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# Global well-posedness for dissipative Korteweg-de Vries equations

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**Abstract.** This paper is devoted to the well-posedness for dissipative KdV equations  $u_t + u_{xxx} + |D_x|^{2\alpha}u + uu_x = 0$ ,  $0 < \alpha \leq 1$ . An optimal bilinear estimate is obtained in Bourgain's type spaces, which provides global well-posedness in  $H^s(\mathbb{R})$ ,  $s > -3/4$  for  $\alpha \leq 1/2$  and  $s > -3/(5-2\alpha)$  for  $\alpha > 1/2$ .

## 1 Introduction

We study the initial value problem (IVP) for the dissipative KdV equations

$$\begin{cases} u_t + u_{xxx} + |D_x|^{2\alpha}u + uu_x = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

with  $0 < \alpha \leq 1$  and where  $|D_x|^{2\alpha}$  denotes the Fourier multiplier with symbol  $|\xi|^{2\alpha}$ . These equations can be viewed as a combinaison of the KdV equation

$$u_t + u_{xxx} + uu_x = 0 \quad (1.2)$$

and Burgers equation

$$u_t - u_{xx} + uu_x = 0, \quad (1.3)$$

involving both nonlinear dispersion and dissipation effects.

The Cauchy problem for the KdV equation has been studied by many authors. In [2], Bourgain introduced new functional spaces adapted to the linear symbol  $\tau - \xi^3$  and showed that the IVP associated to (1.2) is locally well-posed in  $L^2(\mathbb{R})$ . Due to the second conservation law, this result extends

globally in time. Then, working in these spaces, Kenig, Ponce and Vega obtained local well-posedness in Sobolev spaces  $H^s(\mathbb{R})$  for  $s > -5/8$  in [8] and for  $s > -3/4$  in [9]. More recently, global well-posedness was obtained for  $s > -3/4$  in [5]. It is worth noticing that the index  $-3/4$  is far away from the index  $-3/2$  suggested by standard scaling argument. However,  $-3/4$  is indeed the critical index for well-posedness. In fact, the solution map  $u_0 \mapsto u$  fails to be  $\mathcal{C}^3$  in  $H^s(\mathbb{R})$ ,  $s < -3/4$  (see [3]) and  $\mathcal{C}^2$  in homogeneous spaces  $\dot{H}^s(\mathbb{R})$ ,  $s < -3/4$  (see [15]). Moreover, the bilinear estimate in  $X^{b,s}$  spaces used in [9] to prove local well-posedness is sharp with respect of  $s$  (see also [12]).

Concerning the Cauchy problem for the Burgers equation, the situation is quite different. By using the strong smoothing effect of the semigroup related to the heat equation, one can solve (1.3) in the Sobolev space given by an heuristic scaling argument. In [6], Dix proved local well-posedness of (1.3) in  $H^s(\mathbb{R})$  for  $s > -1/2$ . Then, this result was extended to the case  $s = -1/2$  in [1]. Below this critical index, it has been showed in [6] that uniqueness fails.

When  $\alpha = 1/4$ , equation (1.1) models the evolution of the free surface for shallow water waves damped by viscosity, see [13]. When  $\alpha = 1$ , (1.1) is the so-called KdV-Burgers equation which models the propagation of weakly nonlinear dispersive long waves in some contexts when dissipative effects occur (see [13]). In [10], Molinet and Ribaud treat the KdV-B equation by working in the usual Bourgain space related to the KdV equation, considering only the dispersive part of the equation. They were able to prove global well-posedness for KdV-B in  $H^s(\mathbb{R})$ ,  $s > -3/4 - 1/24$ , getting a lower index than the critical indexes for (1.2) and (1.3). Then, the same authors improved this result in [11] by going down to  $H^s(\mathbb{R})$ ,  $s > -1$ . The main new ingredient is the introduction of a new Bourgain space containing both dispersive and dissipative parts of the equation. For  $s < -1$ , the problem is ill-behaved in the sense that the flow map  $u_0 \mapsto u$  is not  $\mathcal{C}^2$  in  $H^s(\mathbb{R})$ .

Concerning the case  $0 < \alpha < 1$  in (1.1), Molinet and Ribaud established in [10] the global well-posedness for data in  $H^s(\mathbb{R})$ ,  $s > -3/4$ , whatever the value of  $\alpha$ . On the other hand, ill-posedness is known for (1.1) in  $\dot{H}^s(\mathbb{R})$ ,  $s < (\alpha - 3)/(2(2 - \alpha))$ , see [11].

In this paper we improve the results obtained in [10]. We show that the

Cauchy problem (1.1) is globally well-posed in  $H^s(\mathbb{R})$ ,  $s > s_\alpha$  with

$$s_\alpha = \begin{cases} -3/4 & \text{if } 0 < \alpha \leq 1/2, \\ -\frac{3}{5-2\alpha} & \text{if } 1/2 < \alpha \leq 1 \end{cases} .$$

Of course the case  $\alpha \leq 1/2$  is well-known, but our general proofs contain this result. In suitable  $X^{b,s}$  spaces, we are going to perform a fixed point argument on the integral formulation of (1.1). This will be achieved by deriving a bilinear estimate in these spaces. By Plancherel's theorem and duality, it reduces to estimating a weighted convolution of  $L^2$  functions. To recover the lost derivative in the nonlinear term  $\partial_x(u^2)$ , we take advantage of the well-known algebraic smoothing relation (2.5) combined with several methods. On one hand, we use Strichartz's type estimates and some techniques introduced in [9]. On the other hand, these techniques are not sufficient in certain regions to go down below  $-3/4$  and we are lead to use dyadic decomposition. In [14], Tao studied some nonlinear dispersive equations like KdV, Schrödinger or wave equation by using such dyadic decomposition and orthogonality. He obtained sharp estimates on dyadic blocs, which leads to multilinear estimates in the  $X^{b,s}$  spaces, usable in many contexts. Note that very recently, such a method was exploited in [4] for the dissipative modified-KdV equation  $u_t + u_{xxx} + |D_x|^{2\alpha}u + u^2u_x = 0$ .

It is worth pointing out that (1.1) has no scaling invariance, so it is not clear that  $s_\alpha$  is the critical index for well-posedness. However, the fundamental bilinear estimate used to prove our results is optimal.

## 1.1 Notations

For two positive reals  $A$  and  $B$ , we write  $A \lesssim B$  if there exists a constant  $C > 0$  such that  $A \leq CB$ . When this constant is supposed to be sufficiently small, we write  $A \ll B$ . Similarly, we use the notations  $A \gtrsim B$ ,  $A \sim B$  and  $A \gg B$ . When  $x \in \mathbb{R}$ ,  $x^+$  denotes its positive part  $\max(0, x)$ . For  $f \in \mathcal{S}'(\mathbb{R}^N)$ , we define its Fourier transform  $\mathcal{F}(f)$  (or  $\widehat{f}$ ) by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} f(x) dx.$$

The Lebesgue spaces are endowed with the norm

$$\|f\|_{L^p(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty$$

with the usual modification for  $p = \infty$ . We also consider the space-time Lebesgue spaces  $L_x^p L_t^q$  defined by

$$\|f\|_{L_x^p L_t^q} = \left\| \|f\|_{L_x^p(\mathbb{R})} \right\|_{L_t^q(\mathbb{R})}.$$

For  $b, s \in \mathbb{R}$ , we define the Sobolev spaces  $H^s(\mathbb{R})$  and their space-time versions  $H^{b,s}(\mathbb{R}^2)$  by the norms

$$\|f\|_{H^s} = \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2},$$

$$\|u\|_{H^{b,s}} = \left( \int_{\mathbb{R}^2} \langle \tau \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\tau, \xi)|^2 d\tau d\xi \right)^{1/2},$$

with  $\langle \cdot \rangle = (1 + |\cdot|)^{1/2}$ .

Let  $U(t)\varphi$  denote the solution of the Airy equation

$$\begin{cases} u_t + u_{xxx} = 0 \\ u(0) = \varphi \end{cases},$$

that is,

$$\forall t \in \mathbb{R}, \mathcal{F}_x(U(t)\varphi)(\xi) = \exp(i\xi^3 t) \widehat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'.$$

In [11], Molinet and Ribaud introduced the function spaces  $X_\alpha^{b,s}$  related to the linear symbol  $i(\tau - \xi^3) + |\xi|^{2\alpha}$  and defined by the norm

$$\|u\|_{X_\alpha^{b,s}} = \left\| \langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \widehat{u} \right\|_{L^2(\mathbb{R}^2)}.$$

Note that since  $\mathcal{F}(U(-t)u)(\tau, \xi) = \widehat{u}(\tau + \xi^3, \xi)$ , we can re-express the norm of  $X_\alpha^{b,s}$  as

$$\begin{aligned} \|u\|_{X_\alpha^{b,s}} &= \left\| \langle i\tau + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \widehat{u}(\tau + \xi^3, \xi) \right\|_{L^2(\mathbb{R}^2)} \\ &= \left\| \langle i\tau + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \mathcal{F}(U(-t)u)(\tau, \xi) \right\|_{L^2(\mathbb{R}^2)} \\ &\sim \|U(-t)u\|_{H^{b,s}} + \|u\|_{L_t^2 H_x^{s+2\alpha b}}. \end{aligned}$$

We will also work in the restricted spaces  $X_{\alpha,T}^{b,s}$ ,  $T \geq 0$ , equipped with the norm

$$\|u\|_{X_{\alpha,T}^{b,s}} = \inf_{w \in X_\alpha^{b,s}} \{ \|w\|_{X_\alpha^{b,s}}, w(t) = u(t) \text{ on } [0, T] \}.$$

Finally, we denote by  $W_\alpha$  the semigroup associated with the free evolution of (1.1),

$$\forall t \geq 0, \mathcal{F}_x(W_\alpha(t)\varphi)(\xi) = \exp[-|\xi|^{2\alpha} t + i\xi^3 t] \widehat{\varphi}(\xi), \quad \varphi \in \mathcal{S}'.$$

and we extend  $W_\alpha$  to a linear operator defined on the whole real axis by setting

$$\forall t \in \mathbb{R}, \mathcal{F}_x(W_\alpha(t)\varphi)(\xi) = \exp[-|\xi|^{2\alpha}|t| + i\xi^3 t]\widehat{\varphi}(\xi), \varphi \in \mathcal{S}'. \quad (1.4)$$

## 1.2 Main results

Let us first state our crucial bilinear estimate.

**Theorem 1.1** *Given  $s > s_\alpha$ , there exist  $\nu, \delta > 0$  such that for any  $u, v \in X_\alpha^{1/2, s}$  with compact support in  $[-T, +T]$ ,*

$$\|\partial_x(uv)\|_{X_\alpha^{-1/2+\delta, s}} \lesssim T^\nu \|u\|_{X_\alpha^{1/2, s}} \|v\|_{X_\alpha^{1/2, s}}. \quad (1.5)$$

This result is optimal in the following sense.

**Theorem 1.2** *For all  $s \leq s_\alpha$  and  $\nu, \delta > 0$ , there exist  $u, v \in X_\alpha^{1/2, s}$  with compact support in  $[-T, +T]$  such that the estimate (1.5) fails.*

Let  $\psi$  be a cutoff function such that

$$\psi \in \mathcal{C}_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi \equiv 1 \text{ on } [-1, 1],$$

and define  $\psi_T(\cdot) = \psi(\cdot/T)$  for all  $T > 0$ . By Duhamel's principle, the solution to the problem (1.1) can be locally written in the integral form as

$$u(t) = \psi(t) \left[ W_\alpha(t)u_0 - \frac{\chi_{\mathbb{R}_+}(t)}{2} \int_0^t W_\alpha(t-t') \partial_x(\psi_T^2(t')u^2(t')) dt' \right]. \quad (1.6)$$

Clearly, if  $u$  is a solution of (1.6) on  $[-T, +T]$ , then  $u$  solves (1.1) on  $[0, T/2]$ .

As a consequence of Theorem 1.1 together with linear estimates of Section 2.1, we obtain the following global well-posedness result.

**Theorem 1.3** *Let  $\alpha \in (0, 1]$  and  $u_0 \in H^s(\mathbb{R})$  with  $s > s_\alpha$ . Then for any  $T > 0$ , there exists a unique solution  $u$  of (1.1) in*

$$Z_T = \mathcal{C}([0, T], H^s(\mathbb{R})) \cap X_{\alpha, T}^{1/2, s}.$$

Moreover, the map  $u_0 \mapsto u$  is smooth from  $H^s(\mathbb{R})$  to  $Z_T$  and  $u$  belongs to  $\mathcal{C}((0, T], H^\infty(\mathbb{R}))$ .

**Remark 1.1** *Actually, we shall prove Theorems 1.1 and 1.3 in the most difficult case. In the sequel we assume*

$$\begin{cases} s_\alpha < s < -1/2 & \text{if } \alpha \leq 1/2, \\ s_\alpha < s < -3/4 & \text{if } \alpha > 1/2. \end{cases} \quad (1.7)$$

**Remark 1.2** *Theorem 1.3 is known to be sharp in the case  $\alpha = 0$  (KdV equation) and in the case  $\alpha = 1$  (KdV-B equation). On the other hand, as far as we know, most of nonlinear equations for which the multilinear estimate fails in the related  $X^{b,s}$  space are ill-posed in  $H^s$ . Therefore it is reasonable to conjecture that  $s_\alpha$  is really the critical index for (1.1). The fact that  $s_\alpha = -3/4$  for  $\alpha \leq 1/2$  could mean that the dissipative part in (1.1), when becoming small enough, has no effect on the low regularity of the equation.*

**Remark 1.3** *It is an interesting problem to consider the periodic dissipative KdV equation*

$$\begin{cases} u_t + u_{xxx} - |D_x|^{2\alpha}u + uu_x = 0, & t \in \mathbb{R}_+, x \in \mathbb{T}, \\ u(0, x) = u_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.8)$$

*Concerning the KdV equation on  $\mathbb{T}$ , global well-posedness is known in  $H^{1/2}(\mathbb{T})$  (see [5]) and the result is optimal (see [3]). For KdV-B, it is established in [11] that the indexes of the critical spaces are the same on the real line and on the circle. We believe that working in the space  $\tilde{X}_\alpha^{1/2,s}$  endowed with the norm*

$$\|u\|_{\tilde{X}_\alpha^{1/2,s}} = \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} \langle i(\tau - n^3) + |n|^{2\alpha} \rangle |\hat{u}(\tau, n)|^2 d\tau \right)^{1/2},$$

*and using Tao's  $[k; Z]$ -multiplier norm estimates [14], one could get well-posedness results for the IVP (1.8) in  $H^s(\mathbb{T})$ ,  $s > \tilde{s}_\alpha$  with  $\tilde{s}_0 = -1/2$  and  $\tilde{s}_1 = -1$ . We do not pursue this issue here.*

The remainder of this paper is organized as follows. In Section 2, we recall some linear estimates on the operators  $W_\alpha$  and  $L_\alpha$ , and we introduce Tao's  $[k; Z]$ -multiplier norm estimates. Section 3 is devoted to the proof of the bilinear estimate (1.5). Theorem 1.3 is established in Section 4. Finally, we show the optimality of (1.5) in Section 5.

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## 2 Preliminaries

### 2.1 Linear estimates

In this subsection, we collect together several linear estimates on the operators  $W_\alpha$  introduced in (1.4) and  $L_\alpha$  defined by

$$L_\alpha : f \mapsto \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W_\alpha(t-t')f(t')dt'.$$

All the results stated here were proved in [11] for  $\alpha = 1$  and in [4] for the general case.

**Lemma 2.1** *For all  $s \in \mathbb{R}$  and all  $\varphi \in H^s(\mathbb{R})$ ,*

$$\|\psi(t)W_\alpha(t)\varphi\|_{X_\alpha^{1/2,s}} \lesssim \|\varphi\|_{H^s}. \quad (2.1)$$

**Lemma 2.2** *Let  $s \in \mathbb{R}$ .*

(a) *For all  $v \in \mathcal{S}(\mathbb{R}^2)$ ,*

$$\begin{aligned} & \left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W_\alpha(t-t')v(t')dt' \right\|_{X_\alpha^{1/2,s}} \\ & \lesssim \|v\|_{X_\alpha^{-1/2,s}} + \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} \left( \int_{\mathbb{R}} \frac{|\widehat{v}(\tau+\xi^3, \xi)|}{|i\tau+\xi|^{2\alpha}} d\tau \right) d\xi \right)^{1/2}. \end{aligned}$$

(b) *For all  $0 < \delta < 1/2$  and all  $v \in X_\alpha^{-1/2+\delta,s}$ ,*

$$\left\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W_\alpha(t-t')v(t')dt' \right\|_{X_\alpha^{1/2,s}} \lesssim \|v\|_{X_\alpha^{-1/2+\delta,s}}. \quad (2.2)$$

To globalize our solution, we will need the next lemma.

**Lemma 2.3** *Let  $s \in \mathbb{R}$  and  $\delta > 0$ . Then for any  $f \in X_\alpha^{-1/2+\delta,s}$ ,*

$$t \mapsto \int_0^t W_\alpha(t-t')f(t')dt' \in \mathcal{C}(\mathbb{R}_+, H^{s+2\alpha\delta}).$$

Moreover, if  $(f_n)$  is a sequence satisfying  $f_n \rightarrow 0$  in  $X_\alpha^{-1/2+\delta,s}$ , then

$$\left\| \int_0^t W_\alpha(t-t')f_n(t')dt' \right\|_{L^\infty(\mathbb{R}_+, H^{s+2\alpha\delta})} \longrightarrow 0.$$

Finally, we recall the following  $L^4$  Strichartz's type estimate showed in [8, 10].

**Lemma 2.4** *Let  $f \in L^2(\mathbb{R}^2)$  with compact support (in time) in  $[-T, +T]$ . For  $0 \leq \theta \leq 1/8$  and  $\rho > 3/8$ , there exists  $\nu > 0$  such that*

$$\left\| \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^\theta \widehat{f}(\tau, \xi)}{\langle \tau - \xi^3 \rangle^\rho} \right) \right\|_{L_{xt}^4} \lesssim T^\nu \|f\|_{L_{xt}^2}.$$

## 2.2 Tao's $[k; Z]$ -multipliers

Now we turn to Tao's  $[k; Z]$ -multiplier norm estimates. For more details, please refer to [14].

Let  $Z$  be any abelian additive group with an invariant measure  $d\xi$ . For any integer  $k \geq 2$  we define the hyperplane

$$\Gamma_k(Z) = \{(\xi_1, \dots, \xi_k) \in Z^k : \xi_1 + \dots + \xi_k = 0\}$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\xi_1, \dots, \xi_{k-1}, -(\xi_1 + \dots + \xi_{k-1})) d\xi_1 \dots d\xi_{k-1}.$$

A  $[k; Z]$ -multiplier is defined to be any function  $m : \Gamma_k(Z) \rightarrow \mathbb{C}$ . The multiplier norm  $\|m\|_{[k; Z]}$  is defined to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right| \leq \|m\|_{[k; Z]} \prod_{j=1}^k \|f_j\|_{L^2(Z)} \quad (2.3)$$

holds for all test functions  $f_1, \dots, f_k$  on  $Z$ . In other words,

$$\|m\|_{[k; Z]} = \sup_{\substack{f_j \in \mathcal{S}(Z) \\ \|f_j\|_{L^2(Z)} \leq 1}} \left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^k f_j(\xi_j) \right|.$$

In his paper [14], Tao used the following notations. Capitalized variables  $N_j$ ,  $L_j$  ( $j = 1, \dots, k$ ) are presumed to be dyadic, i.e. range over numbers of the form  $2^\ell$ ,  $\ell \in \mathbb{Z}$ . In this paper, we only consider the case  $k = 3$ , which corresponds to the quadratic nonlinearity in the equation. It will be convenient to define the quantities  $N_{max} \geq N_{med} \geq N_{min}$  to be the maximum, median and minimum of  $N_1, N_2, N_3$  respectively. Similarly, define

$L_{max} \geq L_{med} \geq L_{min}$  whenever  $L_1, L_2, L_3 > 0$ . The quantities  $N_j$  will measure the magnitude of frequencies of our waves, while  $L_j$  measures how closely our waves approximate a free solution.

Here we consider  $[3, \mathbb{R} \times \mathbb{R}]$ -multipliers and we parameterize  $\mathbb{R} \times \mathbb{R}$  by  $(\tau, \xi)$  endowed with the Lebesgue measure  $d\tau d\xi$ . If  $\tau, \tau_1, \xi, \xi_1$  are given, we set

$$\sigma = \sigma(\tau, \xi) = \tau - \xi^3, \quad \sigma_1 = \sigma(\tau_1, \xi_1), \quad \sigma_2 = \sigma(\tau - \tau_1, \xi - \xi_1). \quad (2.4)$$

From the identity  $\sigma_1 + \sigma_2 - \sigma = 3\xi\xi_1(\xi - \xi_1)$  one can deduce the well-known smoothing relation

$$\max(|\sigma|, |\sigma_1|, |\sigma_2|) \geq |\xi\xi_1(\xi - \xi_1)| \quad (2.5)$$

which will be extensively used in Section 3.

By a dyadic decomposition of the variables  $\xi_1, \xi_2 = -\xi, \xi_3 = \xi - \xi_1$ , and  $\sigma_1, \sigma_2, \sigma_3 = -\sigma$ , we are lead to consider

$$\left\| \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\sigma_j| \sim L_j} \right\|_{[3; \mathbb{R} \times \mathbb{R}]} \quad (2.6)$$

We can now state the fundamental dyadic estimates for the KdV equation on the real line ([14], Proposition 6.1).

**Lemma 2.5** *Let  $N_1, N_2, N_3, L_1, L_2, L_3$  satisfying*

$$N_{max} \sim N_{med},$$

$$L_{max} \sim \max(N_1 N_2 N_3, L_{med}).$$

1. *((++) Coherence) If  $N_{max} \sim N_{min}$  and  $L_{max} \sim N_1 N_2 N_3$  then we have*

$$(2.6) \lesssim L_{min}^{1/2} N_{max}^{-1/4} L_{med}^{1/4}. \quad (2.7)$$

2. *((+-) Coherence) If  $N_2 \sim N_3 \gg N_1$  and  $N_1 N_2 N_3 \sim L_1 \gtrsim L_2, L_3$  then*

$$(2.6) \lesssim L_{min}^{1/2} N_{max}^{-1} \min(N_1 N_2 N_3, \frac{N_{max}}{N_{min}} L_{med})^{1/2}. \quad (2.8)$$

*Similarly for permutations.*

3. *In all other cases, we have*

$$(2.6) \lesssim L_{min}^{1/2} N_{max}^{-1} \min(N_1 N_2 N_3, L_{med})^{1/2}. \quad (2.9)$$

Because only one region needs to be controlled by using a dyadic approach, we just require the (+-) coherence case. On the other hand, these estimates are sharp. In particular, testing (2.3) with

$$\begin{aligned} f_1(\tau, \xi) &= \chi_{|\xi| \sim N_1; |\tau - 3N_2^2 \xi| \lesssim N_1^2 N_2}, \\ f_2(\tau, \xi) &= \chi_{|\xi - N_2| \lesssim N_1; |\sigma| \lesssim L_2}, \\ f_3(\tau, \xi) &= \chi_{|\xi + N_2| \lesssim N_1; |\sigma| \lesssim L_3}, \end{aligned}$$

one obtain the optimality of bound (2.8) in the case  $N_2 \sim N_3 \gtrsim N_1$  and  $N_1 N_2 N_3 \sim L_1 \gtrsim L_2 \gtrsim L_3$ . This will be crucial in the proof of Theorem 1.2.

### 3 Bilinear estimate

In this section, we derive the bilinear estimate (1.5). To get the required contraction factor  $T^\nu$  in our estimates, the next lemma is very useful (see [7]).

**Lemma 3.1** *Let  $f \in L^2(\mathbb{R}^2)$  with compact support (in time) in  $[-T, +T]$ . For any  $\theta > 0$ , there exists  $\nu = \nu(\theta) > 0$  such that*

$$\left\| \mathcal{F}^{-1} \left( \frac{\widehat{f}(\tau, \xi)}{\langle \tau - \xi^3 \rangle^\theta} \right) \right\|_{L_{xt}^2} \lesssim T^\nu \|f\|_{L_{xt}^2}.$$

We will also need the following elementary calculus inequalities.

**Lemma 3.2** (a) *For  $b, b' \in ]\frac{1}{4}, \frac{1}{2}[$  and  $\alpha, \beta \in \mathbb{R}$ ,*

$$\int_{\mathbb{R}} \frac{dx}{\langle x - \alpha \rangle^{2b} \langle x - \beta \rangle^{2b'}} \lesssim \frac{1}{\langle \alpha - \beta \rangle^{2b+2b'-1}}. \quad (3.1)$$

(b) *For  $b, b' \in ]\frac{1}{4}, \frac{1}{2}[$  and  $\alpha, \beta \in \mathbb{R}$ ,*

$$\int_{|x| \leq |\beta|} \frac{dx}{\langle x \rangle^{2b+2b'-1} \sqrt{|\alpha - x|}} \lesssim \frac{\langle \beta \rangle^{2(1-b-b')}}{\langle \alpha \rangle^{1/2}}. \quad (3.2)$$

**Proof of Theorem 1.1 :** By duality, (1.5) is equivalent to

$$\left| \int_{\mathbb{R}^2} \partial_x(uv)w \right| \lesssim T^\nu \|w\|_{X_\alpha^{1/2-\delta, -s}} \|u\|_{X_\alpha^{1/2, s}} \|v\|_{X_\alpha^{1/2, s}}$$

for all  $w \in X_\alpha^{1/2,s}$ , and setting

$$\begin{aligned}\widehat{f}(\tau, \xi) &= \langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^{1/2} \langle \xi \rangle^s \widehat{u}(\tau, \xi), \\ \widehat{g}(\tau, \xi) &= \langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^{1/2} \langle \xi \rangle^s \widehat{v}(\tau, \xi), \\ \widehat{h}(\tau, \xi) &= \langle i(\tau - \xi^3) + |\xi|^{2\alpha} \rangle^{1/2-\delta} \langle \xi \rangle^{-s} \widehat{w}(\tau, \xi),\end{aligned}$$

it is equivalent to show that

$$I = \int_{\mathbb{R}^4} K(\tau, \tau_1, \xi, \xi_1) \widehat{h}(\tau, \xi) \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) d\tau d\tau_1 d\xi d\xi_1 \lesssim T^\nu \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2} \|h\|_{L_{xt}^2}$$

with

$$K = \frac{|\xi| \langle \xi \rangle^s}{\langle i\sigma + |\xi|^{2\alpha} \rangle^{1/2-\delta}} \frac{\langle \xi_1 \rangle^{-s}}{\langle i\sigma_1 + |\xi_1|^{2\alpha} \rangle^{1/2}} \frac{\langle \xi - \xi_1 \rangle^{-s}}{\langle i\sigma_2 + |\xi - \xi_1|^{2\alpha} \rangle^{1/2}}.$$

By Fubini's theorem, we can always assume  $\widehat{f}, \widehat{g}, \widehat{h} \geq 0$ . By symmetry, one can reduce the integration domain of  $I$  to  $\Omega = \{(\tau, \tau_1, \xi, \xi_1) \in \mathbb{R}^4, |\sigma_1| \geq |\sigma_2|\}$ . Split  $\Omega$  into four regions,

$$\begin{aligned}\Omega_1 &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \leq 1, |\xi - \xi_1| \leq 1\}, \\ \Omega_2 &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \leq 1, |\xi - \xi_1| \geq 1\}, \\ \Omega_3 &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \geq 1, |\xi - \xi_1| \leq 1\}, \\ \Omega_4 &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega : |\xi_1| \geq 1, |\xi - \xi_1| \geq 1\}.\end{aligned}$$

#### Estimate in $\Omega_1$

Using Cauchy-Schwarz inequality and Lemma 3.1, we easily obtain

$$\begin{aligned}I_1 &\lesssim \sup_{\tau, \xi} \left[ \frac{|\xi| \langle \xi \rangle^s}{\langle i\sigma + |\xi|^{2\alpha} \rangle^{1/2-\delta/2}} \left( \int_{\widetilde{\Omega}_1} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle i\sigma_1 + |\xi_1|^{2\alpha} \rangle \langle i\sigma_2 + |\xi - \xi_1|^{2\alpha} \rangle} d\tau_1 d\xi_1 \right)^{1/2} \right] \\ &\quad \times T^\nu \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2} \|h\|_{L_{xt}^2}\end{aligned}\tag{3.3}$$

with  $\widetilde{\Omega}_1 = \{(\tau_1, \xi_1) : \exists \tau, \xi \in \mathbb{R}, (\tau, \tau_1, \xi, \xi_1) \in \Omega_1\}$ .

In  $\Omega_1$ , one has  $|\xi| \leq 2$  and thus if  $K_1$  denotes the term between brackets in (3.3),

$$K_1 \lesssim \left( \int_{\widetilde{\Omega}_1} \frac{d\tau_1 d\xi_1}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} \right)^{1/2} \lesssim \left( \int_{|\xi_1| \leq 1} \left( \int_{\mathbb{R}} \frac{d\tau_1}{\langle \sigma_2 \rangle^2} \right) d\xi_1 \right)^{1/2} \lesssim 1.$$

#### Estimate in $\Omega_2$

We split  $\Omega_2$  into

$$\begin{aligned}\Omega_{21} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_2 : |\sigma| \geq |\sigma_1|\}, \\ \Omega_{22} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_2 : |\sigma_1| \geq |\sigma|\}.\end{aligned}$$

Estimate in  $\Omega_{21}$  : Note that in this region,  $|\xi - \xi_1| \sim \langle \xi - \xi_1 \rangle$  and  $\langle \xi \rangle \gtrsim \langle \xi - \xi_1 \rangle$ . Thus using (3.3) as well as (2.5), it follows that

$$\begin{aligned}
K_{21} &\lesssim \langle \xi \rangle^{1/2+s+\delta/2} \left( \int_{\tilde{\Omega}_{21}} \frac{\langle \xi - \xi_1 \rangle^{-2s-1+\delta}}{|\xi_1|^{1-\delta} \langle \sigma_1 \rangle \langle \sigma_2 \rangle^\varepsilon \langle \xi - \xi_1 \rangle^{2\alpha(1-\varepsilon)}} d\tau_1 d\xi_1 \right)^{1/2} \\
&\lesssim \left( \int_{|\xi_1| \leq 1} \frac{\langle \xi - \xi_1 \rangle^{-2\alpha(1-\varepsilon)+2\delta}}{|\xi_1|^{1-\delta}} \left( \int_{\mathbb{R}} \frac{d\tau_1}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle^\varepsilon} \right) d\xi_1 \right)^{1/2} \\
&\lesssim \left( \int_{|\xi_1| \leq 1} \frac{d\xi_1}{|\xi_1|^{1-\delta}} \right)^{1/2} \\
&\lesssim 1.
\end{aligned}$$

Estimate in  $\Omega_{22}$  : By similar arguments, we estimate

$$\begin{aligned}
K_{22} &\lesssim \frac{|\xi| \langle \xi \rangle^s}{\langle \xi \rangle^{\alpha(1-\delta)}} \left( \int_{\tilde{\Omega}_{22}} \frac{\langle \xi - \xi_1 \rangle^{-2s}}{|\xi \xi_1 (\xi - \xi_1)|^{1-\varepsilon} \langle \sigma_1 \rangle^\varepsilon \langle \sigma_2 \rangle} d\tau_1 d\xi_1 \right)^{1/2} \\
&\lesssim \langle \xi \rangle^{1/2+s-\alpha(1-\delta)+\varepsilon/2} \left( \int_{|\xi_1| \leq 1} \frac{\langle \xi - \xi_1 \rangle^{-2s-1+\varepsilon}}{|\xi_1|^{1-\varepsilon}} d\xi_1 \right)^{1/2} \\
&\lesssim \langle \xi \rangle^{-\alpha(1-\delta)+\varepsilon} \\
&\lesssim 1.
\end{aligned}$$

### Estimate in $\Omega_3$

By symmetry, the desired bound in this region can be obtained in the same way.

### Estimate in $\Omega_4$

Divide  $\Omega_4$  into

$$\begin{aligned}
\Omega_{41} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma_1| \geq |\sigma|\}, \\
\Omega_{42} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_4 : |\sigma| \geq |\sigma_1|\}.
\end{aligned}$$

### Estimate in $\Omega_{41}$

We write  $\Omega_{41} = \Omega_{411} \cup \Omega_{412} \cup \Omega_{413}$  with

$$\begin{aligned}
\Omega_{411} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \leq 100|\xi|\}, \\
\Omega_{412} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \geq 100|\xi|, 3|\xi \xi_1 (\xi - \xi_1)| \leq \frac{1}{2}|\sigma_1|\}, \\
\Omega_{413} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{41} : |\xi_1| \geq 100|\xi|, 3|\xi \xi_1 (\xi - \xi_1)| \geq \frac{1}{2}|\sigma_1|\}.
\end{aligned}$$

Estimate in  $\Omega_{411}$  : We have  $\langle \xi_1 \rangle \lesssim \langle \xi \rangle$  and  $\langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle$  thus with (2.5), we deduce for  $0 < \lambda < 1$ ,

$$\begin{aligned} K_{411} &\lesssim \frac{\langle \xi \rangle^{1/2+s}}{\langle \sigma \rangle^{\lambda/2} \langle \xi \rangle^{\alpha(1-\lambda-2\delta)}} \langle \xi_1 \rangle^{-s-1/2} \frac{\langle \xi - \xi_1 \rangle^{-s-1/2}}{\langle \sigma_2 \rangle^{\lambda/2} \langle \xi - \xi_1 \rangle^{\alpha(1-\lambda)}} \\ &\lesssim \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+}}{\langle \sigma \rangle^{\lambda/2}} \frac{\langle \xi - \xi_1 \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+}}{\langle \sigma_2 \rangle^{\lambda/2}}. \end{aligned}$$

Consequently, using Plancherel's theorem and Hölder inequality,

$$\begin{aligned} I_{411} &\lesssim \int_{\mathbb{R}^2} \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+} \widehat{h}}{\langle \sigma \rangle^{\lambda/2}} \right) \mathcal{F}^{-1}(\widehat{f}) \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+} \widehat{g}}{\langle \sigma \rangle^{\lambda/2}} \right) dt dx \\ &\lesssim \left\| \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+} \widehat{h}}{\langle \sigma \rangle^{\lambda/2}} \right) \right\|_{L_{xt}^4} \|f\|_{L_{xt}^2} \left\| \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda-\delta)]^+} \widehat{g}}{\langle \sigma \rangle^{\lambda/2}} \right) \right\|_{L_{xt}^4}. \end{aligned}$$

Now we choose  $\lambda = 3/4 + \delta$  so that  $\lambda/2 > 3/8$  and for  $s > -3/4 - \alpha/2$  and  $\delta > 0$  small enough,

$$-\frac{s}{2} - \frac{1}{4} - \alpha(1 - \lambda - \delta) = \frac{1}{8} - \frac{1}{2} \left( s + \frac{3}{4} + \frac{\alpha}{2} \right) + 2\alpha\delta \leq \frac{1}{8}.$$

Hence with help of Lemma 2.4,  $I_{411} \lesssim T^\nu \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2} \|h\|_{L_{xt}^2}$ .

Estimate in  $\Omega_{412}$  : Using the same arguments that for (3.3), we show that

$$\begin{aligned} I_{412} &\lesssim \sup_{\tau_1, \xi_1} \left[ \frac{\langle \xi_1 \rangle^{-s}}{\langle i\sigma_1 + |\xi_1|^{2\alpha} \rangle^{1/2}} \left( \int_{\widetilde{\Omega}_{412}} \frac{|\xi|^2 \langle \xi \rangle^{2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle i\sigma + |\xi|^{2\alpha} \rangle^{1-\delta} \langle i\sigma_2 + |\xi - \xi_1|^{2\alpha} \rangle} d\tau d\xi \right)^{1/2} \right] \\ &\quad \times T^\nu \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2} \|h\|_{L_{xt}^2} \end{aligned} \tag{3.4}$$

with  $\widetilde{\Omega}_{412} = \{(\tau, \xi) : \exists \tau_1, \xi_1 \in \mathbb{R}, (\tau, \tau_1, \xi, \xi_1) \in \Omega_{422}\}$ . Moreover, we easily check that in  $\Omega_{412}$ ,

$$|\sigma_1 + 3\xi\xi_1(\xi - \xi_1)| \geq \frac{1}{2}|\sigma_1|$$

and

$$|\xi| \leq |\sigma_1|,$$

which combined with (3.1) and smoothing relation (2.5) yield

$$\begin{aligned}
K_{422} &\lesssim \frac{1}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{412}} \frac{\langle \xi \xi_1 (\xi - \xi_1) \rangle^{-2s} \langle \xi \rangle^{2+4s}}{\langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle^{1-\delta}} d\tau d\xi \right)^{1/2} \\
&\lesssim \langle \sigma_1 \rangle^{-s-1/2} \left( \int_{\Omega'_{412}} \frac{\langle \xi \rangle^{2+4s}}{\langle \sigma_1 + 3\xi \xi_1 (\xi - \xi_1) \rangle^{1-2\delta}} d\xi \right)^{1/2} \\
&\lesssim \langle \sigma_1 \rangle^{-s-1+\delta} \left( \int_{|\xi| \leq |\sigma_1|} \frac{d\xi}{\langle \xi \rangle^{-4s-2}} \right)^{1/2},
\end{aligned}$$

(we have set  $\Omega'_{412} = \{\xi : \exists \tau, \tau_1, \xi_1 \in \mathbb{R}, (\tau, \tau_1, \xi, \xi_1) \in \Omega_{412}\}$ ).  
Now from the assumptions (1.7) on  $s$  we see that

$$K_{422} \lesssim \langle \sigma_1 \rangle^{-s-1+\delta} \langle \sigma_1 \rangle^{2s+3/2} \lesssim 1$$

if  $\alpha \leq 1/2$  and

$$K_{422} \lesssim \langle \sigma_1 \rangle^{-s-1+\delta} \lesssim 1$$

otherwise.

Estimate in  $\Omega_{413}$  : In this domain,  $|\xi_1| \sim |\xi - \xi_1|$  thus using (3.4) it follows that

$$\begin{aligned}
K_{413} &\lesssim \frac{\langle \xi_1 \rangle^{-s-1/2+\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{\tilde{\Omega}_{413}} \frac{\langle \xi \rangle^{2s+1+2\delta} \langle \xi - \xi_1 \rangle^{-2s-1+2\delta}}{\langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle^{1-\delta}} d\tau d\xi \right)^{1/2} \\
&\lesssim \frac{\langle \xi_1 \rangle^{-2s-1+2\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{\Omega'_{413}} \frac{d\xi}{\langle \sigma_1 + 3\xi \xi_1 (\xi - \xi_1) \rangle^{1-2\delta}} \right)^{1/2}.
\end{aligned}$$

Following the works of Kenig, Ponce and Vega [9], we perform the change of variables  $\mu_1 = \sigma_1 + 3\xi \xi_1 (\xi - \xi_1)$ . Thus, since  $d\xi \sim \frac{d\mu_1}{|\xi_1|^{1/2} \sqrt{|4\tau_1 - \xi_1^3 - 4\mu_1|}}$  and in view of (3.2), we bound  $K_{413}$  by

$$\begin{aligned}
K_{413} &\lesssim \frac{\langle \xi_1 \rangle^{-2s-5/4+2\delta}}{\langle \sigma_1 \rangle^\delta} \left( \int_{|\mu_1| \leq 2|\sigma_1|} \frac{d\mu_1}{\langle \mu_1 \rangle^{1-2\delta} \sqrt{|4\tau_1 - \xi_1^3 - 4\mu_1|}} \right)^{1/2} \\
&\lesssim \frac{\langle \xi_1 \rangle^{-2s-5/4+2\delta}}{\langle \sigma_1 \rangle^\delta} \frac{\langle \sigma_1 \rangle^\delta}{\langle 4\tau_1 - \xi_1^3 \rangle^{1/4}} \\
&\lesssim \langle \xi_1 \rangle^{-2s-5/4+2\delta} \langle 4\tau_1 - \xi_1^3 \rangle^{-1/4}.
\end{aligned}$$

Note that in  $\Omega_{413}$ , we have  $|\sigma_1| \leq \frac{12}{100} |\xi_1|^3$ , which leads to

$$3|\xi_1|^3 \leq |4\sigma_1 + 3\xi_1^3| + 4|\sigma_1| \leq |4\tau_1 - \xi_1^3| + \frac{48}{100} |\xi_1|^3$$

and thus  $|\xi_1|^3 \lesssim |4\tau_1 - \xi_1^3|$ . One deduce that for  $-2s - 5/4 + 2\delta > 0$ ,

$$K_{413} \lesssim \langle 4\tau_1 - \xi_1^3 \rangle^{\frac{1}{3}(-2s-5/4+2\delta)-1/4} \lesssim \langle 4\tau_1 - \xi_1^3 \rangle^{-\frac{2}{3}(s+1)+2\delta/3} \lesssim 1.$$

**Estimate in  $\Omega_{42}$**

We split this region in two components :

$$\begin{aligned} \Omega_{421} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \leq 100|\xi|\}, \\ \Omega_{422} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{42} : |\xi_1| \geq 100|\xi|\}. \end{aligned}$$

Estimate in  $\Omega_{421}$  : In  $\Omega_{421}$ ,  $\langle \xi_1 \rangle \lesssim \langle \xi \rangle$  and  $\langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle$ . Then, we bound  $I_{421}$  exactly in the same way that for  $I_{411}$ . After Plancherel and Hölder, we are lead to the estimate

$$\begin{aligned} I_{421} &\lesssim \|h\|_{L_{xt}^2} \left\| \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda)+3\delta/2]^+} \widehat{f}}{\langle \sigma \rangle^{\lambda/2}} \right) \right\|_{L_{xt}^4} \\ &\quad \times \left\| \mathcal{F}^{-1} \left( \frac{\langle \xi \rangle^{[-s/2-1/4-\alpha(1-\lambda)+3\delta/2]^+} \widehat{g}}{\langle \sigma \rangle^{\lambda/2}} \right) \right\|_{L_{xt}^4}. \end{aligned}$$

It suffices to choose  $\lambda = 3/4 + \varepsilon$  to apply Lemma 2.4 with  $s > -3/4 - \alpha/2$ .

Estimate in  $\Omega_{422}$  : We split  $\Omega_{422}$  into three sub-domains

$$\begin{aligned} \Omega_{4221} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : 3|\xi\xi_1(\xi - \xi_1)| \leq \frac{1}{2}|\sigma|\}, \\ \Omega_{4222} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : 3|\xi\xi_1(\xi - \xi_1)| \geq \frac{1}{2}|\sigma|, |\sigma_2| \leq 1\}, \\ \Omega_{4223} &= \{(\tau, \tau_1, \xi, \xi_1) \in \Omega_{422} : 3|\xi\xi_1(\xi - \xi_1)| \geq \frac{1}{2}|\sigma|, |\sigma_2| \geq 1\}. \end{aligned}$$

Estimate in  $\Omega_{4221}$  : In this region one has that

$$|\sigma + 3\xi\xi_1(\xi - \xi_1)| \geq \frac{1}{2}|\sigma|$$

and since  $\sigma_1 + \sigma_2 - \sigma = 3\xi\xi_1(\xi - \xi_1)$  it follows that

$$|\sigma_1| \leq |\sigma| \leq 2|\sigma_1 + \sigma_2| \leq 4|\sigma_1|$$

and  $|\sigma_1| \sim |\sigma| \geq |\xi \xi_1 (\xi - \xi_1)|$ . Therefore using (3.4), one obtain

$$\begin{aligned}
K_{4221} &\lesssim \frac{\langle \xi_1 \rangle^{-s}}{\langle \sigma_1 \rangle^{1/2}} \left( \int_{\tilde{\Omega}_{4221}} \frac{|\xi|^2 \langle \xi \rangle^{2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle} d\tau d\xi \right)^{1/2} \\
&\lesssim \frac{1}{\langle \sigma_1 \rangle^{1-\delta}} \left( \int_{\tilde{\Omega}_{4221}} \frac{\langle \xi \xi_1 (\xi - \xi_1) \rangle^{-2s} \langle \xi \rangle^{2+4s}}{\langle \sigma_2 \rangle^{1+\delta}} d\tau d\xi \right)^{1/2} \\
&\lesssim \langle \sigma_1 \rangle^{-s-1+\delta} \left( \int_{|\xi| \lesssim |\sigma_1|} \frac{d\xi}{\langle \xi \rangle^{-4s-2}} \right)^{1/2} \\
&\lesssim 1
\end{aligned}$$

as for  $K_{422}$ .

Estimate in  $\Omega_{4222}$  : First consider the case  $\alpha \leq 1/2$ . Then,

$$\begin{aligned}
K_{4222} &\lesssim \frac{|\xi| \langle \xi \rangle^s}{\langle \sigma \rangle^{1/2-\delta/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} d\tau_1 d\xi_1 \right)^{1/2} \\
&\lesssim \frac{|\xi|^{1+s} \langle \xi \rangle^s}{\langle \sigma \rangle^{1/2-\delta/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{\langle \xi \xi_1 (\xi - \xi_1) \rangle^{-2s}}{\langle \sigma_1 \rangle^{1-\delta} \langle \sigma_2 \rangle^{1-\delta}} d\tau_1 d\xi_1 \right)^{1/2} \\
&\lesssim |\xi|^{1+s} \langle \xi \rangle^s \langle \sigma \rangle^{-s-1/2+\delta/2} \left( \int_{\Omega'_{4222}} \frac{d\xi_1}{\langle \sigma + 3\xi \xi_1 (\xi - \xi_1) \rangle^{1-2\delta}} \right)^{1/2}.
\end{aligned}$$

The change of variables  $\mu = \sigma + 3\xi \xi_1 (\xi - \xi_1)$  gives the inequalities

$$\begin{aligned}
K_{4222} &\lesssim |\xi|^{3/4+s} \langle \xi \rangle^s \langle \sigma \rangle^{-s-1/2+\delta/2} \left( \int_{|\mu| \leq 2|\sigma|} \frac{d\mu}{\langle \mu \rangle^{1-2\delta} \sqrt{|4\tau - \xi^3 - 4\mu|}} \right)^{1/2} \\
&\lesssim |\xi|^{3/4+s} \langle \xi \rangle^s \langle \sigma \rangle^{-s-1/2+3\delta/2} \langle 4\tau - \xi^3 \rangle^{-1/4},
\end{aligned}$$

which is bounded on  $\mathbb{R}^2$ . If  $\alpha > 1/2$ , we have directly

$$\begin{aligned}
K_{4222} &\lesssim \frac{|\xi| \langle \xi \rangle^s}{\langle \sigma \rangle^{1/2-\delta/2}} \left( \int_{\tilde{\Omega}_{4222}} \frac{\langle \xi_1 \rangle^{-4s}}{\langle \xi_1 \rangle^{2\alpha} \langle \xi - \xi_1 \rangle^{2\alpha} \langle \sigma_2 \rangle^{1+\varepsilon}} d\tau_1 d\xi_1 \right)^{1/2} \\
&\lesssim \langle \xi \rangle^{1/2+s+\delta/2} \left( \int_{\mathbb{R}} \frac{d\xi_1}{\langle \xi_1 \rangle^{4s+2+4\alpha+2\delta}} \right)^{1/2} \\
&\lesssim 1
\end{aligned}$$

for  $s > -1/4 - \alpha$ .

Estimate in  $\Omega_{4223}$  : As we will see below, that is in this sub-domain that the condition  $s > s_\alpha$  appears. Also, to obtain our estimates, we will need tu

use a dyadic decomposition of the variables  $\xi_1, \xi - \xi_1, \xi$  and  $\sigma_1, \sigma_2, \sigma$ . Hence, following the notations introduced in Subsection 2.2, we have to bound

$$I_{4223} = \sum_{\substack{N_3 \ll N_1 \sim N_2 \\ N_1 \gtrsim 1}} \sum_{\substack{L_3 \gtrsim L_1 \gtrsim L_2 \\ L_3 \sim N_3 N_1^2}} \frac{N_3 \langle N_3 \rangle^s N_1^{-s} N_2^{-s}}{\max(L_3, N_3^{2\alpha})^{1/2-2\delta} \max(L_1, N_1^{2\alpha})^{1/2} \max(L_2, N_2^{2\alpha})^{1/2}} \\ \times \int_{\mathbb{R}^4} \frac{\widehat{h}(\tau, \xi)}{\langle \sigma \rangle^\delta} \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) \chi_{|\xi| \sim N_3, |\xi_1| \sim N_1, |\xi - \xi_1| \sim N_2} \chi_{|\sigma| \sim L_3, |\sigma_1| \sim L_1, |\sigma_2| \sim L_2} d\tau d\tau_1 d\xi d\xi_1.$$

Using the (+-) coherence case of Lemma 2.5 as well as Lemma 3.1, we get

$$\int_{\mathbb{R}^4} \frac{\widehat{h}(\tau, \xi)}{\langle \sigma \rangle^\delta} \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) \chi_{|\xi| \sim N_3, |\xi_1| \sim N_1, |\xi - \xi_1| \sim N_2} \chi_{|\sigma| \sim L_3, |\sigma_1| \sim L_1, |\sigma_2| \sim L_2} d\tau d\tau_1 d\xi d\xi_1 \\ \lesssim L_2^{1/2} N_1^{-1} \min(N_3 N_1^2, \frac{N_1}{N_3} L_1)^{1/2} \left\| \mathcal{F}^{-1} \left( \frac{\widehat{h}(\tau, \xi)}{\langle \sigma \rangle^\delta} \right) \right\|_{L_{xt}^2} \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2} \\ \lesssim T^\nu L_2^{1/2} N_1^{-1} \min(N_3 N_1^2, \frac{N_1}{N_3} L_1)^{1/2} \|h\|_{L_{xt}^2} \|f\|_{L_{xt}^2} \|g\|_{L_{xt}^2}.$$

Thus we reduce to show

$$\sum_{\substack{N_3 \ll N_1 \sim N_2 \\ N_1 \gtrsim 1}} \sum_{\substack{L_3 \gtrsim L_1 \gtrsim L_2 \\ L_3 \sim N_3 N_1^2}} \frac{N_3 \langle N_3 \rangle^s N_1^{-s} N_2^{-s}}{\max(L_3, N_3^{2\alpha})^{1/2-2\delta} \max(L_1, N_1^{2\alpha})^{1/2} \max(L_2, N_2^{2\alpha})^{1/2}} \\ \times L_2^{1/2} N_1^{-1} \min(N_3 N_1^2, \frac{N_1}{N_3} L_1)^{1/2} \lesssim 1. \quad (3.5)$$

Recalling that  $N_j = 2^{n_j}$  and  $L_j = 2^{\ell_j}$ ,  $j = 1, 2, 3$ , for all  $\lambda \in [0, 1]$ , the right hand side of (3.5) is bounded by

$$\lesssim \sum_{\substack{N_3 \ll N_1 \sim N_2 \\ N_1 \gtrsim 1}} \sum_{\substack{L_3 \gtrsim L_1 \gtrsim L_2 \\ L_3 \sim N_3 N_1^2}} \frac{N_3 \langle N_3 \rangle^s N_1^{-2s-1} (N_3 N_1^2)^{\lambda/2} (\frac{N_1}{N_3} L_1)^{(1-\lambda)/2}}{L_3^\delta (N_3 N_1^2)^{1/2-3\delta} L_1^{(1-\lambda)/2} N_1^{\alpha\lambda}} \\ \lesssim \sum_{\substack{N_3 \ll N_1 \sim N_2 \\ N_1 \gtrsim 1}} \left( \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{1}{(L_1 L_2 L_3)^{\delta/3}} \right) N_3^{\lambda+3\delta} \langle N_3 \rangle^s N_1^{-2s-3/2+\lambda(1/2-\alpha)+6\delta} \\ \lesssim \sum_{\substack{N_1 \gtrsim 1 \\ N_3 \ll N_1}} N_3^{\lambda+3\delta} \langle N_3 \rangle^s N_1^{-2s-3/2+\lambda(1/2-\alpha)+6\delta}.$$

This last expression is finite if  $\lambda + 3\delta < -s$  and  $-2s - 3/2 + \lambda(1/2 - \alpha) + 6\delta < 0$ . When  $\alpha \leq 1/2$ , it suffices to choose  $\lambda = 0$  and  $\lambda = -s - 4\delta$  otherwise.

## 4 Proof of the main result

In this section, we briefly indicate how the results stated in Section 2.1 and the bilinear estimate (1.5) yield Theorem 1.3 (see for instance [11] for the details).

Actually, local existence of a solution is a consequence of the following modified version of Theorem 1.1.

**Proposition 4.1** *Given  $s_c^+ > s_\alpha$ , there exist  $\nu, \delta > 0$  such that for any  $s \geq s_c^+$  and any  $u, v \in X_\alpha^{1/2, s}$  with compact support in  $[-T, +T]$ ,*

$$\|\partial_x(uv)\|_{X_\alpha^{-1/2+\delta, s}} \lesssim T^\nu (\|u\|_{X_\alpha^{1/2, s_c^+}} \|v\|_{X_\alpha^{1/2, s}} + \|u\|_{X_\alpha^{1/2, s}} \|v\|_{X_\alpha^{1/2, s_c^+}}). \quad (4.1)$$

Estimate (4.1) is obtained thanks to (1.5) and the triangle inequality

$$\forall s \geq s_c^+, \langle \xi \rangle^s \leq \langle \xi \rangle^{s_c^+} \langle \xi_1 \rangle^{s-s_c^+} + \langle \xi \rangle^{s_c^+} \langle \xi - \xi_1 \rangle^{s-s_c^+}.$$

Let  $u_0 \in H^s(\mathbb{R})$  with  $s$  obeying (1.7). Define  $F(u)$  as

$$F(u) = F_{u_0}(u) = \psi(t) \left[ W_\alpha(t) u_0 - \frac{\chi_{\mathbb{R}_+}(t)}{2} \int_0^t W_\alpha(t-t') \partial_x(\psi_T^2(t') u^2(t')) dt' \right].$$

We shall prove that for  $T \ll 1$ ,  $F$  is contraction in a ball of the Banach space

$$Z = \{u \in X_\alpha^{1/2, s}, \|u\|_Z = \|u\|_{X_\alpha^{1/2, s_c^+}} + \gamma \|u\|_{X_\alpha^{1/2, s}} < +\infty\},$$

where  $\gamma$  is defined for all nontrivial  $\varphi$  by

$$\gamma = \frac{\|\varphi\|_{H^{s_c^+}}}{\|\varphi\|_{H^s}}.$$

Combining (2.1), (2.2) as well as (4.1), it is easy to derive that

$$\|F(u)\|_Z \leq C(\|u_0\|_{H^{s_c^+}} + \gamma \|u_0\|_{H^s}) + CT^\nu \|u\|_Z^2$$

and

$$\|F(u) - F(v)\|_Z \leq CT^\nu \|u - v\|_Z \|u + v\|_Z$$

for some  $C, \nu > 0$ . Thus, taking  $T = T(\|u_0\|_{H^{s_c^+}})$  small enough, we deduce that  $F$  is contractive on the ball of radius  $4C\|u_0\|_{H^{s_c^+}}$  in  $Z$ . This proves the existence of a solution  $u$  to  $u = F(u)$  in  $X_{\alpha, T}^{1/2, s}$ .

Following similar arguments of [11], it is not too difficult to see that if  $u_1, u_2 \in X_{\alpha, T}^{1/2, s}$  are solutions of (1.6) and  $0 < \delta < T/2$ , then there exists  $\nu > 0$  such that

$$\|u_1 - u_2\|_{X_{\alpha, \delta}^{1/2, s}} \lesssim T^\nu (\|u_1\|_{X_{\alpha, T}^{1/2, s}} + \|u_2\|_{X_{\alpha, T}^{1/2, s}}) \|u_1 - u_2\|_{X_{\alpha, \delta}^{1/2, s}},$$

which leads to  $u_1 \equiv u_2$  on  $[0, \delta]$ , and then on  $[0, T]$  by iteration. This proves the uniqueness of the solution.

It is straightforward to check that  $W_\alpha(\cdot)u_0 \in \mathcal{C}(\mathbb{R}_+, H^s(\mathbb{R})) \cap \mathcal{C}(\mathbb{R}_+^*, H^\infty(\mathbb{R}))$ . Then it follows from Theorem 1.1, Lemma 2.3 and the local existence of the solution that

$$u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap \mathcal{C}((0, T], H^\infty(\mathbb{R}))$$

for some  $T = T(\|u_0\|_{H^{s_c^+}})$ . By induction, we have  $u \in \mathcal{C}((0, T], H^\infty(\mathbb{R}))$ . Taking the  $L^2$ -scalar product of (1.1) with  $u$ , we obtain that  $t \mapsto \|u(t)\|_{H^{s_c^+}}$  is nonincreasing on  $(0, T]$ . Since the existence time of the solution depends only on the norm  $\|u_0\|_{H^{s_c^+}}$ , this implies that the solution can be extended globally in time.

## 5 Proof of Theorem 1.2

Let  $s \geq s_\alpha$ . To prove Theorem 1.2, it suffices to show that the multiplier

$$\mathcal{M} = \sup_{\substack{f, g, h \in \mathcal{S}(\mathbb{R}^2) \\ \|f\|_{L_{xt}^2}, \|g\|_{L_{xt}^2}, \|h\|_{L_{xt}^2} \lesssim 1}} \left| \int_{\mathbb{R}^4} K(\tau, \tau_1, \xi, \xi_1) \widehat{h}(\tau, \xi) \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) d\tau d\tau_1 d\xi d\xi_1 \right|$$

is infinite. Setting

$$\mathcal{A} = \{f \in \mathcal{S}(\mathbb{R}^2) : \widehat{f} \geq 0, \|f\|_{L_{xt}^2} \lesssim 1\},$$

and performing as previously a dyadic decomposition of the variables  $\xi_1$ ,  $\xi - \xi_1$ ,  $\xi$  and  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma$ , it follows that

$$\begin{aligned}
\mathcal{M} &\gtrsim \sup_{f,g,h \in \mathcal{A}} \sum_{N_1, N_2, N_3} \sum_{L_1, L_2, L_3} \frac{N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle \max(L_3, N_3^{2\alpha}) \rangle^{1/2-\delta} \langle \max(L_1, N_1^{2\alpha}) \rangle^{1/2} \langle \max(L_2, N_2^{2\alpha}) \rangle^{1/2}} \\
&\quad \times \int_{\mathbb{R}^4} \widehat{h}(\tau, \xi) \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) \chi_{|\xi| \sim N_3, |\xi_1| \sim N_1, |\xi - \xi_1| \sim N_2} \chi_{|\sigma| \sim L_3, |\sigma_1| \sim L_1, |\sigma_2| \sim L_2} \\
&\gtrsim \sup_{N_1, N_2, N_3} \sup_{L_1, L_2, L_3} \frac{N_3 \langle N_3 \rangle^s \langle N_1 \rangle^{-s} \langle N_2 \rangle^{-s}}{\langle \max(L_3, N_3^{2\alpha}) \rangle^{1/2-\delta} \langle \max(L_1, N_1^{2\alpha}) \rangle^{1/2} \langle \max(L_2, N_2^{2\alpha}) \rangle^{1/2}} \\
&\quad \times \sup_{f,g,h \in \mathcal{A}} \int_{\mathbb{R}^4} \widehat{h}(\tau, \xi) \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) \chi_{|\xi| \sim N_3, |\xi_1| \sim N_1, |\xi - \xi_1| \sim N_2} \chi_{|\sigma| \sim L_3, |\sigma_1| \sim L_1, |\sigma_2| \sim L_2}.
\end{aligned}$$

Now we localize the previous supremum to the critical region

$$\begin{cases} N_3 \ll N_1 \sim N_2 \\ N_3 N_1^2 \sim L_3 \gtrsim L_1 \gtrsim L_2 \gtrsim 1 \end{cases}$$

which corresponds to a sub-domain of  $\Omega_{4223}$ . In this case, the optimality of (2.8) gives

$$\begin{aligned}
\sup_{f,g,h \in \mathcal{A}} \int_{\mathbb{R}^4} \widehat{h}(\tau, \xi) \widehat{f}(\tau_1, \xi_1) \widehat{g}(\tau - \tau_1, \xi - \xi_1) \chi_{|\xi| \sim N_3, |\xi_1| \sim N_1, |\xi - \xi_1| \sim N_2} \chi_{|\sigma| \sim L_3, |\sigma_1| \sim L_1, |\sigma_2| \sim L_2} \\
\gtrsim L_2^{1/2} N_1^{-1} \min \left( N_3 N_1^2, \frac{N_1}{N_3} L_1 \right)^{1/2}.
\end{aligned}$$

Therefore we have the bound

$$\mathcal{M} \gtrsim \sup_{1 \lesssim N_3 \ll N_1 \sim N_2} \sup_{\substack{1 \lesssim L_2 \lesssim L_1 \lesssim L_3 \\ L_3 \sim N_3 N_1^2}} \frac{N_3^{1+s} N_1^{-2s} L_2^{1/2} N_1^{-1} \min \left( N_3 N_1^2, \frac{N_1}{N_3} L_1 \right)^{1/2}}{\max(N_3 N_1^2, N_3^{2\alpha})^{1/2-\delta} \max(L_1, N_1^{2\alpha})^{1/2} \max(L_2, N_2^{2\alpha})^{1/2}}.$$

First consider the case  $0 \leq \alpha \leq 1/2$ . Then,

$$\begin{aligned}
\mathcal{M} &\gtrsim \sup_{\substack{N_1 \gg 1 \\ N_3 \sim 1}} \sup_{L_1 \sim L_2 \sim N_3^2 N_1} \frac{N_1^{-2s-1} L_2^{1/2} \left( \frac{N_1}{N_3} L_1 \right)^{1/2}}{N_1^{1-2\delta} L_1^{1/2} L_2^{1/2}} \\
&\gtrsim \sup_{N_1 \gg 1} N_1^{-2s-3/2+2\delta} \\
&= +\infty
\end{aligned}$$

for  $s \leq -3/4$ . Now if  $1/2 < \alpha \leq 1$ , we estimate

$$\begin{aligned} \mathcal{M} &\gtrsim \sup_{\substack{N_1 \gg 1 \\ N_3 \sim N_1^{\alpha-1/2}}} \sup_{L_1 \sim L_2 \sim N_1^{2\alpha}} \frac{N_3^{1+s} N_1^{-2s-1} L_2^{1/2} N_1^{3/4+\alpha/2}}{(N_3 N_1^2)^{1/2-\delta} L_2^{1/2} N_1^\alpha} \\ &\gtrsim \sup_{N_1 \gg 1} N_1^{-2s-5/4-\alpha/2+(\alpha-1/2)(s+1/2)+(\alpha+3/2)\delta} \\ &= +\infty \end{aligned}$$

for  $-2s - 5/4 - \alpha/2 + (\alpha - 1/2)(s + 1/2) \geq 0$ , i.e.  $s \leq \frac{-3}{5 - 2\alpha}$ .

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