been also investigated by Chung et al. (1987). In both cases for all values of  $p$  one possible solution is that the sphere remains solid, this is due to incompressibility. On the other hand, for a certain range of loading  $p$  one has another configuration with an internal infinitesimal cavity. But as pointed out by Ball, only specific classes of hyperelastic incompressible materials have the possibility of such cavitation under radial dead load. However if the core of the sphere is made by a damaged material which does not suffer any tensile stress after a critical stretch, we obtain a critical pressure for all classes of hyperelastic incompressible materials. This point generalizes the point of view of Ball and gives new definition of nucleation of defects.

In Sect. 3, the possibility of cavitation is generalized to other classes of materials in small strains.

## 2 Extension of the Ball's approach

Consider a sphere with internal cavity of radius  $R_i$  and external radius  $R_e$ . Under the radial loading, the actual state is defined by the transformation  $r = f(R)$ . The deformation gradient tensor, denoted  $\mathbb{F}$ , is then given by

$$\mathbb{F} = f' e_r \otimes E_R + \frac{f}{R} (e_\theta \otimes E_\Theta + e_\phi \otimes E_\Phi). \quad (1)$$

Taking the incompressibility condition into account, the transformation  $f$  is reduced to

$$r^3 = R^3 + A \quad (2)$$

where  $A$  is a constant whose value depends on the loading.

For an isotropic hyperelastic incompressible material the strain energy  $W$  is a symmetric function of the principal stretches  $\lambda_k$ ,  $k = 1, 2, 3$ :  $W(\lambda_1, \lambda_2, \lambda_3)$ .

### 2.1 The homogeneous sphere

When the sphere is homogeneous, for any dead load  $p$  the stress  $\sigma = -p \mathbb{I}$  is a solution of the equilibrium state equation with  $\lambda_k = 1$ ,  $k = 1, 2, 3$ . There is no strain, that is a consequence of incompressibility of the material.

### 2.2 The porous sphere

When a spherical cavity of radius  $R_i$  exists, the solution has the form (2), the principal stretches are now

$$\lambda_1 = \frac{1}{\lambda^2}, \quad \lambda_2 = \lambda_3 = \frac{r}{R} = \lambda. \quad (3)$$

For this transformation, the strain energy has a value governed only by the stretch  $\lambda$ ,

$$\psi(\lambda) = W\left(\frac{1}{\lambda^2}, \lambda, \lambda\right) \quad (4)$$

and the relation between the stretches and the external load  $P_e$  is given by

$$P_e = \int_{\lambda_i}^{\lambda_e} \frac{d\psi}{d\lambda} \frac{d\lambda}{\lambda^3 - 1} = I(\lambda_i, \lambda_e). \quad (5)$$

This equation is derived from the equilibrium elastic problem for the radial deformation of a sphere when the inner surface is stress free. The inflation of a thick walled sphere is an universal deformation in the sense

of Rivlin-Eriksen and is a classical and well-known solution (Ogden 1997; Ball 1982). The values  $\lambda_i, \lambda_e$  are linked by the relation (2)

$$(\lambda^3(R) - 1)R^3 = (\lambda_i^3 - 1)R_i^3 = (\lambda_e^3 - 1)R_e^3 = A \quad (6)$$

### 2.3 Discussion

We must compare these results with those of the homogeneous sphere. For a given  $P_e$ ,  $A \neq 0$  and the possibility to  $\lambda_e$  to tends towards 1 is that  $R_e$  tends to infinity. Simultaneously when the cavity is reduced to a point,  $\lambda_i$  tends to infinity. In fact the cavity is embedded in an infinite matrix and for some value  $P_e$ , the cavity is not detectable from an external point of view. From this constatation, Ball studies the convergence of the integral  $I$  with the bounds  $\lambda_i = \infty, \lambda_e = 1$ .

For example, let us consider a density of energy of the form  $W = \mu_\alpha \sum_k \lambda_k^\alpha$ , where  $\mu_\alpha$  is a constant modulus associated to the power  $\alpha$  of the principal stretches. In this case, the pressure  $P_e = P_c$  is finite under the condition  $\lambda_i = \infty, \lambda_e = 1$  if  $-1 < \alpha < 2$ .

The point  $(\lambda_e, P_e)$  describes a loading path for some range of  $P_e$  which intersects the loading path of the homogeneous sphere at point  $(1, P_c)$ . This is the critical value for cavitation. But locally the strain energy is not finite and the local solution with fixed  $R$  does not tend to the homogeneous one.

The Neo-Hookean material has a critical pressure equal to  $5\mu_o$ , the Mooney-Rivlin material can not have cavitation under finite pressure.

### 2.4 Additional assumptions and remarks

However, the hyperelastic media cannot suffer infinite stretch without rupture. Assume now that the core of the sphere is made by a damaged material which cannot suffer any tensile stress after some critical value given in terms of stretches. The boundary between the sound material and the damaged one is now a spherical interface of radius  $R_i$  along which the stress vector is null and the stretch  $r_i/R_i$  has the critical value  $\lambda_c$ . The inner surface is stress free.

With these additional assumptions, the external load  $P_e$  is always finite for a finite porous sphere and is given by the preceding results

$$P_e = \int_{\lambda_c}^{\lambda_e} \frac{d\psi}{d\lambda} \frac{d\lambda}{\lambda^3 - 1} = I(\lambda_c, \lambda_e), \quad (7)$$

and the link between the stretches is preserved:

$$A = (\lambda_c^3 - 1)R_i^3 = (\lambda_e^3 - 1)R_e^3 = (\lambda^3(R) - 1)R^3. \quad (8)$$

Now when the radius of the damaged zone  $R_i$  tends to zero,  $A$  tends to zero,  $\lambda_e$  tends to 1 and the pressure  $P_e$  tends to the finite value given by

$$P_c = \int_{\lambda_c}^{\lambda_e} \frac{d\psi}{d\lambda} \frac{d\lambda}{\lambda^3 - 1} = I(\lambda_c, 1). \quad (9)$$

We have now a finite value for all hyperelastic incompressible materials which does not support stretching greater than  $\lambda_c$ . Moreover, when  $A$  tends to zero, it is easy to show that  $\lambda(R)$  tends to 1 at fixed material point  $R$ , the local solution tends to the solution of the homogeneous sphere.

Introducing the concept of local damage extends the point of view of Ball of bifurcated equilibrium solution and gives rise to an upper bound for the critical value for nucleation of a spherical defect in a homogeneous sphere.

We examine now the case of elastic brittle material in small strain.

### 3 Case of elastic brittle material

Consider a composite sphere, whose shell (material 1) and core (material 2) are linear elastic materials with different elastic modulus. The bulk moduli are denoted by  $\kappa_i$  and the shear moduli are  $\mu_i$ ,  $i = 1, 2$ .  $\kappa_1$  is supposed greater than  $\kappa_2$ .

The sphere is submitted to an isotropic loading, the radial displacement is prescribed on the external boundary ( $R = R_e$ ) (Fig. 1). The solution of the elasticity problem is given considering a radial displacement

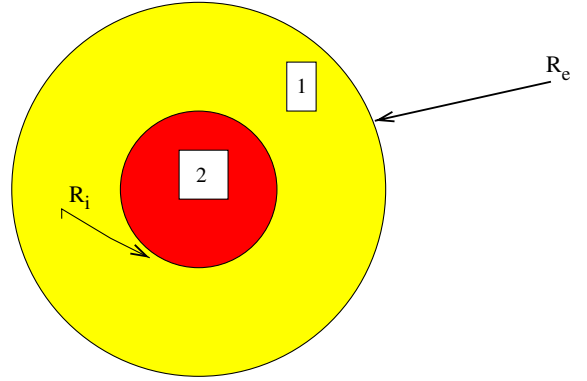
$$u = u_i(R)e_r, u_i(R) = A_i R + \frac{B_i}{R^2}, \quad i = 1, 2. \quad (10)$$

The boundary conditions imply:

$$u_1(R_e) = E R_e, u_2(0) = 0, B_2 = 0. \quad (11)$$

For a given history of  $E$ , the external surface is submitted to a radial force:

$$\sigma_1(R_e) \cdot e_r = \Sigma e_r. \quad (12)$$



**Fig. 1** The composite sphere

For given volume fraction of material 2,  $c = \frac{R_i^3}{R_e^3}$ , the solution of heterogeneous elastic sphere is

$$b = \frac{B_1}{R_i^3} = \frac{3(\kappa_1 - \kappa_2)A_1}{3\kappa_2 + 4\mu_1}$$

$$E = \frac{D(c)}{3\kappa_2 + 4\mu_1} A_1$$

$$\Sigma = \sigma_{rr}(R_e) = \frac{3E}{D(c)} ((3\kappa_2 + 4\mu_1)\kappa_1 - 4\mu_1 c(\kappa_1 - \kappa_2))$$

where

$$D(c) = 3\kappa_2 + 4\mu_1 + 3c(\kappa_1 - \kappa_2), \quad c = \frac{R_i^3}{R_e^3}.$$

The last equation defines the global behaviour of the composite sphere as having an effective bulk modulus

$$\kappa^{eff} = \frac{(3\kappa_2 + 4\mu_1)\kappa_1 - 4\mu_1 c(\kappa_1 - \kappa_2)}{3\kappa_2 + 4\mu_1 + 3c(\kappa_1 - \kappa_2)} \quad (13)$$

Then it is obvious that when  $c$  tends to zero,  $\kappa^{eff}$  tends to  $\kappa_1$ .

When the radius  $R_i$  increases, the rigidity of the composite sphere decreases and some dissipation occurs. The dissipation is given by the rate

$$4\pi R_i^2 \mathcal{G}(R_i, E) \dot{R}_i = -\frac{\partial W}{\partial R_i} \dot{R}_i \quad (14)$$

This defines the analogous way the energy release rate associated with the dissipation along a moving surface (Stolz and Pradeilles-Duval 1997, 2004).

Along the interface  $\Gamma$  the energy release rate has the value

$$\mathcal{G}(R_i, E) = \frac{9E^2}{D^2(c)} (\kappa_1 - \kappa_2)(3\kappa_2 + 4\mu_1)(3\kappa_1 + 4\mu_1) \quad (15)$$

### 3.1 The response under monotonic loading

The loading parameter  $E$  is increasing. Initially, the core does not evolve, the critical value  $G_c$  is not reached. At one time the critical value  $G_c$  is reached, the strain is  $E_c(c_o)$ . After that the actual value of  $R_i$  is determined by the implicit equation

$$\mathcal{G}(R_i(t), E(t)) = G_c \quad (16)$$

this is the consistency condition. During this phase the internal radius  $R_i$  increases monotonical with  $E$  and attains the value  $R_e$  at the value  $E_T$  of the loading.

To any choosen critical value  $G_c$  corresponds a Griffith type local criterion for fracture, and this induces that the local stretch  $u(R_i)/R_i$  is a constant proportional of the square root of  $G_c$ . From Eqs. 15 and 16 we deduce that when the damage occurs

$$\frac{E}{D(c)} = \alpha_c \quad (17)$$

where  $\alpha_c$  is a constant. We remark that  $D$  is an increasing function of  $c$ . During the damage evolution  $A_1 = E_c$

$$\alpha_c = \frac{E_c}{3\kappa_2 + 4\mu_1} = \frac{1}{3\kappa_1 + 4\mu_1} \frac{u(R_i)}{R_i}. \quad (18)$$

At intermediate time  $t < T$  the sphere is not completely transformed,  $\mathcal{G}(R_i(t), E) < G_c$  for any  $E < E(t)$ , then the composite sphere has the answer of an elastic heterogeneous medium with new concentration  $c = R_i^3/R_e^3$ . The global bulk modulus decreases with the transformation.

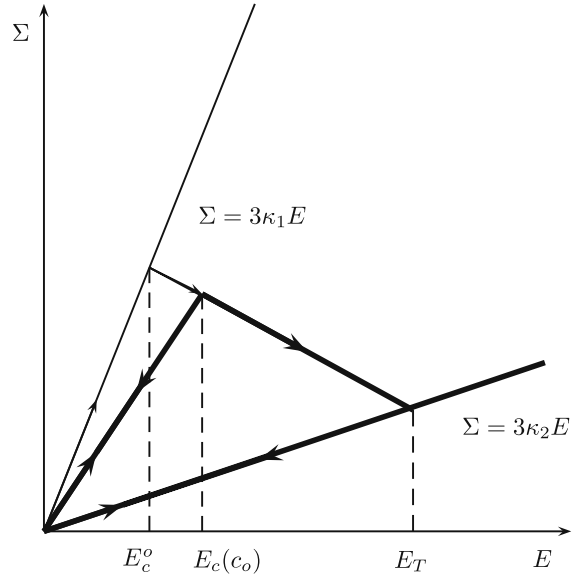
With the given propagation law of the interface, from it's initial position determined by  $c_o = R_i^3(O)/R_e^3$ , we have successively for an increasing function  $E(t)$ :

$$\begin{cases} E(t) < E_c(c_o), & \mathcal{G}(R_i, E(t)) < G_c, & R_i(t) = R_i(0) \\ E(t) \geq E_c(c_o), & \mathcal{G}(R_i(t), E(t)) = G_c, & R_i(t) = f(E(t)), \\ E(t) = E_T, & \mathcal{G}(R_e, E_T) = G_c & R_i(T) = R_e \\ E(t) \geq E_T, & & R_i(t) = R_e \end{cases} \quad (19)$$

and the answer can be plotted as in Fig. 2.

Conserving the value  $\mathcal{G} = G_c$  when the concentration of damaged zone tends to zero ( $c_o \rightarrow O^+$ ), the global bulk modulus tends to  $\kappa_1$ ,  $E_c(c_o) \rightarrow E_c^o$  and  $\Sigma$  tends to  $P_c = 3\kappa_1 E_c^o$ . The local energy remains finite and the local stresses at fixed  $R$  tend to the response of the homogeneous sphere.

This defines a new manner to consider a critical value for nucleation of defects. Let us assume the existence



**Fig. 2** The response of the composite sphere

of a small defect of volume  $\omega$  with a given evolution law. Conserving the evolution law, when the volume  $\omega$  tends to zero, we consider the applied load as the critical load for nucleation of defect if this value coincides with the value of the original volume without defect for the same value of global strain.

In this sense, we cannot distinguish at the global level for this state of equilibrium if there exists or not an infinitesimal defect.

## 4 Conclusion

We have investigated the approach of a critical value for nucleation of defects inside an homogeneous body based on a bifurcation analysis of equilibrium solutions.

The nucleation of defects is considered with two different points of view. One is the point of view of Ball and other authors, with locally infinite stretches and energy. This point of view defines different classes of materials, on one side we can have for same range cavitation, on the other side the cavitation is prohibited.

By introducing the concept of local damage, the nucleation of defects is possible for any materials, whose strains are limited in amplitude. Using this characteristic for governing the propagation of the damaged zone, a new point of view has been presented for critical value of defects nucleation.

We point out that this point of view gives for each configuration an upper bound for the critical value of nucleation because the approach is based on an analysis of bifurcation, then the value depends on the geometry and of the path of the loading.

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