HAL
open science

On the derived category of 1-motives, I

Luca Barbieri-Viale, Bruno Kahn

To cite this version:

Luca Barbieri-Viale, Bruno Kahn. On the derived category of 1-motives, I. 2007. hal-00153780

HAL Id: hal-00153780
https://hal.science/hal-00153780

Preprint submitted on 11 Jun 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

hal-00153780, version 1 - 11 Jun 2007

ON THE DERIVED CATEGORY OF 1-MOTIVES, I

LUCA BARBIERI-VIALE AND BRUNO KAHN

Abstract. We consider the category of Deligne 1-motives over a perfect field k of exponential characteristic p and its derived category for a suitable exact structure after inverting p. As a first result, we provide a fully faithful embedding into an étale version of Voevodsky's triangulated category of geometric motives. Our second main result is that this full embedding "almost" has a left adjoint, that we call LAlb. Applied to the motive of a variety we thus get a bounded complex of 1-motives, that we compute fully for smooth varieties and partly for singular varieties. As an application we give motivic proofs of Roĭtman type theorems (in characteristic 0).

Contents

Introduction 2

Part 1. The universal realisation functor 9
1. The derived category of 1-motives 9
2. Universal realisation 23
3. 1-motivic sheaves and the homotopy t-structure 28
4. Comparing two dualities 43

Part 2. The functors LAlb **and** RPic 53
5. A left adjoint to the universal realisation functor 53
6. LAlb and RPic with rational coefficients 56
7. A tensor structure and an internal Hom on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$ 58

Part 3. Some computations 62
8. The Albanese complexes and their basic properties 62
9. Borel-Moore variants 66
10. Computing $\mathrm{LAlb}(X)$ and $\mathrm{RPic}(X)$ for smooth X 69
11. 1-motivic homology and cohomology of singular schemes 74
12. 1-motivic homology and cohomology of curves 79
13. Comparison with $\mathrm{Pic}^+, \mathrm{Pic}^-, \mathrm{Alb}^+$ and Alb^- 83

Date: June 8, 2007.

14. Generalisations of Roĭtman's theorem 100

Appendices 109
Appendix A. Homological algebra 109
Appendix B. Torsion objects in additive categories 112
Appendix C. 1-motives with torsion 117
Appendix D. Homotopy invariance for étale sheaves with transfers 135
Index 140
References 141

Introduction

While Grothendieck's construction of pure motives associated to smooth projective varieties over a field k is now classical [38, 17, 32, 52], the construction of mixed motives associated to arbitrary k-varieties is still largely work in progress. In this direction, the first concrete step was taken by Deligne in [16] where he defined 1-motives, which should ultimately be *mixed motives of level or dimension* ≤ 1. They form a category that we shall denote by $\mathcal{M}_1(k)$ or $\mathcal{M}_1$.

Deligne's definition was motivated by Hodge theory, and he asked if some Hodge-theoretic constructions could be described as realisations of *a priori* constructed 1-motives. In this direction, the first author and Srinivas associated in [5] homological and cohomological Albanese and Picard 1-motives $\mathrm{Alb}^-(X)$, $\mathrm{Alb}^+(X)$, $\mathrm{Pic}^-(X)$ and $\mathrm{Pic}^+(X)$ to an algebraic scheme X in characteristic zero, providing extensions of the classical Picard and Albanese varieties. This work was pursued in [4], where part of Deligne's conjecture was reformulated and proven rationally (see also [50]).

A different step towards mixed motives was taken by Voevodsky who defined in [63] a *triangulated category* of motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. Taken with rational coefficients, this category is conjectured to have a "motivic" t-structure whose heart should be the searched-for abelian category of mixed motives.

Since $\mathcal{M}_1(k)$ is expected to be contained in such a heart, it is only natural to try and relate Deligne's and Voevodsky's ideas. This is what Voevodsky did in [63, p. 218] (see also [60, Pretheorem 0.0.18]). Denote by $\mathcal{M}_1(k) \otimes \mathbb{Q}$ the abelian category of 1-motives up to isogeny over k. When k is perfect, Voevodsky said that there exists a *fully*

faithful functor

$$D^b(\mathcal{M}_1(k)\otimes\mathbb{Q})\hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)\otimes\mathbb{Q}$$

whose essential image is the thick subcategory $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)\otimes\mathbb{Q}\subseteq \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)\otimes\mathbb{Q}$ generated by motives of smooth curves.

In fact, a 1-motive may be regarded as a length 1 complex of homotopy invariant étale sheaves with transfers, so that it defines an object of Voevodsky's triangulated category $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)$ of étale motivic complexes [63, §3.3] to which $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ maps. This defines a functor

$$\mathcal{M}_1(k)\to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k).$$

F. Orgogozo justified Voevodsky's assertion in [45] by extending a rational version of the above functor to $D^b(\mathcal{M}_1(k)\otimes\mathbb{Q})$.

In this paper, we develop the above results as follows. First of all, let us stress that one has to invert the exponential characteristic p of the base field k throughout. This is due to several reasons:

- Since the category $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)$ is $\mathbb{Z}[1/p]$-linear by [63, Prop. 3.3.3 2)], we cannot expect better comparison results.
- To be in the spirit of Voevodsky, we want to use only the étale topology and not the fppf topology which would be more natural from the viewpoint of 1-motives. Trying to prove anything meaningful without inverting p in this context seems doomed to failure.

Anyway, the basic reason why p is inverted in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)$ is homotopy invariance (the Artin-Schreier exact sequence). But if one wants to deal with non homotopy invariant phenomena, Deligne 1-motives are not sufficient and one should enlarge them to include $\mathbb{G}_a$ factors as in Laumon's 1-motives (*cf.* [6], [2]). See [8] for work in this direction.

0.1. **The derived category of 1-motives, p-integrally.** While the $\mathbb{Z}[1/p]$-linear category $\mathcal{M}_1[1/p]$ is not an abelian category, it fully embeds into the abelian category ${}^t\mathcal{M}_1[1/p]$ of 1-*motives with torsion* introduced in [4] (in characteristic 0), which makes it an exact category in the sense of Quillen (see §1.5). Its derived category $D^b(\mathcal{M}_1[1/p])$ with respect to this exact structure makes sense, and moreover the functor $D^b(\mathcal{M}_1[1/p])\to D^b({}^t\mathcal{M}_1[1/p])$ turns out to be an equivalence (Theorem 1.6.1).

0.2. **p-integral equivalence.** Let $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}=\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k)$ be the thick subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)$ generated by the image of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (see Definition 2.1.1) and $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ the thick subcategory of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ generated by motives of smooth curves. In Theorem 2.1.2, we refine the

Voevodsky-Orgogozo equivalence to an equivalence of categories

$$D^b(\mathcal{M}_1(k)[1/p]) \xrightarrow{\sim} d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}} \tag{0.1}$$

0.3. **Duality.** Deligne's extension of Cartier duality to 1-motives [16] provides the category of 1-motives with a natural involution $M \mapsto M^*$ which extends to $D^b(\mathcal{M}_1(k)[1/p])$: see Proposition 1.8.4. This duality exchanges the category ${}^t\mathcal{M}_1[1/p]$ of §0.1 with an abelian category ${}_t\mathcal{M}_1[1/p]$ of 1-*motives with cotorsion* (see §1.8).

We show in Theorem 4.4.3 that, under Tot, Deligne's Cartier duality is transformed into the involution $M \mapsto \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))$ on $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k)$ given by the internal (effective) Hom. Of course, this result involves biextensions.

0.4. **Left adjoint.** Composing (0.1) with the inclusion into $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k)$, we obtain a "universal realisation functor"

$$\mathrm{Tot} : D^b(\mathcal{M}_1(k)[1/p]) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k).$$

It was conjectured by Voevodsky ([61]; this is also implicit in [60, Preth. 0.0.18]) that, rationally, Tot has a left adjoint. We prove this in Section 6.

It is shown in Remark 5.2.3 that Tot does not have a left adjoint integrally. There is nevertheless an integral statement, which involves an interplay between the étale and the *Nisnevich* topology. Let $\alpha^* : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k)$ be the change of topology functor. We find a functor

$$\mathrm{LAlb} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to D^b(\mathcal{M}_1(k)[1/p])$$

verifying the following universal property: if $(M, N) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \times D^b(\mathcal{M}_1[1/p])$, then there is a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k)}(\alpha^* M, \mathrm{Tot}(N)) \simeq \mathrm{Hom}_{D^b(\mathcal{M}_1[1/p])}(\mathrm{LAlb}(M), N). \tag{0.2}$$

We give its construction in Sect. 5.

The point is that, applying LAlb to various motives, we get interesting and intrinsically-defined 1-motives. For example, applying it to the motive $M(X)$ of a smooth variety X, we get the *homological Albanese complex* $\mathrm{LAlb}(X) := \mathrm{LAlb}(M(X))$ of X. Its homology 1-motives $\mathrm{L}_i\mathrm{Alb}(X) := {}_tH_i(\mathrm{LAlb}(X))$ relative to the t-structure on $D^b(\mathcal{M}_1[1/p])$ with heart ${}_t\mathcal{M}_1[1/p]$ (see §0.3) are 1-motives (with cotorsion) functorially attached to X.

0.5. **Smooth schemes.** We then proceed to compute $\mathrm{LAlb}(X)$ for a smooth scheme X: in principle this determines LAlb on the whole of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, since this category is generated by the $M(X)$. It is related with the "Albanese scheme" $\mathcal{A}_{X/k}$ of [49] (extending the Serre Albanese variety of [54]) in the following way: $\mathrm{LAlb}(X)$ is a "3-extension" of $\mathcal{A}_{X/k}$ by the Cartier dual of the Néron-Severi group of X, that we define as the étale sheaf represented by cycles of codimension 1 on X modulo algebraic equivalence. (See Theorem 10.2.2.) We deduce that $\mathrm{L}_1\mathrm{Alb}(X)$ is isomorphic to the 1-motive $\mathrm{Alb}^-(X)$ of [5].

0.6. LAlb **and** RPic**.** Composing LAlb with duality, we obtain a contravariant functor

$$\mathrm{RPic} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to D^b(\mathcal{M}_1(k)[1/p])$$

such that

$$\mathrm{R}^i\mathrm{Pic}(M) := {}^tH^i(\mathrm{RPic}(M)) \simeq {}_tH_i(\mathrm{LAlb}(M))^*$$

for any $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. Here, ${}^tH^i$ is defined with respect to the t-structure with heart ${}^t\mathcal{M}_1[1/p]$. We call RPic the *motivic Picard* functor. We define the *cohomological Picard complex* by $\mathrm{RPic}(X) := \mathrm{RPic}(M(X))$.

0.7. **Singular schemes.** When k is of characteristic 0, the motive and motive with compact support $M(X)$ and $M^c(X)$ are defined for any variety X as objects of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, so that $\mathrm{LAlb}(X)$ and the *Borel-Moore Albanese complex* $\mathrm{LAlb}^c(X) := \mathrm{LAlb}(M^c(X))$ make sense. Still in characteristic 0 we further define, for an equidimensional scheme X of dimension n, the *cohomological Albanese complex* $\mathrm{LAlb}^*(X) := \mathrm{LAlb}(M(X)^*(n)[2n])$. We define similarly $\mathrm{RPic}^c(X) := \mathrm{RPic}(M^c(X))$ and $\mathrm{RPic}^*(X) := \mathrm{RPic}(M(X)^*(n)[2n])$. We describe some properties of these complexes in Sect. 8.

We then give some general qualitative estimates for $\mathrm{L}_i\mathrm{Alb}(X)$ in Proposition 11.3.2 (see also Proposition 13.6.1) as well as $\mathrm{L}_i\mathrm{Alb}^c(X) := {}_tH_i(\mathrm{LAlb}^c(X))$ in Proposition 11.5.2. Sect. 13 is devoted to a detailed study of $\mathrm{L}_1\mathrm{Alb}(X)$, $\mathrm{L}_1\mathrm{Alb}^c(X)$ and $\mathrm{L}_1\mathrm{Alb}^*(X)$; the main results are summarised in the introduction of this section. In particular, we prove that $\mathrm{L}_1\mathrm{Alb}(X)$ is canonically isomorphic to the 1-motive $\mathrm{Alb}^-(X)$ of [5] if X is normal or proper. Here, the interplay between LAlb and RPic (duality between Picard and Albanese) plays an essential rôle. We also prove that $\mathrm{L}_1\mathrm{Alb}^*(X) \simeq \mathrm{Alb}^+(X)$ for any X (in fact, $\mathrm{R}^1\mathrm{Pic}^*(X) \simeq \mathrm{Pic}^-(X)$ in Theorem 13.12.6).

It is quite striking that $\mathrm{L}_i\mathrm{Alb}(X), \mathrm{L}_i\mathrm{Alb}^c(X)$ and $\mathrm{L}_i\mathrm{Alb}^*(X)$ are actually Deligne 1-motives for $i \leq 1$, but not in general for $i \geq 2$ (already for X smooth projective).

We also completely compute $\mathrm{L}_i\mathrm{Alb}(X)$ for any curve X, showing that $M(X)$ has a "Chow-Künneth decomposition" in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}(k) \otimes \mathbb{Q}$ and that $\mathrm{L}_i\mathrm{Alb}(X)$ coincide with Deligne-Lichtenbaum motivic homology of the curve X (see Theorem 12.2.1, *cf.* [34] and [11]). Finally, we completely compute $\mathrm{L}_i\mathrm{Alb}^c(X)$ of a smooth curve X (see Theorem 12.3.1), showing that $\mathrm{L}_1\mathrm{Alb}^c(X) = H^1_m(X)(1)$ is Deligne's motivic H^1 in [16]. Dually, we recover Deligne's 1-motivic H^1 of any curve. With a little more effort, one should be able to identify our computations with those of Lichtenbaum in [34] and [35].

0.8. **Roĭtman's torsion theorem.** The isomorphism (0.2) comes with a functorial map ("motivic Albanese map")

$$a_M : \alpha^* M \to \mathrm{Tot}\,\mathrm{LAlb}(M) \tag{0.3}$$

for any $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. If X is smooth projective, this canonical map applied to $M = M(X)$ gives back the Albanese map from the 0-th Chow group to the rational points of the Albanese variety. This thus translates very classical mathematics to the motivic setting. When X is only smooth, we recover a generalised Albanese map from Suslin homology

$$a^{sing}_X : H^{sing}_0(X;\mathbb{Z})[1/p] \to \mathcal{A}_{X/k}(k)[1/p]$$

which was first constructed by Ramachandran [48] and Spieß-Szamuely [57].[1] The map a^{sing}_X is an isomorphism if $\dim(X) \leq 1$ (see Proposition 14.1.2).

We then get a very natural proof of the classical theorem of Roĭtman, and even of its generalisation to open smooth varieties by Spieß-Szamuely [57, Th. 1.1] (removing their hypothesis on the existence of a smooth compactification): see Theorem 14.2.5.

We also deal with singular schemes when $\mathrm{char}\, k = 0$, see Proposition 14.3.1 and its corollaries. Here there is an overlap with recent work of Geisser [22]. The works may be compared as follows: Geisser works in arbitrary characteristic and can handle p-torsion in characteristic p, but he works only with proper schemes, while the use of DM forces us to work in characteristic 0 for singular schemes, but we do handle open schemes.

Still in characteristic 0, we get a Borel-Moore version of Roĭtman's theorem as well, see Proposition 14.4.1 and its corollary.

[1] The observation that Suslin homology is related to 1-motives is initially due to Lichtenbaum [34].

Notably, we obtain a "cohomological" Roĭtman theorem, involving torsion in a motivic cohomology group: see Corollary 14.5.4. In a recent work, Mallick [36] proves a parallel cohomological theorem, involving torsion in the Levine-Weibel Chow group (*cf.* [5, Th. 6.4.1]). Mallick works with projective schemes, but in any characteristic. Hopefully, the two theorems are compatible, see Remark 14.5.6.

Moreover, the consideration of Voevodsky's categories provides us with some nonobvious extra structures on $D^b(\mathcal{M}_1[1/p])$:

0.9. **The homotopy t-structure.** It turns out that the homotopy t-structure on $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ and the equivalence of categories (0.1) induce a *third* t-structure on $D^b(\mathcal{M}_1[1/p])$, that we also call the homotopy t-structure (Theorem 3.9.1; see also Corollary 3.9.2). Its heart is formed of so-called 1-*motivic sheaves*: their consideration is very useful for the computation of $\mathrm{LAlb}(X)$ for smooth X.

0.10. **Tensor structure and internal Hom.** Similarly, the functor LAlb turns out to transport the tensor structure on $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \otimes \mathbb{Q}$ to a tensor structure on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. This tensor structure is exact (for the standard t-structure), respects the weight filtration and may be computed explicitly. There is also an exact internal Hom. See Sect. 7.

0.11. **Realisations.** For X smooth over $k = \mathbb{C}$ the complex numbers, one can easily check that the 1-motive $\mathrm{R}^i\mathrm{Pic}(X)$ has a Hodge realisation abstractly isomorphic to $H^i_{(1)}(X_{\mathrm{an}}, \mathbb{Z}(1))$, the largest 1-motivic part of the mixed Hodge structure on $H^i(X_{\mathrm{an}}, \mathbb{Z})$ Tate twisted by 1. We expect that for any scheme X over a field k of characteristic zero the 1-motives $\mathrm{R}^i\mathrm{Pic}(X)$ are isogenous to the 1-motives $M_i(X)$ constructed in [4]. For X normal or proper, this also holds for $i = 1$ by the comparison results indicated in 0.7 and the results of [5].

In the second part of this work, we plan to deal with Hodge and ℓ-adic realisations, thereby providing a canonical version of the isomorphism of the previous paragraph. More generally, this should give a more conceptual (and hopefully more integral) proof of the main results of [4], and more, *e.g.* see [2] where the expected formulas are displayed (up to isogeny).

0.12. **Caveat.** While one might hope that these results are a partial template for a future theory of mixed motives, we should stress that some of them are definitely special to level ≤ 1. Namely:

- It is succintly pointed out in [63, §3.4 p. 215] that the non finite generation of the Griffiths group prevents higher-dimensional analogues of LAlb to exist. (This goes against [60, Conj. 0.0.19].)
- Contrary to Theorem 3.9.1, the homotopy t-structure does not induce a t-structure on $d_{\leq n}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}$ for $n \geq 2$. This can already be seen on $\mathbb{Z}(2)$, although here the homotopy sheaves are conjecturally ind-objects of $d_{\leq 2}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}$ (see [60, §6]). The situation seems to be similar for a surface; it would be interesting to work out a conjectural picture in general.

These two issues seem related in a mysterious way!

0.13. **A small reading guide.** Since this article is rather long, we would like to offer some suggestions to the reader, hoping that they will be helpful.

One might start by quickly brushing through §1.1 to review the definition of Deligne's 1-motives, look up §1.5 to read the definition of $D^b(\mathcal{M}_1[1/p])$ and then proceed directly to Theorem 2.1.2 (full embedding), referring to Section 1 ad libitum to read the proof of this theorem. The lengths of Sections 3 and 4 are necessary evils; they may very well be skipped at first reading with just a look at their main results (Theorem 3.9.1, the homotopy t-structure, and Theorem 4.4.3, agreement of the two Cartier dualities).

One may then read Section 5 on the construction of LAlb (which hopefully will be pleasant enough), glance through Section 6 (the rational version of LAlb) and have a look in passing at Section 7 for the tensor structure and internal Hom on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. After this, the reader might fly over the mostly formal sections 8 and 9, jump to Theorem 10.2.2 which computes $\mathrm{LAlb}(X)$ for a smooth scheme X, read Sections 11 and 13 on LAlb of singular schemes where he or she will have a few surprises, read Section 14 on Roĭtman's theorem and its generalisations, finally have a well-earned rest in recovering familiar objects in Section 12 (the case of curves). And never look at the appendices.

The reader will also find an index of notations at the end.

Acknowledgements. This work has been done during periods of stay of the first author at the IHÉS and the "Institut de Mathématiques de Jussieu" University of Paris 7 and of the second author at the Universities of Padova and Roma "La Sapienza". We like to thank these institutions for their hospitality and financial support. We also thank A. Bertapelle, M. Hindry, L. Illusie, P. Jossen, M. Ojanguren, M. Saito, T. Szamuely, C. Voisin for helpful comments or suggestions regarding

this work. The first author is grateful to all his coauthors in the subject of 1-motives; the second author would like to acknowledge the inspiration provided by the paper of Spieß-Szamuely [57]. Finally, our intellectual debt to A. Grothendieck, P. Deligne and V. Voevodsky is evident.

In all this paper, k is a perfect field of exponential characteristic p. We write $Sm(k)$ for the category of smooth schemes of finite type and $Sch(k)$ for the category of all separated schemes of finite type. Since we ignore characteristic p phenomena in this paper, we invert p in the Hom groups of all categories constructed out of commutative group schemes and étale sheaves from Subsection 1.4 onwards.

Part 1. The universal realisation functor

1. The derived category of 1-motives

The main reference for (integral, free) 1-motives is [16, §10], see also [5, §1]. We also provide an Appendix C on 1-motives with torsion which were introduced in [4, §1]. For the derived category of 1-motives up to isogeny we refer to [63, Sect. 3.4] and [45]: here we are interested in the integral version.

1.1. Deligne 1-motives.

The following terminology is handy:

1.1.1. **Definition.** a) An abelian sheaf L on $(Sm(k))_{\text{ét}}$ is *discrete* if it is locally constant $\mathbb{Z}$-constructible (*i.e.* with finitely generated geometric fibres). The full subcategory of discrete abelian sheaves on $(Sm(k))_{\text{ét}}$ is denoted by ${}^t\mathcal{M}_0(k) = {}^t\mathcal{M}_0$.
b) A *lattice* is a k-group scheme locally constant for the étale topology, with geometric fibre(s) isomorphic to a finitely generated free abelian group, *i.e.* representing a torsion-free discrete sheaf. The full subcategory of lattices is denoted by $\mathcal{M}_0(k) = \mathcal{M}_0$.

A *Deligne* 1*-motive* over k is a complex of group schemes

$$M = [L \xrightarrow{u} G]$$

where L is a lattice and G is a semi-abelian k-scheme. Thus G can be represented by an extension

$$0 \to T \to G \to A \to 0$$

where T is a k-torus and A is an abelian k-scheme.

As a complex, *we shall place L in degree* 0 *and G in degree* 1. Note that this convention is only partially shared by the existing literature.

A map from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ L' & \xrightarrow{u'} & G' \end{array}$$

in the category of group schemes. Denote by $(f, g) : M \to M'$ such a map. The natural composition of squares makes up the category of Deligne's 1-motives. We shall denote this category by $\mathcal{M}_1(k)$. We shall usually write $\mathcal{M}_1$ instead of $\mathcal{M}_1(k)$, unless it is necessary to specify k. The following lemma is immediate:

1.1.2. **Lemma.** *$\mathcal{M}_1$ is an idempotent complete additive category.* □

1.1.3. **Definition.** Let R be a commutative ring. For any additive category $\mathcal{A}$, we denote by $\mathcal{A} \otimes R$ the R-linear category obtained from $\mathcal{A}$ by tensoring morphisms by R, and by $\mathcal{A} \boxtimes R$ the pseudo-abelian hull (idempotent completion) of $\mathcal{A} \otimes R$.

This distinction is useful as $\mathcal{A} \otimes R$ may not be idempotent complete even if $\mathcal{A}$ is.

We shall also use the following category, which is technically very useful:

1.1.4. **Definition.** Let $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ denote the category given by complexes of group schemes $[L \to G]$ where L is discrete and G is a commutative algebraic group whose connected component of the identity G^0 is semi-abelian. It contains $\mathcal{M}_1$ as a full subcategory.

This category is studied in more detail in §C.8.

1.1.5. **Proposition** (*cf.* [45, 3.2.2])**.** *The inclusion $\mathcal{M}_1 \to \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ induces an equivalence of categories*

$$e : \mathcal{M}_1 \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}} \otimes \mathbb{Q}.$$

In particular, the category $\mathcal{M}_1 \otimes \mathbb{Q}$ is abelian, hence $\mathcal{M}_1 \otimes \mathbb{Q} = \mathcal{M}_1 \boxtimes \mathbb{Q}$.

Proof. (See also Lemma B.1.3.) It is enough to show that e is essentially surjective. But if $[L \to G] \in \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$, then we have a diagram

$$\begin{array}{ccc} [L^0 \to G^0] & \longrightarrow & [L^0_{\mathrm{fr}} \to G^0/u(L^0_{\mathrm{tor}})] \\ \downarrow & & \\ [L \xrightarrow{u} G] & & \end{array}$$

where the vertical (*resp.* horizontal) map is a pull-back (*resp.* a push-out) and $L^0_{\mathrm{fr}} := L^0/L^0_{\mathrm{tor}}$ where L^0_{tor} is the torsion subgroup of L^0. Both

maps are isomorphisms in $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}} \otimes \mathbb{Q}$. The last assertion follows from the fact that $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}} \otimes \mathbb{Q}$ is abelian (Proposition C.8.4). $\square$

1.1.6. **Remarks.** 1 (see also Def. B.1.1 c)). An *isogeny* between Deligne's 1-motives, from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ in $\mathcal{M}_1(k)$, is a diagram of group schemes with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] \\
0 \arrow[d] & F \arrow[d] \\
L \arrow[r, "u"] \arrow[d, "f"'] & G \arrow[d, "g"'] \\
L' \arrow[r, "u'"] \arrow[d] & G' \arrow[d] \\
E \arrow[d] & 0 \\
0 &
\end{tikzcd}$$

where F and E are finite. Isogenies become invertible in $\mathcal{M}_1 \otimes \mathbb{Q}$.

2. The category $\mathcal{M}_1$ of Deligne's 1-motives has kernels and cokernels (see Proposition C.1.3) but it is not abelian. This easily follows from the diagram hereabove: an isogeny has vanishing kernel and cokernel but it is not an isomorphism in $\mathcal{M}_1$.

1.2. **Weights and cohomological dimension.** Recall that $M = [L \to G] \in \mathcal{M}_1$ has an increasing filtration by sub-1-motives as follows:

$$W_i(M) = \begin{cases} M & i \geq 0 \\ G & i = -1 \\ T & i = -2 \\ 0 & i \leq -3 \end{cases}$$

We then have $\mathrm{gr}^W_{-2}(M) = T[-1]$, $\mathrm{gr}^W_{-1}(M) = A[-1]$ and $\mathrm{gr}^W_0(M) = L$ (according to our convention of placing L in degree zero). We say that M is *pure of weight* i if $\mathrm{gr}^W_j M = 0$ for all $j \neq i$. Note that for two pure 1-motives M, M', $\mathrm{Hom}(M, M') \neq 0$ only if they have the same weight.

1.2.1. **Proposition** ([45, 3.2.4]). *The category $\mathcal{M}_1 \otimes \mathbb{Q}$ is of cohomological dimension ≤ 1, i.e. if $\mathrm{Ext}^i(M, M') \neq 0$, for $M, M' \in \mathcal{M}_1 \otimes \mathbb{Q}$, then $i = 0$ or 1.*

Recall a sketch of the proof in [45]: one first checks that $\mathrm{Ext}^1(M, M') = 0$ if M, M' are pure of weights i, i' and $i \leq i'$. This formally reduces the issue to checking that if M, M', M'' are pure respectively of weights $0, -1, -2$, then the Yoneda product of two classes $(e_1, e_2) \in \mathrm{Ext}^1(M, M') \times \mathrm{Ext}^1(M', M'')$ is 0. Of course we may assume e_1 and e_2 integral. By a transfer argument, one may further reduce to k algebraically closed. Then the point is that e_1 and e_2 "glue" into a 1-motive, so are induced by a 3 step filtration on a complex of length 1; after that, it is formal to deduce that $e_2 \cdot e_1 = 0$ (*cf.* [SGA7, IX, Prop. 9.3.8 c)]).

1.2.2. **Remark.** We observe that Proposition 1.2.1 can be regarded as an algebraic version of a well-known property of $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$. Namely, $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ can be realised as a thick abelian sub-category of $\mathbb{Q}$-mixed Hodge structures, see [16]. Since the latter is of cohomological dimension ≤ 1, so is $\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}$ (use [37, Ch. III, Th. 9.1]).

1.3. **Group schemes and sheaves with transfers.**

1.3.1. **Definition** (*cf.* Def. D.1.2). We denote by $\mathrm{HI}_{\text{ét}} = \mathrm{HI}_{\text{ét}}(k)$ the category of homotopy invariant étale sheaves with transfers over $Sm(k)$: this is the full subcategory of the category $\mathrm{EST}(k) = \mathrm{Shv}_{\text{ét}}(SmCor(k))$ from [63, §3.3] consisting of those étale sheaves with transfers that are homotopy invariant.

Let G be a commutative k-group scheme. We shall denote by $\underline{G}$ the associated étale sheaf of abelian groups. In fact, under a minor assumption, $\underline{G}$ is an *étale sheaf with transfers*, as explained by Spieß-Szamuely [57, Proof of Lemma 3.2], *cf.* also Orgogozo [45, 3.1.2]. Both references use symmetric powers, hence deal only with smooth quasi-projective varieties. Here is a cheap way to extend their construction to arbitrary smooth varieties: this avoids to have to prove that $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ may be presented in terms of smooth quasi-projective varieties, *cf.* [45, beg. of §1].

1.3.2. **Lemma.** *Suppose that the neutral component G^0 is quasi-projective. Then the étale sheaf $\underline{G}$ is provided with a canonical structure of presheaf with transfers. Moreover, if G^0 is a semi-abelian variety, then $\underline{G}$ is homotopy invariant.*

Proof. For two smooth k-varieties X, Y, we have to provide a pairing

$$c(X, Y) \otimes \underline{G}(X) \to \underline{G}(Y)$$

with the obvious compatibilities. As in [39, Ex. 2.4], it is enough to construct a good transfer $f_* : \underline{G}(W) \to \underline{G}(X)$ for any finite surjective map $f : W \to X$ with X a normal k-variety. For X and W quasi-projective, this is done in [57] or [45][2]. In general, cover X by affine opens U_i and let $V_i = f^{-1}(U_i)$. Since f is finite, V_i is also affine, hence transfers $\underline{G}(V_i) \to \underline{G}(U_i)$ and $\underline{G}(V_i \cap V_j) \to \underline{G}(V_i \cap V_j)$ are defined; the commutative diagram

$$\begin{array}{ccccc} 0 \to \underline{G}(W) & \longrightarrow & \prod \underline{G}(V_i) & \longrightarrow & \prod \underline{G}(V_i \cap V_j) \\ & & {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ 0 \to \underline{G}(X) & \longrightarrow & \prod \underline{G}(U_i) & \longrightarrow & \prod \underline{G}(U_i \cap U_j) \end{array}$$

uniquely defines the desired f_*.

The second statement of the lemma is well-known (*e.g.* [45, 3.3.1]). □

Actually, the proof of [57, Lemma 3.2] defines a homomorphism in $\mathrm{HI}_{\text{ét}}$

$$\sigma : L_{\text{ét}}(G) \to \underline{G}$$

which is split by the obvious morphism of sheaves

$$\gamma : \underline{G} \to L_{\text{ét}}(G)$$

given by the graph of a morphism. Therefore σ is an epimorphism of sheaves. (One should be careful, however, that γ is not additive.) When $\underline{G}$ is homotopy invariant, one deduces from it as in [57, Remark 3.3] a morphism in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k)$

$$M_{\text{ét}}(G) = C_*(L_{\text{ét}}(G)) \to \underline{G}. \tag{1.1}$$

Let $\mathrm{HI}^{[0,1]}_{\text{ét}}$ be the category of complexes of length 1 of objects of $\mathrm{HI}_{\text{ét}}$ (concentrated in degrees 0 and 1): this is an abelian category. Lemma 1.3.2 gives us a functor

$$\begin{aligned} \mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}} &\to \mathrm{HI}^{[0,1]}_{\text{ét}} \\ M &\mapsto \underline{M} \end{aligned}$$

hence, by composing with the embedding $\mathcal{M}_1 \to \mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}$ of Proposition 1.1.5, another functor

$$\mathcal{M}_1 \to \mathrm{HI}^{[0,1]}_{\text{ét}}. \tag{1.2}$$

[2]For the symmetric powers of G to exist as schemes, it suffices that G^0 be quasi-projective.

1.4. 1-motives with torsion and an exact structure on $\mathcal{M}_1[1/p]$.

Recall that, from now on, we invert the exponential characteristic p in the Hom groups of all categories constructed out of commutative k-group schemes or étale k-sheaves. This does nothing in characteristic 0.

The reader can check that most of the statements below become false if $p > 1$ is not inverted. We hope that statements integral at p may be recovered in the future by considering some kind of non-homotopy invariant motives and cohomology theories.

We start with:

1.4.1. **Proposition.** *Let $M^{\cdot}$ be a complex of objects of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$. The following conditions are equivalent:*

(i) *The total complex $\mathrm{Tot}(M^{\cdot})$ in $C(\mathrm{HI}_{\text{ét}})[1/p]$ (see Definition 1.3.1 and Lemma 1.3.2) is acyclic.*
(ii) *For any $q \in \mathbb{Z}$, $H^q(M^{\cdot})$ is of the form $[F^q = F^q]$, where F^q is finite.*

Proof. (ii) $\Rightarrow$ (i) is obvious. For the converse, let $M^q = [L^q \to G^q]$ for all q. Let $L^{\cdot}$ and $\underline{G}^{\cdot}$ be the two corresponding "column" complexes of sheaves. Then we have a long exact sequence in $\mathrm{HI}_{\text{ét}}$:

$$\cdots \to H^q(L^{\cdot}) \to H^q(\underline{G}^{\cdot}) \to H^q(\mathrm{Tot}(M^{\cdot})) \to H^{q+1}(L^{\cdot}) \to \dots$$

The assumption implies that $H^q(L^{\cdot}) \xrightarrow{\sim} H^q(\underline{G}^{\cdot})$ for all q. Since $H^q(L^{\cdot})$ is discrete and $H^q(\underline{G}^{\cdot})$ is representable by a commutative algebraic group, both must be finite. □

We now restrict to complexes of $\mathcal{M}_1[1/p]$.

1.4.2. **Definition.** A complex of $\mathcal{M}_1[1/p]$ is *acyclic* if it satisfies the equivalent conditions of Proposition 1.4.1. An acyclic complex of the form $0 \to N' \xrightarrow{i} N \xrightarrow{j} N'' \to 0$ is called a *short exact sequence.*

Recall that in [4] a category of 1-motives with torsion was introduced. We shall denote it here by ${}^t\mathcal{M}_1$ in order to distinguish it from $\mathcal{M}_1$. Denote by ${}^t\mathcal{M}_1^{\mathrm{eff}}$ the effective 1-motives with torsion: ${}^t\mathcal{M}_1^{\mathrm{eff}}$ is the full subcategory of the category $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}$ of Definition 1.1.4 consisting of the objects $[L \to G]$ where G is connected. Then ${}^t\mathcal{M}_1$ is the localisation of ${}^t\mathcal{M}_1^{\mathrm{eff}}$ with respect to quasi-isomorphisms.

The main properties of ${}^t\mathcal{M}_1$ are recalled in Appendix C. In particular, the category ${}^t\mathcal{M}_1[1/p]$ is abelian (Theorem C.5.3) and by Proposition C.7.1 we have a full embedding

$$\mathcal{M}_1[1/p] \hookrightarrow {}^t\mathcal{M}_1[1/p] \tag{1.3}$$

which makes $\mathcal{M}_1[1/p]$ an exact subcategory of ${}^t\mathcal{M}_1[1/p]$. The following lemma is clear:

1.4.3. **Lemma.** *A complex* $0 \to N' \xrightarrow{i} N \xrightarrow{j} N'' \to 0$ *in* $\mathcal{M}_1[1/p]$ *is a short exact sequence in the sense of Definition 1.4.2 if and only if it is a short exact sequence for the exact structure given by Proposition C.7.1.*

1.4.4. **Remarks.** 1) There is another, much stronger, exact structure on $\mathcal{M}_1[1/p]$, induced by its full embedding in $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$: it amounts to require a complex $[L^{\cdot} \to G^{\cdot}]$ to be exact if and only if both complexes $L^{\cdot}$ and $\underline{G}^{\cdot}$ are acyclic. We shall not use this exact structure in the sequel. (See also Remark 1.8.5.)

2) Clearly, the complexes of Definition 1.4.2 do not provide $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$ with an exact structure. It is conceivable, however, that they define an exact structure on the localisation of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$/homotopies with respect to morphisms with acyclic kernel and cokernel.

1.5. **The derived category of 1-motives.**

1.5.1. **Lemma.** *A complex in* $C(\mathcal{M}_1[1/p])$ *is acyclic in the sense of Definition 1.4.2 if and only if it is acyclic with respect to the exact structure of* $\mathcal{M}_1[1/p]$ *provided by Lemma 1.4.3 in the sense of* [7, 1.1.4] *or* [44, §1].

Proof. Let $X^{\cdot} \in C(\mathcal{M}_1[1/p])$. Viewing $X^{\cdot}$ as a complex of objects of $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$, we define $D^n = \mathrm{Im}(d^n : X^n \to X^{n+1})$. Note that the D^n are Deligne 1-motives. Let $e_n : X^n \to D^n$ be the projection and $m_n : D^n \to X^{n+1}$ be the inclusion. We have half-exact sequences

$$0 \to D^{n-1} \xrightarrow{m_{n-1}} X^n \xrightarrow{e_n} D^n \to 0 \tag{1.4}$$

with middle cohomology equal to $H^n(X^{\cdot})$. Thus, if $X^{\cdot}$ is acyclic in the sense of Definition 1.4.2, the sequences (1.4) are short exact which means that $X^{\cdot}$ is acyclic with respect to the exact structure of $\mathcal{M}_1[1/p]$. Conversely, suppose that $X^{\cdot}$ is acyclic in the latter sense. Then, by definition, we may find D'^n, e'_n, m'_n such that $d^n = m'_n e'_n$ and that the sequences analogous to (1.4) are short exact. Since $\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p]$ is abelian (Proposition C.8.4), $D'^n = D^n$ and we are done. □

From now on, we shall only say "acyclic" without further precision.

Let $K(\mathcal{M}_1[1/p])$ be the homotopy category of $C(\mathcal{M}_1[1/p])$. By [44, Lemmas 1.1 and 1.2], the full subcategory of $K(\mathcal{M}_1[1/p])$ consisting of acyclic complexes is triangulated and thick (the latter uses the fact that $\mathcal{M}_1[1/p]$ is idempotent-complete, *cf.* Lemma 1.1.2). Thus one may define the derived category of $\mathcal{M}_1[1/p]$ in the usual way:

1.5.2. **Definition.** a) The *derived category of* 1*-motives* is the localisation $D(\mathcal{M}_1[1/p])$ of the homotopy category $K(\mathcal{M}_1[1/p])$ with respect to the thick subcategory $A(\mathcal{M}_1[1/p])$ consisting of acyclic complexes. Similarly for $D^{\pm}(\mathcal{M}_1[1/p])$ and $D^b(\mathcal{M}_1[1/p])$.
b) A morphism in $C(\mathcal{M}_1[1/p])$ is a *quasi-isomorphism* if its cone is acyclic.

1.6. **Torsion objects in the derived category of** 1**-motives.** Let $\mathcal{M}_0$ be the category of lattices (see Definition 1.1.1): the inclusion functor $\mathcal{M}_0[1/p] \xrightarrow{A} \mathcal{M}_1[1/p]$ provides it with the structure of an exact subcategory of $\mathcal{M}_1[1/p]$. Moreover, the embedding

$$\mathcal{M}_0[1/p] \xrightarrow{B} {}^t\mathcal{M}_0[1/p]$$

is clearly exact, where ${}^t\mathcal{M}_0$ is the abelian category of discrete étale sheaves (see Definition 1.1.1 again). In fact, we also have an exact functor

$$\begin{aligned} {}^t\mathcal{M}_0[1/p] &\xrightarrow{C} {}^t\mathcal{M}_1[1/p] \\ L &\mapsto [L \to 0]. \end{aligned}$$

Hence an induced diagram of triangulated categories:

$$\begin{tikzcd} D^b(\mathcal{M}_0[1/p]) \arrow[r, "B"] \arrow[d, "A"'] & D^b({}^t\mathcal{M}_0[1/p]) \arrow[d, "C"'] \\ D^b(\mathcal{M}_1[1/p]) \arrow[r, "D"] & D^b({}^t\mathcal{M}_1[1/p]). \end{tikzcd}$$

1.6.1. **Theorem.** *In the above diagram*
a) B and D are equivalence of categories.
b) A and C are fully faithful; restricted to torsion objects they are equivalences of categories.

(For the notion of torsion objects, see Proposition B.2.1.)

Proof. a) For B, this follows from Proposition A.1.2 provided we check that any object M in ${}^t\mathcal{M}_0[1/p]$ has a finite left resolution by objects in $\mathcal{M}_0[1/p]$. In fact M has a length 1 resolution: let E/k be a finite Galois extension of group Γ such that the Galois action on M factors through Γ. Since M is finitely generated, it is a quotient of some power of $\mathbb{Z}[\Gamma]$, and the kernel is a lattice. Exactly the same argument works for D.

b) By a) it is sufficient to prove that C is fully faithful. It suffices to verify that the criterion of Proposition A.1.4 is verified by the full embedding ${}^t\mathcal{M}_0[1/p] \to {}^t\mathcal{M}_1[1/p]$.

Let $[L \to 0] \hookrightarrow [L' \to G']$ be a monomorphism in ${}^t\mathcal{M}_1[1/p]$. We may assume that it is given by an effective map. The assumption implies that $L \to L'$ is mono: it then suffices to compose with the projection $[L' \to G'] \to [L' \to 0]$.

It remains to show that A is essentially surjective on torsion objects. Let $X = [C^{\cdot} \to G^{\cdot}] \in D^b(\mathcal{M}_1[1/p])$, and let $n > 0$ be such that $n1_X = 0$. Arguing as in the proof of Proposition 1.4.1, this implies that the cohomology sheaves of both $C^{\cdot}$ and $G^{\cdot}$ are killed by some possibly larger integer m. We have an exact triangle

$$[0 \to G^{\cdot}] \to X \to [C^{\cdot} \to 0] \stackrel{+1}{\to}$$

which leaves us to show that $[0 \to G^{\cdot}]$ is in the essential image of C. Let q be the smallest integer such that $G^q \neq 0$: we have an exact triangle

$$\{G^q \to \operatorname{Im} d^q\} \to G^{\cdot} \to \{0 \to G^{q+1}/\operatorname{Im} d^q \to \dots\} \stackrel{+1}{\to}$$

(here we use curly braces in order to avoid confusion with the square braces used for 1-motives). By descending induction on q, the right term is in the essential image, hence we are reduced to the case where $G^{\cdot}$ is of length 1. Then $d^q : G^q \to G^{q+1}$ is epi and $\mu := \operatorname{Ker} d^q$ is finite and locally constant[3]. Consider the diagram in $K^b(\mathcal{M}^{\mathrm{eff}}_{\mathrm{anc}}[1/p])$

$$\left[\begin{smallmatrix} 0 \\ \downarrow \\ 0 \end{smallmatrix} \to \begin{smallmatrix} G^q \\ \downarrow \\ G^{q+1} \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} 0 \\ \downarrow \\ \mu \end{smallmatrix} \to \begin{smallmatrix} G^q \\ \| \\ G^q \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} L_1 \\ \downarrow \\ L_0 \end{smallmatrix} \to \begin{smallmatrix} G^q \\ \| \\ G^q \end{smallmatrix}\right] \to \left[\begin{smallmatrix} L_1 \\ \downarrow \\ L_0 \end{smallmatrix} \to \begin{smallmatrix} 0 \\ \downarrow \\ 0 \end{smallmatrix}\right]$$

where $L_1 \to L_0$ is a resolution of μ by lattices (see proof of a)). Clearly all three maps are quasi-isomorphisms, which implies that the left object is quasi-isomorphic to the right one on $D^b(\mathcal{M}_1[1/p])$. □

1.6.2. **Corollary.** *Let A be a subring of $\mathbb{Q}$ containing $1/p$. Then the natural functor*

$$D^b(\mathcal{M}_1[1/p]) \otimes A \to D^b(\mathcal{M}_1 \otimes A)$$

is an equivalence of categories. These categories are idempotent-complete for any A.

Proof. By Proposition B.4.1, this is true by replacing the category $\mathcal{M}_1[1/p]$ by ${}^t\mathcal{M}_1[1/p]$. On the other hand, the same argument as above shows that the functor $D^b(\mathcal{M}_1 \otimes A) \to D^b({}^t\mathcal{M}_1 \otimes A)$ is an equivalence. This shows the first statement; the second one follows from the fact that D^b of an abelian category is idempotent-complete. □

[3]Note that this is true even if m is divisible by the characteristic of k, since we only consider sheaves over *smooth* k-schemes.

1.7. **Discrete sheaves and permutation modules.** The following proposition will be used in §2.6.a.

1.7.1. **Proposition.** *Let G be a profinite group. Denote by $D^b_c(G)$ the derived category of finitely generated (topological discrete) G-modules. Then $D^b_c(G)$ is thickly generated by $\mathbb{Z}$-free permutation modules.*

Proof. The statement says that the smallest thick subcategory $\mathcal{T}$ of $D^b_c(G)$ which contains permutation modules is equal to $D^b_c(G)$. Let M be a finitely generated G-module: to prove that $M \in \mathcal{T}$, we immediately reduce to the case where G is finite. Let $\overline{M} = M/M_{\mathrm{tors}}$. Realise $\overline{M} \otimes \mathbb{Q}$ as a direct summand of $\mathbb{Q}[G]^n$ for n large enough. Up to scaling, we may assume that the image of $\overline{M}$ in $\mathbb{Q}[G]^n$ is contained in $\mathbb{Z}[G]^n$ and that there exists a submodule N of $\mathbb{Z}[G]^n$ such that $\overline{M} \cap N = 0$ and $\overline{M} \oplus N$ is of finite index in $\mathbb{Z}[G]^n$. This reduces us to the case where M is *finite*. Moreover, we may assume that M is ℓ-primary for some prime ℓ.

Let S be a Sylow ℓ-subgroup of G. Recall that there exist two inverse isomorphisms

$$\varphi : \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M)$$
$$\varphi(g \otimes m)(\gamma) = \begin{cases} \gamma g m & \text{if } \gamma g \in S \\ 0 & \text{if } \gamma g \notin S. \end{cases}$$
$$\psi : \mathrm{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M) \xrightarrow{\sim} \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M$$
$$\psi(f) = \sum_{g \in S \backslash G} g^{-1} \otimes f(g).$$

On the other hand, we have the obvious unit and counit homomorphisms

$$\eta : M \to \mathrm{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M)$$
$$\eta(m)(g) = gm$$
$$\varepsilon : \mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M \to M$$
$$\varepsilon(g \otimes m) = gm.$$

It is immediate that

$$\varepsilon \circ \psi \circ \eta = (G : S).$$

Since $(G : S)$ is prime to ℓ, this shows that M is a direct summand of the induced module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[S]} M \simeq \mathrm{Hom}_{\mathbb{Z}[S]}(\mathbb{Z}[G], M)$. But it is well-known (see *e.g.* [56, §8.3, cor. to Prop. 26]) that M, as an S-module, is a successive extension of trivial S-modules. Any trivial

torsion S-module has a length 1 resolution by trivial torsion-free S-modules. Since the "induced module" functor is exact, this concludes the proof. □

1.8. **Cartier duality and 1-motives with cotorsion.** We now introduce a new category ${}_t\mathcal{M}_1$:

1.8.1. **Definition.** We denote by ${}_t\mathcal{M}_1^{\mathrm{eff}}$ the full subcategory of $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ consisting of those $[L \to G]$ such that L is a lattice and G is an extension of an abelian variety by a group of multiplicative type, and by ${}_t\mathcal{M}_1$ its localisation with respect to quasi-isomorphisms. An object of ${}_t\mathcal{M}_1$ is called a 1-*motive with cotorsion.*

Recall that Deligne [16, §10.2.11-13] (*cf.* [5, 1.5]) defined a self-duality on the category $\mathcal{M}_1$, that he called *Cartier duality.* The following facts elucidate the introduction of the category ${}_t\mathcal{M}_1$.

1.8.2. **Lemma.** *Let Γ be a group of multiplicative type, L its Cartier dual and A an abelian variety (over $k = \overline{k}$). We have an isomorphism*

$$\tau : \mathrm{Ext}(A, \Gamma) \xrightarrow{\simeq} \mathrm{Hom}(L, \mathrm{Pic}^0(A))$$

given by the canonical "pushout" mapping.

Proof. Displaying L as an extension of L_{fr} by L_{tor} denote the corresponding torus by $T := \mathrm{Hom}(L_{\mathrm{fr}}, \mathbb{G}_m)$ and let $F := \mathrm{Hom}(L_{\mathrm{tor}}, \mathbb{G}_m)$ be the dual finite group. We obtain a map of short exact sequences

$$\begin{array}{ccccc} 0 \to \mathrm{Ext}(A, T) & \to & \mathrm{Ext}(A, \Gamma) & \to & \mathrm{Ext}(A, F) \to 0 \\ \downarrow{\scriptstyle \tau_{\mathrm{fr}}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_{\mathrm{tor}}} \\ 0 \to \mathrm{Hom}(L_{\mathrm{fr}}, \mathrm{Pic}^0(A)) & \to & \mathrm{Hom}(L, \mathrm{Pic}^0(A)) & \to & \mathrm{Hom}(L_{\mathrm{tor}}, \mathrm{Pic}^0(A)) \to 0. \end{array}$$

Now τ_{fr} is an isomorphism by the classical Weil-Barsotti formula, *i.e.* $\mathrm{Ext}(A, \mathbb{G}_m) \cong \mathrm{Pic}^0(A)$, and τ_{tor} is an isomorphism since the Néron-Severi group of A is free: $\mathrm{Hom}(L_{\mathrm{tor}}, \mathrm{Pic}^0(A)) = \mathrm{Hom}(L_{\mathrm{tor}}, \mathrm{Pic}(A)) = H^1(A, F) = \mathrm{Ext}(A, F)$ (*cf.* [40, 4.20]). □

1.8.3. **Lemma.** *Cartier duality on $\mathcal{M}_1$ extends to a contravariant additive functor*

$$(\ \ )^* : {}^t\mathcal{M}_1^{\mathrm{eff}} \to {}_t\mathcal{M}_1^{\mathrm{eff}}$$

which sends a q.i. *to a* q.i.

Proof. The key point is that $\mathrm{Ext}(-, \mathbb{G}_m)$ vanishes on discrete sheaves (*cf.* [40, 4.17]), hence Cartier duality extends to an exact duality between discrete sheaves and groups of multiplicative type.

To define the functor, we proceed as usual (see [5, 1.5]): starting with $M = [L \xrightarrow{u} A] \in {}^t\mathcal{M}_1^{\mathrm{eff}}$, le G^u be the extension of the dual abelian

variety A^* by the Cartier dual L^* of L given by Lemma 1.8.2 (note that G^u may be described as the group scheme which represents the functor associated to $\mathrm{Ext}(M, \mathbb{G}_m)$). We define $M^* = [0 \to G^u] \in {}_t\mathcal{M}_1^{\mathrm{eff}}$. For a general $M = [L \xrightarrow{u} G] \in {}^t\mathcal{M}_1^{\mathrm{eff}}$, with G an extension of A by T, the extension M of $[L \xrightarrow{\overline{u}} A]$ by the toric part $[0 \to T]$ provides the corresponding extension $G^{\overline{u}}$ of A' by L^* and a boundary map

$$u^* : \mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Ext}([L \xrightarrow{\overline{u}} A], \mathbb{G}_m) = G^{\overline{u}}(k)$$

which defines $M^* \in {}_t\mathcal{M}_1^{\mathrm{eff}}$.

For a quasi-isomorphism $M \twoheadrightarrow M'$ with kernel $[F \xrightarrow{=} F]$ for a finite group F, *cf.* (C.2), the quotient $[L \xrightarrow{\overline{u}} A] \twoheadrightarrow [L' \xrightarrow{\overline{u}'} A']$ has kernel $[F \twoheadrightarrow F_A]$ where $F_A := \mathrm{Ker}(A \twoheadrightarrow A')$ and the following is a pushout

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}(T', \mathbb{G}_m) & \to & \mathrm{Hom}(T, \mathbb{G}_m) & \to & \mathrm{Hom}(F_T, \mathbb{G}_m) \to 0 \\
{\scriptstyle (u')^*}\downarrow & & {\scriptstyle u^*}\downarrow & & \|\downarrow \\
0 \to \mathrm{Ext}([L' \xrightarrow{\overline{u}'} A'], \mathbb{G}_m) & \to & \mathrm{Ext}([L \xrightarrow{\overline{u}} A], \mathbb{G}_m) & \to & \mathrm{Ext}([F \twoheadrightarrow F_A], \mathbb{G}_m) \to 0
\end{array}$$

where $F_T := \mathrm{Ker}(T \twoheadrightarrow T')$. □

1.8.4. **Proposition.** *a) The functor of Lemma 1.8.3 induces an anti-equivalence of categories*

$$(\ \ )^* : {}^t\mathcal{M}_1[1/p] \xrightarrow{\sim} {}_t\mathcal{M}_1[1/p].$$

b) The category ${}_t\mathcal{M}_1[1/p]$ is abelian and the two functors of a) are exact.
c) Cartier duality on $\mathcal{M}_1[1/p]$ is an exact functor, hence induces a triangulated self-duality on $D^b(\mathcal{M}_1[1/p])$.

Proof. a) The said functor exists by Lemma 1.8.3, and it is clearly additive. Let us prove that it is i) essentially surjective, i) faithful and iii) full.

i) We proceed exactly as in the proof of Lemma 1.8.3, taking an $[L' \to G'] \in {}_t\mathcal{M}_1[1/p]$, and writing G' explicitly as an extension of an abelian variety by a group of multiplicative type.

ii) We reduce to show that the functor of Lemma 1.8.3 is faithful by using that Lemma C.2.3 is also true in ${}_t\mathcal{M}_1^{\mathrm{eff}}[1/p]$ (dual proof). By additivity, we need to prove that if $f : M_0 \to M_1$ is mapped to 0, then $f = 0$. But, by construction, f^* sends the mutiplicative type part of M_1^* to that of M_0^*.

iii) Let $M_0 = [L_0 \to G_0]$, $M_1 = [L_1 \to G_1]$ in ${}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$, and let $f : M_1^* \to M_0^*$ be (for a start) an effective map. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_1 & \longrightarrow & G_1' & \longrightarrow & A_1' & \longrightarrow & 0 \\ & & & & {\scriptstyle f_G}\big\downarrow & & & & \\ 0 & \longrightarrow & \Gamma_0 & \longrightarrow & G_0' & \longrightarrow & A_0' & \longrightarrow & 0 \end{array}$$

where $M_i^* = [L_i' \to G_i']$, A_i' is the dual of the abelian part of M_i and Γ_i is the dual of L_i. If f_G maps Γ_1 to Γ_2, there is no difficulty to get an (effective) map $g : M_0 \to M_1$ such that $g^* = f$. In general we reduce to this case: let μ be the image of $f_G(\Gamma_1)$ in A_0': this is a finite group. Let now $A_2' = A_0'/\mu$, so that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_0 & \longrightarrow & G_0' & \longrightarrow & A_0' & \longrightarrow & 0 \\ & & \big\downarrow & & {\scriptstyle \|}\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \Gamma_2 & \longrightarrow & G_0' & \longrightarrow & A_2' & \longrightarrow & 0 \end{array}$$

where $\mu = \mathrm{Ker}(A_0' \to A_2') = \mathrm{Coker}(\Gamma_0 \to \Gamma_2)$. By construction, f_G induces maps $f_\Gamma : \Gamma_1 \to \Gamma_2$ and $f_A : A_1' \to A_2'$.

Consider the object $M_2 = [L_2 \to G_2] \in {}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$ obtained from (L_0', Γ_2, A_2') and the other data by the same procedure as in the proof of Lemma 1.8.3. We then have a q.i. $s : M_2 \to M_0$ with kernel $[\mu = \mu]$ and a map $g : M_2 \to M_1$ induced by (f_L, f_Γ, f_A), and $(gs^{-1})^* = f$.

If f is a q.i., clearly g is a q.i.; this concludes the proof of fullness.

b) Since ${}^t\mathcal{M}_1[1/p]$ is abelian, ${}_t\mathcal{M}_1[1/p]$ is abelian by a). Equivalences of abelian categories are automatically exact.

c) One checks as for ${}^t\mathcal{M}_1[1/p]$ that the inclusion of $\mathcal{M}_1[1/p]$ into ${}_t\mathcal{M}_1[1/p]$ induces the exact structure of $\mathcal{M}_1[1/p]$. Then, thanks to b), Cartier duality preserves exact sequences of $\mathcal{M}_1[1/p]$, which means that it is exact on $\mathcal{M}_1[1/p]$. □

1.8.5. **Remarks.** 1) Cartier duality does not preserve the strong exact structure of Remark 1.4.4 1). For example, let A be an abelian variety, $a \in A(k)$ a point of order $m > 1$ and $B = A/\langle a \rangle$. Then the sequence

$$0 \to [\mathbb{Z} \to 0] \xrightarrow{m} [\mathbb{Z} \xrightarrow{f} A] \to [0 \to B] \to 0,$$

with $f(1) = a$, is exact in the sense of Definition 1.4.2 but not in the sense of Remark 1.4.4. However, its dual

$$0 \to [0 \to B^*] \to [0 \to G] \to [0 \to \mathbb{G}_m] \to 0$$

is exact in the strong sense. Taking the Cartier dual of the latter sequence, we come back to the former.

2) One way to better understand what happens in Lemma 1.8.3 and Proposition 1.8.4 would be to introduce a category ${}_t\widetilde{\mathcal{M}_1}^{\mathrm{eff}}$, whose objects are quintuples (L, u, G, A, Γ) with L a lattice, Γ a group of multiplicative type, A an abelian variety, G an extension of A by Γ and u a morphism from L to G. Morphisms in ${}_t\widetilde{\mathcal{M}_1}^{\mathrm{eff}}$ are additive and respect all these structures. There is an obvious functor $(L, u, G, A, \Gamma) \mapsto [L \xrightarrow{u} G]$ from ${}_t\widetilde{\mathcal{M}_1}^{\mathrm{eff}}$ to ${}_t\mathcal{M}_1^{\mathrm{eff}}$, the functor of Lemma 1.8.3 lifts to an anti-isomorphism of categories ${}^t\mathcal{M}_1^{\mathrm{eff}} \xrightarrow{\sim} {}_t\widetilde{\mathcal{M}_1}^{\mathrm{eff}}$ and the localisation of ${}_t\widetilde{\mathcal{M}_1}^{\mathrm{eff}}[1/p]$ with respect to the images of q.i. of ${}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$ is equivalent to ${}_t\mathcal{M}_1[1/p]$. We leave details to the interested reader.

Dually to Theorem 1.6.1, we now have:

1.8.6. **Theorem.** *The natural functor $\mathcal{M}_1[1/p] \to {}_t\mathcal{M}_1[1/p]$ is fully faithful and induces an equivalence of categories*

$$D^b(\mathcal{M}_1[1/p]) \xrightarrow{\sim} D^b({}_t\mathcal{M}_1[1/p]).$$

Moreover, Cartier duality exchanges ${}^t\mathcal{M}_1[1/p]$ and ${}_t\mathcal{M}_1[1/p]$ inside the derived category $D^b(\mathcal{M}_1[1/p])$.

Proof. This follows from Theorem 1.6.1 and Proposition 1.8.4. □

1.8.7. **Notation.** For $C \in D^b(\mathcal{M}_1[1/p])$, we write ${}^tH^n(C)$ (*resp.* ${}_tH^n(C)$ for its cohomology objects relative to the t-structure with heart ${}^t\mathcal{M}_1[1/p]$ (*resp.* ${}_t\mathcal{M}_1[1/p]$). We also write ${}^tH_n$ for ${}^tH^{-n}$ and ${}_tH_n$ for ${}_tH^{-n}$.

Thus we have *two* t-structures on $D^b(\mathcal{M}_1[1/p])$ which are exchanged by Cartier duality; naturally, these two t-structures coincide after tensoring with $\mathbb{Q}$. In Section 3, we shall introduce a third t-structure, of a completely different kind: see Corollary 3.9.2.

We shall also come back to Cartier duality in Section 4.

1.9. **How not to invert p.** This has been done by Alessandra Bertapelle [8]. She defines a larger variant of ${}^t\mathcal{M}_1$ by allowing finite connected k-group schemes in the component of degree 0. Computing in the fppf topology, she checks that the arguments provided in Appendix C carry over in this context and yield in particular an integral analogue to Theorem C.5.3. Also, the analogue of (1.3) is fully faithful integrally, hence an exact structure on $\mathcal{M}_1$; she also checks that the analogue of Theorem 1.6.1 holds integrally.

In particular, her work provides an exact structure on $\mathcal{M}_1$, hence an integral definition of $D^b(\mathcal{M}_1)$. One could check that this exact structure can be described *a priori* using Proposition 1.4.1 and Lemma 1.4.3, and working with the fppf topology.

It is likely that the duality results of §1.8 also extend to Bertapelle's context.

2. Universal realisation

2.1. **Statement of the theorem.** The derived category of 1-motives up to isogeny can be realised in Voevodsky's triangulated category of motives. With rational coefficients, this is part of Voevodsky's Pretheorem 0.0.18 in [60] and claimed in [63, Sect. 3.4, on page 218]. Details of this fact appear in Orgogozo [45]. In this section we shall give a p-integral version of this theorem, where p is the exponential characteristic of k, using the étale version of Voevodsky's category.

By Lemma 1.3.2, any 1-motive $M = [L \to G]$ may be regarded as a complex of homotopy invariant étale sheaves with transfers. By Lemma D.1.3, $M[1/p] := M \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is a complex of strictly homotopy invariant étale sheaves with transfers; this defines a functor

$$\begin{aligned} \mathcal{M}_1(k) &\to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k) \\ M &\mapsto M[1/p]. \end{aligned} \tag{2.1}$$

(see [63, Sect. 3] for motivic complexes).

From now on, we will usually drop the mention of k from the notation for the various categories of motives encountered.

2.1.1. **Definition.** We denote by $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ the thick subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ generated by the image of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ under the "change of topology" functor

$$\alpha^* : \mathrm{DM}^{\mathrm{eff}}_{-} \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$$

of [63, §3.3]. We set $M_{\text{ét}}(X) := \alpha^* M(X)$.

2.1.2. **Theorem.** *Let p be the exponential characteristic of k. The functor* (2.1) *extends to a fully faithful triangulated functor*

$$T : D^b(\mathcal{M}_1[1/p]) \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$$

where the left hand side was defined in §1.4. Its essential image is the thick subcategory $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ generated by motives of smooth curves.

The proof is in several steps.

2.2. **Construction of T.** We follow Orgogozo. Clearly, the embedding (1.2) extends to a functor

$$C^b(\mathcal{M}_1) \to C^b(\mathrm{HI}^{[0,1]}_{\text{ét}}).$$

By Lemma A.2.1, we have a canonical functor $C^b(\mathrm{HI}_{\text{ét}}^{[0,1]}) \xrightarrow{\mathrm{Tot}} D^b(\mathrm{HI}_{\text{ét}})$, and there is a canonical composite functor

$$D^b(\mathrm{HI}_{\text{ét}}) \xrightarrow{\otimes_{\mathbb{Z}}\mathbb{Z}[1/p]} D^b(\mathrm{HI}_{\text{ét}}^s) \to \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}$$

where $\mathrm{HI}_{\text{ét}}^s$ is the category of strictly homotopy invariant étale sheaves with transfers (see Def. D.1.2 and Proposition D.1.4). To get T, we are therefore left to prove

2.2.1. **Lemma.** *The composite functor*

$$C^b(\mathcal{M}_1) \to C^b(\mathrm{HI}_{\text{ét}}^{[0,1]}) \xrightarrow{\mathrm{Tot}} D^b(\mathrm{HI}_{\text{ét}})$$

factors through $D^b(\mathcal{M}_1[1/p])$.

Proof. It is a general fact that a homotopy in $C^b(\mathcal{M}_1)$ is mapped to a homotopy in $C^b(\mathrm{HI}_{\text{ét}}^{[0,1]})$, and therefore goes to 0 in $D^b(\mathrm{HI}_{\text{ét}})$, so that the functor already factors through $K^b(\mathcal{M}_1)$. The lemma now follows from Lemma 1.5.1. □

2.3. **Full faithfulness.** It is sufficient by Proposition B.2.4 to show that $T \otimes \mathbb{Q}$ and T_{tors} are fully faithful.

For the first fact, we reduce to [45, 3.3.3 ff]. We have to be a little careful since Orgogozo's functor is not quite the same as our functor: Orgogozo sends C to $\mathrm{Tot}(C)$ while we send it to $\mathrm{Tot}(C)[1/p]$, but the map $\mathrm{Tot}(C) \to \mathrm{Tot}(C)[1/p]$ is an isomorphism in $\mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}} \otimes \mathbb{Q}$ by Proposition 6.1.1 (see also Remark 2.7.2 2)).

For the reader's convenience we sketch the proof of [45, 3.3.3 ff]: it first uses the equivalence of categories

$$\mathrm{DM}_{-}^{\mathrm{eff}} \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}} \otimes \mathbb{Q}$$

of [63, Prop. 3.3.2] (*cf.* Proposition 6.1.1). One then reduces to show that the morphisms

$$\mathrm{Ext}^i(M, M') \to \mathrm{Hom}(\mathrm{Tot}(M), \mathrm{Tot}(M')[i])$$

are isomorphisms for any pure 1-motives M, M' and any $i \in \mathbb{Z}$. This is done by a case-by-case inspection, using the fact [63, 3.1.9 and 3.1.12] that in $\mathrm{DM}_{-}^{\mathrm{eff}} \otimes \mathbb{Q}$

$$\mathrm{Hom}(M(X), C) \otimes \mathbb{Q} = \mathbb{H}^0_{\mathrm{Zar}}(X, C) \otimes \mathbb{Q}$$

for any smooth variety X. The key points are that 1) for such X we have $H^i_{\mathrm{Zar}}(X, \mathbb{G}_m) = 0$ for $i > 1$ and for an abelian variety A, $H^i_{\mathrm{Zar}}(X, A) = 0$ for $i > 0$ because the sheaf A is flasque, and 2) that any abelian variety is up to isogeny a direct summand of the Jacobian

of a curve. This point will also be used for the essential surjectivity below.

For the second fact, the argument in the proof of [63, Prop. 3.3.3 1] shows that the functor $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}} \to D^-(\mathrm{Shv}((\operatorname{Spec} k)_{\text{ét}}))$ which takes a complex of sheaves on $Sm(k)_{\text{ét}}$ to its restriction to $(\operatorname{Spec} k)_{\text{ét}}$ is an equivalence of categories on the full subcategories of objects of prime-to-p torsion. The conclusion then follows from Proposition 1.6.1.

2.4. **Gersten's principle.** We want to formalise here an important computational method which goes back to Gersten's conjecture but was put in a wider perspective and systematic use by Voevodsky. For the étale topology it replaces advantageously (but not completely) the recourse to proper base change.

2.4.1. **Proposition.** *a) Let C be a complex of presheaves with transfers on $Sm(k)$ with homotopy invariant cohomology presheaves. Suppose that $C(K) := \varinjlim_{k(U)=K} C(U)$ is acyclic for any function field K/k. Then the associated complex of Zariski sheaves C_{Zar} is acyclic.*
b) Let $f : C \to D$ be a morphism of complex of presheaves with transfers on $Sm(k)$ with homotopy invariant cohomology presheaves. Suppose that for any function field K/k, $f(K) : C(K) \to D(K)$ is a quasi-isomorphism. Then $f_{\mathrm{Zar}} : C_{\mathrm{Zar}} \to D_{\mathrm{Zar}}$ is a quasi-isomorphism.
c) The conclusions of a) and b) hold for the étale topology if their hypotheses are weakened by replacing K by K_s, a separable closure of K.

Proof. a) Let $F = H^q(C)$ for some $q \in \mathbb{Z}$, and let X be a smooth k-variety with function field K. By [62, Cor. 4.18], $F(\mathcal{O}_{X,x}) \hookrightarrow F(K)$ for any $x \in X$, hence $F_{\mathrm{Zar}} = 0$. b) follows from a) by considering the cone of f. c) is seen similarly. □

2.5. **An important computation.** Recall that the category $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ is provided with a partial internal Hom denoted by $\underline{\mathrm{Hom}}_{\text{ét}}$, defined on pairs (M, M') with $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$: it is defined analogously to the one of [63, Prop. 3.2.8] for the Nisnevich topology. We need:

2.5.1. **Definition.** Let $X \in Sch(k)$. We denote by $\pi_0(X)$ the largest étale k-scheme such that the structural map $X \to \operatorname{Spec} k$ factors through $\pi_0(X)$.

(The existence of $\pi_0(X)$ is obvious, for example by Galois descent.)

2.5.2. **Proposition.** *Let $f : C \to \operatorname{Spec} k$ be a smooth projective k-curve. Then, in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$:*

a) There is a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\text{ét}}(M_{\text{ét}}(C), \mathbb{Z}(1)[2]) \simeq R_{\text{ét}} f_* \mathbb{G}_m[1/p][1].$$

b) we have

$$R^q_{\text{ét}} f_* \mathbb{G}_m[1/p] = \begin{cases} R_{\pi_0(C)/k}\mathbb{G}_m[1/p] & \textit{for } q = 0 \\ \underline{\mathrm{Pic}}_{C/k}[1/p] & \textit{for } q = 1 \\ 0 & \textit{else.} \end{cases}$$

Here, $R_{\pi_0(C)/k}$ denotes the Weil restriction of scalars from $\pi_0(C)$ to k.
c) The morphism

$$M_{\text{ét}}(C) \to \underline{\mathrm{Hom}}_{\text{ét}}(M_{\text{ét}}(C), \mathbb{Z}(1)[2])$$

induced by the class $\Delta_C \in \mathrm{Hom}(M_{\text{ét}}(C) \otimes M_{\text{ét}}(C), \mathbb{Z}(1)[2])$ of the diagonal is an isomorphism.

Proof. This is [45, Cor. 3.1.6] with three differences: 1) the fppf topology should be replaced by the étale topology; p must be inverted (*cf.* Corollary D.1.6); 3) the truncation is not necessary since C is a curve.

a) is the étale analogue of [63, Prop. 3.2.8] since $\mathbb{Z}_{\text{ét}}(1) = \mathbb{G}_m[1/p][-1]$ (see Corollary D.1.6) and $f^*(\mathbb{G}_{m,k}) = \mathbb{G}_{m,C}$ for the big étale sites. In b), the isomorphisms for $q = 0, 1$ are clear; for $q > 2$, we reduce by Gersten's principle (Prop. 2.4.1) to stalks at separably closed fields, and then the result is classical [SGA4, IX (4.5)].

It remains to prove c). Recall that its Nisnevich analogue is true in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ ([63, Th. 4.3.2 and Cor. 4.3.6], but see [26, App. B] to avoid resolution of singularities). Let $\alpha^* : \mathrm{DM}^{\mathrm{eff}}_{-} \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ be the change of topology functor (*cf.* [39, Remark 14.3]). By b), the natural morphism

$$\alpha^* \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(C), \mathbb{Z}(1)) \to \underline{\mathrm{Hom}}_{\text{ét}}(\alpha^* M(C), \mathbb{Z}(1)) \tag{2.2}$$

is an isomorphism. Hence the result. □

2.6. **Essential image.** We proceed in two steps:

2.6.a. *The essential image of T is contained in $\mathcal{T} := d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$.* It is sufficient to prove that $T(N) \in \mathcal{T}$ for N a 1-motive of type $[L \to 0]$, $[0 \to G]$ (G a torus) or $[0 \to A]$ (A an abelian variety). For the first type, this follows from Proposition 1.7.1. For the second type, Proposition 1.7.1 applied to the character group of G shows that $T([0 \to G])$ is contained in the thick subcategory generated by permutation tori, which is clearly contained in $\mathcal{T}$.

It remains to deal with the third type. If $A = J(C)$ for a smooth projective curve C having a rational k-point c, then $T([0 \to A]) = A[-1]$ is the direct summand of $M(C)[-1]$ (determined by c) corresponding

to the pure motive $h^1(C)$, so belongs to $\mathcal{T}$. If $A \to A'$ is an isogeny, then Proposition 1.7.1 implies that $A[-1] \in \mathcal{T} \iff A'[-1] \in \mathcal{T}$. In general we may write A as the quotient of a jacobian $J(C)$. Let B be the connected part of the kernel: by complete reducibility there exists a third abelian variety $B' \subseteq J(C)$ such that $B + B' = J(C)$ and $B \cap B'$ is finite. Hence $B \oplus B' \in \mathcal{T}$, $B' \in \mathcal{T}$ and finally $A \in \mathcal{T}$ since it is isogenous to B'.

2.6.b. *The essential image of T contains $\mathcal{T}$.* It suffices to show that $M(X)$ is in the essential image of T if X is smooth projective irreducible of dimension 0 or 1. Let E be the field of constants of X. If $X = \operatorname{Spec} E$, $M(X)$ is the image of $[R_{E/k}\mathbb{Z} \to 0]$. If X is a curve, we apply Proposition 2.5.2: by c) it suffices to show that the sheaves of b) are in the essential image of T. We have already observed that $R_{E/k}\mathbb{G}_m[1/p]$ is in the essential image of T. We then have a short exact sequence

$$0 \to R_{E/k}J(X)[1/p] \to \underline{\mathrm{Pic}}_{X/k}[1/p] \to R_{E/k}\mathbb{Z}[1/p] \to 0.$$

Both the kernel and the cokernel in this extension belong to the image of T, and the proof is complete. □

2.7. **The universal realisation functor.**

2.7.1**. Definition.** Define the *universal realisation functor*

$$\mathrm{Tot} : D^b(\mathcal{M}_1[1/p]) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}}$$

to be the composition of the equivalence of categories of Theorem 2.1.2 and the embedding $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}}$.

2.7.2**. Remarks.** 1) In view of Theorem 1.8.6, the equivalence of Theorem 2.1.2, yields *two* "motivic" t-structures on $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}}$: one with heart ${}^t\mathcal{M}_1[1/p]$ and the other with heart ${}_t\mathcal{M}_1[1/p]$. We shall describe a third one, the homotopy t-structure, in Theorem 3.9.1.

2) In what follows we shall frequently commit an abuse of notation in writing $\underline{G}$ rather that $\underline{G}[1/p]$, etc. for the image of (say) a semi-abelian variety in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}}$ by the functor Tot. This is to keep notation light. A more mathematical justification is that, according to Proposition D.1.5, the functor T is naturally isomorphic to the composition

$$\begin{aligned} D^b(\mathcal{M}_1[1/p]) \to D^b(\mathrm{HI}^{[0,1]}_{\acute{e}t}[1/p]) \to D^b(\mathrm{HI}_{\acute{e}t}[1/p]) \\ \to D^-(\mathrm{Shv}_{\acute{e}t}(Sm(k))[1/p]) \xrightarrow{C_*} \mathrm{DM}^{\mathrm{eff}}_{-,\acute{e}t} \end{aligned}$$

which (apparently) does not invert p on objects.

3. 1-motivic sheaves and the homotopy t-structure

We recall the blanket assumption that p is inverted in all Hom groups.

3.1. **Some useful lemmas.** Except for Proposition 3.1.7, this subsection is in the spirit of [53, Ch. VII].

Let G be a commutative k-group scheme, and let us write $\underline{G}$ for the associated sheaf of abelian groups for a so far unspecified Grothendieck topology. Let also $\mathcal{F}$ be another sheaf of abelian groups. We then have:

- $\mathrm{Ext}^1(\underline{G}, \mathcal{F})$ (an Ext of sheaves);
- $H^1(G, \mathcal{F})$ (cohomology of the scheme G);
- $\overline{H}^2(G, \mathcal{F})$: this is the homology of the complex
$$\mathcal{F}(G) \xrightarrow{d^1} \mathcal{F}(G \times G) \xrightarrow{d^2} \mathcal{F}(G \times G \times G)$$
where the differentials are the usual ones.

3.1.1. **Proposition.** *There is an exact sequence (defining A)*
$$0 \to A \to \mathrm{Ext}^1(\underline{G}, \mathcal{F}) \xrightarrow{b} H^1(G, \mathcal{F}) \xrightarrow{c} H^1(G \times G, \mathcal{F})$$
and an injection
$$0 \to A \xrightarrow{a} \overline{H}^2(G, \mathcal{F}).$$

Proof. Let us first define the maps a, b, c:

- c is given by $p_1^* + p_2^* - \mu^*$, where μ is the group law of G.
- For b: let $\mathcal{E}$ be an extension of $\underline{G}$ by $\mathcal{F}$. We have an exact sequence
$$\mathcal{E}(G) \to \underline{G}(G) \to H^1(G, \mathcal{F}).$$
Then $b([\mathcal{E}])$ is the image of 1_G by the connecting homomorphism. Alternatively, we may think of $\mathcal{E}$ as an $\mathcal{F}$-torsor over G by forgetting its group structure.
- For a: we have $b([\mathcal{E}]) = 0$ if and only if 1_G has an antecedent $s \in \mathcal{E}(G)$. By Yoneda, this s determines a section $s : \underline{G} \to \mathcal{E}$ of the projection. The defect of s to be a homomorphism gives a well-defined element of $\overline{H}^2(G, \mathcal{F})$ by the usual cocycle computation: this is $a([\mathcal{E}])$.

Exactness is checked by inspection. □

3.1.2. **Remark.** It is not clear whether a is surjective.

3.1.3. **Proposition.** *Suppose that the map*
$$\mathcal{F}(G) \oplus \mathcal{F}(G) \xrightarrow{(p_1^*, p_2^*)} \mathcal{F}(G \times G)$$
is surjective. Then $\overline{H}^2(G, \mathcal{F}) = 0$.

Proof. Let $\gamma \in \mathcal{F}(G \times G)$ be a 2-cocycle. We may write $\gamma = p_1^*\alpha + p_2^*\beta$. The cocycle condition implies that α and β are constant. Hence γ is constant, and it is therefore a 2-coboundary (of itself). □

3.1.4. **Example.** $\mathcal{F}$ locally constant, G smooth, the topology = the étale topology. Then the condition of Proposition 3.1.3 is verified. We thus get an isomorphism

$$\mathrm{Ext}^1(\underline{G}, \mathcal{F}) \xrightarrow{\sim} H^1_{\text{ét}}(G, \mathcal{F})_{mult}$$

with the group of multiplicative classes in $H^1_{\text{ét}}(G, \mathcal{F})$.

3.1.5. **Lemma.** *Let G be a semi-abelian k-variety and L a locally constant $\mathbb{Z}$-constructible étale sheaf with torsion-free geometric fibres. Then* $\mathrm{Ext}^1(\underline{G}, L) = 0$.

Proof. By the Ext spectral sequence, it suffices to show that $\mathcal{H}om\,(\underline{G}, L) = \mathcal{E}xt\,(\underline{G}, L) = 0$. This reduces us to the case $L = \mathbb{Z}$. Then the first vanishing is obvious and the second follows from Example 3.1.4. □

3.1.6. **Lemma.** *Let $\mathcal{E} \in \mathrm{Ext}^1(\underline{G}, \mathbb{G}_m)$ and let $g \in G(k)$. Denote by τ_g the left translation by g. Then $\tau_g^* b(\mathcal{E}) = b(\mathcal{E})$. Here b is the map of Proposition 3.1.1.*

Proof. By Hilbert's theorem 90, g lifts to an $e \in \mathcal{E}(k)$. Then τ_e induces a morphism from the $\mathbb{G}_m$-torsor $b(\mathcal{E})$ to the $\mathbb{G}_m$-torsor $\tau_g^* b(\mathcal{E})$: this morphism must be an isomorphism. □

For the proof of Theorem 3.3.1 below we shall need the case $i = 2$ of the following proposition, which unfortunately cannot be proven with the above elementary methods.

3.1.7. **Proposition.** *Let G be a smooth commutative algebraic k-group and L a discrete k-group scheme. Let $\mathcal{A} = \mathrm{Shv}_{\text{ét}}(Sm(k))$ be the category of abelian étale sheaves on the category of smooth k-varieties. Then, for any $i \geq 2$, the group $\mathrm{Ext}^i_{\mathcal{A}}(\underline{G}, L)$ is torsion.*

Proof. Considering the connected part G^0 of G, we reduce to the case where G is connected, hence geometrically connected. We now turn to the techniques of [15][4]: using essentially the Eilenberg-Mac Lane spectrum associated to $\underline{G}$, Breen gets two spectral sequences $'E_r^{p,q}$ and $''E_r^{p,q}$ converging to the same abutment, with

- $''E_2^{p,1} = \mathrm{Ext}^p_{\mathcal{A}}(\underline{G}, L)$;
- $''E_2^{p,q}$ is torsion for $q \neq 1$;
- $'E_2^{p,q}$ is the p-th cohomology group of a complex involving terms of the form $H^q_{\text{ét}}(G^a, L)$.

[4]We thank L. Illusie for pointing out this reference.

(In [15], Breen works with the fppf topology but his methods carry over here without any change: see remark in *loc. cit.* top of p. 34.) It follows from [18, (2.1)] that $H^q_{\text{ét}}(G^a, L)$ is torsion for any $q > 0$: to see this easily, reduce to the case where L is constant by a transfer argument involving a finite extension of k. Hence $'E_2^{p,q}$ is torsion for $q > 0$. On the other hand, since G is geometrically connected, so are its powers G^a, which implies that $H^0(G^a, L) = H^0(k, L)$ for any a. Since the complex giving $'E_2^{*,0}$ is just the bar complex, we get that $'E_2^{0,0} = L(k)$ and $'E_2^{p;0} = 0$ for $p > 0$. Thus all degree > 0 terms of the abutment are torsion, and the conclusion follows. □

3.2. 1-motivic sheaves.

3.2.1. **Definition.** An étale sheaf $\mathcal{F}$ on $Sm(k)$ is 1-*motivic* if there is a morphism of sheaves

$$\underline{G} \xrightarrow{b} \mathcal{F} \tag{3.1}$$

where G is a semi-abelian variety and $\operatorname{Ker} b, \operatorname{Coker} b$ are discrete (see Definition 1.1.1).
We denote by Shv_0 the full subcategory of $\mathrm{Shv}_{\text{ét}}(Sm(k))[1/p]$ consisting of discrete sheaves and by Shv_1 the full subcategory of $\mathrm{Shv}_{\text{ét}}(Sm(k))[1/p]$ consisting of 1-motivic sheaves.

3.2.2. **Remark.** The category Shv_0 is equivalent to the category ${}^t\mathcal{M}_0[1/p]$ of Definition 1.1.1.

3.2.3. **Proposition.** *a) In Definition 3.2.1 we may choose b such that* $\operatorname{Ker} b$ *is torsion-free: we then say that b is* normalised.
b) Given two 1*-motivic sheaves $\mathcal{F}_1, \mathcal{F}_2$, normalised morphisms $b_i : \underline{G}_i \to \mathcal{F}_i$ and a map $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$, there exists a unique homomorphism of group schemes $\varphi_G : G_1 \to G_2$ such that the diagram*

$$\begin{array}{ccc} \underline{G}_1 & \xrightarrow{b_1} & \mathcal{F}_1 \\ {\scriptstyle \varphi_G}\downarrow & & {\scriptstyle \varphi}\downarrow \\ \underline{G}_2 & \xrightarrow{b_2} & \mathcal{F}_2 \end{array}$$

commutes.
c) Given a 1*-motivic sheaf $\mathcal{F}$, a pair (G, b) with b normalised is uniquely determined by $\mathcal{F}$.*
d) The categories Shv_0 *and* Shv_1 *are exact abelian subcategories of* $\mathrm{Shv}_{\text{ét}}(Sm(k))$.

Proof. a) If $\operatorname{Ker} b$ is not torsion-free, simply divide G by the image of its torsion.

b) We want to construct a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "a_1"] \arrow[d, "\varphi_L"'] & \underline{G}_1 \arrow[r, "b_1"] \arrow[d, "\varphi_G"'] & \mathcal{F}_1 \arrow[r, "c_1"] \arrow[d, "\varphi"'] & E_1 \arrow[r] \arrow[d, "\varphi_E"'] & 0 \\
0 \arrow[r] & L_2 \arrow[r, "a_2"] & \underline{G}_2 \arrow[r, "b_2"] & \mathcal{F}_2 \arrow[r, "c_2"] & E_2 \arrow[r] & 0
\end{tikzcd}
\tag{3.2}
$$

where $L_i = \operatorname{Ker} b_i$ and $E_i = \operatorname{Coker} b_i$. It is clear that $c_2\varphi b_1 = 0$: this proves the existence of φ_E. We also get a homomorphism of sheaves $\underline{G}_1 \to \underline{G}_2/L_2$, which lifts to $\varphi_G : \underline{G}_1 \to \underline{G}_2$ by Lemma 3.1.5, hence φ_L.

From the construction, it is clear that φ_E is uniquely determined by φ and that φ_L is uniquely determined by φ_G. It remains to see that φ_G is unique. Let φ'_G be another choice. Then $b_2(\varphi_G - \varphi'_G) = 0$, hence $(\varphi_G - \varphi'_G)(\underline{G}_1) \subseteq L_2$, which implies that $\varphi_G = \varphi'_G$.

c) Follows from b).

d) The case of Shv_0 is obvious. For Shv_1, given a map φ as in b), we want to show that $\mathcal{F}_3 = \operatorname{Ker}\varphi$ and $\mathcal{F}_4 = \operatorname{Coker}\varphi$ are 1-motivic. Let $G_3 = (\operatorname{Ker}\varphi_G)^0$ and $G_4 = \operatorname{Coker}\varphi_G$: we get induced maps $b_i : \underline{G}_i \to \mathcal{F}_i$ for $i = 3, 4$. An easy diagram chase shows that $\operatorname{Ker} b_i$ and $\operatorname{Coker} b_i$ are both discrete. $\square$

Here is an extension of Proposition 3.2.3 which elucidates the structure of Shv_1 somewhat:

3.2.4. **Theorem.** *a) Let* SAb *be the category of semi-abelian k-varieties. Then the inclusion functor*

$$
\begin{aligned}
\mathrm{SAb} &\to \mathrm{Shv}_1 \\
G &\mapsto \underline{G}
\end{aligned}
$$

has a right adjoint/left inverse γ; the counit of this adjunction is given by (3.1) *(with b normalised). The functor γ is faithful and "exact up to isogenies". For a morphism $\varphi \in \mathrm{Shv}_1$, $\gamma(\varphi) = \varphi_G$ is an isogeny if and only if* $\operatorname{Ker}\varphi$ *and* $\operatorname{Coker}\varphi \in \mathrm{Shv}_0$. *In particular, γ induces an equivalence of categories*

$$
\mathrm{Shv}_1 \,/\, \mathrm{Shv}_0 \xrightarrow{\sim} \mathrm{SAb} \otimes \mathbb{Q}
$$

where $\mathrm{SAb} \otimes \mathbb{Q}$ *is the category of semi-abelian varieties up to isogenies.*
b) The inclusion functor $\mathrm{Shv}_0 \to \mathrm{Shv}_1$ *has a left adjoint/left inverse π_0; the unit of this adjunction is given by* $\operatorname{Coker} b$ *in* (3.1). *The right exact functor*

$$
(\pi_0)_{\mathbb{Q}} : \mathrm{Shv}_1 \to \mathrm{Shv}_0 \otimes \mathbb{Q}
$$

has one left derived functor $(\pi_1)_{\mathbb{Q}}$ given by $\operatorname{Ker} b$ *in* (3.1).

Proof. a) The only delicate thing is the exactness of γ up to isogenies. This means that, given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of 1-motivic sheaves, the sequence

$$0 \to \gamma(\mathcal{F}') \to \gamma(\mathcal{F}) \to \gamma(\mathcal{F}'') \to 0$$

is half exact and the middle homology is finite. This follows from a chase in the diagram

$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r, "a'"] \arrow[d] & \underline{G}' \arrow[r, "b'"] \arrow[d] & \mathcal{F}' \arrow[r, "c'"] \arrow[d] & E' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, "a"] \arrow[d] & \underline{G} \arrow[r, "b"] \arrow[d] & \mathcal{F} \arrow[r, "c'"] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'' \arrow[r, "a''"] & \underline{G}'' \arrow[r, "b''"] & \mathcal{F}'' \arrow[r, "c''"] & E'' \arrow[r] & 0
\end{tikzcd}$$

of which we summarize the main points: (1) $G' \to G$ is injective because its kernel is the same as $\mathrm{Ker}(L' \to L)$. (2) $G \to G''$ is surjective because (i) $\mathrm{Hom}(\underline{G}'' \to \mathrm{Coker}(E' \to E)) = 0$ and (ii) if $L'' \to \mathrm{Coker}(\underline{G} \to \underline{G}'')$ is onto, then this cokernel is 0. (3) The middle homology is finite because the image of $\mathrm{Ker}(\underline{G}' \to \underline{G}) \to E'$ must be finite.

In b), the existence and characterisation of $(\pi_1)_{\mathbb{Q}}$ follows from the exactness of γ in a). □

3.2.5. **Remark.** One easily sees that π_1 does not exist integrally. Rather, it exists as a functor to the category of pro-objects of Shv_0. (Actually to a finer subcategory: compare [55].)

3.3. **Extensions of 1-motivic sheaves.** The aim of this subsection is to prove:

3.3.1. **Theorem.** *The categories* Shv_0 *and* Shv_1 *are thick in the abelian category* $\mathrm{Shv}_{\text{ét}}(Sm(k))[1/p]$.

Proof. For simplicity, let us write $\mathcal{A} := \mathrm{Shv}_{\text{ét}}(Sm(k))[1/p]$ as in Proposition 3.1.7. The statement is obvious for Shv_0. Let us now show that Shv_1 is closed under extensions in $\mathcal{A}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be as in (3.2) (no map given between them). We have to show that the injection

$$\mathrm{Ext}^1_{\mathrm{Shv}_1}(\mathcal{F}_2, \mathcal{F}_1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F}_2, \mathcal{F}_1) \tag{3.3}$$

is surjective. This is certainly so in the following special cases:

(1) $\mathcal{F}_1$ and $\mathcal{F}_2$ are semi-abelian varieties;
(2) $\mathcal{F}_2$ is semi-abelian and $\mathcal{F}_1$ is discrete (see Example 3.1.4).

For $m > 1$, consider

$$\mathcal{F}^m = \mathrm{Coker}(L_1 \xrightarrow{(a_1,m)} \underline{G}_1 \oplus L_1)$$

so that we have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{G}_1 & \xrightarrow{(1_{G_1},0)} & \mathcal{F}^m & \longrightarrow & L_1/m & \longrightarrow & 0 \\ 0 & \longrightarrow & L_1 & \xrightarrow{(a_1,0)} & \mathcal{F}^m & \longrightarrow & \underline{G}_1/L_1 \oplus L_1/m & \longrightarrow & 0. \end{array}$$

The first one shows that (3.3) is surjective for $(\mathcal{F}_2, \mathcal{F}_1) = (\underline{G}_2, \mathcal{F}^m)$. Let us now consider the commutative diagram with exact rows associated to the second one, for an unspecified m:

(3.4)

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathrm{Shv}_1}(\underline{G}_2, \mathcal{F}^m) & \to & \mathrm{Ext}^1_{\mathrm{Shv}_1}(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) & \longrightarrow & \mathrm{Ext}^2_{\mathrm{Shv}_1}(\underline{G}_2, L_1) \\ \wr\downarrow & & \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \mathcal{F}^m) & \to & \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) & \xrightarrow{\delta^m} & \mathrm{Ext}^2_{\mathcal{A}}(\underline{G}_2, L_1). \end{array}$$

Note that the composition

$$\mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \underline{G}_1/L_1) \to \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \underline{G}_1/L_1 \oplus L_1/m) \xrightarrow{\delta^m} \mathrm{Ext}^2_{\mathcal{A}}(\underline{G}_2, L_1)$$

coincides with the boundary map δ associated to the exact sequence

$$0 \to L_1 \to \underline{G}_1 \to \underline{G}_1/L_1 \to 0.$$

Let $e \in \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \underline{G}_1/L_1)$. By Proposition 3.1.7, $f = \delta(e)$ is torsion. Choose now m such that $mf = 0$. Then there exists $e' \in \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, L_1/m)$ which bounds to f via the Ext exact sequence associated to the exact sequence of sheaves

$$0 \to L_1 \xrightarrow{m} L_1 \to L_1/m \to 0.$$

Since $\delta^m(e, -e') = 0$, (3.4) shows that $(e, -e')$ comes from the left, which shows that (3.3) is surjective for $(\mathcal{F}_2, \mathcal{F}_1) = (\underline{G}_2, \underline{G}_1/L_1)$.

By Lemma 3.1.5, in the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{Shv}_1}(\underline{G}_2, \underline{G}_1/L_1) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{Shv}_1}(\underline{G}_2, \mathcal{F}_1) \\ \wr\downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \underline{G}_1/L_1) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(\underline{G}_2, \mathcal{F}_1) \end{array}$$

the horizontal maps are isomorphisms. Hence (3.3) is surjective for $\mathcal{F}_2 = \underline{G}_2$ and any $\mathcal{F}_1$.

To conclude, let $\mathcal{F}$ be an extension of $\mathcal{F}_2$ by $\mathcal{F}_1$ in $\mathcal{A}$. By the above, $\mathcal{F}' := b_2^*\mathcal{F}$ is 1-motivic as an extension of $\underline{G}_2$ by $\mathcal{F}_1$, and we have an exact sequence

$$0 \to L_2 \to \mathcal{F}' \to \mathcal{F} \to E_2 \to 0.$$

Let $b' : \underline{G} \to \mathcal{F}'$ be a normalised map (in the sense of Proposition 3.2.3) from a semi-abelian variety to $\mathcal{F}'$ and let $b : \underline{G} \to \mathcal{F}$ be its composite with the above map. It is then an easy exercise to check that $\operatorname{Ker} b$ and $\operatorname{Coker} b$ are both discrete. Hence $\mathcal{F}$ is 1-motivic. $\square$

3.3.2. **Remark.** We may similarly define 1-motivic sheaves for the fppf topology over $\operatorname{Spec} k$; as one easily checks, all the above results hold equally well in this context. This is also the case for §3.7 below.

In fact, let $\operatorname{Shv}_1^{\mathrm{fppf}}$ be the $\mathbb{Z}[1/p]$-linear category of fppf 1-motivic sheaves and $\pi : (\operatorname{Spec} k)_{\mathrm{fppf}} \to Sm(k)_{\text{ét}}$ be the projection functor. Then the functors π^* and π_* induce *quasi-inverse equivalences of categories* between Shv_1 and $\operatorname{Shv}_1^{\mathrm{fppf}}$. Indeed it suffices to check that $\pi_*\pi^*$ is naturally isomorphic to the identity on Shv_1: if $\mathcal{F} \in \operatorname{Shv}_1$ and we consider its normalised representation, then in the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & L & \longrightarrow & \underline{G} & \longrightarrow & \mathcal{F} & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\pi^*L & \longrightarrow & \pi_*\pi^*\underline{G} & \longrightarrow & \pi_*\pi^*\mathcal{F} & \longrightarrow & \pi_*\pi^*E & \longrightarrow & 0
\end{array}$$

the first, second and fourth vertical maps are isomorphisms and the lower sequence is still exact: both facts follow from [40, p. 14, Th. III.3.9].

In particular the restriction of π_* to $\operatorname{Shv}_1^{\mathrm{fppf}}$ is exact. Actually, $(R^q\pi_*)_{|\operatorname{Shv}_1^{\mathrm{fppf}}} = 0$ for $q > 0$ (use same reference).

3.4. **A basic example.**

3.4.1. **Proposition.** *Let $X \in Sm(k)$. Then the sheaf* $\operatorname{Pic}_{X/k}$ *is* 1-*motivic.*

Proof. Suppose first that X is smooth projective. Then $\operatorname{Pic}_{X/k}$ is an extension of the discrete sheaf $\operatorname{NS}_{X/k}$ (Néron-Severi) by the abelian variety $\operatorname{Pic}^0_{X/k}$ (Picard variety).

In general, we apply de Jong's theorem [27, Th. 4.1]: there exists a diagram

$$\begin{array}{ccc}
\widetilde{U} & \longrightarrow & \overline{X} \\
{\scriptstyle p}\downarrow & & \\
U & \longrightarrow & X
\end{array}$$

where the horizontal maps are open immersions, $\overline{X}$ is smooth projective and the vertical map is finite étale. Then we get a corresponding

diagram of Pics

$$\begin{array}{ccc} \mathrm{Pic}_{\widetilde{U}/k} & \longleftarrow & \mathrm{Pic}_{\overline{X}/k} \\ {\scriptstyle p^*}\uparrow & & \\ \mathrm{Pic}_{U/k} & \longleftarrow & \mathrm{Pic}_{X/k}\,. \end{array}$$

The horizontal morphisms are epimorphisms and their kernels are lattices. This already shows by Proposition 3.2.3 d) that $\mathrm{Pic}_{\widetilde{U}/k} \in \mathrm{Shv}_1$.

Consider the Čech spectral sequence associated to the étale cover p. It yields an exact sequence

$$0 \to \check{H}^1(p, \underline{H}^0_{\text{ét}}(\widetilde{U}, \mathbb{G}_m)) \to \mathrm{Pic}_{U/k} \to \check{H}^0(p, \mathrm{Pic}_{\widetilde{U}/k}) \\ \to \check{H}^2(p, \underline{H}^0_{\text{ét}}(\widetilde{U}, \mathbb{G}_m)).$$

All the $\check{H}^i$ are cohomology sheaves of complexes of objects of the abelian category Shv_1, hence belong to Shv_1; it then follows from Theorem 3.3.1 that $\mathrm{Pic}_{U/k} \in \mathrm{Shv}_1$, as well as $\mathrm{Pic}_{X/k}$. □

3.5. **Application: the Néron-Severi group of a smooth scheme.**

3.5.1. **Definition.** Let $X \in Sm(k)$.
a) Suppose that k is algebraically closed. Then we write $\mathrm{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence.
b) In general, we define $\mathrm{NS}_{X/k}$ as the étale sheaf on $Sm(k)$ given by

$$\mathrm{NS}_{X/k}(U) = \mathrm{NS}(X \times_k \overline{k(U)})^G$$

where $U \in Sm(k)$ is irreducible, $\overline{k(U)}$ is a separable closure of $k(U)$ and $G = Gal(\overline{k(U)}/k(U))$.

3.5.2. **Proposition.** *The natural map* $e : \underline{\mathrm{Pic}}_{X/k} \to \mathrm{NS}_{X/k}$ *identifies* $\mathrm{NS}_{X/k}$ *with* $\pi_0(\underline{\mathrm{Pic}}_{X/k})$ *(cf. Theorem 3.2.4 b)). In particular,* $\mathrm{NS}_{X/k} \in \mathrm{Shv}_0$.

Proof. It is well-known that cycles modulo algebraic equivalence are invariant by extension of algebraically closed base field. By Proposition 3.6.2 b), this implies that e induces a map $\overline{e} : \pi_0(\underline{\mathrm{Pic}}_{X/k}) \to \mathrm{NS}_{X/k}$, which is evidently epi. But let $\underline{\mathrm{Pic}}^0_{X/k} = \mathrm{Ker}\, e$: by [12, Lemma 7.10], $\mathrm{Pic}^0(X_{\overline{k}}) = \underline{\mathrm{Pic}}^0_{X/k}(\overline{k})$ is divisible, which forces $\overline{e}$ to be an isomorphism. □

3.5.3. **Remark.** In particular, $\mathrm{NS}(X)$ is finitely generated if k is algebraically closed: this was proven in [29, Th. 3] in a quite different way.

3.6. **Technical results on 1-motivic sheaves.**

3.6.1. **Proposition.** *The functor*

$$ev : \mathrm{Shv}_1 \to Ab$$
$$\mathcal{F} \mapsto \mathcal{F}(\overline{k})$$

to the category Ab of abelian groups is faithful, hence (cf. [14, Ch. 1, p. 44, prop. 1]*) "faithfully exact": a sequence $\mathcal{E}$ is exact if and only if $ev(\mathcal{E})$ is exact.*

Proof. The exactness of ev is clear. For faithfulness, let $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ be such that $ev(\varphi) = 0$. In $ev(3.2)$, we have $\varphi_G(\underline{G}_1(\overline{k})) \subseteq L_2(\overline{k})$; since the former group is divisible and the latter is finitely generated, $ev(\varphi_G) = 0$. Hence $\varphi_G = 0$. On the other hand, $ev(\varphi_E) = 0$, hence $\varphi_E = 0$. This implies that φ is of the form ψc_1 for $\psi : E_1 \to \mathcal{F}_2$. But $ev(\psi) = 0$, which implies that $\psi = 0$. □

The following strengthens Theorem 3.2.4 b):

3.6.2. **Proposition.** *a) Let G be a commutative algebraic k-group and let E be a $Gal(\overline{k}/k)$-module, viewed as an étale sheaf over $Sm(k)$ (E is not supposed to be constructible). Then* $\mathrm{Hom}(\underline{G}, E) = 0$.
b) Let $\mathcal{F} \in \mathrm{Shv}_1$ and E as in a). Then any morphism $\mathcal{F} \to E$ factors canonically through $\pi_0(\mathcal{F})$.

Proof. a) Thanks to Proposition 3.6.1 we may assume k algebraically closed. By Yoneda, $\mathrm{Hom}(\underline{G}, E)$ is a subgroup of $E(G)$ (it turns out to be the subgroup of multiplicative sections but we don't need this). Since $E(k) \xrightarrow{\sim} E(G)$, any homomorphism from $\underline{G}$ to E is constant, hence 0.

b) follows immediately from a) and Proposition 3.2.3. □

3.6.3. **Lemma.** *Let $\mathcal{F} \in \mathrm{Shv}_1$, K a separably closed extension of k and M/K an algebraic extension. Then the map $\mathcal{F}(K) \to \mathcal{F}(M)$ is injective.*

Proof. Consider a normalised representation of $\mathcal{F}$:

$$0 \to L \to \underline{G} \xrightarrow{b} \mathcal{F} \to E \to 0. \tag{3.5}$$

The lemma then follows from an elementary chase in the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L(K) & \longrightarrow & G(K) & \longrightarrow & \mathcal{F}(K) & \longrightarrow & E(K) & \longrightarrow & 0 \\
 & & \wr\downarrow & & \text{mono}\downarrow & & \downarrow & & \wr\downarrow & & \\
0 & \longrightarrow & L(M) & \longrightarrow & G(M) & \longrightarrow & \mathcal{F}(M) & \longrightarrow & E(M) & \longrightarrow & 0.
\end{array}$$

□

3.6.4. **Definition.** We denote by ${}^t\mathrm{AbS}(k) = {}^t\mathrm{AbS}$ the $\mathbb{Z}[1/p]$-linear category of commutative k-group schemes G such that G^0 is semi-abelian and $\pi_0(G)$ is discrete. An object of ${}^t\mathrm{AbS}$ is called a *semi-abelian scheme with torsion.*

3.6.5. **Proposition.** *The functor*

$$\begin{aligned} {}^t\mathrm{AbS} &\to \mathrm{Shv}_1 \\ G &\mapsto \underline{G} \end{aligned}$$

has a left adjoint/left inverse Ω.

Proof. Let $\mathcal{F} \in \mathrm{Shv}_1$ with normalised representation (3.5). As the set of closed subgroups of $H \subseteq G$ is Artinian, there is a minimal H such that the composition

$$L \to \underline{G} \to \underline{G}/\underline{H}$$

is trivial. Then $\mathcal{F}/b(\underline{H})$ represents an object $\Omega(\mathcal{F})$ of ${}^t\mathrm{AbS}$ and it follows from Proposition 3.2.3 b) that the universal property is satisfied. (In other words, $\Omega(\mathcal{F})$ is the quotient of $\mathcal{F}$ by the Zariski closure of L in G.) $\square$

3.6.6. **Proposition.** *Let* $f : \mathcal{F}_1 \to \mathcal{F}_2$ *be a morphism in* Shv_1. *Assume that for any* $n > 1$ *prime to* p, f *is an isomorphism on* n*-torsion and injective on* n*-cotorsion. Then* f *is injective with lattice cokernel. If* f *is even bijective on* n*-cotorsion, it is an isomorphism.*

Proof. a) We first treat the special case where $\mathcal{F}_1 = 0$. Consider multiplication by n on the normalised presentation of $\mathcal{F}_2$:

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "n_L"'] & G \arrow[r] \arrow[d, "n_G"'] & \mathcal{F}_2 \arrow[r] \arrow[d, "n"'] & E \arrow[r] \arrow[d, "n_E"'] & 0 \\ 0 \arrow[r] & L \arrow[r] & G \arrow[r] & \mathcal{F}_2 \arrow[r] & E \arrow[r] & 0. \end{tikzcd}$$

Since L is torsion-free, n_G is injective for all n, hence $G = 0$ and $\mathcal{F}_2 = E$. If moreover multiplication by n is surjective for any n, we have $\mathcal{F}_2 = 0$ since E is finitely generated.

b) The general case. Split f into two short exact sequences:

$$\begin{aligned} 0 \to K \to \mathcal{F}_1 \to I \to 0 \\ 0 \to I \to \mathcal{F}_2 \to C \to 0. \end{aligned}$$

We get torsion/cotorsion exact sequences

$$\begin{aligned} 0 \to {}_nK \to {}_n\mathcal{F}_1 \to {}_nI \to K/n \to \mathcal{F}_1/n \to I/n \to 0 \\ 0 \to {}_nI \to {}_n\mathcal{F}_2 \to {}_nC \to I/n \to \mathcal{F}_2/n \to C/n \to 0. \end{aligned}$$

A standard diagram chase successively yields ${}_nK = 0$, ${}_n\mathcal{F}_1 \xrightarrow{\sim} {}_nI \xrightarrow{\sim} {}_n\mathcal{F}_2$, $\mathcal{F}_1/n \xrightarrow{\sim} I/n$, $K/n = 0$ and ${}_nC = 0$. By a), we find $K = 0$ and C a lattice, which is what we wanted. □

3.7. **Presenting 1-motivic sheaves by group schemes.** In this subsection, we give another description of the category Shv_1; it will be used in the next subsection.

3.7.1. **Definition.** Let AbS be the $\mathbb{Z}[1/p]$-linear category of commutative k-group schemes G such that $\pi_0(G)$ is a lattice and G^0 is a semi-abelian variety (it is a full subcategory of ${}^t\mathrm{AbS}$). We denote by S_1^{eff} the full subcategory of ${}^t\mathrm{AbS}^{[-1,0]}$ consisting of those complexes $F_\cdot = [F_1 \to F_0]$ such that

(i) F_1 is discrete (*i.e.* in ${}^t\mathcal{M}_0$);
(ii) F_0 is of the form $L_0 \oplus G$, with $L_0 \in {}^t\mathcal{M}_0$ and $G \in \mathrm{SAb}$;
(iii) $F_1 \to F_0$ is a monomorphism;
(iv) $\mathrm{Ker}(F_1 \to L_0)$ is free.

We call S_1^{eff} the *category of presentations.*

We shall view S_1^{eff} as a full subcategory of $\mathrm{Shv}_1^{[-1,0]}$ via the functor $G \mapsto \underline{G}$ which sends a group scheme to the associated representable sheaf. In this light, $F_\cdot$ may be viewed as a *presentation* of $\mathcal{F} := H_0(\underline{F_\cdot})$. In the next definition, quasi-isomorphisms are also understood from this viewpoint.

3.7.2. **Definition.** We denote by Σ the collection of quasi-isomorphisms of S_1^{eff}, by $\overline{S}_1^{\mathrm{eff}}$ the homotopy category of S_1^{eff} (Hom groups quotiented by homotopies) and by $S_1 = \Sigma^{-1}\overline{S}_1^{\mathrm{eff}}$ the localisation of $\overline{S}_1^{\mathrm{eff}}$ with respect to (the image of) Σ.

The functor $F_\cdot \mapsto H_0(F_\cdot)$ induces a functor

$$h_0 : S_1 \to \mathrm{Shv}_1\,. \tag{3.6}$$

Let $F_\cdot = (F_1, L_0, G)$ be a presentation of $\mathcal{F} \in \mathrm{Shv}_1$. Let $L = \mathrm{Ker}(F_1 \to L_0)$ and $E = \mathrm{Coker}(F_1 \to L_0)$. Then we clearly have an exact sequence

$$0 \to L \to \underline{G} \to \mathcal{F} \to E \to 0. \tag{3.7}$$

3.7.3. **Lemma.** *Let $F_\cdot = (F_1, L_0, G) \in S_1^{\mathrm{eff}}$. Then, for any finite Galois extension ℓ/k such that L_0 is constant over ℓ, there exists a* q.i. $\widetilde{F}_\cdot \to F_\cdot$, *with $\widetilde{F}_\cdot = [\widetilde{F}_1 \xrightarrow{u_0} \widetilde{L}_0 \oplus G]$ such that u_0 is diagonal and $\widetilde{L}_0$ is a free $Gal(\ell/k)$-module.*

Proof. Just take for $\widetilde{L}_0$ a free module projecting onto L_0 and for $\widetilde{F}_1 \to \widetilde{L}_0$ the pull-back of $F_1 \to L_0$. □

3.7.4. **Lemma.** *The set Σ admits a calculus of right fractions within $\overline{S}_1^{\mathrm{eff}}$ in the sense of (the dual of)* [20, Ch. I, §2.3].

Proof. The statement is true by Lemma A.2.2 if we replace S_1^{eff} by $\mathrm{Shv}_1^{[-1,0]}$; but one easily checks that the constructions in the proof of Lemma A.2.2 preserve S_1^{eff}. □

3.7.5. **Proposition.** *The functor h_0 of* (3.6) *is an equivalence of categories. In particular, S_1 is abelian.*

Proof. Step 1. h_0 is essentially surjective. Let $\mathcal{F} \in \mathrm{Shv}_1$ and let (3.7) be the exact sequence attached to it by Proposition 3.2.3 b). We shall construct a presentation of $\mathcal{F}$. from (3.7). Choose elements $f_1, \dots, f_r \in \mathcal{F}(\overline{k})$ whose images generate $E(\overline{k})$. Let ℓ/k be a finite Galois extension such that all f_i belong to $\mathcal{F}(\ell)$, and let $\Gamma = Gal(\ell/k)$. Let $\widetilde{L}_0 = \mathbb{Z}[\Gamma]^r$ and define a morphism of sheaves $\widetilde{L}_0 \to \mathcal{F}$ by mapping the i-th basis element to f_i. Then $\mathrm{Ker}(\widetilde{L}_0 \to E)$ maps to $\underline{G}/L$. Let M_0 be the kernel of this morphism, and let $L_0 = \widetilde{L}_0/M_0$. Then $\widetilde{L}_0 \twoheadrightarrow E$ factors into a morphism $L_0 \twoheadrightarrow E$, whose kernel K injects into $\underline{G}/L$.

Pick now elements $g_1, \dots, g_s \in G(\overline{k})$ whose image in $G(\overline{k})/L(\overline{k})$ generate the image of $K(\overline{k})$, and $g_{s+1}, \dots, g_t \in G(\overline{k})$ be generators of the image of $L(\overline{k})$. Let ℓ'/k be a finite Galois extension such that all the g_i belong to $G(\ell')$, and let $\Gamma' = Gal(\ell'/k)$. Let $\widetilde{F}_1 = \mathbb{Z}[\Gamma']^t$, and define a map $f : \widetilde{F}_1 \to G$ by mapping the i-th basis element to g_i. By construction, $f^{-1}(L) = \mathrm{Ker}(\widetilde{F}_1 \twoheadrightarrow K)$ and $f' : f^{-1}(L) \to L$ is onto. Let M_1 be the kernel of f' and $F_1 = \widetilde{F}_1/M_1$: then $\widetilde{F}_1 \to K$ factors through F_1 and $\mathrm{Ker}(F_1 \twoheadrightarrow K) = \mathrm{Ker}(F_1 \to L_0) \xrightarrow{\sim} L$. In particular, condition (iii) of Definition 3.7.1 is verified.

Step 2. h_0 is faithful. Let $f : F_. \to F'_.$ be a map in S_1 such that $h_0(f) = 0$. By Lemma 3.7.4, we may assume that f is an effective map (*i.e.* comes from S_1^{eff}). We have $f(L_0 \oplus G) \subseteq \mathrm{Im}(L'_1 \to L'_0 \oplus G')$, hence $f_{|G} = 0$ and $f(L_0)$ is contained in $\mathrm{Im}(L'_1 \to L'_0 \oplus G')$. Pick a finite Galois extension ℓ/k such that L_0 and L'_1 are constant over ℓ. By Lemma 3.7.3, take a q.i. $u : [\widetilde{F}_1 \to \widetilde{L}_0] \to [F_1 \to L_0]$ such that $\widetilde{L}_0$ is $Gal(\ell/k)$-free. Then the composition $\widetilde{L}_0 \to L_0 \to \mathrm{Im}(L'_1 \to L'_0 \oplus G')$ lifts to a map $s : \widetilde{L}_0 \to L'_1$, which defines a homotopy between 0 and fu.

Step 3. h_0 is full. Let $F_., F'_. \in S_1$ and let $\varphi : \mathcal{F} \to \mathcal{F}'$, where $\mathcal{F} = h_0(F_.)$ and $\mathcal{F}' = h_0(F'_.)$. In particular, we get a map $\varphi_G : \underline{G} \to \underline{G}'$

and a map $\psi : L_0 \to L'_0 \oplus \underline{G}'/F'_1$. Let ℓ/k be a finite Galois extension such that F'_1 is constant over ℓ. Pick a q.i. $u : \widetilde{F}_. \to F_.$ as in Lemma 3.7.3 such that $\widetilde{L}_0$ is $Gal(\ell/k)$-free. Then $\psi \circ u$ lifts to a map $\widetilde{\psi} : \widetilde{L}_0 \to L'_0 \oplus \underline{G}'$. The map

$$f = (\widetilde{\psi}, \varphi_G) : \widetilde{L}_0 \oplus G \to L'_0 \oplus G'$$

sends $\widetilde{F}_1$ into F'_1 by construction, hence yields a map $f : \widetilde{F}_. \to F'_.$ such that $h_0(fu^{-1}) = \varphi$. □

3.7.6. **Corollary.** *The obvious functor*

$$S_1 \to D^b(\mathrm{Shv}_1)$$

is fully faithful.

Proof. The composition of this functor with H_0 is the equivalence h_0 of Proposition 3.7.5. Therefore it suffices to show that the restriction of H_0 to the image of S_1 is faithful. This is obvious, since the objects of this image are homologically concentrated in degree 0. □

3.8. **The transfer structure on 1-motivic sheaves.** Recall the category AbS from Definition 3.7.1. Lemma 1.3.2 provides a functor

$$\rho : \mathrm{AbS} \to \mathrm{HI}_{\text{ét}}[1/p].$$

The aim of this subsection is to prove:

3.8.1. **Proposition.** *This functor extends to a full embedding*

$$\rho : \mathrm{Shv}_1 \hookrightarrow \mathrm{HI}_{\text{ét}}[1/p]$$

where $\mathrm{HI}_{\text{ét}}$ *is the category of Definitions 1.3.1 and D.1.2. This functor is exact with thick image (i.e. stable under extensions).*

Proof. By Proposition 3.7.5, it suffices to construct a functor $\rho : S_1 \to \mathrm{HI}_{\text{ét}}[1/p]$. First define a functor $\widetilde{\rho} : S_1^{\text{eff}} \to \mathrm{HI}_{\text{ét}}[1/p]$ by

$$\widetilde{\rho}([F_1 \to F_0]) = \mathrm{Coker}(\rho(F_1) \to \rho(F_0)).$$

Note that the forgetful functor $f : \mathrm{HI}_{\text{ét}} \to \mathrm{Shv}_{\text{ét}}(Sm(k))$ is faithful and exact, hence conservative. This first gives that $\widetilde{\rho}$ factors into the desired ρ.

Proposition 3.2.3 d) says that $f\rho$ is (fully faithful and) exact. Since f is faithful, ρ is fully faithful and exact.

It remains to show that ρ is thick. Recall that Shv_1 is thick in $\mathrm{Shv}_{\text{ét}}(Sm(k))[1/p]$ by Theorem 3.3.1. Since f is exact, we are then left to show:

3.8.2. **Lemma.** *The transfer structure on a sheaf* $\mathcal{F} \in \mathrm{Shv}_1$ *is unique.*

Proof. Let μ be the transfer structure on $\mathcal{F}$ given by the beginning of the proof of Proposition 3.8.1, and let μ' be another transfer structure. Thus, for $X, Y \in Sm(k)$, we have two homomorphisms

$$\mu, \mu' : \mathcal{F}(X) \otimes c(Y, X) \to \mathcal{F}(Y)$$

and we want to show that they are equal. We may clearly assume that Y is irreducible.

Let $F = k(Y)$ be the function field of Y. Since $\mathcal{F}$ is a homotopy invariant Zariski sheaf with transfers, the map $\mathcal{F}(Y) \to \mathcal{F}(F)$ is injective by [62, Cor. 4.19]. Thus we may replace Y by F.

Moreover, it follows from the fact that $\mathcal{F}$ is an étale sheaf and from Lemma 3.6.3 that $\mathcal{F}(F) \hookrightarrow \mathcal{F}(\overline{F})$, where $\overline{F}$ is an algebraic closure of F. Thus, we may even replace Y by $\overline{F}$.

Then the group $c(Y, X)$ is replaced by $c(\overline{F}, X) = Z_0(X_{\overline{F}})$. Since $\overline{F}$ is algebraically closed, all closed points of $X_{\overline{F}}$ are rational, hence all finite correspondences from $\operatorname{Spec} \overline{F}$ to X are linear combinations of morphisms. Therefore μ and μ' coincide on them. □

This concludes the proof of Proposition 3.8.1. □

3.9. **1-motivic sheaves and** DM**.** Recall from Definition D.1.2 the subcategory $\mathrm{HI}^s_{\text{ét}} \subset \mathrm{HI}_{\text{ét}}$ of strictly homotopy invariant étale sheaves with transfers: this is a full subcategory of $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$. By Proposition D.1.4, we have

$$\mathrm{HI}^s_{\text{ét}} = \{\mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \mid \mathcal{F} \in \mathrm{HI}_{\text{ét}}\}.$$

The introduction of Shv_1 is now made clear by the following

3.9.1. **Theorem.** *Let* $\mathrm{Shv}^s_1 \subseteq \mathrm{HI}^s_{\text{ét}}$ *be the full subcategory image of* Shv_1 *by the functor* $\mathcal{F} \mapsto \mathcal{F}[1/p]$ *of Lemma D.1.3. Then* $\mathrm{Shv}_1 \to \mathrm{Shv}^s_1$ *is an equivalence of categories. Moreover, let* $M \in d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$. *Then for all* $i \in \mathbb{Z}$, $\mathcal{H}_i(M) \in \mathrm{Shv}^s_1$. *In particular, there is a t-structure on* $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$, *with heart* Shv^s_1*; it is induced by the homotopy t-structure of Corollary D.3.3 on* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{fr},\text{ét}}$ *(see Definition D.2.1 and Theorem D.2.2).*

Proof. The first assertion is clear since $\mathrm{HI}^s_{\text{ét}} \to \mathrm{HI}_{\text{ét}} \to \mathrm{HI}_{\text{ét}}[1/p]$ is an equivalence of categories by Proposition D.1.4. By Proposition 3.2.3 d), we reduce to the case $M = M(C)$, $C \xrightarrow{p} \operatorname{Spec} k$ a smooth projective curve. By Proposition 2.5.2, the cohomology sheaves of $M(C)$ belong to Shv^s_1: for $\mathcal{H}^1$ this is clear and for $\mathcal{H}^2$ it is a (trivial) special case of Proposition 3.4.1. □

Note that the functor $\mathcal{M}_1 \to \mathrm{HI}^{[0,1]}_{\text{ét}}$ refines to an functor $\mathcal{M}_1 \to \mathrm{Shv}_1^{[0,1]}$, hence, using Lemma A.2.1 again, we get a composed triangulated functor

$$\mathrm{tot} : D^b(\mathcal{M}_1[1/p]) \to D^b(\mathrm{Shv}_1^{[0,1]}) \to D^b(\mathrm{Shv}_1) \tag{3.8}$$

refining the one from Lemma 2.2.1 (same proof). We then get:

3.9.2. **Corollary.** *The two functors*

$$D^b(\mathcal{M}_1[1/p]) \xrightarrow{\mathrm{tot}} D^b(\mathrm{Shv}_1) \to d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$$

are equivalences of categories.

Proof. For the composition, this is Theorem 2.1.2. This implies that the second functor is full and essentially surjective, and to conclude, it suffices by Lemma A.1.1 to see that it is conservative. But this follows immediately from Theorem 3.9.1. Note that by Proposition D.1.4 we have that $\mathrm{Shv}_1^s \simeq \mathrm{Shv}_1$ (see Theorem 3.9.1 for the definition of Shv_1^s). □

3.9.3. **Definition.** We call the t-structure defined on $D^b(\mathcal{M}_1[1/p])$ or on $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ by Corollary 3.9.2 the *homotopy t-structure.*

3.9.4. **Remark** (*cf.* §1.9)**.** In [8], A. Bertapelle defines an *integral* version of the category $\mathrm{Shv}_1^{\mathrm{fppf}}$ of Remark 3.3.2 and constructs an equivalence of categories

$$D^b(\mathcal{M}_1) \simeq D^b(\mathrm{Shv}_1^{\mathrm{fppf}})$$

without inverting p (not going via DM). Hence the homotopy t-structure of Definition 3.9.3 exists integrally.

3.10. **Comparing t-structures.** In this subsection, we want to compare the homotopy t-structure of Definition 3.9.3 with the motivic t-structure of Theorem 1.6.1 a).

Let $C \in D^b(\mathcal{M}_1[1/p])$. Recall from 1.8.7 the notation ${}^tH_n(C) \in {}^t\mathcal{M}_1[1/p]$ for its homology relative to the torsion 1-motivic t-structure from Theorem 1.6.1. We also write $\mathcal{H}^n(C) \in \mathrm{Shv}_1$ for its cohomology objects of relative to the homotopy t-structure.

Consider the functor tot of (3.8). Let $\mathcal{F}$ be a 1-motivic sheaf and (G, b) its associated normalised pair (see Proposition 3.2.3 a)). Let $L = \operatorname{Ker} b$ and $E = \operatorname{Coker} b$. In $D^b(\mathcal{M}_1[1/p])$, we have an exact triangle

$$[L \to G][1] \to \mathrm{tot}^{-1}(\mathcal{F}) \to [E \to 0] \xrightarrow{+1}$$

(see Corollary 3.9.2). This shows:

3.10.1. **Lemma.** *We have*

$$\begin{aligned} {}^tH_0(\mathrm{tot}^{-1}(\mathcal{F})) &= [E \to 0] \\ {}^tH_1(\mathrm{tot}^{-1}(\mathcal{F})) &= [L \to G] \\ {}^tH_q(\mathrm{tot}^{-1}(\mathcal{F})) &= 0 \textit{ for } q \neq 0, 1. \quad \square \end{aligned}$$

On the other hand, given a 1-motive (with torsion or cotorsion) $M = [L \xrightarrow{f} G]$, we clearly have

$$\begin{aligned} \mathcal{H}^0(M) &= \mathrm{Ker}\, f \\ \mathcal{H}^1(M) &= \mathrm{Coker}\, f \\ \mathcal{H}^q(M) &= 0 \text{ for } q \neq 0, 1. \end{aligned} \tag{3.9}$$

by considering it as a complex of length 1 of 1-motivic sheaves.

In particular, ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1 = \mathrm{Shv}_0$, ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1[-1]$ consists of quotients of semi-abelian varieties by discrete subsheaves and ${}^t\mathcal{M}_1[1/p] \cap \mathrm{Shv}_1[q] = 0$ for $q \neq 0, -1$.

Here is a more useful result relating $\mathcal{H}^i$ with the two motivic t-structures:

3.10.2. **Proposition.** *Let $C \in D^b(\mathcal{M}_1[1/p])$; write $[L_i \xrightarrow{u_i} G_i]$ for ${}_tH_i(C)$ and $[L^i \xrightarrow{u^i} G^i]$ for ${}^tH^i(C)$*[5]*. Then we have exact sequences in* Shv_1*:*

$$\begin{aligned} \cdots \to L_{i+1} \xrightarrow{u_{i+1}} G_{i+1} \to \mathcal{H}_i(C) \to L_i \xrightarrow{u_i} G_i \to \dots \\ \cdots \to L^{i-1} \xrightarrow{u^{i-1}} G^{i-1} \to \mathcal{H}^i(C) \to L^i \xrightarrow{u^i} G^i \to \dots \end{aligned}$$

Proof. For the first one, argue by induction on the length of C with respect to the motivic t-structure with heart ${}_t\mathcal{M}_1[1/p]$ (the case of length 0 is (3.9)). For the second one, same argument with the other motivic t-structure. $\square$

Note finally that the homotopy t-structure is far from being invariant under Cartier duality: this can easily be seen by using Proposition 3.7.5.

4. Comparing two dualities

In this section, we show that the classical Cartier duality for 1-motives is compatible with a "motivic Cartier duality" on triangulated motives, described in Definition 4.4.2 below.

[5]Note that (L_i, G_i) and (L^i, G^i) are determined only up to the relevant q.i.'s.

4.1. **Biextensions of 1-motives.** This material is presumably well-known to experts, and the only reason why we write it up is that we could not find it in the literature. Exceptionally, we put 1-motives in degrees -1 and 0 in this subsection and in the next one, for compatibility with Deligne's conventions in [16].

Recall (see [16, §10.2]) that for $M_1 = [L_1 \xrightarrow{u_1} G_1]$ and $M_2 = [L_2 \xrightarrow{u_2} G_2]$ two complexes of abelian sheaves over some site $\mathcal{S}$, concentrated in degrees -1 and 0, a *biextension* of M_1 and M_2 by an abelian sheaf H is given by a (Grothendieck) biextension P of G_1 and G_2 by H and a pair of compatible trivializations of the biextensions of $L_1 \times G_2$ and $G_1 \times L_2$ obtained by pullbacks. Let $\mathrm{Biext}(M_1, M_2; H)$ denote the group of isomorphism classes of biextensions. We have the following fundamental formula (see [16, §10.2.1]):

$$\mathrm{Biext}(M_1, M_2; H) = \mathbb{E}\mathrm{xt}^1_{\mathcal{S}}(M_1 \overset{L}{\otimes} M_2, H). \tag{4.1}$$

Suppose now that M_1 and M_2 are two Deligne 1-motives. Since G_1 and G_2 are smooth, we may compute biextensions by using the étale topology. Hence, we shall take here

$$\mathcal{S} = Sm(k)_{\text{ét}}.$$

Let M_2^* denote the Cartier dual of M_2 as constructed by Deligne (see [16, §10.2.11] and [5, §0]) along with the Poincaré biextension $P_{M_2} \in \mathrm{Biext}(M_2, M_2^*; \mathbb{G}_m)$. We also have the transpose ${}^tP_{M_2} = P_{M_2^*} \in \mathrm{Biext}(M_2^*, M_2; \mathbb{G}_m)$. Pulling back ${}^tP_{M_2}$ yields a map

$$\begin{aligned} \gamma_{M_1,M_2} : \mathrm{Hom}(M_1, M_2^*) &\to \mathrm{Biext}(M_1, M_2; \mathbb{G}_m) \\ \varphi &\mapsto (\varphi \times 1_{M_1})^*({}^tP_{M_2}) \end{aligned} \tag{4.2}$$

which is clearly additive and natural in M_1.

4.1.1. **Proposition.** *The map γ_{M_1,M_2} yields a natural equivalence of functors from 1-motives to abelian groups, i.e. the functor*

$$M_1 \mapsto \mathrm{Biext}(M_1, M_2; \mathbb{G}_m)$$

on 1-motives is representable by the Cartier dual M_2^*. Moreover, γ_{M_1,M_2} is also natural in M_2.

Proof. We start with a few lemmas:

4.1.2. **Lemma.** *For $q \leq 0$, we have*

$$\mathrm{Hom}_{\overline{k}}(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[q]) = 0.$$

Proof. For $q < 0$ this is trivial and for $q = 0$ this is [16, Lemma 10.2.2.1]. □

4.1.3. **Lemma.** *Let $\overline{k}$ be an algebraic closure of k and $G = Gal(\overline{k}/k)$. Then*

$$\mathrm{Hom}_k(M_1, M_2^*) \xrightarrow{\sim} \mathrm{Hom}_{\overline{k}}(M_1, M_2^*)^G$$
$$\mathrm{Biext}_k(M_1, M_2; \mathbb{G}_m) \xrightarrow{\sim} \mathrm{Biext}_{\overline{k}}(M_1, M_2; \mathbb{G}_m)^G.$$

Proof. The first isomorphism is obvious. For the second, thanks to (4.1) we may use the spectral sequence

$$H^p(G, \mathrm{Hom}_{\overline{k}}(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[q])) \Rightarrow \mathrm{Hom}_k(M_1 \overset{L}{\otimes} M_2, \mathbb{G}_m[p+q]).$$

(This is the only place in the proof of Proposition 4.1.1 where we shall use (4.1).) The assertion then follows from Lemma 4.1.2. □

Lemma 4.1.3, reduces the proof of Proposition 4.1.1 to the case where *k is algebraically closed*, which we now assume. The following is a special case of this proposition:

4.1.4. **Lemma.** *The map γ_{M_1,M_2} is an isomorphism when M_1 and M_2 are abelian varieties A_1 and A_2, and is natural in A_2.*

Again this is certainly well-known and mentioned explicitly as such in [SGA7, VII, p. 176, (2.9.6.2)]. Unfortunately we have not been able to find a proof in the literature, so we provide one for the reader's convenience.

Proof. We shall use the universal property of the Poincaré bundle [42, Th. p. 125]. Let $P \in \mathrm{Biext}(A_1, A_2)$. Then

(1) $P_{|A_1 \times \{0\}}$ is trivial;
(2) $P_{|\{a\} \times A_2} \in \mathrm{Pic}^0(A_2)$ for all $a \in A_1(k)$.

Indeed, (1) follows from the multiplicativity of P on the A_2-side. For (2) we offer two proofs (note that they use multiplicativity on different sides):

- By multiplicativity on the A_1-side, $a \mapsto P_{|\{a\} \times A_2}$ gives a homomorphism $A_1(k) \to \mathrm{Pic}(A_2)$. Composing with the projection to $\mathrm{NS}(A_2)$ gives a homomorphism from a divisible group to a finitely generated group, which must be trivial.
- (More direct but more confusing): we have to prove that $T_b^* P_{|\{a\} \times A_2} = P_{|\{a\} \times A_2}$ for all $b \in A_2(k)$. Using simply a to

denote the section $\operatorname{Spec} k \to A_1$ defined by a, we have a commutative diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{a\times 1_{A_2}} & A_1 \times A_2 \\ {\scriptstyle T_b}\downarrow & & \downarrow{\scriptstyle 1_{A_1}\times T_b} \\ A_2 & \xrightarrow{a\times 1_{A_2}} & A_1 \times A_2. \end{array}$$

Let $\pi_1 : A_1 \to \operatorname{Spec} k$ and $\pi_2 : A_2 \to \operatorname{Spec} k$ be the two structural maps. Then by multiplicativity on the A_2-side, an easy computation gives

$$(1_{A_1} \times T_b)^* P = P \otimes (1_{A_1} \times (\pi_2 \circ b))^* P.$$

Applying $(a \times 1_{A_2})^*$ to this gives the result since $(a \times 1_{A_2})^* \circ (1_{A_1} \times (\pi_2 \circ b))^* P = \pi_{A_2}^* P_{a,b}$ is trivial.

By the universal property of the Poincaré bundle, there exists a unique morphism[6] $f : A_1 \to A_2'$ such that $P \simeq (f \times 1_{A_2})^*({}^tP_{A_2})$. It remains to see that f is a homomorphism: for this it suffices to show that $f(0) = 0$. But

$$\begin{aligned} \mathcal{O}_{A_2} \simeq P_{|\{0\}\times A_2} &= (0 \times 1_{A_2})^* \circ (f \times 1_{A_2})^*({}^tP_{A_2}) \\ &= (f(0) \times 1_{A_2})^*({}^tP_{A_2}) = (P_{A_2})_{|A_2\times\{f(0)\}} = f(0) \end{aligned}$$

where the first isomorphism holds by multiplicativity of P on the A_1-side.

Finally, the naturality in A_2 reduces to the fact that, if $f : A_1 \to A_2'$, then $(f \times 1_{A_2})^*({}^tP_{A_2}) \simeq (1_{A_1} \times f')^*(P_{A_1})$. This follows from the description of f' on k-points as the pull-back by f of line bundles. □

We also have the following easier

4.1.5. **Lemma.** *Let L be a lattice and A an abelian variety. Then the natural map*

$$\begin{aligned} \operatorname{Hom}(L, A') &\to \operatorname{Biext}(L[0], A[0]; \mathbb{G}_m) \\ f &\mapsto (1 \times f)^*({}^tP_A) \end{aligned}$$

is bjiective.

Proof. Reduce to $L = \mathbb{Z}$; then the right hand side can be identified with $\operatorname{Ext}(A, \mathbb{G}_m)$ and the claim comes from the Weil-Barsotti formula. □

[6]For convenience we denote here by A' the dual of an abelian variety A and by f' the dual of a homomorphism f of abelian varieties.

Let us now come back to our two 1-motives M_1, M_2. We denote by L_i, T_i and A_i the discrete, toric and abelian parts of M_i for $i = 1, 2$. Let us further denote by $u'_i : L'_i \to A'_i$ the map corresponding to G_i under the isomorphism $\mathrm{Ext}(A_i, T_i) \simeq \mathrm{Hom}(L'_i, A'_i)$ where $L'_i = \mathrm{Hom}(T_i, \mathbb{G}_m)$ and $A'_i = \mathrm{Pic}^0(A_i)$.

We shall use the symmetric avatar $(L_i, A_i, L'_i, A'_i, \psi_i)$ of M_i (see [16, 10.2.12] or [5, p. 17]): recall that ψ_i denotes a certain section of the Poincaré biextension $P_{A_i} \in \mathrm{Biext}(A_i, A'_i; \mathbb{G}_m)$ over $L_i \times L'_i$. The symmetric avatar of the Cartier dual is $(L'_i, A'_i, L_i, A_i, \psi_i^t)$. By *loc. cit.* a map of 1-motives $\varphi : M_1 \to M_2^*$ is equivalent to a homomorphism $f : A_1 \to A'_2$ of abelian varieties and, if f' is the dual of f, liftings g and g' of fu_1 and $f'u_2$ respectively, *i.e.* to the following commutative squares

$$\begin{tikzcd} L_1 \arrow[r, "g"] \arrow[d, "u_1"'] & L'_2 \arrow[d, "u'_2"'] \\ A_1 \arrow[r, "f"] & A'_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L_2 \arrow[r, "g'"] \arrow[d, "u_2"'] & L'_1 \arrow[d, "u'_1"'] \\ A_2 \arrow[r, "f'"] & A'_1 \end{tikzcd} \tag{4.3}$$

under the condition that

$$(1_{L_1} \times g')^* \psi_1 = (g \times 1_{L_2})^{*t} \psi_2 \text{ on } L_1 \times L_2. \tag{4.4}$$

Now let (P, τ, σ) be a biextension of M_1 and M_2 by $\mathbb{G}_m$, *i.e.* a biextension $P \in \mathrm{Biext}(G_1, G_2; \mathbb{G}_m)$, a section τ on $L_1 \times G_2$ and a section σ on $G_1 \times L_2$ such that

$$\tau \mid_{L_1 \times L_2} = \sigma \mid_{L_1 \times L_2} . \tag{4.5}$$

We have to show that $(P, \tau, \sigma) = (\varphi \times 1)^*({}^tP_{A_2}, \tau_2, \sigma_2)$ for a unique $\varphi : M_1 \to M_2^*$, where τ_2 and σ_2 are the universal trivializations.

Recall that $\mathrm{Biext}(G_1, G_2; \mathbb{G}_m) = \mathrm{Biext}(A_1, A_2; \mathbb{G}_m)$ (*cf.* [16, 10.2.3.9]) so that, by Lemma 4.1.4, P is the pull-back to $G_1 \times G_2$ of $(f \times 1_{A_2})^*({}^tP_{A_2})$ for a unique homomorphism $f : A_1 \to A'_2$. We thus have obtained the map f and its dual f' in (4.3), and we now want to show that the extra data (τ, σ) come from a pair (g, g') in a unique way.

We may view $E = (fu_1 \times 1_{A_2})^*({}^tP_{A_2})$ as an extension of $L_1 \otimes A_2$ by $\mathbb{G}_m$. Consider the commutative diagram of exact sequences

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & L_1 \otimes T_2 \arrow[r, equal] \arrow[d] & L_1 \otimes T_2 \arrow[r] \arrow[d, "1_{L_1} \otimes i_2"'] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r, "i"] \arrow[d, equal] & Q \arrow[r, "\pi'"] \arrow[d, "\pi"'] & L_1 \otimes G_2 \arrow[r] \arrow[d, "1_{L_1} \otimes p_2"'] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & L_1 \otimes A_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{4.6}
$$

where i_2 (*resp.* p_2) is the inclusion $T_2 \hookrightarrow G_2$ (*resp.* the projection $G_2 \to A_2$). The section τ yields a retraction $\widetilde{\tau} : Q \to \mathbb{G}_m$ whose restriction to $L_1 \otimes T_2$ yields a homomorphism

$$\widetilde{g} : L_1 \otimes T_2 \to \mathbb{G}_m$$

which in turn defines a homomorphism as in (4.3). We denote the negative of this morphism by g.

4.1.6. **Lemma.** *With this choice of g, the left square of* (4.3) *commutes and $\tau = (g \times 1_{G_2})^*\tau_2$.*

Proof. To see the first assertion, we may apply $\mathrm{Ext}^*(-, \mathbb{G}_m)$ to (4.6) and then apply [10, Lemma 2.8] to the corresponding diagram. Here is a concrete description of this argument: via the map of Lemma 4.1.5, $u_2'g$ goes to the following pushout

$$
\begin{tikzcd}
0 \arrow[r] & L_1 \otimes T_2 \arrow[r, "1 \otimes i_2"] \arrow[d, "-\widetilde{g}"'] & L_1 \otimes G_2 \arrow[r, "1 \otimes p_2"] \arrow[d, "\pi \circ \tau"'] & L_1 \otimes A_2 \arrow[r] \arrow[d, "||"'] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] & E \arrow[r] & L_1 \otimes A_2 \arrow[r] & 0.
\end{tikzcd}
$$

because, due to the relation $i\widetilde{\tau} + \tau\pi' = 1$, the left square in this diagram commutes.

In particular, we have

$$
\begin{aligned}
Q = (1 \otimes p_2)^*(fu_1 \otimes 1)^{*t}P_{A_2} &= (fu_1 \otimes p_2)^{*t}P_{A_2} \\
&= (u_2'g \otimes p_2)^{*t}P_{A_2} = (g \otimes 1)^*(u_2' \otimes p_2^*)^tP_{A_2}.
\end{aligned}
$$

For the second assertion, since $\mathrm{Hom}(L_1 \otimes A_2, \mathbb{G}_m) = 0$ it suffices to check the equality after restricting to $L_1 \otimes T_2$. This is clear because under the isomorphism $\mathrm{Hom}(L_2' \otimes T_2, \mathbb{G}_m) = \mathrm{Hom}(L_2', L_2')$, the canonical trivialization ${}^t\psi_2$ corresponds to the identity. □

Note that if we further pullback we obtain that

$$\tau \mid_{L_1 \times L_2} = \psi_2^t \mid_{L_1 \times L_2} . \tag{4.7}$$

The same computation with σ yields a map

$$g' : L_2 \to L_1'$$

and the same argument as in Lemma 4.1.6 shows that with this choice of g' the right square of (4.3) commutes. We now use that $P = (1_{A_1} \times f')^*(P_{A_1})$, which follows from the naturality statement in Lemma 4.1.4. As in the proof of Lemma 4.1.6, this implies that its trivialization σ on $G_1 \times L_2$ is the pullback of the canonical trivialization ψ_1 on $G_1 \times L_1'$ along $1_{G_1} \times g' : G_1 \times L_2 \to G_1 \times L_1'$. In particular:

$$\sigma \mid_{L_1 \times L_2} = \psi_1 \mid_{L_1 \times L_2} . \tag{4.8}$$

Put together, (4.5), (4.7) and (4.8) show that Condition (4.4) is verified: thus we get a morphism $\varphi : M_1 \to M_2^*$. Let $h : G_1 \to G_2'$ be its group component. It remains to check that $\sigma = (h \times 1_{L_2})^* \sigma_2$. As in the proof of Lemma 4.1.6 we only need to check this after restriction to $T_1 \otimes L_2$. But the restriction of h to the toric parts is the Cartier dual of g', so we conclude by the same argument.

Finally, let us show that γ_{M_1,M_2} is natural in M_2. This amounts to comparing two biextensions. For the bitorsors this follows from Lemma 4.1.4 and for the sections we may argue again as in the proof of Lemma 4.1.6. □

4.2. **Biextensions of complexes of 1-motives.** Let $\mathcal{A}$ be a category of abelian sheaves, and consider two bounded complexes C_1, C_2 of objects of $\mathcal{A}^{[-1,0]}$. Let $H \in \mathcal{A}$. We have a double complex

$$\underline{\mathrm{Biext}}(C_1, C_2; H)^{p,q} := \mathrm{Biext}(C_1^p, C_2^q; H).$$

4.2.1. **Definition.** A *biextension of C_1 and C_2 by H* is an element of the group of cycles

$$\mathrm{Biext}(C_1, C_2; H) := Z^0(\mathbf{Tot}\underline{\mathrm{Biext}}(C_1, C_2; H)).$$

Here **Tot** denotes the total complex associated to a double complex.

Concretely: such a biextension P is given by a collection of biextensions $P_p \in \mathrm{Biext}(C_1^p, C_2^{-p}; H)$ such that, for any p,

$$(d_1^p \otimes 1)^* P_{p+1} = (1 \otimes d_2^{-p-1})^* P_p$$

where $d_1^{\cdot}$ (*resp.* $d_2^{\cdot}$) are the differentials of C_1 (*resp.* of C_2).

Now suppose that $\mathcal{A}$ is the category of fppf sheaves, that $H = \mathbb{G}_m$ and that all the C_i^j are Deligne 1-motives. By Lemma 4.1.2, we have

$$\mathbb{E}\mathrm{xt}^i(C_1^p, C_2^q; \mathbb{G}_m) = 0 \text{ for } i \leq 0.$$

Therefore, a spectral sequence argument yields an edge homomorphism

$$\mathrm{Biext}(C_1, C_2; \mathbb{G}_m) \to \mathbb{E}\mathrm{xt}^1(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m). \tag{4.9}$$

Recall that Deligne's Cartier duality [16] provides an exact functor

$$M \mapsto M^* : \mathcal{M}_1[1/p] \to \mathcal{M}_1[1/p]$$

yielding by Proposition 1.8.4 a triangulated functor

$$(\ \ )^* : D^b(\mathcal{M}_1[1/p]) \to D^b(\mathcal{M}_1[1/p]). \tag{4.10}$$

Note that for a complex of 1-motives

$$C = (\cdots \to M^i \to M^{i+1} \to \cdots)$$

we can compute C^* by means of the complex

$$C^* = (\cdots \to (M^{i+1})^* \to (M^i)^* \to \cdots)$$

of Cartier duals here placed in degrees ..., $-i-1$, $-i$, etc.

Let us now take in (4.9) $C_1 = C$, $C_2 = C^*$. For each $p \in \mathbb{Z}$, we have the Poincaré biextension $P_p \in \mathrm{Biext}(C^p, (C^p)^*; \mathbb{G}_m)$. By Proposition 4.1.1, the $\{P_p\}$ define a class in $\mathrm{Biext}(C, C^*; \mathbb{G}_m)$.

4.2.2. **Definition.** This class P_C is the *Poincaré biextension of the complex* C.

Let $C_1, C_2 \in C^b(\mathcal{M}_1)$. As in Subsection 4.1, pulling back ${}^tP_{C_1} = P_{C_1^*} \in \mathrm{Biext}(C_1, C_1^*; \mathbb{G}_m)$ yields a map generalising (4.2):

$$\begin{aligned} \gamma_{C_1,C_2} : \mathrm{Hom}(C_1, C_2^*) &\to \mathrm{Biext}(C_1, C_2; \mathbb{G}_m) \\ \varphi &\mapsto (\varphi \times 1_{C_1})^*({}^tP_{C_2}). \end{aligned} \tag{4.11}$$

which is clearly additive and natural in C_1. We then have the following trivial extension of the functoriality in Proposition 4.1.1:

4.2.3. **Proposition.** γ_{C_1,C_2} *is also natural in* C_2. □

4.3. **Comparing two Ext groups.** The aim of this subsection is to prove:

4.3.1. **Proposition.** *Let $C_1, C_2 \in D^b(\mathcal{M}_1[1/p])$. Then the forgetful triangulated functors*

$$\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}(k) \to D^-(\mathrm{Shv}_{\text{ét}}(SmCor(k))) \to D^-(\mathrm{Shv}_{\text{ét}}(Sm(k)))$$

induce an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(\mathrm{Tot}\, C_1 \otimes \mathrm{Tot}\, C_2, \mathbb{Z}(1)[q]) \xrightarrow{\sim} \mathrm{Hom}_{Sm(k)_{\text{ét}}}(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m[q+1])$$

for any $q \in \mathbb{Z}$.

Proof. Each of the two functors in Proposition 4.3.1 has a left adjoint. For the first one see [63, Prop. 3.2.3]; we shall denote it by RC as in *loc. cit.* The second may be constructed using [63, Remark 1 p. 202]: we shall denote it by Φ. In both cases, the construction is done for the Nisnevich topology but carries over for the étale topology as well (see also [39]). The tensor product $\overset{L}{\otimes}_{tr}$ in $D^-(\mathrm{EST})$ is defined from the formula

$$L(X) \otimes L(Y) = L(X \times Y)$$

see [63, p. 206]. For $X \in Sm(k)$, let $\mathbb{Z}(X)$ be the $\mathbb{Z}$-free étale sheaf on the representable sheaf $Y \mapsto \mathrm{Map}_k(Y, X)$. It is clear that

$$\Phi\mathbb{Z}(X) = L(X)$$
$$\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y).$$

From this it follows that one has natural isomorphisms

$$\Phi(A \overset{L}{\otimes} B) = \Phi(A) \overset{L}{\otimes}_{tr} \Phi(B).$$

On the other hand, the tensor product in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ is defined by descent of $\overset{L}{\otimes}_{tr}$ via RC [63, p. 210]. Hence we get an adjunction isomorphism

$$\begin{aligned}\mathrm{Hom}_{Sm(k)_{\text{ét}}}(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m[q+1]) &\simeq \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(RC \circ \Phi(C_1 \overset{L}{\otimes} C_2), \mathbb{Z}(1)[q]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(RC \circ \Phi(C_1) \otimes RC \circ \Phi(C_2)), \mathbb{Z}(1)[q]).\end{aligned}$$

Now, since the components of C_1 and C_2 belong to $\mathrm{HI}_{\text{ét}}$, the counit maps $RC \circ \Phi(C_1) \to \mathrm{Tot}(C_1)$ and $RC \circ \Phi(C_2) \to \mathrm{Tot}(C_2)$ are isomorphisms. This concludes the proof. $\square$

4.4. **Two Cartier dualities.** Recall the internal Hom $\underline{\mathrm{Hom}}_{\text{ét}}$ from §2.5. We define

$$D^{\text{ét}}_{\leq 1}(M) := \underline{\mathrm{Hom}}_{\text{ét}}(M, \mathbb{Z}(1)) \tag{4.12}$$

for any object $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$.

We now want to compare the duality (4.10) with the following duality on triangulated 1-motives:

4.4.1. **Proposition.** *The functor $D^{\text{ét}}_{\leq 1}$ restricts to a self-duality $(\ )^\vee$ (anti-equivalence of categories) on $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$.*

Proof. It suffices to compute on motives of smooth projective curves $M(C)$. Then it is obvious in view of Proposition 2.5.2 c). □

4.4.2. **Definition.** For $M \in d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$, we say that $M^\vee$ is the *motivic Cartier dual* of M.

Note that motivic Cartier duality exchanges Artin motives and Tate motives, *e.g.* $\mathbb{Z}^\vee = \mathbb{Z}(1)$. We are going to compare it with the Cartier duality on $D^b(\mathcal{M}_1[1/p])$ (see Proposition 1.8.4) via Theorem 2.1.2.

For two complexes of 1-motives C_1 and C_2, by composing (4.11) and (4.9) and applying Proposition 4.2.3, we get a bifunctorial morphism

$$\mathrm{Hom}(C_1, C_2^*) \to \mathrm{Biext}(C_1, C_2; \mathbb{G}_m) \to \mathrm{Hom}(C_1 \overset{L}{\otimes} C_2, \mathbb{G}_m[-1]) \tag{4.13}$$

where the right hand side is computed in the derived category of étale sheaves. This natural transformation trivially factors through $D^b(\mathcal{M}_1[1/p])$.

From Proposition 4.3.1, it follows that the map (4.13) may be reinterpreted as a natural transformation

$$\mathrm{Hom}_{D^b(\mathcal{M}_1[1/p])}(C_1, C_2^*) \to \mathrm{Hom}_{d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}}(\mathrm{Tot}(C_1), \mathrm{Tot}(C_2)^\vee).$$

Now we argue à la Yoneda: taking $C_1 = C$ and $C_2 = C^*$, the image of the identity yields a canonical morphism of functors:

$$\eta_C : \mathrm{Tot}(C^*) \to \mathrm{Tot}(C)^\vee.$$

4.4.3. **Theorem.** *The natural transformation η is an isomorphism of functors.*

Proof. It suffices to check this on 1-motives, since they are dense in the triangulated category $D^b(\mathcal{M}_1[1/p])$. Using Yoneda again and the previous discussion, it then follows from Theorem 2.1.2 and the isomorphisms (4.1) and (4.2) (the latter being proven in Proposition 4.1.1).

The following diagram explains this:

$$\begin{array}{ccc} \mathrm{Hom}(N, M^*) & \xrightarrow[\sim]{\text{Th. 2.1.2}} & \mathrm{Hom}(\mathrm{Tot}(N), \mathrm{Tot}(M^*)) \\ \wr \Big\downarrow (4.1)+(4.2) & & \eta_* \Big\downarrow \\ \mathbb{E}\mathrm{xt}^1_{Sm(k)_{\text{ét}}}(N \overset{L}{\otimes} M, \mathbb{G}_m) & \xleftarrow[\sim]{\text{Prop. 4.3.1}} & \mathrm{Hom}(\mathrm{Tot}(N), \mathrm{Tot}(M)^\vee). \end{array}$$

□

Part 2. The functors LAlb and RPic

5. A left adjoint to the universal realisation functor

The aim of this section is to construct the closest approximation to a left adjoint of the full embedding Tot of Definition 2.7.1: see Theorem 5.2.1 (and Remark 5.2.3 for a caveat). In order to work it out, we first recollect some ideas from [60].

In Section 6, we shall show that the functor LAlb of Theorem 5.2.1 does provide a left adjoint to Tot after we tensor Hom groups with $\mathbb{Q}$: this will provide a proof of Pretheorems announced in [60, Preth. 0.0.18] and [61].

5.1. **Motivic Cartier duality.** Recall the functor $D^{\text{ét}}_{\leq 1} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}} \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ of (4.12). On the other hand, by Corollary D.3.3 and Theorem D.2.2, we may consider truncation on $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ with respect to the homotopy t-structure. We have:

5.1.1. **Lemma.** *Let $p : X \to \operatorname{Spec} k$ be a smooth variety. Then the truncated complex $\tau_{\leq 2} D^{\text{ét}}_{\leq 1}(M_{\text{ét}}(X))$ belongs to $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ (here we set $M_{\text{ét}}(X) := \alpha^* M(X)$).*

Proof. The same computation as in the proof of Proposition 2.5.2 yields that the nonvanishing cohomology sheaves are $\mathcal{H}^1 = R_{\pi_0(X)/k}\mathbb{G}_m[1/p]$ and $\mathcal{H}^2 = \mathrm{Pic}_{X/k}[1/p]$. Both belong to Shv_1 (the latter by Proposition 3.4.1), hence the claim follows from Theorem 3.9.1. □

Unfortunately, $\mathcal{H}^i(D^{\text{ét}}_{\leq 1}(M_{\text{ét}}(X)))$ does not belong to $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ for $i > 2$ in general: indeed, it is well-known that this is a torsion sheaf of cofinite type, with nonzero divisible part in general (for $i \geq 3$ and in characteristic 0, its corank is equal to the i-th Betti number of X). It might be considered as an ind-object of $d_{\leq 0}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$, but this would take us too far. To get around this problem, we shall restrict to the standard category of geometric triangulated motives of [63], $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

Let us denote by $D^{\mathrm{Nis}}_{\leq 1}$ the same functor as $D^{\text{ét}}_{\leq 1}$ in the category $\mathrm{DM}^{\mathrm{eff}}_{-}$, defined with the Nisnevich topology. Let as before $\alpha^* : \mathrm{DM}^{\mathrm{eff}}_{-} \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ denote the "change of topology" functor.

5.1.2. **Lemma.** *a) For any smooth X with motive $M(X) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, we have*

$$\alpha^* D^{\mathrm{Nis}}_{\leq 1} M(X) \xrightarrow{\sim} \tau_{\leq 2} D^{\text{ét}}_{\leq 1} \alpha^* M(X).$$

b) The functor $\alpha^* D^{\mathrm{Nis}}_{\leq 1}$ induces a triangulated functor

$$\alpha^* D^{\mathrm{Nis}}_{\leq 1} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to d_{\leq 1} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}.$$

Proof. a) This is the weight 1 case of the Beilinson-Lichtenbaum conjecture (here equivalent to Hilbert's theorem 90.) b) follows from a) and Lemma 5.1.1. □

5.1.3. **Definition.** We denote by $d_{\leq 1} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to d_{\leq 1} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$ the composite functor $D^{\text{ét}}_{\leq 1} \circ \alpha^* \circ D^{\mathrm{Nis}}_{\leq 1}$.

Thus, for $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, we have

$$d_{\leq 1}(M) = \underline{\mathrm{Hom}}_{\text{ét}}(\alpha^* \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M, \mathbb{Z}(1)), \mathbb{Z}(1)). \tag{5.1}$$

The evaluation map $M \otimes \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M, \mathbb{Z}(1)) \to \mathbb{Z}(1)$ then yields a canonical map

$$a_M : \alpha^* M \to d_{\leq 1}(M) \tag{5.2}$$

for any object $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

5.1.4. **Proposition.** *The restriction of* (5.2) *to $d_{\leq 1} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is an isomorphism of functors. In particular, we have an equality*

$$\alpha^* D_{\leq 1}(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) = \alpha^* d_{\leq 1} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}.$$

Proof. For the first claim, we reduce to the case $M = M(C)$ where C is a smooth proper curve. The argument is then exactly the same as in Proposition 4.4.1, using (2.2). The other claim is then clear. □

5.2. **Motivic Albanese.**

5.2.1. **Theorem.** *Let $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Then a_M induces an isomorphism*

$$\mathrm{Hom}(d_{\leq 1} M, M') \xrightarrow{\sim} \mathrm{Hom}(\alpha^* M, M')$$

for any $M' \in d_{\leq 1} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}$,

Proof. By Proposition 4.4.1, M' can be written as $N^\vee = D^{\text{ét}}_{\leq 1}(N)$ for some $N \in d_{\leq 1}\,\mathrm{DM}^{\text{eff}}_{\text{gm,ét}}$. We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(\alpha^* M, D^{\text{ét}}_{\leq 1}(N)) & = & \mathrm{Hom}(\alpha^* M \otimes N, \mathbb{Z}(1)) & = & \mathrm{Hom}(N, D^{\text{ét}}_{\leq 1}(\alpha^* M)) \\
{\scriptstyle a_M^*}\uparrow & & {\scriptstyle (a_M \otimes 1_N)^*}\uparrow & & {\scriptstyle D^{\text{ét}}_{\leq 1}(a_M)_*}\uparrow \\
\mathrm{Hom}(d_{\leq 1} M, D^{\text{ét}}_{\leq 1}(N)) & = & \mathrm{Hom}(d_{\leq 1} M \otimes N, \mathbb{Z}(1)) & = & \mathrm{Hom}(N, D^{\text{ét}}_{\leq 1}(d_{\leq 1} M)).
\end{array}$$

But $D^{\text{ét}}_{\leq 1}(a_M) \circ a_{D^{\text{Nis}}_{\leq 1} M} = 1_{D^{\text{ét}}_{\leq 1}\alpha^* M}$ and $a_{D^{\text{Nis}}_{\leq 1} M}$ is an isomorphism by Proposition 5.1.4, which proves the claim. □

5.2.2. **Definition.** The *motivic Albanese functor*

$$\mathrm{LAlb} : \mathrm{DM}^{\text{eff}}_{\text{gm}} \to D^b(\mathcal{M}_1[1/p])$$

is the composition of $d_{\leq 1}$ with a quasi-inverse to the equivalence of categories of Theorem 2.1.2.

By Theorems 2.1.2 and 5.2.1, the relationship between LAlb and the functor Tot of Definition 2.7.1 is as follows: for $M \in \mathrm{DM}^{\text{eff}}_{\text{gm}}$ and $N \in D^b(\mathcal{M}_1[1/p])$, the map a_M of (5.2) induces an isomorphism

$$\mathrm{Hom}(\mathrm{LAlb}\, M, N) \xrightarrow{\sim} \mathrm{Hom}(\alpha^* M, \mathrm{Tot}(N)). \tag{5.3}$$

We call a_M the *motivic Albanese map* associated to M for reasons that will appear later.

5.2.3. **Remark.** The above suggests that LAlb might actually extend to a left adjoint of $\mathrm{Tot} : D^b(\mathcal{M}_1[1/p]) \to \mathrm{DM}^{\text{eff}}_{\text{gm,ét}}$. Unfortunately this is not the case, and in fact Tot *does not have a left adjoint.*

Indeed, suppose that such a left adjoint exists, and let us denote it by $\mathrm{LAlb}^{\text{ét}}$. For simplicity, suppose k algebraically closed. Let $n \geq 2$. For any $m > 0$, the exact triangle in $\mathrm{DM}^{\text{eff}}_{\text{gm,ét}}$

$$\mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n) \to \mathbb{Z}/m(n) \xrightarrow{+1}$$

must yield an exact triangle

$$\mathrm{LAlb}^{\text{ét}}\, \mathbb{Z}(n) \xrightarrow{m} \mathrm{LAlb}^{\text{ét}}\, \mathbb{Z}(n) \to \mathrm{LAlb}^{\text{ét}}\, \mathbb{Z}/m(n) \xrightarrow{+1}.$$

Since Tot is an equivalence on torsion objects, so must be $\mathrm{LAlb}^{\text{ét}}$. Since k is algebraically closed, $\mathbb{Z}/m(n) \simeq \mu_m^{\otimes n}$ is constant, hence we must have $\mathrm{LAlb}^{\text{ét}}\, \mathbb{Z}/m(n) \simeq [\mathbb{Z}/m \to 0]$. Hence, multiplication by m must be bijective on the 1-motives $H^q(\mathrm{LAlb}^{\text{ét}}(\mathbb{Z}(n)))$ for all $q \neq 0, 1$, which forces these 1-motives to vanish. For $q = 0, 1$ we must have exact

sequences

$$0 \to H^0(\mathrm{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \xrightarrow{m} H^0(\mathrm{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \to [\mathbb{Z}/m \to 0]$$
$$\to H^1(\mathrm{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \xrightarrow{m} H^1(\mathrm{LAlb}^{\text{ét}}(\mathbb{Z}(n))) \to 0$$

which force either $H^0 = [\mathbb{Z} \to 0]$, $H^1 = 0$ or $H^0 = 0$, $H^1 = [0 \to \mathbb{G}_m]$. But both cases are impossible as one easily sees by computing

$$\mathrm{Hom}(M(\mathbb{P}^n), \mathrm{Tot}([\mathbb{Z} \to 0])[2n+1]) = H^{2n+1}_{\text{ét}}(\mathbb{P}^n, \mathbb{Z})[1/p]$$
$$\simeq H^{2n}_{\text{ét}}(\mathbb{P}^n, (\mathbb{Q}/\mathbb{Z})') \simeq (\mathbb{Q}/\mathbb{Z})'$$

via the trace map, where $(\mathbb{Q}/\mathbb{Z})' = \bigoplus_{l \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

Presumably, $\mathrm{LAlb}^{\text{ét}}$ does exist with values in a suitable pro-category containing $D^b(\mathcal{M}_1[1/p])$, and sends $\mathbb{Z}(n)$ to the complete Tate module of $\mathbb{Z}(n)$ for $n \geq 2$. Note that, by 8.1.c below, $\mathrm{LAlb}(\mathbb{Z}(n)) = 0$ for $n \geq 2$, so that the natural transformation $\mathrm{LAlb}^{\text{ét}}(\alpha^* M) \to \mathrm{LAlb}(M)$ will not be an isomorphism of functors in general.

We shall show in Section 6 that LAlb does yield a left adjoint of Tot after Hom groups have been tensored with $\mathbb{Q}$.

5.3. **Motivic Pic.**

5.3.1**. Definition.** The *motivic Picard functor* (a contravariant functor) is the functor

$$\mathrm{RPic} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to D^b(\mathcal{M}_1[1/p])$$

given by $\mathrm{Tot}^{-1}\, \alpha^* D^{\mathrm{Nis}}_{\leq 1}$ (*cf.* Definition 5.2.2).

For $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ we then have the following tautology

$$(\mathrm{Tot}\,\mathrm{RPic}(M))^\vee = \mathrm{Tot}\,\mathrm{LAlb}(M).$$

Actually, from Theorem 4.4.3 we deduce:

5.3.2**. Corollary.** *For $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ we have*

$$\mathrm{RPic}(M)^* = \mathrm{LAlb}(M). \quad \square$$

Therefore we get ${}^tH^i(\mathrm{RPic}(M)) = ({}_tH_i(\mathrm{LAlb}(M)))^*$.

6. LAlb and RPic with rational coefficients

Throughout this section, we use the notations $\otimes\mathbb{Q}$ and $\boxtimes\mathbb{Q}$ from Definition 1.1.3.

6.1. **Rational coefficients revisited.** To give a rational version of Theorem 5.2.1, we have to be a little careful. Let $\mathrm{DM}_-^{\mathrm{eff}}(k;\mathbb{Q})$ and $\mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k;\mathbb{Q})$ denote the full subcategories of $\mathrm{DM}_-^{\mathrm{eff}}(k)$ and $\mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k)$ formed of those complexes whose cohomology sheaves are uniquely divisible. Recall that by [63, Prop. 3.3.2], the change of topology functor

$$\alpha^* : \mathrm{DM}_-^{\mathrm{eff}}(k) \to \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k)$$

induces an equivalence of categories

$$\mathrm{DM}_-^{\mathrm{eff}}(k;\mathbb{Q}) \xrightarrow{\sim} \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k;\mathbb{Q}).$$

Beware that in *loc. cit.* , these two categories are respectively denoted by $\mathrm{DM}_-^{\mathrm{eff}}(k)\otimes\mathbb{Q}$ and $\mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k)\otimes\mathbb{Q}$, while this notation is used here according to Definition 1.1.3. The composite functors (with our notation)

$$\mathrm{DM}_-^{\mathrm{eff}}(k;\mathbb{Q}) \to \mathrm{DM}_-^{\mathrm{eff}}(k) \to \mathrm{DM}_-^{\mathrm{eff}}(k)\otimes\mathbb{Q}$$
$$\mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k;\mathbb{Q}) \to \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k) \to \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k)\otimes\mathbb{Q}$$

are fully faithful but not essentially surjective. The functor $\alpha^*\otimes\mathbb{Q}$ is not essentially surjective, nor (a priori) fully faithful. Nevertheless, these two composite functors have a left adjoint/left inverse $C \mapsto C\otimes\mathbb{Q}$, and

6.1.1. **Proposition.** *a) The compositions*

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)\otimes\mathbb{Q} \to \mathrm{DM}_-^{\mathrm{eff}}(k)\otimes\mathbb{Q} \xrightarrow{\otimes\mathbb{Q}} \mathrm{DM}_-^{\mathrm{eff}}(k;\mathbb{Q})$$
$$\mathrm{DM}_{\mathrm{gm},\text{ét}}^{\mathrm{eff}}(k)\otimes\mathbb{Q} \to \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k)\otimes\mathbb{Q} \xrightarrow{\otimes\mathbb{Q}} \mathrm{DM}_{-,\text{ét}}^{\mathrm{eff}}(k;\mathbb{Q})$$

are fully faithful.
b) Via these full embeddings, the functor α^* induces equivalences of categories

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)\boxtimes\mathbb{Q} \xrightarrow{\sim} \mathrm{DM}_{\mathrm{gm},\text{ét}}^{\mathrm{eff}}(k)\boxtimes\mathbb{Q}$$
$$d_{\leq 1}\,\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)\boxtimes\mathbb{Q} \xrightarrow{\sim} d_{\leq 1}\,\mathrm{DM}_{\mathrm{gm},\text{ét}}^{\mathrm{eff}}(k)\boxtimes\mathbb{Q}.$$

Here $d_{\leq 1}\,\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is the thick subcategory of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ generated by motives of smooth curves.

Proof. a) We shall give it for the first composition (for the second one it is similar). Let $M, N \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$: we have to prove that the obvious map

$$\mathrm{Hom}(M,N)\otimes\mathbb{Q} \to \mathrm{Hom}(M\otimes\mathbb{Q}, N\otimes\mathbb{Q})$$

is an isomorphism. We shall actually prove this isomorphism for any $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ and any $N \in \mathrm{DM}_-^{\mathrm{eff}}(k)$. By adjunction, the right hand side coincides with $\mathrm{Hom}(M, N\otimes\mathbb{Q})$ computed in $\mathrm{DM}_-^{\mathrm{eff}}(k)$. We may

reduce to $M = M(X)$ for X smooth. By [63, Prop. 3.2.8], we are left to see that the map

$$H^q_{\mathrm{Nis}}(X, N) \otimes \mathbb{Q} \to H^q_{\mathrm{Nis}}(X, N \otimes \mathbb{Q})$$

is an isomorphism for any $q \in \mathbb{Z}$. By hypercohomology spectral sequences, we reduce to the case where N is a sheaf concentrated in degree 0; then the assertion follows from the fact that Nisnevich cohomology commutes with filtering direct limits of sheaves.

b) It is clear that the two compositions commute with α^*, which sends $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \otimes \mathbb{Q}$ into $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}(k) \otimes \mathbb{Q}$. By a) and [63, Prop. 3.3.2], this functor is fully faithful, and the induced functor on the $\boxtimes$ categories remains so and is essentially surjective by definition of the two categories. Similarly for the $d_{\leq 1}$ categories. □

6.1.2. **Remarks.** 1) In fact, $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}(k) \otimes \mathbb{Q} = d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}(k) \boxtimes \mathbb{Q}$ thanks to Corollary 1.6.2 and Theorem 2.1.2. We don't know whether the same is true for the other categories.
2) See [51, A.2.2] for a different, more general approach to Proposition 6.1.1.

6.2. **The functor** $\mathrm{LAlb}^{\mathbb{Q}}$**.** We now get a rational version of Theorem 5.2.1 by taking (5.1) with rational coefficients according with the results collected in 1.6.2 and 6.1.1.

6.2.1. **Corollary.** *The functor $d_{\leq 1}$ of* (5.1) *induces a left adjoint to the embedding $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \boxtimes \mathbb{Q} \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \boxtimes \mathbb{Q}$. The Voevodsky-Orgogozo full embedding* $\mathrm{Tot} : D^b(\mathcal{M}_1 \otimes \mathbb{Q}) \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \boxtimes \mathbb{Q}$ *has a left adjoint/left inverse* $\mathrm{LAlb}^{\mathbb{Q}}$*.* □

6.2.2. **Remark.** In this case, the formula $d_{\leq 1} = D^{\mathrm{ét}}_{\leq 1} \circ \alpha^* \circ D^{\mathrm{Nis}}_{\leq 1}$ collapses into the simpler formula

$$d_{\leq 1} = D^2_{\leq 1}$$

with $D_{\leq 1} = D^{\mathrm{Nis}}_{\leq 1} = D^{\mathrm{ét}}_{\leq 1}$.

7. A tensor structure and an internal Hom on $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$

In this section, coefficients are tensored with $\mathbb{Q}$ and we use the functor $\mathrm{LAlb}^{\mathbb{Q}}$ of Corollary 6.2.1.

7.1. Tensor structure.

7.1.1. **Lemma.** *Let G_1, G_2 be two semi-abelian varieties. Then, we have in* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}} \boxtimes \mathbb{Q}$*:*

$$\mathcal{H}^q(D_{\leq 1}(\underline{G}_1[-1] \otimes \underline{G}_2[-1])) = \begin{cases} \mathrm{Biext}(G_1, G_2; \mathbb{G}_m) & \textit{if } q = 0 \\ 0 & \textit{else.} \end{cases}$$

Proof. By Gersten's principle (Proposition 2.4.1), it is enough to show that the isomorphisms are valid over function fields K of smooth k-varieties and that $\mathcal{H}^0$ comes from the small étale site of $\operatorname{Spec} k$. Since we work up to torsion, we may even replace K by its perfect closure. Thus, without loss of generality, we may assume $K = k$ and we have to show the lemma for sections over k.

For $q \leq 0$, we use Proposition 4.3.1: for $q < 0$ this follows from Lemma 4.1.2, while for $q = 0$ it follows from the isomorphisms (4.1) and (4.2) (see Proposition 4.1.1), which show that $\operatorname{Biext}(G_1, G_2; \mathbb{G}_m)$ is rigid.

For $q > 0$, we use the formula

$$D_{\leq 1}(\underline{G}_1[-1] \otimes \underline{G}_2[-1]) \simeq \underline{\operatorname{Hom}}(\underline{G}_1[-1], \operatorname{Tot}([0 \to G_2]^*))$$

coming from Theorem 4.4.3. Writing $[0 \to G_2]^* = [L_2 \to A_2]$ with L_2 a lattice and A_2 an abelian variety, we are left to show that

$$\operatorname{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}} \boxtimes \mathbb{Q}}(\underline{G}_1, L_2[q+1]) = 0 \text{ for } q > 0$$
$$\operatorname{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}} \boxtimes \mathbb{Q}}(\underline{G}_1, \underline{A}_2[q]) = 0 \text{ for } q > 0.$$

For this, we may reduce to the case where G_1 is either an abelian variety or $\mathbb{G}_m$. If $G_1 = \mathbb{G}_m$, $\underline{G}_1$ is a direct summand of $M(\mathbb{P}^1)[-1]$ and the result follows. If G_1 is an abelian variety, it is isogenous to a direct summand of $J(C)$ for C a smooth projective geometrically irreducible curve. Then $\underline{G}_1$ is a direct summand of $M(C)$, and the result follows again since L_2 and $\underline{A}_2$ define locally constant (flasque) sheaves for the Zariski topology. □

7.1.2. **Proposition.** *a) The functor* $\mathrm{LAlb}^{\mathbb{Q}} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \boxtimes \mathbb{Q} \to D^b(\mathcal{M}_1 \otimes \mathbb{Q})$ *is a localisation functor; it carries the tensor structure* $\otimes$ *of* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \boxtimes \mathbb{Q}$ *to a tensor structure* $\otimes_1$ *on* $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$.
b) For $(M, N) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \boxtimes \mathbb{Q} \times D^b(\mathcal{M}_1 \otimes \mathbb{Q})$, *we have*

$$\mathrm{LAlb}^{\mathbb{Q}}(M \otimes \operatorname{Tot}(N)) \simeq \mathrm{LAlb}^{\mathbb{Q}}(M) \otimes_1 N.$$

c) We have

$$[\mathbb{Z} \to 0] \otimes_1 C = C$$

for any $C \in D^b(\mathcal{M}_1 \otimes \mathbb{Q})$*;*

$$N_1 \otimes_1 N_2 = [L \to G]$$

for two Deligne 1*-motives* $N_1 = [L_1 \to G_1]$, $N_2 = [L_2 \to G_2]$, *where*

$$L = L_1 \otimes L_2;$$

there is an extension

$$0 \to \operatorname{Biext}(G_1, G_2; \mathbb{G}_m)^* \to G \to L_1 \otimes G_2 \oplus L_2 \otimes G_1 \to 0.$$

d) The tensor product $\otimes_1$ is exact with respect to the motivic t-structure and respects the weight filtration. Moreover, it is right exact with respect to the homotopy t-structure.
e) For two 1-motives N_1, N_2 and a semi-abelian variety G, we have

$$\mathrm{Hom}(N_1 \otimes_1 N_2, [0 \to G]) \simeq \mathrm{Biext}(N_1, N_2; G) \otimes \mathbb{Q}.$$

Proof. a) The first statement is clear since LAlb is left adjoint to the fully faithful functor Tot. For the second, it suffices to see that if $\mathrm{LAlb}^{\mathbb{Q}}(M) = 0$ then $\mathrm{LAlb}^{\mathbb{Q}}(M \otimes N) = 0$ for any $N \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \otimes \mathbb{Q}$. We may check this after applying Tot. Note that, by Proposition 4.4.1 and Remark 6.2.2, $\mathrm{Tot}\,\mathrm{LAlb}^{\mathbb{Q}}(M) = d_{\leq 1}(M) = 0$ is equivalent to $D_{\leq 1}(M) = 0$. We have:

$$D_{\leq 1}(M \otimes N) = \underline{\mathrm{Hom}}(M \otimes N, \mathbb{Z}(1)) = \underline{\mathrm{Hom}}(N, \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))) = 0.$$

b) Let $M' = fibre(\mathrm{Tot}\,\mathrm{LAlb}^{\mathbb{Q}}(M) \to M)$: then $\mathrm{LAlb}^{\mathbb{Q}}(M') = 0$. By definition of $\otimes_1$ we then have

$$\begin{aligned}\mathrm{LAlb}^{\mathbb{Q}}(M) \otimes_1 N &= \mathrm{LAlb}^{\mathbb{Q}}(\mathrm{Tot}\,\mathrm{LAlb}^{\mathbb{Q}}(M) \otimes \mathrm{Tot}(N)) \\ &\xrightarrow{\sim} \mathrm{LAlb}^{\mathbb{Q}}(M \otimes \mathrm{Tot}(N)).\end{aligned}$$

c) The first formula is obvious. For the second, we have an exact triangle

$$\begin{aligned}G_1[-1] \otimes G_2[-1] &\to \mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2) \\ &\to \mathrm{Tot}([L_1 \otimes L_2 \to L_1 \otimes G_2 \oplus L_2 \otimes G_1]) \xrightarrow{+1}\end{aligned}$$

hence an exact triangle

$$\begin{aligned}&\underline{\mathrm{Hom}}(\mathrm{Tot}([L_1 \otimes L_2 \to L_1 \otimes G_2 \oplus L_2 \otimes G_1], \mathbb{Z}(1)) \\ &\to \underline{\mathrm{Hom}}(\mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2), \mathbb{Z}(1)) \to \underline{\mathrm{Hom}}(G_1[-1] \otimes G_2[-1], \mathbb{G}_m) \xrightarrow{+1}\end{aligned}$$

By Lemma 7.1.1, the last term is $\mathrm{Biext}(G_1, G_2; \mathbb{G}_m)$, hence the claim.

d) Exactness and compatibility with weights follow from the second formula of b); right exactness for the homotopy t-structure holds because it holds on $\mathrm{DM}^{\mathrm{eff}}_{-} \otimes \mathbb{Q}$.

e) We have:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{M}_1 \otimes \mathbb{Q}}(N_1 \otimes_1 N_2, [0 \to G]) = \mathrm{Hom}_{d_{\leq 1}\,\mathrm{DM} \otimes \mathbb{Q}}(\mathrm{Tot}(N_1 \otimes_1 N_2), G[-1]) \\ &= \mathrm{Hom}_{\mathrm{DM} \otimes \mathbb{Q}}(\mathrm{Tot}(N_1) \otimes \mathrm{Tot}(N_2), G[-1]) = \mathrm{Biext}(N_1, N_2; G) \otimes \mathbb{Q}\end{aligned}$$

by Proposition 4.3.1 and formula (4.1). □

7.1.3. **Remarks.** 1) It would be interesting to try and define $\otimes_1$ a priori, with integral coefficients, and to see whether it is compatible with the tensor product of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ via the integral LAlb.

2) It is likely that Proposition 7.1.2 e) generalises to an isomorphism

$$\mathrm{Hom}_{\mathcal{M}_1\otimes\mathbb{Q}}(N_1 \otimes_1 N_2, N) = \mathrm{Biext}(N_1, N_2; N) \otimes \mathbb{Q}$$

for three 1-motives N_1, N_2, N, where the right hand side is the biextension group introduced by Cristiana Bertolin [9], but we have not tried to check it. This would put in perspective her desire to interpret these groups as Hom groups in the (future) tannakian category generated by 1-motives.

More precisely, one expects that $\mathrm{DM}_{\mathrm{gm}} \boxtimes \mathbb{Q}$ carries a motivic t-structure whose heart $\mathcal{MM}$ would be the searched-for abelian category of mixed motives. Then $\mathcal{M}_1 \otimes \mathbb{Q}$ would be a full subcategory of $\mathcal{MM}$ and we might consider the thick tensor subcategory $\mathcal{M}_1^{\otimes} \subseteq \mathcal{MM}$ generated by $\mathcal{M}_1 \otimes \mathbb{Q}$ and the Tate motive (inverse to the Lefschetz motive): this is the putative category Bertolin has in mind.

Since the existence of the abelian category of mixed Tate motives (to be contained in $\mathcal{M}_1^{\otimes}$!) depends on the truth of the Beilinson-Soulé conjecture, this basic obstruction appears here too.

Extrapolating from Corollary 6.2.1 and Proposition 7.1.2, it seems that the embedding $\mathcal{M}_1 \otimes \mathbb{Q} \hookrightarrow \mathcal{MM}^{\mathrm{eff}}$ (where $\mathcal{MM}^{\mathrm{eff}}$ is to be the intersection of $\mathcal{MM}$ with $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \boxtimes \mathbb{Q}$) is destined to have a left adjoint/left inverse $\mathrm{Alb}^{\mathbb{Q}} = H_0 \circ \mathrm{LAlb}^{\mathbb{Q}}_{|\mathcal{MM}^{\mathrm{eff}}}$, which would carry the tensor product of $\mathcal{MM}^{\mathrm{eff}}$ to $\otimes_1$. Restricting $\mathrm{Alb}^{\mathbb{Q}}$ to $\mathcal{M}_1^{\otimes} \cap \mathcal{MM}^{\mathrm{eff}}$ would provide the link between Bertolin's ideas and Proposition 7.1.2 e).

7.2. **Internal Hom.**

7.2.1. **Proposition.** *a) The formula*

$$\underline{\mathrm{Hom}}(C_1, C_2) = (C_1 \otimes_1 C_2^*)^*$$

defines an internal Hom *on* $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$*, right adjoint to the tensor product of Proposition 7.1.2.*
b) This internal Hom *is exact for the motivic* t*-structure.*

Proof. a) We have to get a natural isomorphism

$$\mathrm{Hom}(C_1 \otimes_1 C_2, C_3) \simeq \mathrm{Hom}(C_1, \underline{\mathrm{Hom}}(C_2, C_3))$$

for three objects $C_1, C_2, C_3 \in D^b(\mathcal{M}_1 \otimes \mathbb{Q})$.

Let us still write $\otimes_1$ for the tensor product in $d_{\leq 1}\,\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}} \otimes \mathbb{Q}$ corresponding to the tensor product in $D^b(\mathcal{M}_1 \otimes \mathbb{Q})$. By definition, we have

for $M_1, M_2, M \in d_{\leq 1}\,\mathrm{DM}$

$$\begin{aligned}\mathrm{Hom}_{d_{\leq 1}\,\mathrm{DM}}(M_1 \otimes_1 M_2, M) &\simeq \mathrm{Hom}_{\mathrm{DM}}(M_1 \otimes M_2, M)\\ &= \mathrm{Hom}_{\mathrm{DM}}(M_1, \underline{\mathrm{Hom}}(M_2, M)).\end{aligned}$$

In view of Theorem 4.4.3, we are left to show that $\underline{\mathrm{Hom}}(M_2, M) \simeq (M_2 \otimes_1, M^\vee)^\vee$. By duality, we may replace M by $M^\vee$. Then:

$$\begin{aligned}\underline{\mathrm{Hom}}(M_2, M^\vee) &= \underline{\mathrm{Hom}}(M_2, \underline{\mathrm{Hom}}(M, \mathbb{Z}(1))\\ &\simeq \underline{\mathrm{Hom}}(M_2 \otimes M, \mathbb{Z}(1)) \simeq \underline{\mathrm{Hom}}(M_2 \otimes_1 M, \mathbb{Z}(1)) = (M_2 \otimes_1 M)^\vee\end{aligned}$$

where the second isomorphism follows from Proposition 7.1.2 b).

b) This follows from the formula of a), Proposition 7.1.2 d) and Proposition 1.8.4 c). □

Part 3. Some computations

8. The Albanese complexes and their basic properties

We introduce homological and Borel-Moore Albanese complexes of an algebraic variety providing a computation of their 1-motivic homology.

We also consider a slightly more sophisticated cohomological Albanese complex $\mathrm{LAlb}^*(X)$ which is only contravariantly functorial for maps between schemes of the same dimension. All these complexes coincide for smooth proper schemes.

8.1. **The homological Albanese complex.** Let $p : X \to \operatorname{Spec} k$ be a smooth variety. Recall that X has a motive $M(X) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ [63]: M is a covariant functor from the category $Sm(k)$ of smooth k-schemes of finite type to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. The image of $M(X)$ via the full embedding $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to \mathrm{DM}^{\mathrm{eff}}_{-}$ is given by the Suslin complex C_* of the representable Nisnevich sheaf with transfers $L(X)$ associated to X.

For X an arbitrary k-scheme of fnite type, the formula $M(X) = C_*(L(X))$ still defines an object of $\mathrm{DM}^{\mathrm{eff}}_{-}$; if $\operatorname{char} k = 0$, this object is in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ by [63, §4.1].

Convention. In this subsection, "scheme" means k-scheme of finite type if $\operatorname{char} k = 0$ and smooth k-scheme of finite type if $\operatorname{char} k > 0$.

8.1.1. **Definition.** We define the *homological Albanese complex* of X by

$$\mathrm{LAlb}(X) := \mathrm{LAlb}(M(X)).$$

Define, for $i \in \mathbb{Z}$

$$\mathrm{L}_i\mathrm{Alb}(X) := {}_tH_i(\mathrm{LAlb}(X))$$

the 1-motives with cotorsion (see Definition 1.8.1 and Notation 1.8.7) determined by the homology of the Albanese complex.

The functor LAlb has the following properties, easily deduced from [63, 2.2]:

8.1.a. *Homotopy invariance.* For any scheme X the map

$$\mathrm{LAlb}(X \times \mathbb{A}^1) \to \mathrm{LAlb}(X)$$

is an isomorphism, thus

$$\mathrm{L}_i\mathrm{Alb}(X \times \mathbb{A}^1) \xrightarrow{\sim} \mathrm{L}_i\mathrm{Alb}(X)$$

for all $i \in \mathbb{Z}$.

8.1.b. *Mayer-Vietoris.* For a scheme X and an open covering $X = U \cup V$ there is a distinguished triangle

$$\begin{tikzcd} \mathrm{LAlb}(U \cap V) \arrow[rr] & & \mathrm{LAlb}(U) \oplus \mathrm{LAlb}(V) \arrow[dl] \\ & \mathrm{LAlb}(X) \arrow[ul, "+1"] & \end{tikzcd}$$

and therefore a long exact sequence of 1-motives

$$\cdots \to \mathrm{L}_i\mathrm{Alb}(U \cap V) \to \mathrm{L}_i\mathrm{Alb}(U) \oplus \mathrm{L}_i\mathrm{Alb}(V) \to \mathrm{L}_i\mathrm{Alb}(X) \to \cdots$$

8.1.c. *Tate twists.* If X is a smooth scheme and $n > 0$, then

$$\mathrm{Tot}\,\mathrm{LAlb}(M(X)(n)) = \begin{cases} 0 & \text{if } n > 1 \\ M(\pi_0(X))(1) & \text{if } n = 1 \end{cases}$$

where $\pi_0(X)$ is the scheme of constants of X, see Definition 2.5.1. Indeed

$$\begin{aligned} \mathrm{Tot}\,\mathrm{LAlb}(M(X)(n)) &= \underline{\mathrm{Hom}}_{\text{ét}}\alpha^*(\underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(X)(n), \mathbb{Z}(1)), \mathbb{Z}(1)) \\ &= \underline{\mathrm{Hom}}_{\text{ét}}\alpha^*(\underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(X)(n-1), \mathbb{Z}), \mathbb{Z}(1)) \end{aligned}$$

by the cancellation theorem [64]. Now

$$\underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(X)(n-1), \mathbb{Z}) = \begin{cases} 0 & \text{if } n > 1 \\ \underline{\mathrm{Hom}}_{\mathrm{Nis}}(M(\pi_0(X)), \mathbb{Z}) & \text{if } n = 1. \end{cases}$$

The last formula should follow from [28, Lemma 2.1 a)] but the formulation there is wrong; however, the formula immediately follows from the argument in the proof of *loc. cit.* , *i.e.* considering the Zariski cohomology of X with coefficients in the flasque sheaf $\mathbb{Z}$.

This gives

$$\mathrm{L}_i\mathrm{Alb}(M(X)(1)) = \begin{cases} [0 \to R_{\pi_0(X)/k}\mathbb{G}_m] & \text{if } i = 0 \\ 0 & \text{else} \end{cases} \tag{8.1}$$

where $R_{L/k}(-)$ is Weil's restriction of scalars.

From this computation we formally deduce:

8.1.2. **Proposition.** *For any $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, we have*

- $\mathrm{LAlb}(M(n)) = 0$ *for* $n \geq 2$*;*
- $\mathrm{LAlb}(M(1))$ *is a complex of toric* 1*-motives.* □

8.1.d. *Gysin.* Let Z be a closed smooth subscheme purely of codimension n of a smooth scheme X, and let $U = X - Z$. Then

$$\mathrm{LAlb}(U) \xrightarrow{\sim} \mathrm{LAlb}(X) \text{ if } n > 1$$

and if $n = 1$ we have an exact triangle

$$\mathrm{LAlb}(U) \to \mathrm{LAlb}(X) \to [0 \to R_{\pi_0(Z)/k}\mathbb{G}_m][2] \to \mathrm{LAlb}(U)[1]$$

hence a long exact sequence of 1-motives

$$\begin{aligned} 0 \to \mathrm{L}_2\mathrm{Alb}(U) \to \mathrm{L}_2\mathrm{Alb}(X) \to [0 \to R_{\pi_0(X)/k}\mathbb{G}_m] \\ \to \mathrm{L}_1\mathrm{Alb}(U) \to \mathrm{L}_1\mathrm{Alb}(X) \to 0 \end{aligned}$$

and an isomorphism $\mathrm{L}_0\mathrm{Alb}(U) \to \mathrm{L}_0\mathrm{Alb}(X)$.

8.1.e. *Blow ups.* If X is a scheme and $Z \subseteq X$ is a closed subscheme, denote by $p : \widetilde{X} \to X$ a proper surjective morphism such that $p^{-1}(X - Z) \to X - Z$ is an isomorphism, *e.g.* the blow up of X at Z. Then there is a distinguished triangle

$$\begin{tikzcd} \mathrm{LAlb}(\widetilde{Z}) \arrow[rr] & & \mathrm{LAlb}(\widetilde{X}) \oplus \mathrm{LAlb}(Z) \arrow[dl] \\ & \mathrm{LAlb}(X) \arrow[ul, "+1"] & \end{tikzcd}$$

with $\widetilde{Z} = p^{-1}(Z)$, yielding a long exact sequence of 1-motives

$$\cdots \to \mathrm{L}_i\mathrm{Alb}(\widetilde{Z}) \to \mathrm{L}_i\mathrm{Alb}(\widetilde{X}) \oplus \mathrm{L}_i\mathrm{Alb}(Z) \to \mathrm{L}_i\mathrm{Alb}(X) \to \cdots \tag{8.2}$$

If X and Z are smooth, we get (using [63, Prop. 3.5.3] and the above)

$$\mathrm{LAlb}(\widetilde{X}) = \mathrm{LAlb}(X) \oplus [0 \to R_{\pi_0(Z)/k}\mathbb{G}_m][2]$$

and corresponding formulas for homology.

8.1.f. *Albanese map.* If X is a scheme we have the natural map (5.2) in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,ét}}$

$$a_X : \alpha^* M(X) \to \mathrm{Tot}\,\mathrm{LAlb}(X)$$

inducing isomorphisms on étale motivic cohomology

$$\mathrm{Hom}(M(X), \mathbb{Z}(1)[j]) = \mathrm{Hom}(\mathrm{LAlb}(X), \mathbb{G}_m[j-1]).$$

8.2. **Cohomological Pic.** Dual to 8.1.1 we set:

8.2.1**. Definition.** Define the *cohomological Picard complex* of X by

$$\mathrm{RPic}(X) := \mathrm{RPic}(M(X)).$$

Define, for $i \in \mathbb{Z}$

$$\mathrm{R}^i\mathrm{Pic}(X) := {}^tH^i(\mathrm{RPic}(X))$$

the 1-motives with torsion determined by the cohomology of the Picard complex (see Notation 1.8.7).

The functor RPic has similar properties to LAlb, deduced by duality. Homotopical invariance, Mayer-Vietoris, Gysin and the distinguished triangle for abstract blow-ups are clear, and moreover we have

$$\mathrm{RPic}(M(X)(n)) = \begin{cases} 0 & \text{if } n > 1 \\ [\mathbb{Z}\pi_0(X) \to 0] & \text{if } n = 1. \end{cases}$$

We also have that $\mathrm{RPic}(X) = \mathrm{LAlb}(X)^\vee$, hence

$$\mathrm{R}^i\mathrm{Pic}(X) = \mathrm{L}_i\mathrm{Alb}(X)^\vee.$$

We shall complete §8.1.d by

8.2.a. RPic *and* LAlb *with supports.* Let $X \in Sm(k)$, U a dense open subset of X and $Z = X - U$ (reduced structure). In DM, we have the *motive with supports* $M^Z(X)$ fitting in an exact triangle

$$M(U) \to M(X) \to M^Z(X) \xrightarrow{+1}$$

hence the cohomological complex with supports

$$\mathrm{RPic}_Z(X) := \mathrm{RPic}(M^Z(X))$$

fitting in an exact triangle

$$\mathrm{RPic}_Z(X) \to \mathrm{RPic}(X) \to \mathrm{RPic}(U) \xrightarrow{+1}.$$

Dually, we have $\mathrm{LAlb}^Z(X) := \mathrm{LAlb}(M^Z(X))$.

8.2.2**. Lemma.** *If* $\dim X = d$, $\mathrm{RPic}_Z(X) \simeq [\underline{CH}_{d-1}(Z) \to 0][-2]$, *where* $\underline{CH}_{d-1}(Z)$ *is the lattice corresponding to the Galois module* $CH_{d-1}(Z_{\overline{k}})$.

Note that $CH_{d-1}(Z_{\overline{k}})$ is simply the free abelian group with basis the irreducible components of $Z_{\overline{k}}$ which are of codimentsion 1 in $X_{\overline{k}}$.

Proof. This follows readily from the exact sequence

$$0 \to \Gamma(X_{\overline{k}}, \mathbb{G}_m) \to \Gamma(U_{\overline{k}}, \mathbb{G}_m) \to CH_{d-1}(Z_{\overline{k}})$$
$$H^1(X_{\overline{k}}, \mathbb{G}_m) \to H^1(U_{\overline{k}}, \mathbb{G}_m) \to 0.$$

□

8.3. **Relative** LAlb **and** RPic**.** For $f : Y \to X$ a map of schemes we let $M(X, Y)$ denote the cone of $M(Y) \to M(X)$. Note that for a closed embedding f in a proper scheme X we have $M(X, Y) = M^c(X - Y)$.

We denote by $\mathrm{LAlb}(X, Y)$ and $\mathrm{RPic}(X, Y)$ the resulting complexes of 1-motives.

9. Borel-Moore variants

9.1. **The Borel-Moore Albanese complex.** Let $p : X \to \operatorname{Spec} k$ be a scheme of finite type over a field k which admits resolution of singularities. Recall that the motive with compact support of X, denoted $M^c(X) \in \mathrm{DM}_-^{\mathrm{eff}}$, has also been defined in [63, Sect. 4]. It is the Suslin complex C_* of the representable presheaf with transfers $L^c(X)$ given by quasi-finite correspondences. Since finite implies quasi-finite we have a canonical map $M(X) \to M^c(X)$ which is an isomorphism if X is proper over k.

In general, $M^c : Sch^c(k) \to \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ is a covariant functor from the category of schemes of finite type over k and proper maps between them.

9.1.1**. Definition.** We define the *Borel-Moore Albanese complex* of X by

$$\mathrm{LAlb}^c(X) := \mathrm{LAlb}(M^c(X)).$$

Define, for $i \in \mathbb{Z}$

$$\mathrm{L}_i\mathrm{Alb}^c(X) := {}_tH_i(\mathrm{LAlb}^c(X))$$

the 1-motivic homology of this complex.

Note that we have the following properties:

9.1.a. *Functoriality.* The functor $X \mapsto \mathrm{LAlb}^c(X)$ is covariant for proper maps and contravariant with respect to flat morphisms of relative dimension zero, for example étale morphisms. We have a canonical, covariantly functorial map

$$\mathrm{LAlb}(X) \to \mathrm{LAlb}^c(X)$$

which is an isomorphism if X is proper.

9.1.b. *Localisation triangle.* For any closed subscheme Y of a scheme X we have a triangle

$$\begin{array}{ccc} \mathrm{LAlb}^c(Y) & \longrightarrow & \mathrm{LAlb}^c(X) \\ {}^{+1}\nwarrow & & \swarrow \\ & \mathrm{LAlb}^c(X-Y) & \end{array}$$

and therefore a long exact sequence of 1-motives

(9.1)

$$\cdots \to \mathrm{L}_i\mathrm{Alb}^c(Y) \to \mathrm{L}_i\mathrm{Alb}^c(X) \to \mathrm{L}_i\mathrm{Alb}^c(X-Y) \to \mathrm{L}_{i-1}\mathrm{Alb}^c(Y) \to \dots$$

In particular, let X be a scheme obtained by removing a divisor Y from a proper scheme $\overline{X}$, *i.e.* $X = \overline{X} - Y$. Then

$$\begin{aligned} \cdots \to \mathrm{L}_1\mathrm{Alb}(Y) \to \mathrm{L}_1\mathrm{Alb}(\overline{X}) \to \mathrm{L}_1\mathrm{Alb}^c(X) \to \mathrm{L}_0\mathrm{Alb}(Y) \\ \to \mathrm{L}_0\mathrm{Alb}(\overline{X}) \to \mathrm{L}_0\mathrm{Alb}^c(X) \to \dots . \end{aligned}$$

9.1.c. *Albanese map.* We have the following natural map (5.2)

$$a^c_X : \alpha^* M^c(X) \to \mathrm{Tot}\,\mathrm{LAlb}^c(X)$$

which is an isomorphism if $\dim(X) \le 1$. In general, for any X, a^c_X induces an isomorphism on motivic cohomology with compact supports, *i.e.* $H^j_c(X, \mathbb{Q}(1)) = \mathrm{Hom}(\mathrm{LAlb}^c(X), \mathbb{G}_m[j-1])$.

9.2. **Cohomological Albanese complex.**

9.2.1. **Lemma.** *Suppose $p = 1$ (i.e. $\mathrm{char}\, k = 0$), and let $n \ge 0$. For any X of dimension $\le n$, the motive $M(X)^*(n)[2n]$ is effective. (Here, contrary to the rest of the paper, $M(X)^*$ denotes the "usual" dual $\mathrm{Hom}(M(X), \mathbb{Z})$ in $\mathrm{DM}_{\mathrm{gm}}$.)*

Proof. First assume X irreducible. Let $\widetilde{X} \to X$ be a resolution of singularities of X. With notation as in §8.1.e, we have an exact triangle

$$M(X)^*(n) \to M(\widetilde{X})^*(n) \oplus M(Z)^*(n) \oplus M(\widetilde{Z})^*(n) \stackrel{+1}{\to} .$$

Since $\widetilde{X}$ is smooth, $M(\widetilde{X})^*(n) \simeq M^c(X)[-2n]$ is effective by [63, Th. 4.3.2]; by induction on n, so are $M(Z)^*(n)$ and $M(\widetilde{Z})^*(n)$ and therefore $M(X)^*(n)$ is effective.

In general, let $X_1, \dots, X_r$ be the irreducible components of X. Suppose $r \geq 2$ and let $Y = X_2 \cap \dots \cap X_n$: since (X_1, Y) is a cdh cover of X, we have an exact triangle

$$M(X)^*(n) \to M(X_1)^*(n) \oplus M(Y)^*(n) \oplus M(X_1 \cap Y)^*(n) \stackrel{+1}{\to}.$$

The same argument then shows that $M(X)^*(n)$ is effective, by induction on r. □

We can therefore apply our functor LAlb and obtain another complex $\mathrm{LAlb}(M(X)^*(n)[2n])$ of 1-motives. If X is smooth this is just the Borel-Moore Albanese.

9.2.2. **Definition.** We define the *cohomological Albanese complex* of a scheme X of dimension n by

$$\mathrm{LAlb}^*(X) := \mathrm{LAlb}(M(X^{(n)})^*(n)[2n])$$

where $X^{(n)}$ is the union of the n-dimensional components of X. Define, for $i \in \mathbb{Z}$

$$\mathrm{L}_i\mathrm{Alb}^*(X) := {}_tH_i(\mathrm{LAlb}^*(X))$$

the 1-motivic homology of this complex.

9.2.3. **Lemma.** *a) If $Z_1, \dots Z_n$ are the irreducible components of dimension n of X, then the cone of the natural map*

$$\mathrm{LAlb}^*(X) \to \bigoplus \mathrm{LAlb}^*(Z_i)$$

is a complex of groups of multiplicative type.
b) If X is integral and $\widetilde{X}$ is a desingularisation of X, then the cone of the natural map

$$\mathrm{LAlb}^*(X) \to \mathrm{LAlb}^*(\widetilde{X})$$

is a complex of groups of multiplicative type.

Proof. a) and b) follow from dualising the abstract blow-up exact triangles of [63, 2.2] and applying Proposition 8.1.2. □

9.3. **Compactly supported and homological Pic.** We now consider the dual complexes of the Borel-Moore and cohomological Albanese.

9.3.1. **Definition.** Define the *compactly supported Picard complex* of any scheme X by

$$\mathrm{RPic}^c(X) := \mathrm{RPic}(M^c(X))$$

and the *homological Picard complex* of an equidimensional scheme X of dimension n by

$$\mathrm{RPic}^*(X) := \mathrm{RPic}(M(X)^*(n)[2n]).$$

Denote $\mathrm{R}^i\mathrm{Pic}^c(X) := {}^tH^i(\mathrm{RPic}^c(X))$ and $\mathrm{R}^i\mathrm{Pic}^*(X) := {}^tH^i(\mathrm{RPic}^*(X))$ the 1-motives with torsion determined by the homology of these Picard complexes.

Recall that $\mathrm{RPic}^c(X) = \mathrm{RPic}(X)$ if X is proper and $\mathrm{RPic}^c(X) = \mathrm{RPic}^*(X)$ if X is smooth.

9.4. **Topological invariance.** To conclude this section and the previous one, we note the following useful

9.4.1. **Lemma.** *Suppose that $f : Y \to X$ is a universal topological homeomorphism, in the sense that $1_U \times f : U \times Y \to U \times X$ is a homeomorphism of topological spaces for any smooth U (in particular f is proper). Then f induces isomorphisms* $\mathrm{LAlb}(Y) \xrightarrow{\sim} \mathrm{LAlb}(X)$, $\mathrm{RPic}(X) \xrightarrow{\sim} \mathrm{RPic}(Y)$, $\mathrm{LAlb}^c(Y) \xrightarrow{\sim} \mathrm{LAlb}^c(X)$ *and* $\mathrm{RPic}^c(X) \xrightarrow{\sim} \mathrm{RPic}^c(Y)$. *Similarly,* $\mathrm{LAlb}^*(X) \xrightarrow{\sim} \mathrm{LAlb}^*(Y)$ *and* $\mathrm{RPic}^*(Y) \xrightarrow{\sim} \mathrm{RPic}^*(X)$. *This applies in particular to Y = the semi-normalisation of X.*

Proof. It suffices to notice that f induces isomorphisms $L(Y) \xrightarrow{\sim} L(X)$ and $L^c(Y) \to L^c(X)$, since by definition these sheaves only depend on the underlying topological structures. □

Lemma 9.4.1 implies that in order to compute $\mathrm{LAlb}(X)$, etc., we may always assume X semi-normal if we wish so.

10. Computing $\mathrm{LAlb}(X)$ and $\mathrm{RPic}(X)$ for smooth X

10.1. **The Albanese scheme.** Let X be a reduced k-scheme of finite type. "Recall" ([49, Sect. 1], [57]) the Albanese scheme $\mathcal{A}_{X/k}$ fitting in the following extension

$$0 \to \mathcal{A}^0_{X/k} \to \mathcal{A}_{X/k} \to \mathbb{Z}[\pi_0(X)] \to 0 \tag{10.1}$$

where $\mathcal{A}^0_{X/k}$ is Serre's generalised Albanese semi-abelian variety, and $\pi_0(X)$ is the scheme of constants of X viewed as an étale sheaf on $Sm(k)$.[7] In particular,

$$\mathcal{A}_{X/k} \in \mathrm{AbS} \ \text{(see Definition 3.7.1)}.$$

There is a canonical morphism

$$\overline{a}_X : X \to \mathcal{A}_{X/k} \tag{10.2}$$

[7] In the said references, $\mathcal{A}^0_{X/k}$ is denoted by Alb_X and $\mathcal{A}_{X/k}$ is denoted by $\widetilde{Alb}_X$.

which is universal for morphisms from X to group schemes of the same type.

For the existence of $\mathcal{A}_{X/k}$, the reference [49, Sect. 1] is sufficient if X is a variety (integral k-scheme of finite type), hence if X is normal (for example smooth): this will be sufficient in this section. For the general case, see §11.1.

We shall denote the object $\mathrm{Tot}^{-1}(\underline{\mathcal{A}}_{X/k}) \in D^b(\mathcal{M}_1[1/p])$ simply by $\mathcal{A}_{X/k}$. As seen in Lemma 3.10.1, we have

$$H_i(\mathcal{A}_{X/k}) = \begin{cases} [\mathbb{Z}[\pi_0(X)] \to 0] & \text{for } i = 0 \\ [0 \to \mathcal{A}^0_{X/k}] & \text{for } i = 1 \\ 0 & \text{for } i \neq 0, 1. \end{cases}$$

10.2. **The main theorem.** Suppose X smooth. Via (1.1), (10.2) induces a composite map

$$M_{\text{ét}}(X) \to M_{\text{ét}}(\mathcal{A}_{X/k}) \to \mathcal{A}_{X/k}. \tag{10.3}$$

Theorem 4.4.3 gives:

10.2.1**. Lemma.** *We have an exact triangle*

$$\mathbb{Z}[\pi_0(X)]^*[0] \to \underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \to (\mathcal{A}^0_{X/k})^*[-2] \xrightarrow{+1} \quad \square$$

By Lemma 10.2.1, the map

$$\underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \to \underline{\mathrm{Hom}}(M_{\text{ét}}(X), \mathbb{Z}(1))$$

deduced from (10.3) factors into a map

$$\underline{\mathrm{Hom}}(\mathcal{A}_{X/k}, \mathbb{Z}(1)) \to \tau_{\leq 2}\underline{\mathrm{Hom}}(M_{\text{ét}}(X), \mathbb{Z}(1)). \tag{10.4}$$

Applying Proposition 4.4.1 and Lemma 5.1.2, we therefore get a canonical map in $D^b(\mathcal{M}_1[1/p])$

$$\mathrm{LAlb}(X) \to \mathcal{A}_{X/k}. \tag{10.5}$$

10.2.2**. Theorem.** *Suppose X smooth. Then the map* (10.5) *sits in an exact triangle*

$$[0 \to \mathrm{NS}^*_{X/k}][2] \to \mathrm{LAlb}(X) \to \mathcal{A}_{X/k} \xrightarrow{+1}$$

where $\mathrm{NS}^*_{X/k}$ denotes the group of multiplicative type dual to $\mathrm{NS}_{X/k}$ (cf. Definition 3.5.1 and Proposition 3.5.2).

This theorem says in particular that, on the object $\mathrm{LAlb}(X)$, the motivic t-structure and the homotopy t-structure are compatible in a strong sense.

10.2.3. **Corollary.** *For X smooth over k we have*

$$\mathrm{L}_i\mathrm{Alb}(X) = \begin{cases} [\mathbb{Z}[\pi_0(X)] \to 0] & \text{if } i = 0 \\ [0 \to \mathcal{A}^0_{X/k}] & \text{if } i = 1 \\ [0 \to \mathrm{NS}^*_{X/k}] & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

10.2.4. **Corollary.** *For X smooth, $\mathrm{L}_1\mathrm{Alb}(X)$ is isomorphic to the homological Albanese 1-motive $\mathrm{Alb}^-(X)$ of* [5]. □

10.3. **Reformulation of Theorem 10.2.2.** It is sufficient to get an exact triangle after application of $D_{\leq 1} \circ \mathrm{Tot}$, so that we have to compute the cone of the morphism (10.4) in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm,\acute{e}t}}$. We shall use:

10.3.1. **Lemma.** *For $\mathcal{F} \in \mathrm{HI}^s_{\text{ét}}$, the morphism b of Proposition 3.1.1 is induced by* (1.1).

Proof. This is clear by construction, since $\mathrm{Hom}_{\mathrm{DM}_{\text{ét}}}(M(G), \mathcal{F}[1]) = H^1_{\text{ét}}(G, \mathcal{F})$ [63, Prop. 3.3.1]. □

Taking the cohomology sheaves of (10.4), we get morphisms

$$\mathcal{H}om\,(\mathcal{A}_{X/k}, \mathbb{G}_m) \to p_*\mathbb{G}_{m,X} \tag{10.6}$$

$$f : \mathcal{E}xt\,(\mathcal{A}_{X/k}, \mathbb{G}_m) \xrightarrow{b} \underline{\mathrm{Pic}}_{\mathcal{A}_{X/k}/k} \xrightarrow{\overline{a}^*} \underline{\mathrm{Pic}}_{X/k} \tag{10.7}$$

where in (10.7), b corresponds to the map of Proposition 3.1.1 thanks to Lemma 10.3.1. Thanks to Proposition 3.6.1, Theorem 10.2.2 is then equivalent to the following

10.3.2. **Theorem.** *Suppose k algebraically closed; Then*
a) (10.6) *yields an isomorphism* $\mathrm{Hom}(\mathcal{A}_{X/k}, \mathbb{G}_m) \xrightarrow{\sim} \Gamma(X, \mathbb{G}_m)$.
b) (10.7) *defines a short exact sequence*

$$0 \to \mathrm{Ext}(\underline{\mathcal{A}}_{X/k}, \mathbb{G}_m) \xrightarrow{f} \mathrm{Pic}(X) \xrightarrow{e} \mathrm{NS}(X) \to 0 \tag{10.8}$$

where e is the natural map.

Before proving Theorem 10.3.2, it is convenient to prove Lemma 10.3.3 below. Let $A_{X/k}$ be the abelian part of $\mathcal{A}^0_{X/k}$; then the sheaf $\mathcal{E}xt\,(\underline{A}_{X/k}, \mathbb{G}_m)$ is represented by the dual abelian variety $A^*_{X/k}$. Composing with the map f of (10.7), we get a map of 1-motivic sheaves

$$\underline{A}^*_{X/k} \to \underline{\mathrm{Pic}}_{X/k}. \tag{10.9}$$

10.3.3. **Lemma.** *The map* (10.9) *induces an isogeny in* SAb

$$A^*_{X/k} \twoheadrightarrow \gamma(\underline{\mathrm{Pic}}_{X/k}) = \underline{\mathrm{Pic}}^0_{X/k}$$

where γ is the adjoint functor appearing in Theorem 3.2.4 a).

Proof. We proceed in 3 steps:

(1) The lemma is true if X is smooth projective: this follows from the representability of Pic^0_X and the duality between the Picard and the Albanese varieties.
(2) Let $j : U \to X$ be an open immersion: then the lemma is true for X if and only if it is true for U. This is clear since $\underline{\mathrm{Pic}}_{X/k} \to \underline{\mathrm{Pic}}_{U/k}$ is an epimorphism with discrete kernel.
(3) Let $\varphi : Y \to X$ be an étale covering. If the lemma is true for Y, then it is true for X. This follows from the existence of transfer maps $\varphi_* : A^*_{Y/k} \to A^*_{X/k}$, $\underline{\mathrm{Pic}}_{Y/k} \to \underline{\mathrm{Pic}}_{X/k}$ commuting with the map of the lemma, plus the usual transfer argument.

We conclude by de Jong's theorem [27, Th. 4.1]. □

10.4. **Proof of Theorem 10.3.2.** We may obviously suppose that X is irreducible.

a) is obvious from the universal property of $\mathcal{A}_{X/k}$. For b) we proceed in two steps:

(1) Verification of $ef = 0$.
(2) Proof that the sequence is exact.

(1) As above, let $A = A_{X/k}$ be the abelian part of $\mathcal{A}^0_{X/k}$. In the diagram

$$\mathrm{Ext}(A, \mathbb{G}_m) \to \mathrm{Ext}(\mathcal{A}^0_{X/k}, \mathbb{G}_m) \leftarrow \mathrm{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m)$$

the first map is surjective and the second map is an isomorphism, hence we get a surjective map

$$v : \mathrm{Ext}(A, \mathbb{G}_m) \to \mathrm{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m).$$

Choose a rational point $x \in X(\overline{k})$. We have a diagram

$$\begin{array}{ccc}
\mathrm{Ext}(A, \mathbb{G}_m) & \xrightarrow{v} & \mathrm{Ext}(\mathcal{A}_{X/k}, \mathbb{G}_m) \\
{\scriptstyle a}\downarrow & & {\scriptstyle f}\downarrow \\
\mathrm{Pic}(A) & \xrightarrow{x^*} & \mathrm{Pic}(X) \\
{\scriptstyle c}\downarrow & & {\scriptstyle e}\downarrow \\
\mathrm{NS}(A) & \xrightarrow{x^*} & \mathrm{NS}(X)
\end{array}$$

in which

(i) a is given by [53, p. 170, prop. 5 and 6] (or by Proposition 3.1.1).
(ii) $ca = 0$ (ibid., p. 184, th. 6;).
(iii) x^* is induced by the "canonical" map $X \to A$ sending x to 0.

Lemma 3.1.6 applied to $G = \mathcal{A}_{X/k}$ implies that the top square commutes (the bottom one trivially commutes too). Moreover, since v is surjective and $ca = 0$, we get $ef = 0$.

(2) In the sequence (10.8), the surjectivity of e is clear. Let us prove the injectivity of f: suppose that $f(\mathcal{E})$ is trivial. In the pull-back diagram

$$\begin{array}{ccc} \overline{a}^*\mathcal{E} & \xrightarrow{\pi'} & X \\ {\scriptstyle \overline{a}'}\downarrow & & {\scriptstyle \overline{a}}\downarrow \\ \mathcal{E} & \xrightarrow{\pi} & \mathcal{A}_{X/k} \end{array}$$

π' has a section σ'. Observe that $\mathcal{E}$ is a locally semi-abelian scheme: by the universal property of $\mathcal{A}_{X/k}$, the morphism $\overline{a}'\sigma'$ factors canonically through $\overline{a}$. In other words, there exists $\sigma : \mathcal{A}_{X/k} \to \mathcal{E}$ such that $\overline{a}'\sigma' = \sigma\overline{a}$. Then

$$\pi\sigma\overline{a} = \pi\overline{a}'\sigma' = \overline{a}\pi'\sigma' = \overline{a}$$

hence $\pi\sigma = 1$ by reapplying the universal property of $\mathcal{A}_{X/k}$, and $\mathcal{E}$ is trivial. Finally, exactness in the middle follows immediately from Proposition 3.5.2 and Lemma 10.3.3. □

10.4.1. **Corollary.** *The isogeny of Lemma 10.3.3 is an isomorphism.*

Proof. This follows from the injectivity of f in (10.8). □

10.5. **An application.**

10.5.1. **Corollary.** *Let X be a smooth k-variety of dimension d, U a dense open subset and $Z = X - U$ (reduced structure). Then the morphism $\mathcal{A}_{U/k} \to \mathcal{A}_{X/k}$ is epi; its kernel $T_{X/U,k}$ is a torus whose character group fits into a short exact sequence*

$$0 \to T^*_{X/U,k} \to \underline{\mathrm{CH}}_{d-1}(Z) \to \mathrm{NS}_Z(X) \to 0$$

where $\underline{\mathrm{CH}}_{d-1}(Z)$ is as in Lemma 8.2.2 and $\mathrm{NS}_Z(X) = \mathrm{Ker}(\mathrm{NS}(X) \to \mathrm{NS}(U)$.

Proof. To see that $\mathcal{A}_{U/k} \to \mathcal{A}_{X/k}$ is epi with kernel of multiplicative type, it is sufficient to see that $\pi_0(\mathcal{A}_{U/k}) \xrightarrow{\sim} \pi_0(\mathcal{A}_{X/k})$ and that $A_{U/k} \xrightarrow{\sim} A_{X/k}$. The first isomorphism is obvious and the second one follows from [41, Th. 3.1]. The characterisation of $T_{X/U,k}$ is then an immediate consequence of Theorem 10.2.2 and Lemma 8.2.2; in particular, it is a torus. □

10.6. $\mathrm{RPic}(X)$**.** Recall that for X smooth projective $\mathcal{A}^0_{X/k} = A_{X/k}$ is the classical Albanese abelian variety $\mathrm{Alb}(X)$. In the case where X is obtained by removing a divisor Y from a smooth proper scheme $\overline{X}$, $\mathcal{A}^0_{X/k}$, can be described as follows (*cf.* [5]). Consider the (cohomological Picard) 1-motive $\mathrm{Pic}^+(X) := [\mathrm{Div}^0_Y(\overline{X}) \to \mathrm{Pic}^0(\overline{X})]$: its Cartier dual is $\mathcal{A}^0_{X/k}$ which can be represented as a torus bundle

$$0 \to T_{\overline{X}/X,k} \to \mathcal{A}^0_{X/k} \to \mathrm{Alb}(\overline{X}) \to 0$$

where $T_{\overline{X}/X,k}$ has character group $\mathrm{Div}^0_Y(\overline{X})$ according to Corollary 10.5.1.

From the previous remarks and Corollary 10.2.4, we deduce:

10.6.1. **Corollary.** *If X is smooth, $\mathrm{R}^1\mathrm{Pic}(X)$ is isomorphic to the 1-motive $\mathrm{Pic}^+(X)$ of* [5] *(the Cartier dual of $\mathrm{Alb}^-(X)$). If $\overline{X}$ is a smooth compactification of X, then*

$$\mathrm{R}^i\mathrm{Pic}(X) = \begin{cases} [0 \to \mathbb{Z}[\pi_0(X)]^*] & \text{if } i = 0 \\ [\mathrm{Div}^0_Y(\overline{X}) \to \mathrm{Pic}^0(\overline{X})] & \text{if } i = 1 \\ [\mathrm{NS}(X) \to 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $Y = \overline{X} - X$.

11. 1-motivic homology and cohomology of singular schemes

11.1. $\mathcal{A}_{X/k}$ **for** $X \in Sch(k)$**.** In this subsection, we extend the construction of $\mathcal{A}_{X/k}$ to arbitrary reduced k-schemes of finite type, starting from the case where X is integral (which is treated in [49, Sect. 1]). So far, k may be of any characteristic.

To make the definition clear:

11.1.1. **Definition.** Let $X \in Sch(k)$. We say that $\mathcal{A}_{X/k}$ *exists* if the functor

$$\begin{aligned} \mathrm{AbS} &\to Ab \\ G &\mapsto G(X) \end{aligned}$$

is corepresentable.

First note that $\mathcal{A}_{X/k}$ does not exist (as a semi-abelian scheme, at least) if X is not reduced. For example, for $X = \operatorname{Spec} k[\varepsilon]$ with $\varepsilon^2 = 0$, we have an exact sequence

$$0 \to \mathbb{G}_a(k) \to \mathrm{Map}_k(X, \mathbb{G}_m) \to \mathbb{G}_m(k) \to 0$$

which cannot be described by $\mathrm{Hom}(\mathcal{A}, \mathbb{G}_m)$ for any semi-abelian scheme $\mathcal{A}$.

On the other hand, $M(X) = M(X_{\mathrm{red}})$ for any $X \in Sch(k)$, where X_{red} is the reduced subscheme of X (see proof of Lemma 9.4.1), so we are naturally led to neglect nonreduced schemes.

11.1.2. **Lemma.** *Let $Z \in Sch(k)$, $G \in \mathrm{AbS}$ and $f_1, f_2 : Z \rightrightarrows G$ two morphisms which coincide on the underlying topological spaces (thus, $f_1 = f_2$ if Z is reduced). Then there exists a largest quotient $\overline{G}$ of G such that $\pi_0(G) \xrightarrow{\sim} \pi_0(\overline{G})$ and the two compositions*

$$Z \rightrightarrows G \to \overline{G}$$

coincide.

Proof. The set S of such quotients $\overline{G}$ is in one-to-one correspondence with the set of closed subgroups $H^0 \subseteq G^0$. Clearly $\pi_0(G) \in S$, and if $\overline{G}_1 = G/H_1^0 \in S$, $\overline{G}_2 = G/H_2^0 \in S$, then $G_3 = G/(H_1^0 \cap H_2^0) \in S$. Therefore S has a smallest element, since it is Artinian (compare proof of Proposition 3.6.5). □

11.1.3. **Proposition.** *$\mathcal{A}_{X/k}$ exists for any reduced $X \in Sch(k)$.*

Proof. When X is integral, this is [49, Sect. 1]. Starting from this case, we argue by induction on $\dim X$. Let $Z_1, \ldots, Z_n$ be the irreducible components of X and $Z_{ij} = Z_i \cap Z_j$.

By induction, $\mathcal{A}_{ij} := \mathcal{A}_{(Z_{ij})_{\mathrm{red}}/k}$ exists for any (i, j). Consider

$$\mathcal{A} = \mathrm{Coker}\left(\bigoplus \mathcal{A}_{ij} \to \bigoplus \mathcal{A}_i\right)$$

with $\mathcal{A}_i = \mathcal{A}_{Z_i/k}$. Let $Z = \coprod Z_{ij}$ and $f_1, f_2 : Z \rightrightarrows \coprod Z_i$ the two inclusions: the compositions $f_1, f_2 : Z \rightrightarrows \mathcal{A}$ verify the hypothesis of Lemma 11.1.2. Hence there is a largest quotient $\mathcal{A}'$ of $\mathcal{A}$ with $\pi_0(\mathcal{A}) \xrightarrow{\sim} \pi_0(\mathcal{A}')$, equalising f_1 and f_2. Then the composition

$$\coprod Z_i \to \bigoplus \mathcal{A}_i \to \mathcal{A}'$$

glues down to a morphism $X \to \mathcal{A}'$. It is clear that $\mathcal{A}' = \mathcal{A}_{X/k}$ since, for any commutative group scheme G, the sequence

$$0 \to \mathrm{Map}_k(X, G) \to \bigoplus \mathrm{Map}_k(Z_i, G) \to \bigoplus \mathrm{Map}_k(Z_{ij}, G)$$

is exact. □

Unfortunately this result is only useful to understand $\mathrm{L}_1\mathrm{Alb}(X)$ for X "strictly reduced", as we shall see below. In general, we shall have to consider Albanese schemes for the éh topology.

11.2. **The** éh **topology.** In this subsection and the next ones, we assume that k is of characteristic 0. Recall that $\mathrm{HI}_{\text{ét}} = \mathrm{HI}^s_{\text{ét}}$ in this case by Proposition D.1.4.

The following étale analogue of the cdh topology was first considered by Thomas Geisser [21]:

11.2.1**. Definition.** The éh *topology* on $Sch(k)$ is the topology generated by the étale topology and coverings defined by abstract blow-ups (it is the same as [63, Def. 4.1.9] by replacing the Nisnevich topology by the étale topology).

As in [63, Th. 4.1.10] (see [39, Proof of Th. 14.20] for more details), one has:

11.2.2**. Proposition.** *Let $C \in \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$. Then, for any $X \in Sch(k)$ and any $q \in \mathbb{Z}$ one has*

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(M_{\text{ét}}(X), C[q]) \simeq H^q_{\text{éh}}(X, C_{\text{éh}}[q]).$$

In particular, if X is smooth then $H^q_{\text{ét}}(X, C_{\text{ét}}) \xrightarrow{\sim} H^q_{\text{éh}}(X, C_{\text{éh}})$. □

(See [21, Th. 4.3] for a different proof of the second statement.)

The following lemma will be used many times:

11.2.3**. Lemma** (Blow-up induction)**.** *Let $\mathcal{A}$ be an abelian category.*
a) Let $\mathcal{B} \subseteq \mathcal{A}$ be a thick subcategory and $H^* : Sch(k)^{op} \to \mathcal{A}^{(\mathbb{N})}$ a functor with the following property: given an abstract blow-up as in §8.1.e, we have a long exact sequence

$$\cdots \to H^i(X) \to H^i(\widetilde{X}) \oplus H^i(Z) \to H^i(\widetilde{X}) \to H^{i+1}(X) \to \dots$$

Let $n \geq 0$, and assume that $H^i(X) \in \mathcal{B}$ for $i \leq n$ and $X \in Sm(k)$. Then $H^i(X) \in \mathcal{B}$ for $i \leq n$ and all $X \in Sch(k)$.
b) Let H^*_1, H^*_2 be two functors as in a) and $\varphi^* : H^*_1 \to H^*_2$ be a natural transformation. Let $n \geq 0$, and suppose that φ^i_X is an isomorphism for all $X \in Sm(k)$ and $i \leq n$. Then φ^i_X is an isomorphism for all $X \in Sch(k)$ and $i \leq n$.
We get the same statements as a) and b) by replacing "$i \leq n$" by "$i \geq n + \dim X$".

Proof. Induction on $\dim X$ in two steps: 1) if X is integral, choose a resolution of singularities $\widetilde{X} \to X$; 2) in general, if $Z_1, \dots, Z_r$ are the irreducible components of X, choose $\widetilde{X} = \coprod Z_i$ and $Z = \bigcup_{i \neq j} Z_i \cap Z_j$. □

11.2.4**. Examples.** 1) Thanks to [63, Th. 4.1.10] and Proposition 11.2.2, cdh or éh cohomology with coefficients in an object of $\mathrm{DM}^{\mathrm{eff}}_-$ or

$\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ satisfy the hypothesis of a) (here $\mathcal{A}$ = abelian groups). (See also [21, Prop. 3.2] for a different proof.)

2) Étale cohomology with torsion coefficients satisfies the hypothesis of a) by [46, Prop. 2.1] (recall that the proof of *loc. cit.* relies on the proper base change theorem).

Here is a variant of Lemma 11.2.3:

11.2.5. **Proposition.** *a) Let $X \in Sch(k)$ and $X_\bullet \to X$ be a hyper-envelope in the sense of Gillet-Soulé* [23, 1.4.1]. *Let $\tau = \mathrm{cdh}$ or* éh. *Then, for any (bounded below) complex of sheaves C over $Sch(X)_\tau$, the augmentation map*

$$H^*_\tau(X, C) \to H^*_\tau(X_\bullet, C)$$

is an isomorphism.
b) Suppose that X_0 and X_1 are smooth and $\mathcal{F}$ is a homotopy invariant Nisnevich (if $\tau = \mathrm{cdh}$) or étale (if $\tau =$ éh) sheaf with transfers. Then we have

$$\mathcal{F}_\tau(X) = \mathrm{Ker}(\mathcal{F}(X_0) \to \mathcal{F}(X_1)).$$

Proof. a) By [39, Lemma 12.26], $X_\bullet$ is a proper τ-hypercovering (*cf.* [13, p. 46]). Therefore the proposition follows from the standard theory of cohomological descent.

b) Let us take $C = \mathcal{F}_\tau[0]$. By a) we have a descent spectral sequence which gives a short exact sequence

$$0 \to \mathcal{F}_\tau(X) \to \mathcal{F}_\tau(X_0) \to \mathcal{F}_\tau(X_1)$$

and the conclusion now follows from 11.2.2. □

11.3. $\mathrm{L}_i\mathrm{Alb}(X)$ **for X singular.** The following is a general method for computing the 1-motivic homology of $\mathrm{LAlb}^{\mathbb{Q}}(X)$:

11.3.1. **Proposition.** *If* $\mathrm{char}\, k = 0$ *and $X \in Sch(k)$ consider cdh cohomology groups $\mathbb{H}^i_{\mathrm{cdh}}(X, \pi^*(N))_{\mathbb{Q}}$, where $\pi : Sch(k)_{\mathrm{cdh}} \to Sch(k)_{\mathrm{Zar}}$ is the canonical map from the cdh site to the big Zariski site. Then we have short exact sequences, for all $i \in \mathbb{Z}$*

$$0 \to \mathrm{Ext}^1(\mathrm{L}_{i-1}\mathrm{Alb}^{\mathbb{Q}}(X), N) \to \mathbb{H}^i_{\mathrm{cdh}}(X, \pi^*(N))_{\mathbb{Q}} \\ \to \mathrm{Hom}(\mathrm{L}_i\mathrm{Alb}^{\mathbb{Q}}(X), N) \to 0$$

$$0 \to \mathrm{Ext}^1(N, \mathrm{L}_{i+1}\mathrm{Alb}^{\mathbb{Q}}(X)) \to \mathbb{E}\mathrm{xt}^{-i}(N, \mathrm{LAlb}^{\mathbb{Q}}(X)) \\ \to \mathrm{Hom}(N, \mathrm{L}_i\mathrm{Alb}(X)) \to 0.$$

Proof. For any 1-motive $N \in \mathcal{M}_1$ we have a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\mathrm{L}_q\mathrm{Alb}(X), N) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}(\mathrm{LAlb}(X), N) \tag{11.1}$$

yielding the following short exact sequence

$$0 \to \mathrm{Ext}^1(\mathrm{L}_{i-1}\mathrm{Alb}^{\mathbb{Q}}(X), N) \to \mathbb{E}\mathrm{xt}^i(\mathrm{LAlb}^{\mathbb{Q}}(X), N) \to \\ \mathrm{Hom}(\mathrm{L}_i\mathrm{Alb}^{\mathbb{Q}}(X), N) \to 0$$

because of Proposition 1.2.1. By adjunction we also obtain

$$\mathbb{E}\mathrm{xt}^i(\mathrm{LAlb}^{\mathbb{Q}}(X), N) = \mathrm{Hom}(\mathrm{LAlb}^{\mathbb{Q}}(X), N[i]) \cong \mathrm{Hom}(M(X), \mathrm{Tot}\, N[i]).$$

Now from [63, Thm. 3.2.6 and Cor. 3.2.7], for X smooth we have

$$\mathrm{Hom}(M(X), \mathrm{Tot}\, N[i]) \cong \mathbb{H}^i_{\mathrm{Zar}}(X, N)_{\mathbb{Q}}.$$

If k is of characteristic 0 and X is arbitrary, we get the same isomorphism with cdh hypercohomology by [63, Thm. 4.1.10].

The proof for the second short exact sequence is similar. □

One gets similar computations integrally by replacing the cdh topology by the éh topology, but here the spectral sequence does not degenerate. In any case we shall obtain the following integral results directly (except for Proposition 13.6.4).

The following proposition folllows readily by blow-up induction (Lemma 11.2.3) from Corollary 10.2.3 and the exact sequences (8.2):

11.3.2. **Proposition.** *For any $X \in Sch(k)$ of dimension d in characteristic* 0*, we have*
a) $\mathrm{L}_i\mathrm{Alb}(X) = 0$ *if* $i < 0$.
b) $\mathrm{L}_0\mathrm{Alb}(X) = [\mathbb{Z}[\pi_0(X)] \to 0]$.
c) $\mathrm{L}_i\mathrm{Alb}(X) = 0$ *for* $i > \max(2, d+1)$.
d) $\mathrm{L}_{d+1}\mathrm{Alb}(X)$ *is a group of multiplicative type.* □

11.4. **The cohomological 1-motives** $\mathrm{R}^i\mathrm{Pic}(X)$**.** If $X \in Sch(k)$, we quote the following variant of Proposition 11.3.1:

11.4.1. **Lemma.** *Let $N \in \mathcal{M}_1 \otimes \mathbb{Q}$ and $X \in Sch(k)$. We have a short exact sequence, for all $i \in \mathbb{Z}$*

$$0 \to \mathrm{Ext}(N, \mathrm{R}^{i-1}\mathrm{Pic}(X)) \to \mathbb{H}^i_{\mathrm{cdh}}(X, \pi^*(N^*))_{\mathbb{Q}} \\ \to \mathrm{Hom}(N, \mathrm{R}^i\mathrm{Pic}(X)) \to 0$$

here $\pi : Sch(k)_{\mathrm{cdh}} \to Sch(k)_{\mathrm{Zar}}$ and N^* is the Cartier dual.

Proof. The spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(N, \mathrm{R}^q\mathrm{Pic}(X)) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}(N, \mathrm{RPic}(X))$$

yields the following short exact sequence

$$0 \to \mathrm{Ext}(N, \mathrm{R}^{i-1}\mathrm{Pic}(X)) \to \mathbb{E}\mathrm{xt}^i(N, \mathrm{RPic}(X)) \to \mathrm{Hom}(N, \mathrm{R}^i\mathrm{Pic}(X)) \to 0$$

and by Cartier duality, the universal property and [63, Thm. 4.1.10] we obtain:

$$\mathbb{E}\mathrm{xt}^i(N, \mathrm{RPic}(X)) = \mathrm{Hom}(N, \mathrm{RPic}(X)[i]) \cong \mathrm{Hom}(\mathrm{LAlb}(X), N^*[i]) = \mathrm{Hom}(M(X), N^*[i]) \cong \mathbb{H}^i_{\mathrm{cdh}}(X, \pi^*(N^*))_{\mathbb{Q}}.$$

□

On the other hand, here is a dual to Proposition 11.3.2:

11.4.2. **Proposition.** *For any $X \in Sch(k)$ of dimension d in characteristic 0, we have*
a) $\mathrm{R}^i\mathrm{Pic}(X) = 0$ *if* $i < 0$.
b) $\mathrm{R}^0\mathrm{Pic}(X) = [0 \to \mathbb{Z}[\pi_0(X)]^*]$.
c) $\mathrm{R}^i\mathrm{Pic}(X) = 0$ *for* $i > \max(2, d+1)$.
d) $\mathrm{R}^{d+1}\mathrm{Pic}(X)$ *is discrete.* □

11.5. **Borel-Moore variants.**

11.5.1. **Definition.** For $X \in Sch(k)$, we denote by $\pi_0^c(X)$ the disjoint union of $\pi_0(Z_i)$ where Z_i runs through the *proper* connected components of X: this is the *scheme of proper constants.*

11.5.2. **Proposition.** *Let $X \in Sch(k)$ of dimension d. Then:*
a) $\mathrm{L}_i\mathrm{Alb}^c(X) = 0$ *if* $i < 0$.
b) $\mathrm{L}_0\mathrm{Alb}^c(X) = [\mathbb{Z}[\pi_0^c(X)] \to 0]$. *In particular,* $\mathrm{L}_0\mathrm{Alb}^c(X) = 0$ *if no connected component is proper.*
c) $\mathrm{L}_i\mathrm{Alb}^c(X) = 0$ *for* $i > \max(2, d+1)$.
d) $\mathrm{L}_{d+1}\mathrm{Alb}^c(X)$ *is a group of multiplicative type.*

Proof. If X is proper, this is Proposition 11.3.2. In general, we may choose a compactification $\overline{X}$ of X; if $Z = \overline{X} - X$, with $\dim Z < \dim X$, the claim follows inductively by the long exact sequence (9.1). □

We leave it to the reader to formulate the dual of this proposition for $\mathrm{RPic}^c(X)$.

12. 1-motivic homology and cohomology of curves

12.1. **"Chow-Künneth" decomposition for a curve.** Note that for any curve C, the map a_C is an isomorphism by Proposition 5.1.4. Moreover, since the category of 1-motives up to isogeny is of cohomological dimension 1 (see Prop. 1.2.1), the complex $\mathrm{LAlb}^{\mathbb{Q}}(C)$ can be

represented by a complex with zero differentials. Using Proposition 11.3.2 c), we then have:

12.1.1. **Corollary.** *If C is a curve then the motive $M(C)$ decomposes in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \otimes \mathbb{Q}$ as*

$$M(C) = M_0(C) \oplus M_1(C) \oplus M_2(C)$$

where $M_i(C) := \operatorname{Tot} \mathrm{L}_i\mathrm{Alb}^{\mathbb{Q}}(C)[i]$.

12.2. $\mathrm{L}_i\mathrm{Alb}$ **and** $\mathrm{R}^i\mathrm{Pic}$ **of curves.** Here we shall complete the computation of Proposition 11.3.2 in the case of a curve C.

Let $\widetilde{C}$ denote the normalisation of C. Let $\overline{C}$ be a smooth compactification of $\widetilde{C}$ so that $F = \overline{C} - \widetilde{C}$ is a finite set of closed points. Consider the following cartesian square

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow \\ S & \longrightarrow & C \end{array}$$

where S denote the singular locus. Let $\overline{S}$ denote $\widetilde{S}$ regarded in $\overline{C}$. Note that $S = \pi_0(S)$, $\widetilde{S} = \pi_0(\widetilde{S})$ and $\pi_0(\widetilde{S}) \xrightarrow{=} \pi_0(S)$ if $\widetilde{C} \to C$ is radicial, yielding $M(\widetilde{C}) \xrightarrow{=} M(C)$ in this case. In general, we have the following.

12.2.1. **Theorem.** *Let C, $\widetilde{C}$, $\overline{C}$, S, $\widetilde{S}$, $\overline{S}$ and F as above. Then*

$$\mathrm{L}_i\mathrm{Alb}(C) = \begin{cases} [\mathbb{Z}[\pi_0(C)] \to 0] & \text{if } i = 0 \\ [\mathrm{Div}^0_{\overline{S}/S}(\overline{C}, F) \xrightarrow{u} \mathrm{Pic}^0(\overline{C}, F)] & \text{if } i = 1 \\ [0 \to \mathrm{NS}^*_{\widetilde{C}/k}] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where: $\mathrm{Div}^0_{\overline{S}/S}(\overline{C}, F) = \mathrm{Div}^0_{\widetilde{S}/S}(\widetilde{C})$ here is the free group of degree zero divisors generated by $\widetilde{S}$ having trivial push-forward on S and the map u is the canonical map (cf. [5, Def. 2.2.1]*); $\mathrm{NS}_{\widetilde{C}/k}$ is the sheaf associated to the free abelian group on the proper irreducible components of $\widetilde{C}$.*

Proof. We use the long exact sequence (8.2)

$$\cdots \to \mathrm{L}_i\mathrm{Alb}(\widetilde{S}) \to \mathrm{L}_i\mathrm{Alb}(\widetilde{C}) \oplus \mathrm{L}_i\mathrm{Alb}(S) \to \mathrm{L}_i\mathrm{Alb}(C) \to \mathrm{L}_{i-1}\mathrm{Alb}(\widetilde{S}) \to \dots$$

Since S and $\widetilde{S}$ are 0-dimensional we have $\mathrm{L}_i\mathrm{Alb}(\widetilde{S}) = \mathrm{L}_i\mathrm{Alb}(S) = 0$ for $i > 0$, therefore

$$\mathrm{L}_i\mathrm{Alb}(C) = \mathrm{L}_i\mathrm{Alb}(\widetilde{C}) \text{ for } i \geq 2$$

and by 10.2.3 we get the claimed vanishing and description of $\mathrm{L}_2\mathrm{Alb}(C)$. For $i = 0$ see Corollary 11.3.2. If $i = 1$ then $\mathrm{L}_1\mathrm{Alb}(C)$ is here represented as an element of $\mathrm{Ext}([\Lambda \to 0], \mathrm{L}_1\mathrm{Alb}(\widetilde{C}))$ where $\Lambda := \mathrm{Ker}(\mathbb{Z}[\pi_0(\widetilde{S})] \to \mathbb{Z}[\pi_0(\widetilde{C})] \oplus \mathbb{Z}[\pi_0(S)]$. Recall, see 10.2.3, that $\mathrm{L}_1\mathrm{Alb}(\widetilde{C}) = [0 \to \mathcal{A}^0_{\widetilde{C}/k}]$ thus $\mathrm{Ext}(\Lambda, \mathrm{L}_1\mathrm{Alb}(\widetilde{C})) = \mathrm{Hom}_k(\Lambda, \mathcal{A}^0_{\widetilde{C}/k})$ and

$$\mathrm{L}_1\mathrm{Alb}(C) = [\Lambda \xrightarrow{u} \mathcal{A}^0_{\widetilde{C}/k}].$$

Now $\Lambda = \mathrm{Div}^0_{\overline{S}/S}(\overline{C}, F)$, $\mathcal{A}^0_{\widetilde{C}/k} = \mathrm{Pic}^0(\overline{C}, F)$ and the map u is induced by the following canonical map.

Consider $\varphi_{\widetilde{C}} : \widetilde{C} \to \mathrm{Pic}(\overline{C}, F)$ where $\varphi_{\widetilde{C}}(P) := (\mathcal{O}_{\overline{C}}(P), 1)$ yielding $\mathcal{A}_{\widetilde{C}/k} = \mathrm{Pic}(\overline{C}, F)$ and such that

$$0 \to \mathrm{Div}^0_F(\overline{C})^* \to \mathrm{Pic}^0(\overline{C}, F) \to \mathrm{Pic}^0(\overline{C}) \to 0.$$

Thus $\mathrm{L}_1\mathrm{Alb}(\widetilde{C}) = [0 \to \mathrm{Pic}^0(\overline{C}, F)]$. Note that $\mathbb{Z}[\pi_0(\widetilde{S})] = \mathrm{Div}_{\widetilde{S}}(\widetilde{C}) = \mathrm{Div}_{\overline{S}}(\overline{C}, F)$, the map $\mathbb{Z}[\pi_0(\widetilde{S})] \to \mathbb{Z}[\pi_0(\widetilde{C})]$ is the degree map and the following map $\mathbb{Z}[\pi_0(\widetilde{S})] \to \mathbb{Z}[\pi_0(S)]$ is the proper push-forward of Weil divisors, *i.e.* $\Lambda = \mathrm{Div}^0_{\overline{S}/S}(\overline{C}, F)$. The map $\varphi_{\widetilde{C}}$ then induces the mapping $u \in \mathrm{Hom}_k(\Lambda, \mathrm{Pic}^0(\overline{C}, F))$ which also is the canonical lifting of the universal map $D \mapsto \mathcal{O}_{\overline{C}}(D)$ as the support of D is disjoint from F (*cf.* [5, Lemma 3.1.3]). □

12.2.2. **Remark.** We remark that $\mathrm{L}_1\mathrm{Alb}(C)$ coincides with the homological Albanese 1-motive $\mathrm{Alb}^-(C)$ ($= \mathrm{Pic}^-(C)$ for curves, see [5]). The $\mathrm{L}_i\mathrm{Alb}(C)$ also coincide with Lichtenbaum-Deligne motivic homology $h_i(C)$ of the curve C, *cf.* [34].

Note that $\mathrm{R}^1\mathrm{Pic}(C) = \mathrm{L}_1\mathrm{Alb}(C)^*$ is Deligne's motivic cohomology $H^1_m(C)(1)$ of the singular curve C by [5, Prop. 3.1.2]. Hence:

12.2.3. **Corollary.** *Let C be a curve, C' its seminormalisation, $\overline{C}'$ a compactification of C', and $F = \overline{C}' - C'$. Let further $\widetilde{C}$ denote the normalisation of C. Then*

$$\mathrm{R}^i\mathrm{Pic}(C) = \begin{cases} [0 \to \mathbb{G}_m[\pi_0(C)]] & \text{if } i = 0 \\ [\mathrm{Div}^0_F(\overline{C}') \to \mathrm{Pic}^0(\overline{C}')] & \text{if } i = 1 \\ [\mathrm{NS}(\widetilde{C}) \to 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{NS}(\widetilde{C}) = \mathbb{Z}[\pi^c_0(\widetilde{C})]$ and $\pi^c_0(\widetilde{C})$ is the scheme of proper constants.

12.3. **Borel-Moore variants.**

12.3.1. **Theorem.** *Let C be a smooth curve, $\overline{C}$ a smooth compactification of C and $F = \overline{C} - C$ the finite set of closed points at infinity. Then*

$$\mathrm{L}_i\mathrm{Alb}^c(C) = \begin{cases} [\mathbb{Z}[\pi_0^c(C)] \to 0] & \text{if } i = 0 \\ [\mathrm{Div}_F^0(\overline{C}) \to \mathrm{Pic}^0(\overline{C})] & \text{if } i = 1 \\ [0 \to \mathrm{NS}^*_{\overline{C}/k}] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{NS}(\overline{C}) = \mathbb{Z}[\pi_0(\overline{C})]$ *and* $\pi_0^c(C)$ *is the scheme of proper constants.*

Proof. It follows from the distinguished triangle

$$\begin{array}{ccc} \mathrm{LAlb}(F) & \longrightarrow & \mathrm{LAlb}(\overline{C}) \\ {}^{+1}\nwarrow & & \swarrow \\ & \mathrm{LAlb}^c(C) & \end{array}$$

and Corollary 10.2.3, yielding the claimed description: $\mathrm{L}_0\mathrm{Alb}^c(C) = \mathrm{Coker}(\mathrm{L}_0\mathrm{Alb}(F) \to \mathrm{L}_0\mathrm{Alb}(\overline{C}))$ moreover we have

$$[\mathrm{Div}_F^0(\overline{C}) \to 0] = \mathrm{Ker}(\mathrm{L}_0\mathrm{Alb}(F) \to \mathrm{L}_0\mathrm{Alb}(\overline{C}))$$

and the following extension

$$0 \to \mathrm{L}_1\mathrm{Alb}(\overline{C}) \to \mathrm{L}_1\mathrm{Alb}^c(C) \to [\mathrm{Div}_F^0(\overline{C}) \to 0] \to 0.$$

Finally, $\mathrm{L}_i\mathrm{Alb}(\overline{C}) = \mathrm{L}_i\mathrm{Alb}^c(C)$ for $i \geq 2$. □

12.3.2. **Corollary.** *Let C be a smooth curve, $\overline{C}$ a smooth compactification of C and $F = \overline{C} - C$ the finite set of closed points at infinity. Then*

$$\mathrm{R}^i\mathrm{Pic}^c(C) = \begin{cases} [0 \to \mathbb{G}_m[\pi_0^c(C)]] & \text{if } i = 0 \\ [0 \to \mathrm{Pic}^0(\overline{C}, F)] & \text{if } i = 1 \\ [\mathrm{NS}(\overline{C}) \to 0] & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{NS}(\overline{C}) = \mathbb{Z}[\pi_0(\overline{C})]$ *and* $\pi_0^c(C)$ *is the scheme of proper constants.*

Here we have that $\mathrm{R}^1\mathrm{Pic}^c(C) = \mathrm{R}^1\mathrm{Pic}^*(C)$ is also the Albanese variety of the smooth curve.

Note that $\mathrm{L}_1\mathrm{Alb}^c(C)$ $(= \mathrm{Pic}^+(C) = \mathrm{Alb}^+(C)$ for curves, see [5]) coincide with Deligne's motivic $H^1_m(C)(1)$ of the smooth curve C. This is due to the Poincaré duality isomorphism $M^c(C) = M(C)^*(1)[2]$.

13. Comparison with $\mathrm{Pic}^+, \mathrm{Pic}^-, \mathrm{Alb}^+$ and Alb^-

In this section, we want to study $\mathrm{L}_1\mathrm{Alb}(X)$ and its variants in more detail. In particular, we show in Proposition 13.6.3 c) that it is always a Deligne 1-motive, and show in Corollaries 13.7.2 and 13.10.2 that, if X is normal or proper, it is canonically isomorphic to the 1-motive $\mathrm{Alb}^-(X)$ of [5]. Precise descriptions of $\mathrm{L}_1\mathrm{Alb}(X)$ are given in Proposition 13.6.4 and Corollary 13.6.5.

We also describe $\mathrm{L}_1\mathrm{Alb}^c(X)$ in Proposition 13.11.2; more precisely, we prove in Theorem 13.11.1 that its dual $\mathrm{R}^1\mathrm{Pic}^c(X)$ is canonically isomorphic to $\mathrm{Pic}^0(\overline{X}, Z)/\mathcal{U}$, where $\overline{X}$ is a compactification of X with complement Z and $\mathcal{U}$ is the unipotent radical of the commutative algebraic group $\mathrm{Pic}^0(\overline{X}, Z)$. Finally, we prove in Theorem 13.12.6 that $\mathrm{L}_1\mathrm{Alb}^*(X)$ is abstractly isomorphic to the 1-motive $\mathrm{Alb}^+(X)$ of [5].

We start with some comparison results between éh and étale cohomology for non smooth schemes.

Let $\varepsilon : Sch_{\text{éh}} \to Sch_{\text{ét}}$ be the obvious morphism of sites. If $\mathcal{F}$ is an étale sheaf on Sch, we denote by $\mathcal{F}_{\text{éh}}$ its éh sheafification (that is, $\mathcal{F}_{\text{éh}} = \varepsilon_*\varepsilon^*\mathcal{F}$). We shall abbreviate $H^*_{\text{éh}}(X, \mathcal{F}_{\text{éh}})$ to $H^*_{\text{éh}}(X, \mathcal{F})$.

13.1. **Torsion sheaves.** The first basic result is a variant of [59, Cor. 7.8 and Th. 10.7]: it follows from Proposition 11.2.2 and Examples 11.2.4 via Lemma 11.2.3 b).

13.1.1. **Proposition.** *Let C be a bounded below complex of torsion sheaves on $(\operatorname{Spec} k)_{\text{ét}}$. Then, for any $X \in Sch$ and any $n \in \mathbb{Z}$, $H^n_{\text{ét}}(X, C) \xrightarrow{\sim} H^n_{\text{éh}}(X, C)$.* □

(See [21, Th. 3.6] for a different proof.)

13.2. **Discrete sheaves.**

13.2.1. **Lemma.** *a) If $\mathcal{F}$ is discrete, then $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_{\text{éh}}$. More precisely, for any $X \in Sch$, $\mathcal{F}(\pi_0(X)) \xrightarrow{\sim} \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$.*
b) If $f : Y \to X$ is surjective with geometrically connected fibres, then $\mathcal{F}_{\text{éh}} \xrightarrow{\sim} f_*f^*\mathcal{F}_{\text{éh}}$.

Proof. a) We may assume X reduced. Clearly it suffices to prove that $\mathcal{F}(\pi_0(X)) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$ for any $X \in Sch$. In the situation of §8.1.e, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}_{\text{éh}}(X) & \longrightarrow & \mathcal{F}_{\text{éh}}(\widetilde{X}) \oplus \mathcal{F}_{\text{éh}}(Z) & \longrightarrow & \mathcal{F}_{\text{éh}}(\widetilde{Z}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathcal{F}(\pi_0(X)) & \longrightarrow & \mathcal{F}(\pi_0(\widetilde{X})) \oplus \mathcal{F}(\pi_0(Z)) & \longrightarrow & \mathcal{F}(\pi_0(\widetilde{Z})).
\end{array}$$

The proof then goes exactly as the one of Proposition 13.1.1. b) follows from a). □

It is well-known that $H^1_{\text{ét}}(X, \mathcal{F}) = 0$ for any geometrically unibranch scheme $X \in Sch$ if $\mathcal{F}$ is constant and torsion-free (*cf.* [SGA4, IX, Prop. 3.6 (ii)]). The following lemma shows that this is also true for the éh topology, at least if X is normal.

13.2.2. **Lemma** (compare [39, Ex. 12.31 and 12.32])**.** *Let $\mathcal{F}$ be a constant torsion-free sheaf on $Sch(k)$.*
a) For any $X \in Sch$, $H^1_{\text{éh}}(X, \mathcal{F})$ is torsion-free. It is finitely generated if $\mathcal{F}$ is a lattice.
b) Let $f : \widetilde{X} \to X$ be a surjective morphism. Then $H^1_{\text{éh}}(X, \mathcal{F}) \to H^1_{\text{éh}}(\widetilde{X}, \mathcal{F})$ is injective in the following cases:

(i) *The geometric fibres of f are connected.*
(ii) *f is finite and flat.*

c) If X is normal, $H^1_{\text{éh}}(X, \mathcal{F}) = 0$.

Proof. a) The first assertion follows immediately from Lemma 13.2.1 (consider the exact sequence of multiplication by n on $\mathcal{F}$). The second assertion follows by blow-up induction from the fact that $H^1_{\text{éh}}(X, \mathcal{F}) = 0$ if X is smooth, by Proposition 11.2.2.

b) In the first case, the Leray spectral sequence yields an injection

$$H^1_{\text{éh}}(X, f_*\mathcal{F}) \hookrightarrow H^1_{\text{éh}}(\widetilde{X}, \mathcal{F})$$

and $f_*\mathcal{F} = \mathcal{F}$ by Lemma 13.2.1 b). In the second case, the theory of trace [SGA4, XVII, Th. 6.2.3] provides $\mathcal{F}$, hence $\mathcal{F}_{\text{éh}}$, with a morphism $Tr_f : f_*\mathcal{F} \to \mathcal{F}$ whose composition with the natural morphism is (on each connected component of X) multiplication by some nonzero integer. This shows that the kernel of $H^1_{\text{éh}}(X, \mathcal{F}) \to H^1_{\text{éh}}(X, f_*\mathcal{F})$ is torsion, hence 0 by a).

c) follows from b) with $\widetilde{X}$ a desingularisation of X: by Proposition 13.1.1, $H^1_{\text{ét}}(\widetilde{X}, \mathcal{F}) \xrightarrow{\sim} H^1_{\text{éh}}(\widetilde{X}, \mathcal{F})$ and it is well-known that the first group is 0; on the other hand, the fibres of f are geometrically connected by Zariski's main theorem. □

The following is a version of [65, Lemma 5.6]:

13.2.3. **Lemma.** *Let $f : \widetilde{X} \to X$ be a finite birational morphism, $i : Z \hookrightarrow X$ a closed subset and $\widetilde{Z} = p^{-1}(Z)$. Then, for any discrete*

sheaf, we have a long exact sequence:

$$\cdots \to H^i_{\text{ét}}(X,\mathcal{F}) \to H^i_{\text{ét}}(\widetilde{X},\mathcal{F}) \oplus H^i_{\text{ét}}(Z,\mathcal{F}) \\ \to H^i_{\text{ét}}(\widetilde{Z},\mathcal{F}) \to H^{i+1}_{\text{ét}}(X,\mathcal{F}) \to \dots$$

Proof. Let $g : \widetilde{Z} \to Z$ be the induced map. Then f_*, i_* and g_* are exact for the étale topology. Thus it suffices to show that the sequence of sheaves

$$0 \to \mathcal{F} \to f_*f^*\mathcal{F} \oplus i_*i^*\mathcal{F} \to (ig)_*(ig)^*\mathcal{F} \to 0$$

is exact. The assertion is local for the étale topology, hence we may assume that X is strictly local. Then $Z, \widetilde{X}$ and $\widetilde{Z}$ are strictly local as well, hence connected, thus the statement is obvious. □

We can now prove:

13.2.4. **Proposition.** *For any $X \in Sch(k)$ and any discrete sheaf $\mathcal{F}$, the map $H^1_{\text{ét}}(X,\mathcal{F}) \to H^1_{\text{éh}}(X,\mathcal{F})$ is an isomorphism.*

Proof. Let $f : \widetilde{X} \to X$ be the normalisation of X, and take for Z the non-normal locus of X in Lemma 13.2.3. The result now follows from comparing the exact sequence of this lemma with the one for éh topology, and using Lemma 13.2.2 c). □

13.2.5. **Corollary.** *The exact sequence of Lemma 13.2.3 holds up to $i = 1$ for a general abstract blow-up.* □

13.3. **Strictly reduced schemes.** If G is a commutative k-group scheme, the associated presheaf $\underline{G}$ is an étale sheaf on reduced k-schemes of finite type. However, $\underline{G}(X) \to \underline{G}_{\text{éh}}(X)$ is not an isomorphism in general if X is not smooth. Nevertheless we have some nice results in Lemma 13.3.4 below.

13.3.1. **Definition.** A k-scheme of finite type X is *strictly reduced* (a recursive definition) if it is reduced and

(i) If X is irreducible: X_{sing}, considered with its reduced structure, is strictly reduced.
(ii) If $Z_1, \dots, Z_n$ are the irreducible components of X: all Z_i are strictly reduced and the scheme-theoretic intersection $Z_i \cap Z_j$ is reduced for any $i \neq j$.

13.3.2. **Examples.** 1) If $\dim X = 0$, X is strictly reduced.
2) The union of a line and a tangent parabola is not strictly reduced.
3) If X is normal and of dimension ≤ 2, it is strictly reduced.
4) M. Ojanguren provided the following example of a normal 3-fold which is not strictly reduced: take the affine hypersurface with equation $uv = x^2(y^2 - x)^2$.

13.3.3. **Lemma.** *Let G be an affine group scheme and $f : Y \to X$ a proper surjective map with geometrically connected fibres. Then $\underline{G}(X) \xrightarrow{\sim} \underline{G}(Y)$, and $H^0_\tau(X, \underline{G}) \xrightarrow{\sim} H^0_\tau(Y, \underline{G})$ for any Grothendieck topology τ stronger than the Zariski topology.*

Proof. The first statement is clear, and the second folllows because the hypothesis on f is stable under any base change. □

13.3.4. **Lemma.** *a) If X is reduced, then the map*

$$\underline{G}(X) \to \underline{G}_{\text{éh}}(X) \tag{13.1}$$

is injective for any semi-abelian k-scheme G.
b) If X is strictly reduced, (13.1) *is an isomorphism.*
c) If X is proper and G is a torus, the maps $\underline{G}(\pi_0(X)) \to \underline{G}_{\text{ét}}(X) \to \underline{G}_{\text{éh}}(X)$ are isomorphisms. If moreover X is reduced, (13.1) *is an isomorphism.*

Proof. a) Let Z_i be the irreducible components of X, and for each i let $p_i : \widetilde{Z}_i \to Z_i$ be a resolution of singularities. We have a commutative diagram

$$\begin{array}{ccccc}
\underline{G}_{\text{éh}}(X) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_i) & \xrightarrow{(p_i^*)} & \bigoplus \underline{G}_{\text{éh}}(\widetilde{Z}_i) \\
\uparrow & & \uparrow & & \uparrow \\
\underline{G}(X) & \longrightarrow & \bigoplus \underline{G}(Z_i) & \xrightarrow{(p_i^*)} & \bigoplus \underline{G}(\widetilde{Z}_i).
\end{array}$$

The bottom horizontal maps are injective; the right vertical map is an isomorphism by Proposition 11.2.2. The claim follows.

b) We argue by induction on $d = \dim X$. If $d = 0$ this is trivial. If $d > 0$, we first assume X irreducible. Let Z be its singular locus, and choose a desingularisation $p : \widetilde{X} \to X$ with p proper surjective, $\widetilde{X}$ smooth, $\widetilde{Z} = p^{-1}(Z)$ a divisor with normal crossings (in particular reduced) and $p_{|\widetilde{X}-\widetilde{Z}}$ an isomorphism. We now have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \underline{G}_{\text{éh}}(X) & \longrightarrow & \underline{G}_{\text{éh}}(Z) \oplus \underline{G}_{\text{éh}}(\widetilde{X}) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(\widetilde{Z}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \underline{G}(X) & \longrightarrow & \underline{G}(Z) \oplus \underline{G}(\widetilde{X}) & \longrightarrow & \bigoplus \underline{G}(\widetilde{Z})
\end{array}$$

where the lower sequence is exact, the middle vertical map is bijective by induction on d and the smooth case (Proposition 11.2.2) and the right vertical map is injective by a). It follows that the left vertical map is surjective.

In general, write $Z_1, \dots, Z_n$ for the irreducible components of X: by assumption, the two-fold intersections Z_{ij} are reduced. The commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{G}_{\text{éh}}(X) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_i) & \longrightarrow & \bigoplus \underline{G}_{\text{éh}}(Z_{ij}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \underline{G}(X) & \longrightarrow & \bigoplus \underline{G}(Z_i) & \longrightarrow & \bigoplus \underline{G}(Z_{ij}) \end{array}$$

then has the same formal properties as the previous one, and we conclude.

For c), same proof as for Lemma 13.2.1 a). (The second statement of c) is true because $\underline{G}(\pi_0(X)) \xrightarrow{\sim} \underline{G}(X)$ if X is proper and reduced.) $\square$

13.4. **Normal schemes.** The main result of this subsection is:

13.4.1. **Theorem.** *Let X be normal. Then, for any $\mathcal{F} \in \mathrm{Shv}_1$, the map $\mathcal{F}(X) \to \mathcal{F}_{\text{éh}}(X)$ is bijective and the map $H^1_{\text{ét}}(X, \mathcal{F}) \to H^1_{\text{éh}}(X, \mathcal{F}_{\text{éh}})$ is injective (with torsion-free cokernel by Proposition 13.1.1).*

Proof. In several steps:

Step 1. The first result implies the second for a given sheaf $\mathcal{F}$: let $\varepsilon : Sch(k)_{\text{éh}} \to Sch(k)_{\text{ét}}$ be the projection morphism. The associated Leray spectral sequence gives an injection

$$H^1_{\text{ét}}(X, \varepsilon_* \mathcal{F}_{\text{éh}}) \hookrightarrow H^1_{\text{éh}}(X, \mathcal{F}_{\text{éh}}).$$

But any scheme étale over X is normal [SGA1, Exp. I, Cor. 9.10], therefore $\mathcal{F} \to \varepsilon_* \mathcal{F}_{\text{éh}}$ is an isomorphism over the small étale site.

Step 2. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence in Shv_1. If the theorem is true for $\mathcal{F}'$ and $\mathcal{F}''$, it is true for $\mathcal{F}$. This follows readily from *Step 1* and a diagram chase.

Step 3. Given the structure of 1-motivic sheaves, *Step 1 - 2* reduce us to prove that $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}_{\text{éh}}(X)$ separately when $\mathcal{F}$ is discrete, a torus or an abelian variety. The discrete case follows from Lemma 13.2.1 a).

Step 4. If G is a torus, let $\pi : \widetilde{X} \to X$ be a desingularisation of X. We have a commutative diagram

$$\begin{array}{ccc} \underline{G}_{\text{éh}}(X) & \longrightarrow & \underline{G}_{\text{éh}}(\widetilde{X}) \\ \uparrow & & \uparrow \\ \underline{G}(X) & \longrightarrow & \underline{G}(\widetilde{X}). \end{array}$$

Here the right vertical map is an isomorphism because $\widetilde{X}$ is smooth and the two horizontal maps are also isomorphisms by Lemma 13.3.3 applied to π (Zariski's main theorem). The result follows.

Step 5. Let finally G be an abelian variety. This time, it is not true in general that $\underline{G}(X) \xrightarrow{\sim} \underline{G}(\widetilde{X})$ for a smooth desingularisation $\widetilde{X}$ of X. However, we get the result from Proposition 11.2.5 b) and the following general lemma. $\square$

13.4.2. **Lemma.** *Let $X \in Sch(k)$ be normal, $p : X_0 \to X$ a proper surjective map such that the restriction of p to a suitable connected component X_0' of X_0 is birational. Let $X_1 \overset{p_0}{\underset{p_1}{\rightrightarrows}} X_0$ be two morphisms such that $pp_0 = pp_1$ and that the induced map $\Psi : X_1 \to X_0 \times_X X_0$ is proper surjective. Let $Y \in Sch(k)$ and let $f : X_0 \to Y$ be such that $fp_0 = fp_1$.*

$$\begin{tikzcd} X_1 \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & \\ X_0 \arrow[d, "p"'] \arrow[dr, "f"] & \\ X & Y \end{tikzcd}$$

Then there exists a unique morphism $\overline{f} : X \to Y$ such that $f = \overline{f}p$.

Proof. We may assume X connected. Since Ψ is proper surjective, the hypothesis is true by replacing X_1 by $X_0 \times_X X_0$, which we shall assume henceforth. Let $x \in X$ and $K = k(x)$. Base-changing by the morphism $\operatorname{Spec} K \to X$, we find (by faithful flatness) that f is constant on $p^{-1}(x)$. Since p is surjective, this defines $\overline{f}$ as a set-theoretic map, and this map is continuous for the Zariski topology because p is also proper.

It remains to show that $\overline{f}$ is a map of locally ringed spaces. Let $x \in X$, $y = \overline{f}(x)$ and $x' \in p^{-1}(x) \cap X_0'$. Then $f^\sharp(\mathcal{O}_{Y,y}) \subseteq \mathcal{O}_{X_0',x'}$. Note that X and X_0' have the same function field L, and $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X_0',x'} \subseteq L$. Now, since X is normal, $\mathcal{O}_{X,x}$ is the intersection of the valuation rings containing it.

Let $\mathcal{O}$ be such a valuation ring, so that x is the centre of $\mathcal{O}$ on X. By the valuative criterion of properness, we may find $x' \in p^{-1}(x) \cap X_0'$ such that $\mathcal{O}_{X_0',x'} \subseteq \mathcal{O}$. This shows that

$$\mathcal{O}_{X,x} = \bigcap_{x' \in p^{-1}(x) \cap X_0'} \mathcal{O}_{X_0',x'}$$

and therefore that $f^\sharp(\mathcal{O}_{Y,y}) \subseteq \mathcal{O}_{X,x}$. Moreover, the corresponding map $\overline{f}^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is local since $f^\sharp$ is.

(Alternatively, observe that f and its topological factorisation induce a map

$$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_{X_0} \simeq \overline{f}_* p_* \mathcal{O}_{X_0} = \overline{f}_* \mathcal{O}_X.)$$

□

13.5. **Some representability results.**

13.5.1. **Proposition.** *Let π^X be the structural morphism of X and $(\pi^X_*)^{\text{éh}}$ the induced direct image morphism on the* éh *sites. For any $\mathcal{F} \in \mathrm{HI}_{\text{ét}}$, let us denote the restriction of $R^q(\pi^X_*)^{\text{éh}}\mathcal{F}_{\text{éh}}$ to Sm by $\underline{R}^q\pi^X_*\mathcal{F}$ (in other words, $R^q(\pi^X_*)^{\text{éh}}\mathcal{F}_{\text{éh}}$ is the sheaf on $Sm(k)_{\text{ét}}$ associated to the presheaf $U \mapsto H^q_{\text{éh}}(X \times U, \mathcal{F}_{\text{éh}})$): it is an object of $\mathrm{HI}_{\text{ét}}$. Then*
a) For any lattice L, $\underline{R}^q\pi^X_*L$ is a ind-discrete sheaf for all $q \geq 0$; it is a lattice for $q = 0, 1$.
b) For any torus T, $\underline{R}^q\pi^X_*\underline{T}$ is 1-motivic for $q = 0, 1$.

Proof. We apply Lemma 11.2.3 a) in the following situation: $\mathcal{A} = \mathrm{HI}_{\text{ét}}$, $\mathcal{B} = \mathrm{Shv}_0$, $H^i(X) = R^i\pi^X_*L$ in case a), $\mathcal{B} = \mathrm{Shv}_1$, $H^i(X) = R^i\pi^X_*\underline{T}$ in case b). The smooth case is trivial in a) and the lattice assertions follow from lemmas 13.2.1 and 13.2.2 a). In b), the smooth case follows from Proposition 3.4.1. □

13.6. $\mathrm{L}_1\mathrm{Alb}(X)$ **and the Albanese schemes.** We now compute the 1-motive $\mathrm{L}_1\mathrm{Alb}(X) = [L_1 \to G_1]$ in important special cases This is done in the following three propositions; in particular, we shall show that it always "is" a Deligne 1-motive. Note that, by definition of a 1-motive with cotorsion, the pair (L_1, G_1) is determined only up to a q.i.: the last sentence means that we may choose this pair such that G_1 is connected.

13.6.1. **Proposition.** *Let $X \in Sch(k)$. Then*
a) $\mathcal{H}_i(\mathrm{LAlb}(X)) = 0$ for $i < 0$.
b) Let $\mathcal{F}_X = \mathcal{H}_0(\mathrm{Tot}\,\mathrm{LAlb}(X))$. Then $\mathcal{F}_X$ corepresents the functor

$$\begin{aligned} \mathrm{Shv}_1 &\to Ab \\ \mathcal{F} &\mapsto \mathcal{F}_{\text{éh}}(X) \ \textit{(see Def. 11.2.1)} \end{aligned}$$

via the composition

$$\alpha^* M(X) \to \mathrm{Tot}\,\mathrm{LAlb}(X) \to \mathcal{F}_X[0].$$

Moreover, we have an exact sequence, for any representative $[L_1 \xrightarrow{u_1} G_1]$ of $\mathrm{L}_1\mathrm{Alb}(X)$:

$$L_1 \xrightarrow{u_1} G_1 \to \mathcal{F}_X \to \mathbb{Z}\pi_0(X) \to 0. \tag{13.2}$$

c) Let $\mathcal{A}^{\text{éh}}_{X/k} := \Omega(\mathcal{F}_X)$ (cf. Proposition 3.6.5). Then $\mathcal{A}^{\text{éh}}_{X/k}$ corepresents the functor

$$\begin{aligned} {}^t\mathrm{AbS} &\to Ab \\ G &\mapsto \underline{G}_{\text{éh}}(X). \end{aligned}$$

Moreover we have an epimorphism

$$\mathcal{A}^{\text{éh}}_{X/k} \twoheadrightarrow \mathcal{A}_{X_{\text{red}}/k}. \tag{13.3}$$

d) If X_{red} is strictly reduced (Def. 13.3.1) or normal, (13.3) *is an isomorphism.*

Proof. a) is proven as in Proposition 11.3.2 by blow-up induction (reduction to the smooth case). If $\mathcal{F} \in \mathrm{Shv}_1$, we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(\alpha^* M(X), \mathcal{F}) = \mathcal{F}_{\text{éh}}(X)$$

by Propositions 3.8.1, 11.2.2 and D.1.4. The latter group coincides with $\mathrm{Hom}_{\mathrm{Shv}_1}(\mathcal{F}_X, \mathcal{F})$ by (5.3) and a), hence b); the exact sequence follows from Proposition 3.10.2. The sheaf $\mathcal{A}^{\text{éh}}_{X/k}$ clearly corepresents the said functor; the map then comes from the obvious natural transformation in G: $G(X_{\text{red}}) \to G_{\text{éh}}(X)$ and its surjectivity follows from Lemma 13.3.4 a), hence c). d) follows from Lemma 13.3.4 b), Theorem 13.4.1 and the universal property of $\mathcal{A}_{X/k}$. □

13.6.2. **Remark.** One could christen $\mathcal{F}_X$ and $\mathcal{A}^{\text{éh}}_{X/k}$ the *universal* 1-*motivic sheaf* and the éh-*Albanese scheme* of X.

13.6.3. **Proposition.** *a) The sheaves $\mathcal{F}_X$ and $\mathcal{A}^{\text{éh}}_{X/k}$ have π_0 equal to $\mathbb{Z}[\pi_0(X)]$; in particular, $\mathcal{A}^{\text{éh}}_{X/k} \in \mathrm{AbS}$.*
b) In (13.2)*, the composition $L_1 \xrightarrow{u_1} G_1 \to \pi_0(G_1)$ is surjective.*
c) One may choose $\mathrm{L_1Alb}(X) \simeq [L_1 \to G_1]$ with G_1 connected (in other words, $\mathrm{L_1Alb}(X)$ is a Deligne 1-motive).

Proof. In a), it suffices to prove the first assertion for $\mathcal{F}_X$: then it follows from its universal property and Lemma 13.2.1 a). The second assertion of a) is obvious.

b) Let $0 \to L_1' \to G_1' \to \mathcal{F}_X \to \mathbb{Z}[\pi_0(X)] \to 0$ be the normalised presentation of $\mathcal{F}_X$ given by Proposition 3.2.3. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1' & \longrightarrow & G_1' & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & || & & || & & \\
0 & \longrightarrow & \overline{u_1(L_1)} & \xrightarrow{\overline{u}_1} & \overline{G}_1 & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & || & & || & & \\
 & & L_1 & \xrightarrow{u_1} & G_1 & \longrightarrow & \mathcal{F}_X & \longrightarrow & \mathbb{Z}[\pi_0(X)] & \longrightarrow & 0
\end{array}$$

with $\overline{u_1(L_1)} = u_1(L_1)/F$ and $\overline{G}_1 = G_1/F$, where F is the torsion subgroup of $u_1(L_1)$. Indeed, $\mathrm{Ext}(G_1', \overline{u_1(L_1)}) = 0$ so we get the downwards

vertical maps as in the proof of Proposition 3.2.3. By uniqueness of the normalised presentation, G_1' maps onto $\overline{G}_1^0$. A diagram chase then shows that the composition

$$\overline{u_1(L_1)} \xrightarrow{\overline{u}_1} \overline{G}_1 \to \pi_0(\overline{G}_1)$$

is onto, and another diagram chase shows the same for u_1.

c) The pull-back diagram

$$\begin{array}{ccc} L_1^0 & \longrightarrow & G_1^0 \\ \downarrow & & \downarrow \\ L_1 & \longrightarrow & G_1 \end{array}$$

is a quasi-isomorphism in ${}_t\mathcal{M}_1^{\mathrm{eff}}$, thanks to b). □

13.6.4. **Proposition.** *Let* $[L_1 \xrightarrow{u_1} G_1]$ *be the Deligne* 1*-motive that lies in the* q.i. *class of* $\mathrm{L_1Alb}(X)$*, thanks to Proposition 13.6.3 c).*
a) We have an isomorphism

$$L_1 \xrightarrow{\sim} \mathcal{H}om\,(\underline{R}^1\pi_*\mathbb{Z}, \mathbb{Z})$$

(cf. Proposition 13.5.1).
b) We have a canonical isomorphism

$$G_1/(L_1)_{\mathrm{Zar}} \xrightarrow{\sim} (\mathcal{A}_{X/k}^{\text{éh}})^0$$

where $(L_1)_{\mathrm{Zar}}$ *is the Zariski closure of the image of* L_1 *in* G_1 *and* $\mathcal{A}_{X/k}^{\text{éh}}$ *was defined in Proposition 13.6.1 (*$(\mathcal{A}_{X/k}^{\text{éh}})^0$ *corepresents the functor* $\mathrm{SAb} \ni G \mapsto G_{\text{éh}}(X)$*).*

Proof. For the computations, we may assume k algebraically closed.

a) Let L be a lattice. We compute:

$$H^1_{\text{éh}}(X, L) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}}}(M_{\text{ét}}(X), L[1]) = \mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[1]).$$

From the spectral sequence (11.1), we get an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1(\mathrm{L_0Alb}(X), L) &\to \mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[1]) \\ &\to \mathrm{Hom}(\mathrm{L_1Alb}(X), L) \to \mathrm{Ext}^2(\mathrm{L_0Alb}(X), L). \end{aligned}$$

Since the two Ext are 0, we get an isomorphism

$$\mathrm{Hom}_{D^b(\mathcal{M}_1)}(\mathrm{LAlb}(X), L[1]) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{L_1Alb}(X), L).$$

Since $[L_1 \to G_1]$ is a Deligne 1-motive, the last group is isomorphic to $\mathrm{Hom}(L_1, L)$. This gives a), since we obviously have $H^1_{\text{éh}}(X, L) = H^0_{\text{ét}}(k, \underline{R}^1\pi_*L) = \underline{R}^1\pi_*\mathbb{Z} \otimes L$ by Proposition 13.5.1 a).

b) This follows directly from the definition of $\mathcal{A}_{X/k}^{\text{éh}}$. □

13.6.5. **Corollary.** *Let* $\mathrm{L}_1\mathrm{Alb}(X) = [L_1 \to G_1]$, *as a Deligne* 1*-motive.*
a) If X *is proper, then* G_1 *is an abelian variety.*
b) If X *is normal, then* $\mathrm{L}_1\mathrm{Alb}(X) = [0 \to \mathcal{A}^0_{X/k}]$.
c) If X *is normal and proper then* $\mathrm{R}^1\mathrm{Pic}(X) = [0 \to \mathrm{Pic}^0_{X/k}]$ *is an abelian variety with dual the Serre Albanese* $\mathrm{L}_1\mathrm{Alb}(X) = [0 \to \mathcal{A}^0_{X/k}]$.

Proof. a) is seen easily by blow-up induction, by reducing to the smooth projective case (Corollary 10.2.3). b) follows from Proposition 13.6.4 a), b), Lemma 13.2.2 c) and Proposition 13.6.1 d). c) follows immediately from a) and b). □

13.7. $\mathrm{L}_1\mathrm{Alb}(X)$ **and** $\mathrm{Alb}^-(X)$ **for** X **normal.** In this subsection, we prove that these two 1-motives are isomorphic in 13.7.2. We begin with a slight improvement of Theorem 13.4.1 in the case of semi-abelian schemes:

13.7.1. **Lemma.** *Let* X *be normal, and let* $X_\bullet$ *be a smooth hyperenvelope (cf. Lemma 13.4.2). Then we have*

$$\mathcal{A}_{X/k} = \mathrm{Coker}(\mathcal{A}_{X_1} \xrightarrow{(p_0)_* - (p_1)_*} \mathcal{A}_{X_0})$$

and

$$(\mathcal{A}_{X/k})^0 = \mathrm{Coker}(\mathcal{A}^0_{X_1} \xrightarrow{(p_0)_* - (p_1)_*} \mathcal{A}^0_{X_0}).$$

Proof. The first isomorphism follows from Lemma 13.4.2 applied with Y running through torsors under semi-abelian varieties. To deduce the second isomorphism, consider the short exact sequence of complexes

$$0 \to \mathcal{A}^0_{X_\bullet/k} \to \mathcal{A}_{X_\bullet/k} \to \mathbb{Z}[\pi_0(X_\bullet)] \to 0$$

and the resulting long exact sequence
(13.4)
$$H_1(\mathbb{Z}[\pi_0(X_\bullet)]) \to H_0(\mathcal{A}^0_{X_\bullet/k}) \to H_0(\mathcal{A}_{X_\bullet/k}) \to H_0(\mathbb{Z}[\pi_0(X_\bullet)]) \to 0.$$

For any $i \geq 0$, $\mathbb{Z}[\pi_0(X_i)]$ is $\mathbb{Z}$-dual to the Galois module $E_1^{i,0}$, where $E_1^{p,q} = H^q_{\text{ét}}(X_p \times_k \overline{k}, \mathbb{Z})$ is the E_1-term associated to the simplicial spectral sequence for $X_\bullet \times_k \overline{k}$. Since $H^1_{\text{ét}}(X_p \times_k \overline{k}, \mathbb{Z}) = 0$ for all $p \geq 0$, we get

$$H_i(\mathbb{Z}[\pi_0(X_\bullet)]) \simeq (H^i_{\text{ét}}(X_\bullet \times_k \overline{k}, \mathbb{Z}))^\vee \text{ for } i = 0, 1.$$

By Proposition 11.2.2, these étale cohomology groups may be replaced by éh cohomology groups. By Proposition 11.2.5, we then have

$$H^i_{\text{ét}}(X_\bullet \times_k \overline{k}, \mathbb{Z}) \simeq H^i_{\text{éh}}(X \times_k \overline{k}, \mathbb{Z}).$$

Now, by Lemma 13.2.1 a), $H^0_{\text{éh}}(X \times_k \overline{k}, \mathbb{Z})$ is $\mathbb{Z}$-dual to $\mathbb{Z}[\pi_0(X)]$, and by Lemma 13.2.2 c), $H^1_{\text{éh}}(X \times_k \overline{k}, \mathbb{Z}) = 0$ because X is normal. Hence (13.4) yields a short exact sequence

$$0 \to H_0(\mathcal{A}^0_{X_\bullet/k}) \to \mathcal{A}_{X/k} \to \mathbb{Z}[\pi_0(X)] \to 0$$

which identifies $H_0(\mathcal{A}^0_{X_\bullet/k})$ with $\mathcal{A}^0_{X/k}$. □

13.7.2. **Proposition.** *If X is normal,* $\mathrm{R}^1\mathrm{Pic}(X)$ *and* $\mathrm{L}_1\mathrm{Alb}(X)$ *are isomorphic, respectively, to the* 1*-motives* $\mathrm{Pic}^+(X)$ *and* $\mathrm{Alb}^-(X)$ *defined in* [5, Ch. 4-5].

Proof. Let $\overline{X}$ be a normal compactification of X; choose a smooth hyperenvelope $X_\bullet$ of X along with $\overline{X}_\bullet$ a smooth compactification with normal crossing boundary $Y_\bullet$ such that $\overline{X}_\bullet \to \overline{X}$ is an hyperenvelope. Now we have, in the notation of [5, 4.2], a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Pic}^0_{\overline{X}/k} & \longrightarrow & \mathrm{Pic}^0_{\overline{X}_0/k} & \longrightarrow & \mathrm{Pic}^0_{\overline{X}_1/k} \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathrm{Div}^0_{Y_\bullet}(\overline{X}_\bullet) & \longrightarrow & \mathrm{Div}^0_{Y_0}(\overline{X}_0) & \longrightarrow & \mathrm{Div}^0_{Y_1}(\overline{X}_1)
\end{array}$$

where $\mathrm{Pic}^+(X) = [\mathrm{Div}^0_{Y_\bullet}(\overline{X}_\bullet) \to \mathrm{Pic}^0_{\overline{X}/k}]$ since $\overline{X}$ is normal. Taking Cartier duals we get an exact sequence of 1-motives

$$[0 \to \mathcal{A}^0_{X_1/k}] \xrightarrow{(p_0)_*-(p_1)_*} [0 \to \mathcal{A}^0_{X_0/k}] \to \mathrm{Alb}^-(X) \to 0.$$

Thus $\mathrm{Alb}^-(X) = [0 \to \mathcal{A}^0_{X/k}]$ by Lemma 13.7.1. We conclude by Corollary 13.6.5 b) since X is normal and $\mathrm{L}_1\mathrm{Alb}(X) = [0 \to \mathcal{A}^0_{X/k}]$. □

13.7.3. **Remarks.** 1) Note that, while $\mathrm{L}_0\mathrm{Alb}(X)$ and $\mathrm{L}_1\mathrm{Alb}(X)$ are Deligne 1-motives, the same is not true of $\mathrm{L}_2\mathrm{Alb}(X)$ in general, already for X smooth projective (see Corollary 10.2.3).
2) One could make use of Proposition 11.3.1 to compute $\mathrm{L}_i\mathrm{Alb}(X)$ for singular X and $i > 1$. However, $H^i_{\text{éh}}(X, \mathbb{G}_m)_{\mathbb{Q}}$ can be non-zero also for $i \geq 2$, therefore a precise computation for X singular and higher dimensional appears to be difficult. We did completely the case of curves in Sect. 12.

13.8. $\mathrm{RPic}(X)$ **and** $H^*_{\text{éh}}(X, \mathbb{G}_m)$**.** By definition of RPic, we have a morphism in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$

$$\begin{aligned}\mathrm{Tot}\,\mathrm{RPic}(X) = \alpha^*\,\mathrm{Hom}_{\mathrm{Nis}}(M(X), \mathbb{Z}(1)) \\ \to \mathrm{Hom}_{\text{ét}}(M_{\text{ét}}(X), \mathbb{Z}(1)) = \underline{R}\pi^X_*\mathbb{G}_m[-1].\end{aligned}$$

This gives homomorphisms

$$\mathcal{H}^i(\operatorname{Tot}\operatorname{RPic}(X)) \to \underline{R}^{i-1}\pi_*^X\mathbb{G}_m, \quad i \geq 0. \tag{13.5}$$

13.8.1. **Proposition.** *For $i \leq 2$, (13.5) is an isomorphism.*

Proof. By blow-up induction, we reduce to the smooth case, where it follows from Hilbert's theorem 90. □

13.9. $H^1_{\text{ét}}(X, \mathbb{G}_m)$ **and** $H^1_{\text{éh}}(X, \mathbb{G}_m)$**.** In this subsection, we assume $\pi^X : X \to \operatorname{Spec} k$ *proper.* Recall that, then, the étale sheaf associated to the presheaf

$$U \mapsto \operatorname{Pic}(X \times U)$$

is representable by a k-group scheme $\operatorname{Pic}_{X/k}$ locally of finite type (Grothendieck-Murre [43]). Its connected component $\operatorname{Pic}^0_{X/k}$ is an extension of a semi-abelian variety by a unipotent subgroup $\mathcal{U}$. By homotopy invariance of $\underline{R}^1\pi_*^X\mathbb{G}_m$, we get a map

$$\operatorname{Pic}_{X/k}/\mathcal{U} \to \underline{R}^1\pi_*^X\mathbb{G}_m. \tag{13.6}$$

Recall that the right hand side is a 1-motivic sheaf by Proposition 13.5.1. We have:

13.9.1. **Proposition.** *This map is injective with lattice cokernel.*

Proof. Consider multiplication by an integer $n > 1$ on both sides. Using the Kummer exact sequence, Proposition 13.1.1 and Lemma 13.3.4 c), we find that (13.6) is an isomorphism on n-torsion and injecitve on n-cotorsion. The conclusion then follows from Proposition 3.6.6. □

13.10. $\mathrm{R}^1\operatorname{Pic}(X)$ **and** $\operatorname{Pic}^+(X)$ **for** X **proper.**

13.10.1. **Theorem.** *For X proper, the composition*

$$\operatorname{Pic}_{X/k}/\mathcal{U} \to \underline{R}^1\pi_*^X\mathbb{G}_m \to \mathcal{H}^2(\operatorname{Tot}\operatorname{RPic}(X))$$

where the first map is (13.6) *and the second one is the inverse of the isomorphism* (13.5)*, induces an isomorphism*

$$\operatorname{Pic}^+(X) \xrightarrow{\sim} \mathrm{R}^1\operatorname{Pic}(X)$$

where $\operatorname{Pic}^+(X)$ *is the* 1*-motive defined in* [5, Ch. 4].

Proof. Proposition 3.10.2 yields an exact sequence

$$L^1 \to G^1 \to \mathcal{H}^2(\operatorname{Tot}\operatorname{RPic}(X)) \to L^2$$

where we write $\mathrm{R}^i\operatorname{Pic}(X) = [L^i \to G^i]$. Propositions 13.8.1 and 13.9.1 then imply that the map of Theorem 13.10.1 induces an isomorphism $\operatorname{Pic}^0_{X/k}/\mathcal{U} \xrightarrow{\sim} G^1$. The conclusion follows, since on the one hand $\operatorname{Pic}^+(X) \simeq [0 \to \operatorname{Pic}^0_{X/k}/\mathcal{U}]$ by [5, Lemma 5.1.2 and Remark 5.1.3], and on the other hand the dual of Corollary 13.6.5 a) says that $L^1 = 0$. □

13.10.2. **Corollary.** *For X proper there is a canonical isomorphism*

$$\mathrm{L}_1\mathrm{Alb}(X) \xrightarrow{\sim} \mathrm{Alb}^-(X). \quad \square$$

13.11. **The Borel-Moore variant.** Let $X \in Sch$ be provided with a compactification $\overline{X}$ and closed complement $Z \xrightarrow{i} \overline{X}$. The relative Picard functor is then representable by a k-group scheme locally of finite type $\mathrm{Pic}_{\overline{X},Z}$, and we shall informally denote by $\mathcal{U}$ its unipotent radical. Similarly to (13.5) and (13.6), we have two canonical maps

$$\mathcal{H}^2(\mathrm{Tot}\,\mathrm{RPic}^c(X)) \to \mathrm{Pic}^{\text{éh}}_{\overline{X},Z} \leftarrow \mathrm{Pic}_{\overline{X},Z}/\mathcal{U} \tag{13.7}$$

where $\mathrm{Pic}^{\text{éh}}_{\overline{X},Z}$ is by definition the 1-motivic sheaf associated to the presheaf $U \mapsto H^1_{\text{éh}}(\overline{X} \times U, (\mathbb{G}_m)_{\overline{X}\times U} \to i_*(\mathbb{G}_m)_{Z\times U})$ (compare [5, 2.1]). Indeed, the latter group is canonically isomorphic to

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(M^c(X \times U), \mathbb{Z}(1)[2])$$

via the localisation exact triangle. From Theorem 13.10.1 and Proposition 11.4.2 b), we then deduce:

13.11.1. **Theorem.** *The maps* (13.7) *induce an isomorphism*

$$\mathrm{R}^1\mathrm{Pic}^c(X) \simeq [0 \to \mathrm{Pic}^0(\overline{X}, Z)/\mathcal{U}]. \quad \square$$

The following is a sequel of Proposition 11.5.2:

13.11.2. **Corollary.** *Let $X \in Sch(k)$ of dimension d. Then:*
a) $\mathrm{L}_1\mathrm{Alb}^c(X) = [L_1 \to A_1]$, *where A_1 is an abelian variety. In particular,* $\mathrm{L}_1\mathrm{Alb}^c(X)$ *is a Deligne* 1*-motive.*
b) If X is normal connected and not proper, let $\overline{X}$ be a normal compactification of X. Then $\mathrm{rank}\, L_1 = \#\pi_0(\overline{X} - X) - 1$.

Proof. a) follows immediately from Theorem 13.11.1. For b), consider the complex of discrete parts associated to the exact sequence (9.1): we get with obvious notation an almost exact sequence

$$L_1(\overline{X}) \to L_1(X) \to L_0(\overline{X} - X) \to L_0(\overline{X}) \to L_0(X)$$

where "almost exact" means that its homology is finite. The last group is 0 and $L_0(\overline{X}) = \mathbb{Z}$; on the other hand, $L_1(\overline{X}) = 0$ by Corollary 13.6.5 b). Hence the claim. $\square$

13.11.3. **Remark.** As a consequence we see that in b), the number of connected components of $\overline{X} - X$ only depends on X. Here is an elementary proof of this fact: let $\overline{X}'$ be another normal compactification and $\overline{X}''$ the closure of X in $\overline{X} \times \overline{X}'$. Then the two maps $\overline{X}'' \to \overline{X}$ and $\overline{X}'' \to \overline{X}'$ have connected fibres by Zariski's main theorem, thus $\overline{X} - X$ and $\overline{X}' - X$ have the same number of connected components as $\overline{X}'' - X$.

(The second author is indebted to Marc Hindry for a discussion leading to this proof.)

We shall also need the following computation in the next subsection.

13.11.4. **Theorem.** *Let $\overline{X}$ be smooth and proper, $Z \subset \overline{X}$ a divisor with normal crossings and $X = \overline{X} - Z$. Let $Z_1, \dots, Z_r$ be the irreducible components of Z and set*

$$Z^{(p)} = \begin{cases} \overline{X} & \text{if } p = 0 \\ \coprod_{i_1 < \dots < i_p} Z_{i_1} \cap \dots \cap Z_{i_p} & \text{if } p > 0. \end{cases}$$

Let $\mathrm{NS}_c^{(p)}(X)$ (resp. $\mathrm{Pic}_c^{(p)}(X)$, $T_c^{(p)}(X)$ be the cohomology (resp. the connected component of the cohomology) in degree p of the complex

$$\dots \to \mathrm{NS}(Z^{(p-1)}) \to \mathrm{NS}(Z^{(p)}) \to \mathrm{NS}(Z^{(p+1)}) \to \dots$$

(resp.

$$\dots \to \mathrm{Pic}^0(Z^{(p-1)}) \to \mathrm{Pic}^0(Z^{(p)}) \to \mathrm{Pic}^0(Z^{(p+1)}) \to \dots$$

$$\dots \to \mathbb{R}_{\pi_0(Z^{(p-1)})/k}\mathbb{G}_m \to \mathbb{R}_{\pi_0(Z^{(p)})/k}\mathbb{G}_m \to \mathbb{R}_{\pi_0(Z^{(p+1)})/k}\mathbb{G}_m \to \dots).$$

Then, for all $n \geq 0$, $\mathrm{R}^n\mathrm{Pic}^c(X)$ is of the form $[\mathrm{NS}_c^{(n-2)}(X) \xrightarrow{u^n} G_c^{(n)}]$, where $G_c^{(n)}$ is an extension of $\mathrm{Pic}_c^{(n-1)}(X)$ by $T_c^{(n)}(X)$.

Proof. A standard argument (compare e.g. [19, 3.3]) yields a spectral sequence of cohomological type in ${}^t\mathcal{M}_1$:

$$E_1^{p,q} = \mathrm{R}^q\mathrm{Pic}^c(Z^{(p)}) \Rightarrow \mathrm{R}^{p+q}\mathrm{Pic}^c(X).$$

By Corollary 10.6.1, we have $E_2^{p,2} = [\mathrm{NS}_c^{(p)}(X) \to 0]$, $E_2^{p,1} = [0 \to \mathrm{Pic}_c^{(p)}(X)]$ and $E_2^{p,0} = [0 \to T_c^{(p)}(X)]$. By Proposition C.9.1, all d_2 differentials are 0, hence the theorem. □

13.11.5. **Corollary.** *With notation as in Theorem 13.11.4, the complex* $\mathrm{RPic}(M^c(X)(1)[2])$ *is* q.i. *to*

$$\dots \to [\mathbb{Z}^{\pi_0(Z^{(p-2)})} \to 0] \to \dots$$

In particular, $\mathrm{R}^0\mathrm{Pic}(M^c(X)(1)[2]) = \mathrm{R}^1\mathrm{Pic}(M^c(X)(1)[2]) = 0$ *and* $\mathrm{R}^2\mathrm{Pic}(M^c(X)(1)[2]) = [\mathbb{Z}^{\pi_0^c(X)} \to 0]$ *(see Definition 11.5.1).*

Proof. This follows from Theorem 13.11.4 via the formula $M^c(X \times \mathbb{P}^1) = M^c(X) \oplus M^c(X)(1)[2]$, noting that $\overline{X} \times \mathbb{P}^1$ is a smooth compactification of $X \times \mathbb{P}^1$ with $\overline{X} \times \mathbb{P}^1 - X \times \mathbb{P}^1$ a divisor with normal

crossings with components $Z_i \times \mathbb{P}^1$, and

$$\begin{aligned}\mathrm{NS}(Z^{(p)} \times \mathbb{P}^1) &= \mathrm{NS}(Z^{(p)}) \oplus \mathbb{Z}^{\pi_0(Z^{(p)})} \\ \mathrm{Pic}^0(Z^{(p)} \times \mathbb{P}^1) &= \mathrm{Pic}^0(Z^{(p)}) \\ \pi_0(Z^{(p)} \times \mathbb{P}^1) &= \pi_0(Z^{(p)}).\end{aligned}$$

□

13.11.6. **Remark.** Let X be arbitrary, and filter it by its successive singular loci, *i.e.*

$$X = X^{(0)} \supset X^{(1)} \supset \dots$$

where $X^{(i+1)} = X^{(i)}_{sing}$. Then we have a spectral sequence of cohomological type in ${}^t\mathcal{M}_1$

$$E_2^{p,q} = \mathrm{R}^{p+q}\mathrm{Pic}^c(X^{(q)} - X^{(q+1)}) \Rightarrow \mathrm{R}^{p+q}\mathrm{Pic}^c(X)$$

in which the E_2-terms involve smooth varieties. This qualitatively reduces the computation of $\mathrm{R}^*\mathrm{Pic}^c(X)$ to the case of smooth varieties, but the actual computation may be complicated; we leave this to the interested reader.

13.12. $\mathrm{L}_1\mathrm{Alb}^*$ **and** Alb^+**.**

13.12.1. **Lemma.** *Let $n > 0$ and $Z \in Sch$ of dimension $< n$; then*

$$\mathrm{R}^i\mathrm{Pic}(M(Z)^*(n)[2n]) = 0 \text{ for } i \leq 1$$

$$\mathrm{R}^2\mathrm{Pic}(M(Z)^*(n)[2n]) = [\mathbb{Z}^{\pi_0^c(Z^{[n-1]})} \to 0]$$

where $Z^{[n-1]}$ is the disjoint union of the irreducible components of Z of dimension $n-1$.

Proof. Let

$$\begin{array}{ccc} \widetilde{T} & \longrightarrow & \widetilde{Z} \\ \downarrow & & \downarrow \\ T & \longrightarrow & Z \end{array}$$

be an abstract blow-up square, with $\widetilde{Z}$ smooth and $\dim T, \dim \widetilde{T} < \dim Z$. By Lemma 9.2.1, $M(T)^*(n-2)[2n-4]$ and $M(\widetilde{T})^*(n-2)[2n-4]$ are effective, so by Proposition 8.1.2, the exact triangle

$$\begin{aligned}\mathrm{RPic}(M(\widetilde{T})^*(n)[2n]) \to \mathrm{RPic}(M(\widetilde{Z})^*(n)[2n]) \oplus \mathrm{RPic}(M(T)^*(n)[2n]) \\ \to \mathrm{RPic}(M(Z)^*(n)[2n]) \xrightarrow{+1}\end{aligned}$$

degenerates into an isomorphism

$$\mathrm{RPic}(M(\widetilde{Z})^*(n)[2n]) \xrightarrow{\sim} \mathrm{RPic}(M(Z)^*(n)[2n]).$$

The lemma now follows from Corollary 13.11.5 by taking for $\widetilde{Z}$ a desingularisation of $Z^{[n-1]}$ and for T the union of the singular locus of Z and its irreducible components of dimension $< n-1$ (note that $M(\widetilde{Z})^*(n)[2n] \simeq M^c(\widetilde{Z})(1)[2]$). □

13.12.2. **Lemma.** *Let $\overline{X}$ a proper smooth scheme with a pair Y and Z of disjoint closed (reduced) subschemes of pure codimension* 1 *in $\overline{X}$. We then have*

$$\mathrm{R}^1\mathrm{Pic}(\overline{X}-Z,Y) \cong \mathrm{Pic}^+(\overline{X}-Z,Y)$$

(see [5, 2.2.1] *for the definition of relative* Pic^+*).*

Proof. The following exact sequence provides the weight filtration

$$0 \to \mathrm{R}^1\mathrm{Pic}(\overline{X},Y) \to \mathrm{R}^1\mathrm{Pic}(\overline{X}-Z,Y) \to \mathrm{R}^2\mathrm{Pic}_Z(\overline{X},Y)$$

where $\mathrm{R}^1\mathrm{Pic}(\overline{X},Y) \cong \mathrm{R}^1\mathrm{Pic}^c(\overline{X}-Y) \cong [0 \to \mathrm{Pic}^0(\overline{X},Y)]$ by Theorem 13.11.1 (here $\mathcal{U}=0$ since $\overline{X}$ is smooth). Also $\mathrm{R}^2\mathrm{Pic}_Z(\overline{X},Y) \cong \mathrm{R}^2\mathrm{Pic}_Z(\overline{X}) = [\mathrm{Div}_Z(\overline{X}) \to 0]$ from 8.2.2: thus the discrete part of $\mathrm{R}^1\mathrm{Pic}(\overline{X}-Z,Y)$ is given by a subgroup D of $\mathrm{Div}_Z(\overline{X}) = \mathrm{Div}_Z(\overline{X},Y)$.

It remains to identify the map $u : D \to \mathrm{Pic}^0(\overline{X},Y)$. Using now the exact sequence

$$\mathrm{R}^0\mathrm{Pic}(Y) \to \mathrm{R}^1\mathrm{Pic}(\overline{X}-Z,Y) \to \mathrm{R}^1\mathrm{Pic}(\overline{X}-Z) \to \mathrm{R}^1\mathrm{Pic}(Y)$$

where $\mathrm{R}^i\mathrm{Pic}(Y)$ is of weight < 0 for $i \le 1$ (11.3.2 and 13.6.5), we get that u is the canonical lifting of the map of the 1-motive $\mathrm{R}^1\mathrm{Pic}(\overline{X}-Z)$ described in 10.6.1. Thus $D = \mathrm{Div}^0_Z(\overline{X},Y)$ and the claimed isomorphism is clear. □

This proof also gives:

13.12.3. **Corollary.** *We have*

$$[\mathrm{Div}^0_Z(\overline{X},Y) \to 0] = \mathrm{Ker}(\mathrm{R}^2\mathrm{Pic}_Z(\overline{X},Y) \to \mathrm{R}^2\mathrm{Pic}(\overline{X},Y)).$$

We shall need:

13.12.4. **Theorem** (Relative duality)**.** *Let $\overline{X}$, Y and Z be as above and further assume that $\overline{X}$ is n-dimensional. Then*

$$M(\overline{X}-Z,Y)^*(n)[2n] \cong M(\overline{X}-Y,Z)$$

and therefore

$$\mathrm{RPic}^*(\overline{X}-Z,Y) \cong \mathrm{RPic}(\overline{X}-Y,Z)$$

and dually for LAlb.

Proof. See [3]. □

13.12.5. **Corollary.** *Let Z be a divisor in $\overline{X}$ such that $Z \cap Y = \emptyset$. There exists a "cycle class" map η fitting in the following commutative diagram*

$$\begin{array}{ccc} \mathrm{R}^2\mathrm{Pic}(M(Z)^*(n)[2n]) & \xrightarrow{\eta} & \mathrm{R}^2\mathrm{Pic}^c(\overline{X}-Y) \\ \| \downarrow & & \| \downarrow \\ \mathrm{R}^2\mathrm{Pic}_Z(\overline{X},Y) & \longrightarrow & \mathrm{R}^2\mathrm{Pic}(\overline{X},Y) \end{array}$$

Writing $Z = \cup Z_i$ as union of its irreducible components we have that η on Z_i is the "fundamental class" of Z_i in $\overline{X}$ modulo algebraic equivalence.

Proof. We have a map $M(Z) \to M(\overline{X}-Y)$, and the vertical isomorphisms in the following commutative square

$$\begin{array}{ccc} M(\overline{X}-Y)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\ \| \uparrow & & \| \uparrow \\ M(\overline{X},Y) & \longrightarrow & M^Z(\overline{X},Y) \end{array}$$

are given by relative duality. □

13.12.6. **Theorem.** *For $X \in Sch$ we have*

$$\mathrm{R}^1\mathrm{Pic}^*(X) \cong \mathrm{Pic}^-(X).$$

Proof. We are left to consider $X \in Sch$ purely of dimension n with the following associated set of data and conditions.

For the irreducible components $X_1, \dots, X_r$ of X we let $\widetilde{X}$ be a desingularisation of $\coprod X_i$, $S := X_{sing} \cup \bigcup_{i \neq j} S_i \cap S_j$ and $\widetilde{S}$ the inverse image of S in $\widetilde{X}$. We let $\overline{X}$ be a smooth proper compactification with normal crossing divisor Y. Let $\overline{S}$ denote the Zariski closure of $\widetilde{S}$ in $\overline{X}$. Assume that $Y + \overline{S}$ is a reduced normal crossing divisor in $\overline{X}$. Finally denote by Z the union of all compact components of divisors in $\widetilde{S}$ (*cf.* [5, 2.2]).

We have an exact sequence coming from the abstract blow-up square associated to the above picture:

$$\begin{aligned} \cdots \to \mathrm{R}^1\mathrm{Pic}^*(\widetilde{X}) \oplus \mathrm{R}^1\mathrm{Pic}(M(S)^*(n)[2n]) \to \mathrm{R}^1\mathrm{Pic}^*(X) \\ \to \mathrm{R}^2\mathrm{Pic}(M(\widetilde{S})^*(n)[2n]) \to \mathrm{R}^2\mathrm{Pic}^*(\widetilde{X}) \oplus \mathrm{R}^2\mathrm{Pic}(M(S)^*(n)[2n]) \end{aligned}$$

Now:

- the first map is injective (Lemma 13.12.1),
- $\mathrm{R}^1\mathrm{Pic}^*(\widetilde{X}) = \mathrm{R}^1\mathrm{Pic}^c(\widetilde{X}) = [0 \to \mathrm{Pic}^0(\overline{X},Y)]$ since $\widetilde{X}$ is smooth (Theorem 13.11.1; note that $\mathcal{U} = 0$ by the smoothness of $\widetilde{X}$),
- $\mathrm{R}^1\mathrm{Pic}(M(S)^*(n)[2n]) = 0$ (Lemma 13.12.1),

- $\mathrm{R}^2\mathrm{Pic}(M(\widetilde{S})^*(n)[2n]) = [\mathbb{Z}^{\pi_0^c(\widetilde{S}^{[n-1]})} \to 0] := [\mathrm{Div}_{\overline{S}}(\overline{X}, Y) \to 0]$ is given by the free abelian group on compact irreducible components of $\widetilde{S}$ (Lemma 13.12.1),
- $\mathrm{R}^2\mathrm{Pic}^*(\widetilde{X}) = \mathrm{R}^2\mathrm{Pic}^c(\widetilde{X}) = \mathrm{R}^2\mathrm{Pic}(\overline{X}, Y) = [\mathrm{NS}_c^{(0)}(\widetilde{X}) \xrightarrow{u^2} G_c^{(2)}]$ (Theorem 13.11.4),
- $\mathrm{R}^2\mathrm{Pic}(M(S)^*(n)[2n]) = [\mathbb{Z}^{\pi_0^c(S^{[n-1]})} \to 0] = [\mathrm{Div}_S(X) \to 0]$ (Lemma 13.12.1).

We may therefore rewrite the above exact sequence as follows:

$$\begin{aligned} 0 \to [0 \to \mathrm{Pic}^0(\overline{X}, Y)] \to \mathrm{R}^1\mathrm{Pic}^*(X) \to [\mathrm{Div}_{\overline{S}}(\overline{X}, Y) \to 0] \\ \xrightarrow{\alpha} [\mathrm{NS}_c^{(0)}(\widetilde{X}) \xrightarrow{u^2} G_c^{(2)}] \oplus [\mathrm{Div}_S(X) \to 0]. \end{aligned}$$

The map $\mathrm{Div}_{\overline{S}}(\overline{X}, Y) \to \mathrm{Div}_S(X)$ induced from $M(\widetilde{S}) \to M(S)$ is clearly the proper push-forward of Weil divisors. The map

$$[\mathrm{Div}_{\overline{S}}(\overline{X}, Y) \to 0] \to [\mathrm{NS}_c^{(0)}(\widetilde{X}) \xrightarrow{u^2} G_c^{(2)}]$$

is the cycle class map described in Corollary 13.12.5. By Corollary 13.12.3 we then get

$$\mathrm{Ker}\,\alpha = [\mathrm{Div}^0_{\overline{S}/S}(\overline{X}, Y) \to 0]$$

where the lattice $\mathrm{Div}^0_{\overline{S}/S}(\overline{X}, Y)$ is from the definition of Pic^- (see [5, 2.2.1]). In other terms, we have

$$\mathrm{R}^1\mathrm{Pic}^*(X) = [\mathrm{Div}^0_{\overline{S}/S}(\overline{X}, Y) \xrightarrow{u} \mathrm{Pic}^0(\overline{X}, Y)]$$

and we are left to check that the mapping u is the one described in [5]. Just observe that, by Lemma 13.12.1 and Theorem 13.12.4,

$$\begin{aligned} \mathrm{R}^1\mathrm{Pic}^*(X) \hookrightarrow \mathrm{R}^1\mathrm{Pic}^*(X, S) \cong \mathrm{R}^1\mathrm{Pic}^*(\widetilde{X}, \widetilde{S}) \cong \mathrm{R}^1\mathrm{Pic}^*(\widetilde{X}, Z) \\ = \mathrm{R}^1\mathrm{Pic}^*(\overline{X} - Y, Z) \cong \mathrm{R}^1\mathrm{Pic}(\overline{X} - Z, Y) \end{aligned}$$

and the latter is isomorphic to $\mathrm{Pic}^+(\overline{X} - Z, Y)$ by 13.12.2. Since, by construction, $\mathrm{Pic}^-(X)$ is a sub-1-motive of $\mathrm{Pic}^+(\overline{X} - Z, Y)$ the isomorphism of 13.12.2 restricts to the claimed one. □

14. Generalisations of Roĭtman's theorem

In this section we give a unified treatment of Roĭtman's theorem on torsion 0-cycles on a smooth projective variety and its various generalisations.

14.1. **Motivic and classical Albanese.** Let $X \in Sch(k)$; we assume X smooth if $p > 1$ and X semi-normal (in particular reduced) if $p = 1$, see Lemma 9.4.1. Recall that Suslin's singular algebraic homology is

$$H_j(X) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_-}(\mathbb{Z}[j], M(X)) = \mathbb{H}^{-j}_{\mathrm{Nis}}(k, C_*(X))$$

for any scheme $p : X \to k$. On the other hand, we may define

$$H^{\text{ét}}_j(X) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(\mathbb{Z}[j], M_{\text{ét}}(X)) = \mathbb{H}^{-j}_{\text{ét}}(k, \alpha^* C_*(X)).$$

We also have versions with torsion coefficients:

$$H^{\text{ét}}_j(X, \mathbb{Z}/n) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(\mathbb{Z}[j], M_{\text{ét}}(X) \otimes \mathbb{Z}/n), (n, p) = 1$$

(this convention is chosen so that we have the usual long exact sequences.)

We shall use the following notation throughout:

14.1.1. **Notation.** For any $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and any abelian group A, we write $H^{(1)}_j(M, A)$ for the abelian group

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}}(\mathbb{Z}[j], \mathrm{Tot}\,\mathrm{LAlb}(M) \otimes A) \simeq \mathbb{H}^{-j}_{\text{ét}}(k, \mathrm{Tot}\,\mathrm{LAlb}(M) \overset{L}{\otimes} A).$$

This is *Suslin* 1*-motivic homology of* M *with coefficients in* A. If $M = M(X)$, we write $H^{(1)}_j(X, A)$ for $H^{(1)}_j(M, A)$. We drop A in the case where $A = \mathbb{Z}$.

The motivic Albanese map (5.2) then gives maps

$$H^{\text{ét}}_j(X) \to H^{(1)}_j(X) \tag{14.1}$$

$$H^{\text{ét}}_j(X, \mathbb{Z}/n) \to H^{(1)}_j(X, \mathbb{Z}/n). \tag{14.2}$$

hence in the limit

$$H^{\text{ét}}_j(X, (\mathbb{Q}/\mathbb{Z})') \to H^{(1)}_j(X, (\mathbb{Q}/\mathbb{Z})') \tag{14.3}$$

where, as usual, $(\mathbb{Q}/\mathbb{Z})' := \bigoplus_{l \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

14.1.2. **Proposition.** *If X is a smooth curve (or any curve in characteristic 0), the maps* (14.1), (14.2) *and* (14.3) *are isomorphisms for any j.*

Proof. This follows immediately from Proposition 5.1.4. □

Note that if $X = \overline{X} - Y$ is a smooth curve obtained by removing a finite set of closed points from a projective smooth curve $\overline{X}$ then $\mathcal{A}_{X/k} = \mathrm{Pic}_{(\overline{X},Y)/k}$ is the relative Picard scheme (see [5] for its representability) and the Albanese map just send a point $P \in X$ to $(\mathcal{O}_{\overline{X}}(P), 1)$ where 1 is the tautological section, trivialising $\mathcal{O}_{\overline{X}}(P)$ on X. We then have the following result (*cf.* [39, Lect. 7, Th. 7.16]).

14.1.3. **Corollary.** *If $X = \overline{X} - Y$ is a smooth curve,*

$$H_0^{\text{ét}}(X) \to \operatorname{Pic}(\overline{X}, Y)[1/p]$$

is an isomorphism.

Now let $\mathcal{A}^{\text{éh}}_{X/k}$ be as in Proposition 13.6.1 and Remark 13.6.2: recall that $\mathcal{A}^{\text{éh}}_{X/k} = \mathcal{A}_{X/k}$ if X is strictly reduced. The map $\operatorname{Tot}\operatorname{LAlb}(X) \to \mathcal{A}^{\text{éh}}_{X/k}$ of *loc. cit.* induces a homomorphism

$$H_0^{(1)}(X) \to \mathcal{A}^{\text{éh}}_{X/k}(k)[1/p] \tag{14.4}$$

which is not an isomorphism in general (but see Lemma 14.2.2).

Finally, composing (14.4), (14.1) and the obvious map $H_0(X)[1/p] \to H_0^{\text{ét}}(X)$, we get a map

$$H_0(X)[1/p] \to \mathcal{A}^{\text{éh}}_{X/k}(k)[1/p]. \tag{14.5}$$

We may further restrict to parts of degree 0, getting a map

$$H_0(X)^0[1/p] \to (\mathcal{A}^{\text{éh}}_{X/k})^0(k)[1/p].$$

If X is smooth, this is the $\mathbb{Z}[1/p]$-localisation of the generalised Albanese map of Spieß-Szamuely [57, (2)].

Dually to Lemma C.6.2, the functor

$$\begin{aligned} D^b(\mathcal{M}_1[1/p]) &\to \mathrm{DM}^{\text{eff}}_{-,\text{ét}} \\ C &\mapsto \operatorname{Tot}(C) \otimes (\mathbb{Q}/\mathbb{Z})' \end{aligned}$$

is exact with respect to the ${}_t\mathcal{M}_1$ t-structure on the left and the homotopy t-structure on the right; in other words:

14.1.4. **Lemma.** *For any $C \in D^b(\mathcal{M}_1[1/p])$, there are canonical isomorphisms of sheaves*

$$\mathcal{H}_j(\operatorname{Tot}(C) \otimes (\mathbb{Q}/\mathbb{Z})') \simeq \operatorname{Tot}({}_tH_j(C)) \otimes (\mathbb{Q}/\mathbb{Z})'$$

(note that the right hand side is a single sheaf!) In particular, for $C = \operatorname{LAlb}(M)$ and k algebraiclly closed:

$$H_j^{(1)}(M, (\mathbb{Q}/\mathbb{Z})') \simeq \Gamma(k, \operatorname{Tot}(\mathrm{L}_j\operatorname{Alb}(M)) \otimes (\mathbb{Q}/\mathbb{Z})').$$

14.2. **A proof of Roĭtman's and Spieß-Szamuely's theorems.** *Until the end of this section, we assume that k is algebraically closed.* In this subsection, we only deal with smooth schemes and the characteristic is arbitrary: we shall show how the results of Section 10 allows us to recover the classical theorem of Roĭtman on torsion 0-cycles up to p-torsion, as well as its generalisation to smooth varieties by Spieß-Szamuely [57]. The reader should compare our argument with theirs (*loc. cit.* , §5).

Since k is algebraically closed, Corollary D.1.6 implies

14.2.1. **Lemma.** *For any $j \in \mathbb{Z}$, $H_j^{\text{ét}}(X) = H_j(X)[1/p]$; similarly with finite or divisible coefficients.* □

Moreover, it is easy to evaluate $H_j^{(1)}(X) = \mathcal{H}_j(\operatorname{Tot} \operatorname{LAlb}(X))(k)$ out of Theorem 10.2.2: if $\mathrm{L}_n\mathrm{Alb}(X) = [L_n \to G_n]$, we have a long exact sequence coming from Proposition 3.10.2

$$\begin{aligned}(14.6)\quad L_{n+1}(k)[1/p] \to G_{n+1}(k)[1/p] \to H_j^{(1)}(X) \\ \to L_n(k)[1/p] \to G_n(k)[1/p] \to \dots\end{aligned}$$

Thus:

14.2.2. **Lemma.** *For X smooth,* (14.4) *is an isomorphism and we have*

$$\begin{aligned}(14.7)\qquad H_1^{(1)}(X) &\simeq \mathrm{NS}^*_{X/k}(k)[1/p] \\ H_j^{(1)}(X) &= 0 \ \textit{if}\ j \neq 0, 1.\end{aligned}$$

Here is now the main lemma:

14.2.3. **Lemma.** *The map* (14.2) *is an isomorphism for $j = 0, 1$ and surjective for $j = 2$.*

Proof. It is sufficient to show that $\operatorname{Hom}((14.2), \mathbb{Z}/n(1))$ is an isomorphism (*resp.* injective). Since k is algebraically closed, by finite group duality (!) this is nothing else than the map

$$H^j(\operatorname{Tot} \operatorname{RPic}(X) \otimes \mathbb{Z}/n) \to H^j_{\text{ét}}(X, \mathbb{Z}/n(1)).$$

By definition, the left hand side is

$$H^j(\alpha^* \underline{\operatorname{Hom}}_{\mathrm{Nis}}(M(X), \mathbb{Z}(1)) \otimes \mathbb{Z}/n) = H^j_{\mathrm{Nis}}(X, \mathbb{Z}/n(1))$$

(using again that k is algebraically closed), hence the result follows from Hilbert's theorem 90. □

From this and Lemma 14.1.4 we deduce:

14.2.4. **Corollary.** *The homomorphism* (14.3)

$$H_j(X, (\mathbb{Q}/\mathbb{Z})') \to H_j^{(1)}(X, (\mathbb{Q}/\mathbb{Z})')$$

(see Lemma 14.2.1) is bijective for $j = 0, 1$ and surjective for $j = 2$.

The following theorem extends in particular [57, Th. 1.1] to all smooth varieties[8].

[8]In *loc. cit.* , X is supposed to be admit an open embedding in a smooth projective variety.

14.2.5. **Theorem.** *a) The map* (14.5) *is an isomorphism on torsion prime to p.*
b) $H_1(X) \otimes (\mathbb{Q}/\mathbb{Z})' = 0$.
c) The map (14.1) *yields a surjection*

$$H_1(X)\{p'\} \twoheadrightarrow \mathrm{NS}^*_{X/k}(k)\{p'\}$$

where $M\{p'\}$ *denotes the torsion prime to* p *in an abelian group* M.

Proof. Lemmas 14.2.1 and 14.2.2 reduce us to show that (14.1) is an isomorphism on torsion for $j = 0$. We have commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & H_j(X) \otimes (\mathbb{Q}/\mathbb{Z})' & \to & H_j(X, (\mathbb{Q}/\mathbb{Z})') & \to & H_{j-1}(X)\{p'\} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^{(1)}_j(X) \otimes (\mathbb{Q}/\mathbb{Z})' & \to & H^{(1)}_j(X, (\mathbb{Q}/\mathbb{Z})') & \to & H^{(1)}_{j-1}(X)\{p'\} & \to & 0 \end{array} \tag{14.8}$$

For $j = 1$, the middle vertical map is an isomorphism by Lemma 14.2.3 or Corollary 14.2.4 and $H^{(1)}_1(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ by (14.7), which gives a) and b). For $j = 2$, the middle map is surjective by the same lemma and corollary, which gives c). The proof is complete. □

14.2.6. **Remark.** If X is smooth projective of dimension n, $H_j(X)$ is isomorphic to the higher Chow group $CH^n(X, j)$. In (14.8) for $j = 2$, the lower left term is 0 by Lemma 14.2.2. The composite map

$$H_2(X, (\mathbb{Q}/\mathbb{Z})') \to H_1(X)\{p'\} \to H^{(1)}_1(X)\{p'\} = \mathrm{NS}^*_{X/k}(k)\{p'\}$$

is "dual" to the map

$$\mathrm{NS}(X) \otimes (\mathbb{Q}/\mathbb{Z})' \to H^2_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(1))$$

whose cokernel is $Br(X)\{p'\}$. Let

$$Br(X)^D = \varinjlim_{(n,p)=1} \mathrm{Hom}({}_nBr(X), \mu_n) :$$

a diagram chase in (14.8) for $j = 2$ then yields an exact sequence

$$\begin{aligned} 0 \to CH^n(X, 2) \otimes (\mathbb{Q}/\mathbb{Z})' &\to Br(X)^D \\ &\to CH^n(X, 1)\{p'\} \to \mathrm{NS}^*_{X/k}(k)\{p'\} \to 0. \end{aligned}$$

Together with $CH^n(X, 1) \otimes (\mathbb{Q}/\mathbb{Z})' = 0$, this should be considered as a natural complement to Roĭtman's theorem.

14.3. **Generalisation to singular schemes.** We now assume $\operatorname{char} k = 0$, and show how the results of Section 11 allow us to extend the results of the previous subsection to singular schemes. By blow-up induction and the 5 lemma, we get:

14.3.1. **Proposition.** *The isomorphisms and surjection of Lemma 14.2.3 and Corollary 14.2.4 extend to all $X \in Sch$.* □

Let $\mathrm{L}_1\mathrm{Alb}(X) = [L_1 \xrightarrow{u_1} G_1]$. Proposition 14.3.1, the exact sequence (14.6) and the snake chase in the proof of Theorem 14.2.5 give:

14.3.2. **Corollary.** *For $X \in Sch$, we have exact sequences*

$$0 \to H_1(X) \otimes \mathbb{Q}/\mathbb{Z} \to \operatorname{Ker}(u_1) \otimes \mathbb{Q}/\mathbb{Z} \to H_0(X)_{\mathrm{tors}} \to \operatorname{Coker}(u_1)_{\mathrm{tors}} \to 0$$

$$0 \to H_1(X) \otimes \mathbb{Q}/\mathbb{Z} \to H_1^{(1)}(X, \mathbb{Q}/\mathbb{Z}) \to H_0(X)_{\mathrm{tors}} \to 0.$$

The second exact sequence is more intrinsic than the first, but note that it does not give information on $H_1(X) \otimes \mathbb{Q}/\mathbb{Z}$.

14.3.3. **Corollary.** *If X is normal, $H_1(X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and there is an isomorphism*

$$\mathcal{A}_{X/k}(k)_{\mathrm{tors}} \xrightarrow{\sim} H_0(X)_{\mathrm{tors}}.$$

Proof. This follows from the previous corollary and Corollary 13.6.5 c). □

14.3.4. **Remark.** Theorem 13.10.2 shows that the second isomorphism of Proposition 14.3.1 coincides with the one of Geisser in [22, Th. 6.2] when X is proper. When X is further normal, the isomorphism of Corollary 14.3.3 also coincides with the one of his Theorem 6.1.

14.3.5. **Remarks.** Note that the reformulation of "Roitman's theorem" involving $\operatorname{Ker} u_1$ is the best possible!
1) Let X be a proper scheme such that $\operatorname{Pic}^0(X)/\mathcal{U} = \mathbb{G}_m^r$ is a torus (more likely such that $\operatorname{Alb}(X_0) = 0$ where $X_0 \to X$ is a resolution, according with the description in [5, pag. 68]). Then $\mathrm{R}^1\operatorname{Pic}(X)^* = \mathrm{L}_1\operatorname{Alb}(X) = [\mathbb{Z}^r \to 0]$ is the character group (*cf.* [5, 5.1.4]). For example, take a nodal projective curve X with resolution $X_0 = \mathbb{P}^1$. In this case the map (14.1) is an isomorphism for all j and thus $\operatorname{Ker}(u_1) \otimes \mathbb{Q}/\mathbb{Z} = H_1(X) \otimes \mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^r$.
2) For Borel-Moore and $\mathrm{L}_1\operatorname{Alb}^c(X) = \mathrm{L}_1\operatorname{Alb}^*(X)$ for X smooth open is Cartier dual of $\operatorname{Pic}^0(\overline{X}, Y)$ then (*cf.* [5, pag. 47] the $\operatorname{Ker} u_1^c$ can be non-zero take $\overline{X} = \mathbb{P}^1$ and $Y =$ finite number of points.

14.4. **Borel-Moore Roĭtman.** We are still in characteristic 0. Recall that the Borel-Moore motivic homology group

$$H^c_j(X,\mathbb{Z}) := \mathrm{Hom}(\mathbb{Z}[j], M^c(X))$$

is canonically isomorphic to Bloch's higher Chow group $CH_0(X,j)$. Similarly to the previous sections, we have maps

$$H^c_j(X,\mathbb{Z}) \to H^{(1)}_j(M^c(X)) =: H^{c,(1)}_j(X)$$
$$H^c_j(X,\mathbb{Q}/\mathbb{Z}) \to H^{(1)}_j(M^c(X),\mathbb{Q}/\mathbb{Z}) =: H^{c,(1)}_j(X,\mathbb{Q}/\mathbb{Z})$$

and

14.4.1. **Proposition.** *The second map is an isomorphism for $j = 0, 1$ and surjective for $j = 2$.*

Proof. By localisation induction, reduce to X proper and use Proposition 14.3.1. □

14.4.2. **Corollary.** *For $X \in Sch$, we have exact sequences*

$$0 \to CH_0(X,1)\otimes\mathbb{Q}/\mathbb{Z} \to \mathrm{Ker}(u^c_1)\otimes\mathbb{Q}/\mathbb{Z} \\ \to CH_0(X)_{\mathrm{tors}} \to \mathrm{Coker}(u^c_1)_{\mathrm{tors}} \to 0$$

$$0 \to CH_0(X,1)\otimes\mathbb{Q}/\mathbb{Z} \to H^{c,(1)}_j(X,\mathbb{Q}/\mathbb{Z}) \to CH_0(X)_{\mathrm{tors}} \to 0$$

where we write $\mathrm{L}_1\mathrm{Alb}^c(X) = [L^c_1 \xrightarrow{u^c_1} G^c_1]$.

14.5. **"Cohomological" Roĭtman.** We are still in characteristic 0.

14.5.1. **Lemma.** *Let $0 < r \le n$. Then for any $Z \in Sch$ of dimension $\le n-r$ and any $i > 2(n-r)$, we have $H^i_{\mathrm{cdh}}(Z,\mathbb{Q}/\mathbb{Z}(n)) = 0$.*

Proof. By blow-up induction we reduce to the case where Z is smooth of pure dimension $n-r$; then $H^i_{\mathrm{cdh}}(Z,\mathbb{Q}/\mathbb{Z}(n)) = H^i_{\mathrm{Nis}}(Z,\mathbb{Q}/\mathbb{Z}(n))$. Since k is algebraically closed, and $n \ge \dim Z$, $H^i_{\mathrm{Nis}}(Z,\mathbb{Q}/\mathbb{Z}(n)) \simeq H^i_{\text{ét}}(Z,\mathbb{Q}/\mathbb{Z}(n))$ by Suslin's theorem [58] and the vanishing follows from the known bound for étale cohomological dimension. □

Now consider the 1-motive $\mathrm{L}_1\mathrm{Alb}^*(X)$ for X of dimension n. This time, we have maps

$$H^{2n-j}_{\mathrm{cdh}}(X,\mathbb{Z}(n)) \to H^{(1)}_j(M(X)^*(n)[2n]) =: H^{2n-j}_{(1)}(X,\mathbb{Z}(n))$$
(14.9)
$$H^{2n-j}_{\mathrm{cdh}}(X,\mathbb{Q}/\mathbb{Z}(n)) \to H^{(1)}_j(M(X)^*(n)[2n],\mathbb{Q}/\mathbb{Z}) =: H^{2n-j}_{(1)}(X,\mathbb{Q}/\mathbb{Z}(n)).$$

14.5.2. **Lemma.** *Let $Z \in Sch$ be of dimension $< n$. Then the map*

$$H^{2n-2}_{\mathrm{cdh}}(Z, \mathbb{Q}/\mathbb{Z}(n)) \to H^{(1)}_2(M(Z)^*(n)[2n], \mathbb{Q}/\mathbb{Z})$$

is an isomorphism.

Proof. For notational simplicity, write $H^*(Y, n)$ for $H^*_{\mathrm{cdh}}(Y, \mathbb{Q}/\mathbb{Z}(n))$ and $F_j(Y)$ for $H^{(1)}_j(M(Y)^*(n)[2n], \mathbb{Q}/\mathbb{Z}))$, where Y is a scheme of dimension $\leq n$. Let $\widetilde{Z}, T, \widetilde{T}$ be as in the proof of Lemma 13.12.1. Then Lemma 14.5.1 and proposition 8.1.2 yield a commutative diagram

$$\begin{array}{ccc} H^{2n-2}(Z, n) & \longrightarrow & H^{2n-2}(\widetilde{Z}, n) \\ \downarrow & & \downarrow \\ F_2(Z) & \longrightarrow & F_2(\widetilde{Z}) \end{array}$$

in which both horizontal maps are isomorphisms. Therefore, it suffices to prove the lemma when Z is smooth quasiprojective of dimension $n-1$.

The motive $\mathrm{R}^2\mathrm{Pic}(M(Z)^*(n)[2n]) \simeq \mathrm{R}^2\mathrm{Pic}(M^c(Z)(1)[2])$ was computed in Corollary 13.11.5: it is $[\mathbb{Z}^{\pi^c_0(Z)} \to 0]$. Therefore, we get

$$F_2(Z) \simeq \mathbb{Q}/\mathbb{Z}(1)[\pi^c_0(Z)].$$

On the other hand, the trace map defines an isomorphism

$$H^{2n-2}(Z, n) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}(1)[\pi^c_0(Z)]$$

and the issue is to prove that the vertical map in the diagram is this isomorphism. For this, we first may reduce to Z projective and connected. Now we propose the following argument: take a chain of smooth closed subvarieties $Z \supset Z_2 \supset \cdots \supset Z_n$, with Z_i of dimension $n-i$ and connected (take multiple hyperplane sections up to Z_{n-1} and then a single point of Z_{n-1} for Z_n. The Gysin exact triangles give commutative diagrams

$$\begin{array}{ccc} H^{2n-2i-2}(Z_{i+1}, n-i) & \longrightarrow & H^{2n-2i}(Z_i, n-i+1) \\ \downarrow & & \downarrow \\ F_2(Z_{i+1}) & \longrightarrow & F_2(Z_i) \end{array}$$

in which both horizontal maps are isomorphisms: thus we are reduced to the case $\dim Z = 0$, where it follows from Proposition 14.1.2 applied to $X = \mathbb{P}^1$. $\square$

14.5.3. **Theorem.** *The map* (14.9) *is an isomorphism for $j = 0, 1$.*

Proof. This is easy and left to the reader for $j = 0$. For $j = 1$, we argue as usual by blowup induction. In the situation of 8.1.e, we then have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} H^{2n-2}(\widetilde{X},n)\oplus H^{2n-2}(Z,n) & \longrightarrow & H^{2n-2}(\widetilde{Z},n) & \longrightarrow & H^{2n-1}(X,n) & \longrightarrow & H^{2n-1}(\widetilde{X},n)\oplus H^{2n-1}(Z,n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_2(\widetilde{X})\oplus F_2(Z) & \longrightarrow & F_2(\widetilde{Z}) & \longrightarrow & F_1(X) & \longrightarrow & F_1(\widetilde{X})\oplus F_1(Z). \end{array}$$

In this diagram, we have $F_1(Z) = 0$ by Lemma 13.12.1 and $H^{2n-1}(Z, n) = 0$ by Lemma 14.5.1, and the same lemmas imply that both rightmost horizontal maps are surjective. The rightmost vertical map is now an isomorphism by Proposition 14.4.1, which also gives the surjectivity of $H^{2n-2}(\widetilde{X}, n) \to F_2(\widetilde{X})$. Finally, Lemma 14.5.2 implies that $H^{2n-2}(\widetilde{Z}, n) \to F_2(\widetilde{Z})$ and $H^{2n-2}(Z, n) \to F_2(Z)$ are isomorphisms, and the conclusion follows from the 5 lemma. □

14.5.4. **Corollary.** *For $X \in Sch$ of dimension n, we have exact sequences*

$$\begin{aligned} 0 \to H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} \to \operatorname{Ker}(u_1^*) \otimes \mathbb{Q}/\mathbb{Z} \\ \to H^{2n}(X, \mathbb{Z}(n))_{\mathrm{tors}} \to \operatorname{Coker}(u_1^*)_{\mathrm{tors}} \to 0 \end{aligned}$$

$$\begin{aligned} 0 \to H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} \to H^{2n-1}_{(1)}(X, \mathbb{Q}/\mathbb{Z}(n)) \\ \to H^{2n}(X, \mathbb{Z}(n))_{\mathrm{tors}} \to 0 \end{aligned}$$

where u_1^* is the map involved in the 1*-motive* $\mathrm{L}_1\mathrm{Alb}^*(X)$ *(which is isomorphic to* $\mathrm{Alb}^+(X)$ *by the dual of Theorem 13.12.6).*

14.5.5. **Corollary.** *If X is a* proper *scheme of dimension n we then get $H^{2n-1}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} = 0$ and an isomorphism*

$$\mathrm{Alb}^+(X)(k)_{\mathrm{tors}} \xrightarrow{\sim} H^{2n}(X, \mathbb{Z}(n))_{\mathrm{tors}}.$$

Proof. If X is proper then $\mathrm{L}_1\mathrm{Alb}^*(X) \cong \mathrm{Alb}^+(X)$ is semiabelian and the claim follows from the previous corollary. □

14.5.6. **Remark.** Marc Levine outlined us how to construct a "cycle map" $c\ell^{\mathrm{cdh}}$ from $CH^n_{LW}(X)$ to $H^{2n}_{\mathrm{cdh}}(X, \mathbb{Z}(n))$, where $CH^n_{LW}(X)$ is the Levine-Weibel cohomological Chow group of zero cycles. This gives a map

$$c\ell^{\mathrm{cdh}}_{\mathrm{tors}} : CH^n_{LW}(X)_{\mathrm{tors}} \to H^{2n}_{\mathrm{cdh}}(X, \mathbb{Z}(n))_{\mathrm{tors}}$$

which most likely fits in a commutative diagram (for X projective)

$$\begin{array}{ccc} CH^n_{LW}(X)_{\mathrm{tors}} & \xrightarrow{c\ell^{\mathrm{cdh}}_{\mathrm{tors}}} & H^{2n}_{\mathrm{cdh}}(X,\mathbb{Z}(n))_{\mathrm{tors}} \\ {\scriptstyle a^+_{\mathrm{tors}}}\downarrow & & \wr\downarrow \\ \mathrm{Alb}^+(X)(k)_{\mathrm{tors}} & \xrightarrow{\sim} & \mathrm{L}_1\mathrm{Alb}^*(X)(k)_{\mathrm{tors}} \end{array}$$

where: the horizontal bottom isomorphism is that induced by Theorem 13.12.6 and the right vertical one comes from the previous Corollary 14.5.4; the left vertical map is the one induced, on torsion, by the universal regular homomorphism $a^+ : CH^n_{LW}(X)_{\deg 0} \to \mathrm{Alb}^+(X)(k)$ constructed in [5, 6.4.1]. This would imply that

$$c\ell^{\mathrm{cdh}}_{\mathrm{tors}} \text{ is an isomorphism } \iff a^+_{\mathrm{tors}} \text{ is an isomorphism.}$$

If X is normal and $k = \overline{k}$ or for any X projective if $k = \mathbb{C}$ then a^+_{tors} is known to be an isomorphism, see [33]. For X projective over any algebraically closed field, see Mallick [36].

We expect that Levine's "cycle map" $c\ell^{\mathrm{cdh}}$ is surjective with uniquely divisible kernel (probably representable by a unipotent group).

Appendices

Appendix A. Homological algebra

A.1. **Some comparison lemmas.** The following lemma is probably well-known:

A.1.1. **Lemma.** *Let $T : \mathcal{T} \to \mathcal{T}'$ be a full triangulated functor between two triangulated categories. Then T is conservative if and only if it is faithful.*

Proof. "If" is obvious. For "only if", let $f : X \to X'$ be a morphism of $\mathcal{T}$ such that $T(f) = 0$. Let $g : X' \to X''$ denote a cone of f. Then $T(g)$ has a retraction ρ. Applying fullness, we get an equality $\rho = T(r)$. Applying conservativity, $u = rg$ is an isomorphism. Then $r' = u^{-1}r$ is a retraction of g, which implies that $f = 0$. □

A.1.2. **Proposition.** *a) Let $i : \underline{\mathrm{E}} \hookrightarrow \underline{\mathrm{A}}$ be an exact full sub-category of an abelian category $\underline{\mathrm{A}}$, closed under kernels. Assume further that for each $A^\bullet \in C^b(\underline{\mathrm{A}})$ there exists $E^\bullet \in C^b(\underline{\mathrm{E}})$ and a quasi-isomorphism $i(E^\bullet) \to A^\bullet$ in $K^b(\underline{\mathrm{A}})$. Then $i : D^b(\underline{\mathrm{E}}) \to D^b(\underline{\mathrm{A}})$ is an equivalence of categories.*
b) The hypothesis of a) is granted when every object in $\underline{\mathrm{A}}$ has a finite left resolution by objects in $\underline{\mathrm{E}}$.

Proof. a) Clearly, the functor $D^b(\underline{\mathrm{E}}) \to D^b(\underline{\mathrm{A}})$ is conservative. The assumption implies that i is essentially surjective: thanks to Lemma A.1.1, in order to conclude it remains to see that i is full.

Let $f \in D^b(\underline{\mathrm{A}})(i(D^\bullet), i(E^\bullet))$. Since $D^b(\underline{\mathrm{A}})$ has left calculus of fractions there exists a quasi-isomorphism s such that $f = f's^{-1}$ where $f' : A^\bullet \to i(E^\bullet)$ is a map in $K^b(\underline{\mathrm{A}})$, which then lifts to a map in $C^b(\underline{\mathrm{A}})$. By hypothesis there exists $F^\bullet \in C^b(\underline{\mathrm{E}})$ and a quasi-isomorphism $s' : i(F^\bullet) \to A^\bullet$. Set $f'' := f's' : i(F^\bullet) \to i(E^\bullet)$. Then $f = f''(ss')^{-1}$ where $ss' : i(F^\bullet) \to i(D^\bullet)$ is a quasi-isomorphism. By conservativity of i, we are reduced to check fullness for effective maps, *i.e.* arising from true maps in $C^b(\underline{\mathrm{A}})$: this easily follows from the fullness of the functor $C^b(\underline{\mathrm{E}}) \hookrightarrow C^b(\underline{\mathrm{A}})$.

b) This follows by adapting the argument in [25, I, Lemma 4.6]. □

A.1.3. **Proposition.** *Let $\underline{\mathrm{E}} \hookrightarrow \underline{\mathrm{A}}$ be an exact category. Let D be a triangulated category and let $T : D^b(\underline{\mathrm{E}}) \to D$ be a triangulated functor such that*

$$\mathrm{Hom}_{D^b(\underline{\mathrm{E}})}(E', E[i]) \xrightarrow{\cong} \mathrm{Hom}_D(T(E'), T(E[i]))$$

for all $E', E \in \underline{\mathrm{E}}$ and $i \in \mathbb{Z}$. Then T is fully faithful.

Proof. Let $C, C' \in C^b(\underline{\mathrm{E}})$: we want to show that the map

$$\mathrm{Hom}_{D^b(\underline{\mathrm{E}})}(C, C') \to \mathrm{Hom}_D(T(C), T(C'))$$

is bijective. We argue by induction on the lengths of C and C'. □

Finally, we have the following very useful criterion for a full embedding of derived categories, that we learned from Pierre Schapira.

A.1.4. **Proposition** ([30, p. 329, Th. 13.2.8])**.** *Let $\mathcal{A} \hookrightarrow \mathcal{B}$ be an exact full embedding of abelian categories. Assume that, given any monomorphism $X' \hookrightarrow X$ in $\mathcal{B}$, with $X' \in \mathcal{A}$, there exists a morphism $X \to X''$, with $X'' \in \mathcal{A}$, such that the composite morphism $X \to X''$ is a monomorphism. Then the functor*

$$D^*(\mathcal{A}) \to D^*(\mathcal{B})$$

is fully faithful for $* = +, b$.

A.2. **The** Tot **construction.**

A.2.1. **Lemma.** *Let $\mathcal{A}$ be an abelian category and let $\mathcal{A}^{[0,1]}$ be the (abelian) category of complexes of length 1 of objects of $\mathcal{A}$. Then the "total complex" functor induces a triangulated functor*

$$D^*(\mathcal{A}^{[0,1]}) \to D^*(\mathcal{A})$$

for any decoration *.

Proof. We may consider a complex of objects of $\mathcal{A}^{[0,1]}$ as a double complex of objects of $\mathcal{A}$ and take the associated total complex. This yields a functor

$$\mathrm{Tot} : C^*(\mathcal{A}^{[0,1]}) \to D^*(\mathcal{A}).$$

(Note that if we consider a complex of objects of $\mathcal{A}^{[0,1]}$

$$M^\bullet = [L^\bullet \xrightarrow{u^\bullet} G^\bullet]$$

as a map $u^\bullet : L^\bullet \to G^\bullet$ of complexes of $\mathcal{A}$, then $\mathrm{Tot}(M^\bullet)$ coincides with the cone of $u^\bullet$.)

This functor factors through a triangulated functor from $D^*(\mathcal{A}^{[0,1]})$: indeed it is easily checked that a) Tot preserves homotopies, b) the induced functor on $K^*(\mathcal{A}^{[0,1]})$ is triangulated; c) Tot of an acyclic complex is 0 (which follows from a spectral sequence argument). $\square$

A.2.2. **Lemma.** *With notation as in Lemma A.2.1, the set of* q.i. *of* $\mathcal{A}^{[0,1]}$ *enjoys a calculus of left and right fractions within the homotopy category* $K(\mathcal{A}^{[0,1]})$ *(same objects, morphisms modulo the homotopy relation).*

Proof. It is enough to show the calculus of right fractions (for left fractions, replace $\mathcal{A}$ by $\mathcal{A}^{op}$).

a) Let

$$\begin{array}{ccc} & & \widetilde{C} \\ & & {\scriptstyle u}\big\downarrow \\ D & \xrightarrow{f} & C \end{array}$$

be a diagram in $\mathcal{A}^{[0,1]}$. Consider the mapping fibre (= shifted mapping cone) F of the map $\widetilde{C} \oplus D \to C$. The complex F is in general concentrated in degrees $[0, 1, 2]$; however, since $F \to D$ is a q.i. , the truncation $\tau_{<2}F$ is q.i. to F; then $\widetilde{D} = \tau_{<2}F$ fills in the square in $K(\mathcal{A}^{[0,1]})$.

b) Let

$$[C^0 \to C^1] \xrightarrow{f} [D^0 \to D^1] \xrightarrow{u} [\widetilde{D}^0 \to \widetilde{D}^1]$$

be a chain of maps of $\mathcal{A}^{[0,1]}$ such that u is a q.i. and uf is homotopic to 0. Let $s : C^1 \to \widetilde{D}^0$ be a corresponding homotopy. Define $\widetilde{C}^1$ as the fibre product of C^1 and D^0 over $\widetilde{D}^0$ (via s and u^0), $\widetilde{s} : \widetilde{C}^1 \to D^0$ the corresponding map and $\widetilde{C}^0$ the fibre product of C^0 and $\widetilde{C}^1$ over C^1. One then checks that $v : [\widetilde{C}^0 \to \widetilde{C}^1] \to [C^0 \to C^1]$ is a q.i. and that $\widetilde{s}$ defines a homotopy from fv to 0. $\square$

A.2.3. **Remark.** One can probably extend these two lemmas to complexes of a fixed length n by the same arguments: we leave this to the interested reader.

Appendix B. Torsion objects in additive categories

B.1. Additive categories.

B.1.1. **Definition.** Let $\mathcal{A}$ be an additive category, and let A be a subring of $\mathbb{Q}$.
a) We write $\mathcal{A} \otimes A$ for the category with the same objects as $\mathcal{A}$ but morphisms

$$(\mathcal{A} \otimes A)(X, Y) := \mathcal{A}(X, Y) \otimes A.$$

b) We denote by $\mathcal{A}\{A\}$ the full subcategory of $\mathcal{A}$:

$$\{X \in \mathcal{A} \mid \exists n > 0 \text{ invertible in } A, n1_X = 0\}.$$

For $A = \mathbb{Q}$, we write $\mathcal{A}\{A\} = \mathcal{A}_{\text{tors}}$. We say that $X \in \mathcal{A}\{A\}$ is an *A-torsion object* (a torsion object if $A = \mathbb{Q}$).
c) A morphism $f : X \to Y$ in $\mathcal{A}$ is an *A-isogeny* (an *isogeny* if $A = \mathbb{Q}$) if there exists a morphism $g : Y \to X$ and an integer n invertible in A such that $fg = n1_Y$ and $gf = n1_X$. We denote by $\Sigma_A(\mathcal{A})$ the collection of A-isogenies of $\mathcal{A}$.
d) We say that two objects $X, Y \in \mathcal{A}$ are *A-isogenous* if they can be linked by a chain of A-isogenies (not necessarily pointing in the same direction).

B.1.2. **Proposition.** *a) The subcategory $\mathcal{A}\{A\}$ is additive and closed under direct summands.*
b) The A-isogenies $\Sigma_A(\mathcal{A})$ form a multiplicative system of morphisms in $\mathcal{A}$, enjoying calculi of left and right fractions. The corresponding localisation of $\mathcal{A}$ is isomorphic to $\mathcal{A} \otimes A$.

Proof. a) is clear. For b), consider the obvious functor $P : \mathcal{A} \to \mathcal{A} \otimes A$. We claim that

$$\Sigma_A(\mathcal{A}) = \{f \mid P(f) \text{ is invertible.}\}$$

One inclusion is clear. Conversely, let $f : X \to Y$ be such that $P(f)$ is invertible. This means that there exists $\gamma \in (\mathcal{A} \otimes A)(Y, X)$ such that $P(f)\gamma = 1_Y$ and $\gamma P(f) = 1_X$. Choose an integer $m \in A - \{0\}$ such that $m\gamma = P(g_1)$ for some g_1. Then there is another integer $n \in A - \{0\}$ such that

$$n(fg_1 - m1_Y) = 0 \text{ and } n(g_1 f - m1_X) = 0.$$

Taking $g = ng_1$ shows that $f \in \Sigma_A(\mathcal{A})$.

It is also clear that homotheties by nonzero integers of A form a cofinal system in $\Sigma_A(\mathcal{A})$. This shows immediately that we have calculi of left and right fractions.

It remains to show that the induced functor

$$\Sigma_A(\mathcal{A})^{-1}\mathcal{A} \to \mathcal{A} \otimes A$$

is an isomorphism of categories; but this is immediate from the well-known formula, in the presence of calculus of fractions:

$$\Sigma_A(\mathcal{A})^{-1}\mathcal{A}(X,Y) = \varinjlim_{X' \xrightarrow{f} X \in \Sigma} \mathcal{A}(X,Y) = \varinjlim_{X \xrightarrow{n} X, n \in A-\{0\}} \mathcal{A}(X,Y).$$

□

The following lemma is clear.

B.1.3. **Lemma.** *Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$, and suppose that every object of $\mathcal{B}$ is A-isogenous to an object of $\mathcal{A}$. Then $\mathcal{B} \otimes A \xrightarrow{\sim} \mathcal{A} \otimes A$.* □

B.2. **Triangulated categories.** (See [51, A.2.1] for a different treatment.)

B.2.1. **Proposition.** *Let $\mathcal{T}$ be a triangulated category. Then*
a) The subcategory $\mathcal{T}\{A\}$ is triangulated and thick.
b) Let $X \in \mathcal{T}$ and $n \in A - \{0\}$. Then "the" cone X/n of multiplication by n on X belongs to $\mathcal{T}\{A\}$.
c) The localised category $\mathcal{T}/\mathcal{T}\{A\}$ is canonically isomorphic to $\mathcal{T} \otimes A$. In particular, $\mathcal{T} \otimes A$ is triangulated.
d) A morphism f of $\mathcal{T}$ belongs to $\Sigma_A(\mathcal{T})$ if and only if cone$(f) \in \mathcal{T}\{A\}$.

Proof. a) It is clear that $\mathcal{T}\{A\}$ is stable under direct summands; it remains to see that it is triangulated. Let $X, Y \in \mathcal{T}\{A\}$, $f : X \to Y$ a morphism and Z a cone of f. We may assume that $n1_X = n1_Y = 0$. The commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle n=0}\downarrow & & {\scriptstyle n}\downarrow & & {\scriptstyle n=0}\downarrow \\ Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

show that multiplication by n on Z factors through Y; this implies that $n^2 1_Z = 0$.

b) Exactly the same argument as in a) shows that multiplication by n on X/n factors through X, hence that $n^2 1_{X/n} = 0$.

c) Let $f \in \mathcal{T}$ be such that $C := cone(f) \in \mathcal{T}\{A\}$, and let $n > 0$ be such that $n1_C = 0$. The same trick as in a) and b) shows that

there exist factorisations $n = ff' = f''f$, hence that $f \in \Sigma_A(\mathcal{T})$. In particular, f becomes invertible under the canonical (additive) functor $\mathcal{T} \to \mathcal{T} \otimes A$. Hence an induced (additive) functor

$$\mathcal{T}/\mathcal{T}\{A\} \to \mathcal{T} \otimes A$$

which is evidently bijective on objects; b) shows immediately that it is fully faithful.

d) One implication has been seen in the proof of c). For the other, if $f \in \Sigma_A(\mathcal{T})$, then f becomes invertible in $\mathcal{A} \otimes \mathbb{Q}$, hence $cone(f) \in \mathcal{T}\{A\}$ by c). □

B.2.2. **Remark.** As is well-known, the stable homotopy category gives a counterexample to the expectation that in fact $n1_{X/n} = 0$ in b) ($X = S^0$, $n = 2$).

B.2.3. **Lemma.** *Let $0 \to \mathcal{T}' \to \mathcal{T} \to \mathcal{T}'' \to 0$ be a short exact sequence of triangulated categories (by definition, this means that $\mathcal{T}'$ is thick in $\mathcal{T}$ and that $\mathcal{T}''$ is equivalent to $\mathcal{T}/\mathcal{T}'$). Then the sequence*

$$0 \to \mathcal{T}' \otimes A \to \mathcal{T} \otimes A \to \mathcal{T}'' \otimes A \to 0$$

is exact.

Proof. We have to show that the functor

$$a : \frac{\mathcal{T} \otimes A}{\mathcal{T}' \otimes A} \to \mathcal{T}'' \otimes A$$

is an equivalence of categories. Since the left hand side is A-linear, the natural functor

$$\mathcal{T}/\mathcal{T}' \to \mathcal{T} \otimes A/\mathcal{T}' \otimes A$$

canonically extends to a functor

$$b : (\mathcal{T}/\mathcal{T}') \otimes A \to \mathcal{T} \otimes A/\mathcal{T}' \otimes A.$$

It is clear that a and b are inverse to each other. □

B.2.4. **Proposition.** *Let $T : \mathcal{S} \to \mathcal{T}$ be a triangulated functor between triangulated categories. Then T is fully faithful if and only if the induced functors $T\{A\} : \mathcal{S}\{A\} \to \mathcal{T}\{A\}$ and $T \otimes A : \mathcal{S} \otimes A \to \mathcal{T} \otimes A$ are fully faithful.*

Proof. "Only if" is obvious; let us prove "if". Let $X, Y \in \mathcal{S}$: we have to prove that $T : \mathcal{S}(X, Y) \to \mathcal{T}(T(X), T(Y))$ is bijective. We do it in two steps:

1) Y is torsion, say $n1_Y = 0$. The claim follows from the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{S}(X/n[1],Y) & \to & \mathcal{S}(X,Y) & \xrightarrow{n=0} & \mathcal{S}(X,Y) & \to & \mathcal{S}(X/n,Y) \\
{\scriptstyle T}\downarrow\wr & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow\wr \\
\mathcal{T}(T(X)/n[1],T(Y)) & \to & \mathcal{T}(T(X),T(Y)) & \xrightarrow{n=0} & \mathcal{T}(T(X),T(Y)) & \to & \mathcal{T}(T(X)/n,T(Y))
\end{array}$$

and the assumption (see Proposition B.2.1 b)).

2) The general case. Let $n > 0$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{S}(X,Y)/n & \to & \mathcal{S}(X,Y/n) & \to & {}_n\mathcal{S}(X,Y[1]) & \to & 0 \\
 & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow\wr & & {\scriptstyle T}\downarrow & & \\
0 & \to & \mathcal{T}(T(X),T(Y))/n & \to & \mathcal{T}(T(X),T(Y)/n) & \to & {}_n\mathcal{T}(T(X),T(Y)[1]) & \to & 0
\end{array}$$

where the middle isomorphism follows from 1). The snake lemma yields an exact sequence

$$\begin{aligned}
0 \to \mathcal{S}(X,Y)/n &\xrightarrow{T} \mathcal{T}(T(X),T(Y))/n \\
&\to {}_n\mathcal{S}(T(X),T(Y)[1]) \xrightarrow{T} {}_n\mathcal{T}(T(X),T(Y)[1]) \to 0.
\end{aligned}$$

Passing to the limit over n, we get another exact sequence

$$\begin{aligned}
\text{(B.1)}\quad 0 \to \mathcal{S}(X,Y)\otimes A/\mathbb{Z} &\xrightarrow{T} \mathcal{T}(T(X),T(Y))\otimes A/\mathbb{Z} \\
&\to \mathcal{S}(T(X),T(Y)[1])\{A\} \xrightarrow{T} \mathcal{T}(T(X),T(Y)[1])\{A\} \to 0.
\end{aligned}$$

Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{S}(X,Y)\{A\} & \to & \mathcal{S}(X,Y) & \to & \mathcal{S}(X,Y)\otimes A & \to & \mathcal{S}(X,Y)\otimes A/\mathbb{Z} & \to 0 \\
 & & {\scriptstyle T}\downarrow \underline{1} & & {\scriptstyle T}\downarrow \underline{2} & & {\scriptstyle T}\downarrow\wr & & {\scriptstyle T}\downarrow \underline{4} & \\
0 & \to & \mathcal{T}(T(X),T(Y))\{A\} & \to & \mathcal{T}(T(X),T(Y)) & \to & \mathcal{T}(T(X),T(Y))\otimes A & \to & \mathcal{T}(T(X),T(Y))\otimes A/\mathbb{Z} & \to 0
\end{array}$$

where the isomorphism is by assumption. By this diagram and (B.1), $\underline{4}$ is an isomorphism. Using this fact in (B.1) applied with $Y[-1]$, we get that $\underline{1}$ is an isomorphism; then $\underline{2}$ is an isomorphism by the 5 lemma, as desired. $\square$

B.3. **Torsion objects in an abelian category.** The proof of the following proposition is similar to that of Proposition B.2.1 and is left to the reader.

B.3.1**. Proposition.** *Let $\mathcal{A}$ be an abelian category. Then*
a) The full subcategory $\mathcal{A}\{A\}$ is thick (a Serre subcategory, in another terminology).

b) Let $X \in \mathcal{A}$ and $n > 0$ invertible in A. Then the kernel and cokernel of multiplication by n on X belong to $\mathcal{A}\{A\}$.
c) The localised category $\mathcal{A}/\mathcal{A}\{A\}$ is canonically isomorphic to $\mathcal{A} \otimes A$. In particular, $\mathcal{A} \otimes A$ is abelian.
d) A morphism $f \in \mathcal{A}$ is in $\Sigma_A(\mathcal{A})$ if and only if $\operatorname{Ker} f \in \mathcal{A}\{A\}$ *and* $\operatorname{Coker} f \in \mathcal{A}\{A\}$. □

The following corollary is a direct consequence of Proposition B.3.1 and Lemma B.1.3:

B.3.2. **Corollary.** *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$, and suppose that every object of $\mathcal{B}$ is A-isogenous to an object of $\mathcal{A}$ (see Definition B.1.1). Then $\mathcal{B} \otimes A$ is abelian, and in particular idempotent-complete.* □

B.4. **Abelian and derived categories.**

B.4.1. **Proposition.** *Let $\mathcal{A}$ be an abelian category. Then the natural functor $D^b(\mathcal{A}) \to D^b(\mathcal{A} \otimes A)$ induces an equivalence of categories*

$$D^b(\mathcal{A}) \otimes A \xrightarrow{\sim} D^b(\mathcal{A} \otimes A).$$

In particular, $D^b(\mathcal{A}) \otimes A$ is idempotent-complete.

Proof. In 3 steps:

1) The natural functor $C^b(\mathcal{A}) \otimes A \to C^b(\mathcal{A} \otimes A)$ is an equivalence of categories. Full faithfulness is clear. For essential surjectivity, take a bounded complex C of objects of $\mathcal{A} \otimes A$. Find a common denominator to all differentials involved in C. Then the corresponding morphisms of $\mathcal{A}$ have torsion composition; since they are finitely many, we may multiply by a common bigger integer so that they compose to 0. The resulting complex of $C^b(\mathcal{A})$ then becomes isomorphic to C in $C^b(\mathcal{A} \otimes A)$.

2) The functor of 1) induces an equivalence of categories $K^b(\mathcal{A}) \otimes A \xrightarrow{\sim} K^b(\mathcal{A} \otimes A)$. Fullness is clear, and faithfulness is obtained by the same technique as in a).

3) The functor of 2) induces the desired equivalence of categories. First, the functor

$$D^b(\mathcal{A})/D^b_{\mathcal{A}\{A\}}(\mathcal{A}) \to D^b(\mathcal{A}/\mathcal{A}\{A\})$$

is obviously conservative. But clearly $D^b_{\mathcal{A}\{A\}}(\mathcal{A}) = D^b(\mathcal{A})\{A\}$. Hence, by Propositions B.2.1 and B.3.1, this functor translates as

$$D^b(\mathcal{A}) \otimes A \to D^b(\mathcal{A} \otimes A).$$

Let $A^b(\mathcal{A})$ denote the thick subcategory of $K^b(\mathcal{A})$ consiting of acyclic complexes. By Lemma B.2.3 we have a commutative diagram of exact

sequences of triangulated categories

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^b(\mathcal{A}) \otimes A & \longrightarrow & K^b(\mathcal{A}) \otimes A & \longrightarrow & D^b(\mathcal{A}) \otimes A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^b(\mathcal{A} \otimes A) & \longrightarrow & K^b(\mathcal{A} \otimes A) & \longrightarrow & D^b(\mathcal{A} \otimes A) & \longrightarrow & 0. \end{array}$$

We have just seen that the right vertical functor is conservative, and by b), the middle one is an equivalence. Hence the left one is essentially surjective, and the result follows. □

Appendix C. 1-motives with torsion

Effective 1-motives which admit torsion are introduced in [4, §1] (in characteristic 0). We investigate some properties (over a perfect field of exponential characteristics $p > 1$) which are not included in op. cit. as a supplement to our Sect. 1.

C.1. **Effective 1-motives.** An *effective* 1*-motive with torsion* over k is a complex of group schemes $M = [L \xrightarrow{u} G]$ where L is finitely generated locally constant for the étale topology *i.e.* a discrete sheaf of Def. 1.1.1, and G is a semi-abelian k-scheme. Therefore L can be represented by an extension

$$0 \to L_{\mathrm{tor}} \to L \to L_{\mathrm{fr}} \to 0$$

where L_{tor} is a finite étale k-group scheme and L_{fr} is free, *i.e.* a lattice. Also G can be represented by an extension of an abelian k-scheme A by a k-torus T.

C.1.1. **Definition.** An *effective* map from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ L' & \xrightarrow{u'} & G' \end{array}$$

in the category of group schemes. Denote by $(f, g) : M \to M'$ such a map. The natural composition of squares makes up a category, denoted by ${}^t\mathcal{M}_1^{\mathrm{eff}}$. We will denote by $\mathrm{Hom}_{\mathrm{eff}}(M, M')$ the abelian group of effective morphisms.

For a given 1-motive $M = [L \xrightarrow{u} G]$ we have (in the abelian category of commutative group schemes) a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}(u) \cap L_{\mathrm{tor}} \arrow[r] \arrow[d] & L_{\mathrm{tor}} \arrow[r, "u"] \arrow[d] & u(L_{\mathrm{tor}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}(u) \arrow[r] & L \arrow[r, "u"] \arrow[d] & G \arrow[d] & \\
 & & L_{\mathrm{fr}} \arrow[r, "\overline{u}"] \arrow[d] & G/u(L_{\mathrm{tor}}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{C.1}$$

with exact rows and columns. We set

- $M_{\mathrm{fr}} := [L_{\mathrm{fr}} \xrightarrow{\overline{u}} G/u(L_{\mathrm{tor}})]$
- $M_{\mathrm{tor}} := [\operatorname{Ker}(u) \cap L_{\mathrm{tor}} \to 0]$
- $M_{\mathrm{tf}} := [L/\operatorname{Ker}(u) \cap L_{\mathrm{tor}} \xrightarrow{u} G]$

considered as effective 1-motives. From Diagram (C.1) there are canonical effective maps $M \to M_{\mathrm{tf}}$, $M_{\mathrm{tor}} \to M$ and $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$.

C.1.2. **Definition.** A 1-motive $M = [L \xrightarrow{u} G]$ is *free* if L is free, *i.e.* if $M = M_{\mathrm{fr}}$. M is *torsion* if L is torsion and $G = 0$, *i.e.* if $M = M_{\mathrm{tor}}$, and *torsion-free* if $\operatorname{Ker}(u) \cap L_{\mathrm{tor}} = 0$, *i.e.* if $M = M_{\mathrm{tf}}$.

Denote by ${}^t\mathcal{M}_1^{\mathrm{eff,fr}}$, ${}^t\mathcal{M}_1^{\mathrm{eff,tor}}$ and ${}^t\mathcal{M}_1^{\mathrm{eff,tf}}$, the full sub-categories of ${}^t\mathcal{M}_1^{\mathrm{eff}}$ given by free, torsion and torsion-free 1-motives respectively.

The category ${}^t\mathcal{M}_1^{\mathrm{eff,fr}}$ is nothing else than the category $\mathcal{M}_1$ of Deligne 1-motives and we shall henceforth use this notation. It is clear that ${}^t\mathcal{M}_1^{\mathrm{eff,tor}}$ is equivalent to the category of finite étale group schemes. If M is torsion-free then Diagram (C.1) is a pull-back, *i.e.* L is the pull-back of L_{fr} along the isogeny $G \to G/L_{\mathrm{tor}}$.

C.1.3. **Proposition.** *The categories ${}^t\mathcal{M}_1^{\mathrm{eff}}$ and $\mathcal{M}_1$ have all finite limits and colimits.*

Proof. Since these are additive categories (with biproducts), it is enough to show that they have kernels, dually cokernels. Now let $\varphi = (f, g) : M \to M'$ be an effective map. We claim that

$$\operatorname{Ker}\varphi = [\operatorname{Ker}^0(f) \xrightarrow{u} \operatorname{Ker}^0(g)]$$

is the pull-back of $\mathrm{Ker}^0(g)$ along u, where $\mathrm{Ker}^0(g)$ is the (reduced) connected component of the identity of the kernel of $g : G \to G'$ and $\mathrm{Ker}^0(f) \subseteq \mathrm{Ker}(f)$. We have to show that the following diagram of effective 1-motives

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}^0(f) \arrow[r, "u"] \arrow[d] & \mathrm{Ker}^0(g) \arrow[d] \\
L \arrow[r, "u"] \arrow[d, "f"'] & G \arrow[d, "g"'] \\
L' \arrow[r, "u'"] & G'
\end{tikzcd}$$

satisfies the universal property for kernels. Suppose that $M'' = [L'' \xrightarrow{u''} G'']$ is mapping to M in such a way that the composition $M'' \to M \to M'$ is the zero map. Then L'' maps to $\mathrm{Ker}(f)$ and G'' maps to $\mathrm{Ker}(g)$. Since G'' is connected, it actually maps to $\mathrm{Ker}^0(g)$ and, by the universal property of pull-backs in the category of group schemes, L'' then maps to $\mathrm{Ker}^0(f)$. Finally note that if L is free then also $\mathrm{Ker}^0(f)$ is free.

For cokernels, we see that

$$[\mathrm{Coker}(f) \xrightarrow{\overline{u}'} \mathrm{Coker}(g)]$$

is an effective 1-motive which is clearly a cokernel of φ.

For $\mathcal{M}_1$, it is enough to take the free part of the cokernel, *i.e.* given $(f, g) : M \to M'$ then $[\mathrm{Coker}(f) \to \mathrm{Coker}(g)]_{\mathrm{fr}}$ meets the universal property for coker of free 1-motives. $\square$

C.2. **Quasi-isomorphisms.** (*cf.* [4, §1]).

C.2.1**. Definition.** An effective morphism of 1-motives $M \to M'$, here $M = [L \xrightarrow{u} G]$ and $M' = [L' \xrightarrow{u} G']$, is a *quasi-isomorphism* (q.i. for

short) of 1-motives if it yields a pull-back diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F \arrow[r, equal] \arrow[d] & F \arrow[d] \\
L \arrow[r, "u"] \arrow[d] & G \arrow[d] \\
L' \arrow[r, "u'"] \arrow[d] & G' \arrow[d] \\
0 & 0
\end{tikzcd} \tag{C.2}$$

where F is a finite étale group.

C.2.2. **Remarks.** 1) Note that kernel and cokernel of a quasi-isomorphism of 1-motives are 0 but, in general, a quasi-isomorphism is not an isomorphism in ${}^t\mathcal{M}_1^{\mathrm{eff}}$. Hence the category ${}^t\mathcal{M}_1^{\mathrm{eff}}$ is not abelian.
2) A q.i. of 1-motives $M \to M'$ is actually a q.i. of complexes of group schemes. In fact, an effective map of 1-motives $M \to M'$ is a q.i. of complexes if and only if we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(u) \arrow[r] \arrow[d, equal] & L \arrow[r, "u"] \arrow[d] & G \arrow[r] \arrow[d] & \mathrm{Coker}(u) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(u') \arrow[r] & L' \arrow[r, "u'"] & G' \arrow[r] & \mathrm{Coker}(u') \arrow[r] & 0.
\end{tikzcd}$$

Therefore Ker and Coker of $L \to L'$ and $G \to G'$ are equal. Then $\mathrm{Coker}(G \to G') = 0$, since it is connected and discrete, and $\mathrm{Ker}(G \to G')$ is a finite group. Conversely, Diagram (C.2) clearly yields a q.i. of complexes. In particular, it easily follows that the class of q.i. of 1-motives is closed under composition of effective morphisms.

C.2.3. **Proposition.** *Quasi-isomorphisms are simplifiable on the left and on the right.*

Proof. The assertion "on the right" is obvious since the two components of a q.i. are epimorphisms. For the left, let $\varphi = (f, g) : M \to M'$ and

$\sigma = (s,t) : M' \to \widetilde{M}$ a q.i. such that $\sigma\varphi = 0$. In the diagram

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & G \arrow[d, "g"'] \\ L' \arrow[r, "u'"] \arrow[d, "s"'] & G' \arrow[d, "s"'] \\ \widetilde{L} \arrow[r, "\widetilde{u}"] & \widetilde{G} \end{tikzcd}$$

we have $\widetilde{L} = L'/F$, $\widetilde{G} = G'/F$, for some finite group F, $\mathrm{Im}(f) \subseteq F$ and $\mathrm{Im}(g) \subseteq F$. Now u' restricts to the identity on F thus $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$ and $\mathrm{Im}(g) = 0$, since $\mathrm{Im}(g)$ is connected, hence $\varphi = 0$. □

C.2.4. **Proposition.** *The class of* q.i. *admits a calculus of right fractions in the sense of (the dual of)* [20, Ch. I, §2.3].

Proof. By [4, Lemma 1.2], the first condition of calculus of right fractions is verified, and Proposition C.2.3 shows that the second one is verified as well. (Note that we here only consider isogenies with étale kernel.) □

C.2.5. **Remark.** The example of the diagram

$$\begin{tikzcd} {[L \to G]} \arrow[r, "\sigma"] \arrow[d, "(1,0)"'] & {[L' \to G']} \\ {[L \to 0]} & \end{tikzcd}$$

where σ is a nontrivial q.i. shows that calculus of left fractions fails in general.

C.2.6. **Lemma.** *Let s, t, u be three maps in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, with $su = t$. If s and t are* q.i. *, then so is u.* □

Proof. Consider the exact sequence of complexes of sheaves

$$0 \to \mathrm{Ker}\, u \to \mathrm{Ker}\, t \to \mathrm{Ker}\, s \to \mathrm{Coker}\, u \to \mathrm{Coker}\, t \to \mathrm{Coker}\, s \to 0.$$

Since s and t are q.i. , the last two terms are 0. Hence $\mathrm{Coker}(u) = [L \to G]$ is a quotient of $\mathrm{Ker}(s)$; since G is connected, we must have $G = 0$. On the other hand, as a cokernel of a map of acyclic complexes of length 1, $\mathrm{Coker}(u)$ is acyclic, hence $L = 0$. Similarly, $\mathrm{Ker}(u)$ is acyclic. □

C.3. **1-motives.** We now define the category of 1-motives with torsion from ${}^t\mathcal{M}_1^{\mathrm{eff}}$ by formally inverting quasi-isomorphisms.

C.3.1. **Definition.** The category ${}^t\mathcal{M}_1$ of 1-*motives with torsion* is the localisation of ${}^t\mathcal{M}_1^{\mathrm{eff}}$ with respect to the multiplicative class $\{\text{q.i.}\}$ of quasi-isomorphisms.

C.3.2. **Remark.** Note that there are no nontrivial q.i. between free (or torsion) 1-motives. However, the canonical map $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$ is a quasi-isomorphism (it is an effective isomorphism when $u(L_{\mathrm{tor}}) = 0$).

It follows from Proposition C.2.4 that the Hom sets in ${}^t\mathcal{M}_1$ are given by the formula

$$\mathrm{Hom}(M, M') = \varinjlim_{\text{q.i.}} \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M')$$

where the limit is taken over the filtering set of all quasi-isomorphisms $\widetilde{M} \to M$. A morphism of 1-motives $M \to M'$ can be represented as a diagram

$$\begin{tikzcd} M & & M' \\ & \widetilde{M} \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & \end{tikzcd}$$

and the composition is given by the existence of a $\widehat{M}$ making the following diagram

$$\begin{tikzcd} M & & M' & & M'' \\ & \widetilde{M} \arrow[ul, "\text{q.i.}"] \arrow[ur] & & \widetilde{M'} \arrow[ul] \arrow[ur, "\text{eff}"'] & \\ & & \widehat{M} \arrow[ul, "\text{q.i.}"] \arrow[ur, "\text{eff}"'] & & \end{tikzcd}$$

commutative. (This $\widehat{M}$ is in fact unique, see [4, Lemma 1.2].)

C.4. **Strict morphisms.** The notion of strict morphism is essential in order to show that the $\mathbb{Z}[1/p]$-linear category of 1-motives is abelian (*cf.* [4, §1]).

C.4.1. **Definition.** We say that an effective morphism $(f, g) : M \to M'$ is *strict* if we have

$$\mathrm{Ker}(f, g) = [\mathrm{Ker}(f) \to \mathrm{Ker}(g)]$$

i.e. if $\mathrm{Ker}(g)$ is (connected) semiabelian.

To get a feeling on the notion of strict morphism, note:

C.4.2. **Lemma.** *Let $\varphi = (f, g) : M \to N$ be a strict morphism, with g onto. Suppose that $\varphi = \sigma\widetilde{\varphi}$, where σ is a* q.i. *Then σ is an isomorphism.*

Conversely, we obtain:

C.4.3. **Proposition** ([4, Prop. 1.3])**.** *Any effective morphism $\varphi \in \mathrm{Hom}_{\mathrm{eff}}(M, M')$ can be factored as follows*

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & & M' \\ & \widetilde{M} \arrow[ur] & \end{tikzcd} \tag{C.3}$$

where $\widetilde{\varphi}$ is a strict morphism and $\widetilde{M} \to M'$ is a q.i. *or a p-power isogeny.*

Proof. (Sketch) Note that if $\varphi = (f, g)$ we always have the following natural factorisation of the map g between semi-abelian schemes

$$\begin{tikzcd} G \arrow[rr, "g"] \arrow[dr] & & G' \\ & G/\operatorname{Ker}^0(g) \arrow[ur] & \end{tikzcd}$$

If g is a surjection we get the claimed factorisation by taking $\widetilde{M} = [\widetilde{L} \to G/\operatorname{Ker}^0(g)]$ where $\widetilde{L}$ is the pull-back of L', the lifting of f is granted by the universal property of pull-backs. In general, we can extend the so obtained isogeny on the image of g to an isogeny of G' (see the proof of Prop. 1.3 in [4] for details). □

C.4.4. **Lemma.** *Let*

$$\begin{tikzcd} M' \arrow[r, "f"] & M \\ N' \arrow[u, "u"] \arrow[r, "h"] & N \arrow[u, "t"'] \end{tikzcd}$$

be a commutative diagram in ${}^t\mathcal{M}_1^{\mathrm{eff}}$, where f is strict and u, t are q.i. *Then the induced map $v : \mathrm{Coker}(h) \to \mathrm{Coker}(f)$ is a* q.i.

Proof. In all this proof, the term "kernel" is taken in the sense of kernel of complexes of sheaves. Let K and K' be the kernels of f and h respectively:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & M' \arrow[r, "f"] & M \arrow[r] & \mathrm{Coker}(f) \\ 0 \arrow[r] & K' \arrow[r] \arrow[u, "w"] & N' \arrow[r, "h"] \arrow[u, "u"] & N \arrow[r] \arrow[u, "t"] & \mathrm{Coker}(h) \arrow[u, "v"] \end{tikzcd}$$

By a diagram chase, we see that v is onto and that $\operatorname{Ker} t \to \operatorname{Ker} v$ is surjective. To conclude, it will be sufficient to show that the sequence of complexes

$$\operatorname{Ker}(u) \to \operatorname{Ker}(t) \to \operatorname{Ker}(v) \tag{C.4}$$

is exact termwise. For this, note that the second component of w is onto because f is strict and by dimension reasons. This implies by a diagram chase that the second component of (C.4) is exact. But then the first component has to be exact too. □

C.5. **Exact sequences of 1-motives.** We have the following basic properties of 1-motives.

C.5.1. **Proposition.** *The canonical functor*

$${}^t\mathcal{M}_1^{\mathrm{eff}} \to {}^t\mathcal{M}_1$$

is left exact and faithful.

Proof. Faithfulness immediately follows from Proposition C.2.3, while left exactness follows from Proposition C.2.4 and (the dual of) [20, Ch. I, Prop. 3.1]. □

C.5.2. **Lemma.** *Let $f : M' \to M$ be an effective map.*

(1) *The canonical projection $\pi : M \to \operatorname{Coker}(f)$ remains an epimorphism in ${}^t\mathcal{M}_1[1/p]$.*
(2) *If f is strict then π remains a cokernel in ${}^t\mathcal{M}_1[1/p]$.*
(3) *Cokernels exist in ${}^t\mathcal{M}_1[1/p]$.*

Proof. To show Part 1 let $\pi : M \to N$ be an effective map. One sees immediately that π is epi in ${}^t\mathcal{M}_1[1/p]$ if and only if for any commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N \\ {\scriptstyle s'}\uparrow & & {\scriptstyle s}\uparrow \\ Q' & \xrightarrow{\pi'} & Q \end{array}$$

with s, s' q.i., the map π' is an epi in the effective category. Now specialise to the case $N = \operatorname{Coker}(f)$ and remark that (up to modding out by $\operatorname{Ker} f$) we may assume f to be a monomorphism as a map of complexes, thus strict. Take π', s, s' as above. We have a commutative

diagram of effective maps

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[rr, "\pi"] & & \mathrm{Coker}(f) \\
& & & & \\
& & \mathrm{Coker}(f') \arrow[uurr, "t"] & & \\
& Q'' \arrow[uu, "s''"] \arrow[r, "f'"] & Q' \arrow[uu, "s'"] \arrow[uur] \arrow[rr, "\pi'"] & & Q. \arrow[uu, "s"] \arrow[uul, "u"']
\end{tikzcd}$$

Here s'' is a q.i. and Q'', f', s'' are obtained by calculus of right fractions (Proposition C.2.4). By Lemma C.4.4, the induced map $t : \mathrm{Coker}(f') \to \mathrm{Coker}(f)$ is a q.i. By Proposition C.2.3, $\pi' f' = 0$, hence the existence of u. By Lemma C.2.6, u is a q.i.. Hence π' is a composition of two epimorphisms and Part 1 is proven.

To show Part 2, let $gt^{-1} : M \to M''$ be such that the composition $M' \to M''$ is zero. By calculus of right fractions we have a commutative diagram

$$\begin{tikzcd}
M' \arrow[r, "f"] & M \arrow[r, "\pi"] & \mathrm{Coker}(f) \\
N''' \arrow[u, "u"] \arrow[r, "h"] & N'' \arrow[u, "t"] \arrow[r] & \mathrm{Coker}(h) \arrow[u, "v"] \arrow[d] \\
& & M'' \arrow[ul, "g"']
\end{tikzcd}$$

where all maps are effective and u is a q.i.. As above we have $gh = 0$, hence the factorisation of g through $\mathrm{Coker}(h)$. Moreover $\mathrm{Coker}(h)$ maps canonically to $\mathrm{Coker}(f)$ via a map v (say), which is a q.i. by Lemma C.4.4. This shows that gt^{-1} factors through $\mathrm{Coker}(f)$ in ${}^t\mathcal{M}_1[1/p]$. Uniqueness of the factorisation is then granted by Part 1.

In a category, the existence of cokernels is invariant by left or right composition by isomorphisms, hence Part 3 is a consequence of Parts 1 and 2 via Proposition C.4.3. □

Now we can show the following key result (*cf.* [4, Prop. 1.3]).

C.5.3. **Theorem.** *The category ${}^t\mathcal{M}_1[1/p]$ is abelian.*

Proof. Existence and description of kernels follows from Propositions C.1.3, C.2.4, C.5.1 and (the dual of) [20, Ch. I, Cor. 3.2], while existence of cokernels has been proven in Lemma C.5.2. We are then left to show that, for any (effective) strict map $\varphi : M \to M'$, the

canonical effective morphism from the coimage of φ to the image of φ is a q.i. of 1-motives, *i.e.* the canonical morphism

$$\operatorname{Coker}(\operatorname{Ker}\varphi \to M) \to \operatorname{Ker}(M' \to \operatorname{Coker}\varphi) \tag{C.5}$$

is a quasi-isomorphism. Since we can split φ in two short exact sequences of complexes in which each term is an effective 1-motive we see that (C.5) is even a isomorphism in ${}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$.

□

C.5.4. **Remark.** Note that (even in characteristic zero) for a given non-strict effective map $(f,g) : M \to M'$ the effective morphism (C.5) is not a q.i. of 1-motives. In fact, the following diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\operatorname{Ker}(f)/\operatorname{Ker}^0(f) \arrow[d] & \subseteq & \operatorname{Ker}(g)/\operatorname{Ker}^0(g) \arrow[d] \\
L/\operatorname{Ker}^0(f) \arrow[r] \arrow[d] & & G/\operatorname{Ker}^0(g) \arrow[d] \\
\operatorname{Im}(f) \arrow[r] \arrow[d] & & \operatorname{Im}(g) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

is not a pull-back, in general. For example, let $g : G \to G'$ be with finite kernel and a proper sub-group $F \subsetneq \operatorname{Ker}(g)$, and consider

$$(0,g) : [F \to G] \to [0 \to G'].$$

C.5.5. **Corollary.** *A short exact sequence of* 1*-motives in* ${}^t\mathcal{M}_1[1/p]$

$$0 \to M' \to M \to M'' \to 0 \tag{C.6}$$

can be represented up to isomorphisms by a strict effective epimorphism $(f,g) : M \to M''$ *with kernel* M'*, i.e. by an exact sequence of complexes.*

C.5.6. **Example.** Let M be a 1-motive with torsion. We then always have a canonical exact sequence in ${}^t\mathcal{M}_1[1/p]$

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{fr}} \to 0 \tag{C.7}$$

induced by (C.1), according to Definition C.1.2. Note that in the following canonical factorisation

$$\begin{array}{ccc} M & \longrightarrow & M_{\mathrm{fr}} \\ & \searrow \quad \nearrow & \\ & M_{\mathrm{tf}} & \end{array}$$

the effective map $M \to M_{\mathrm{tf}}$ is a strict epimorphism with kernel M_{tor} and $M_{\mathrm{tf}} \to M_{\mathrm{fr}}$ is a q.i. (providing an example of Proposition C.4.3).

C.6. **ℓ-adic realisation.** Let $n : M \to M$ be the (effective) multiplication by n on a 1-motive $M = [L \xrightarrow{u} G]$ over a field k where n is prime to the characteristic of k. It is then easy to see, *e.g.* by the description of kernels in Proposition C.1.3, that

$$_nM := \mathrm{Ker}(M \xrightarrow{n} M) = [\mathrm{Ker}(u) \cap {}_nL \to 0].$$

Thus $_nM = 0$ (all n in characteristic zero) if and only if M is torsion-free, *i.e.* $M_{\mathrm{tor}} = 0$. Moreover, by Proposition C.4.3 and Lemma C.5.2, we can see that

$$M/n := \mathrm{Coker}(M \xrightarrow{n} M)$$

is always a torsion 1-motive. If $L = 0$, let simply G denote, as usual, the 1-motive $[0 \to G]$. Then we get an extension in $\mathcal{M}_1[1/p]$

$$0 \to G \xrightarrow{n} G \to {}_nG[1] \to 0 \tag{C.8}$$

where $_nG[1]$ is the torsion 1-motive $[_nG \to 0]$. If $L \neq 0$ then M/n can be regarded as an extension of L/n by $\mathrm{Coker}(_nL \to {}_nG)$, *e.g.* also by applying the snake lemma to the multiplication by n on the following canonical short exact sequence (here $L[1] = [L \to 0]$ as usual)

$$0 \to G \to M \to L[1] \to 0 \tag{C.9}$$

of effective 1-motives (which is also exact in $\mathcal{M}_1[1/p]$ by Corollary C.5.5). Summarizing up from (C.8), (C.9) we then get a long exact sequence in ${}^t\mathcal{M}_1^{\mathrm{tor}}[1/p]$

$$0 \to {}_nM \to {}_nL[1] \to {}_nG[1] \to M/n \to L/n[1] \to 0. \tag{C.10}$$

Let now be $n = \ell^\nu$ where $\ell \neq \mathrm{char}(k)$. Set:

C.6.1. **Definition.** The *ℓ-adic realisation* of a 1-motive M is

$$T_\ell(M) := \text{“}\varprojlim_{\nu}\text{”}\, L_\nu$$

in the category of l-adic sheaves, where $M/\ell^\nu = [L_\nu \to 0]$.

Since the inverse system "$\varprojlim_\nu$"$_{l^\nu}L$ is Mittag-Leffler trivial, we obtain a short exact sequence

$$0 \to T_\ell(G) \to T_\ell(M) \to L \otimes \mathbb{Z}_\ell \to 0$$

where $T_\ell(G)$ is the Tate module of the semiabelian variety G. More generally, using Corollary C.5.5, we have:

C.6.2. **Lemma.** *The functor T_ℓ is exact on ${}^t\mathcal{M}_1[1/p]$, and extends canonically to ${}^t\mathcal{M}_1 \otimes \mathbb{Z}_\ell$.* □

C.7. **Deligne 1-motives.** Let ${}^t\mathcal{M}_1^{\mathrm{fr}}[1/p]$, ${}^t\mathcal{M}_1^{\mathrm{tor}}[1/p]$ and ${}^t\mathcal{M}_1^{\mathrm{tf}}[1/p]$ denote the corresponding full sub-categories of ${}^t\mathcal{M}_1[1/p]$ given by free, torsion and torsion-free effective 1-motives respectively. The following $M \mapsto M_{\mathrm{fr}}$ (resp. $M \mapsto M_{\mathrm{tor}}$) define functors from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1^{\mathrm{fr}}[1/p]$ (resp. from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1^{\mathrm{tor}}[1/p]$). We have (*cf.* [4, (1.1.3)]):

C.7.1. **Proposition.** *The natural functor*

$$\mathcal{M}_1[1/p] \to {}^t\mathcal{M}_1[1/p]$$

from Deligne 1-motives to 1-motives with torsion has a left adjoint/left inverse given by $M \mapsto M_{\mathrm{fr}}$. In particular, it is fully faithful and makes $\mathcal{M}_1[1/p]$ an exact sub-category of ${}^t\mathcal{M}_1[1/p]$ in the sense of Quillen [47, §2]. *The above left adjoint defines equivalences*

$$\mathcal{M}_1[1/p] \cong {}^t\mathcal{M}_1^{\mathrm{fr}}[1/p] \cong {}^t\mathcal{M}_1^{\mathrm{tf}}[1/p].$$

Proof. Consider an effective map $(f,g) : \widetilde{M} \to M'$, to a free 1-motive M', and a q.i. $\widetilde{M} \to M$, *i.e.* $M = [\widetilde{L}/F \to \widetilde{G}/F]$ for a finite group F. Since M' is free then F is contained in the kernel of f and the same holds for g. Thus (f,g) induces an effective map $M \to M'$. Let $M = [L \xrightarrow{u} G]$. Then $L_{\mathrm{tor}} \subseteq \mathrm{Ker}(f)$ and also $u(L_{\mathrm{tor}}) \subseteq \mathrm{Ker}(g)$ yielding an effective map $(f,g) : M_{\mathrm{fr}} \to M'$. This proves the first assertion.

Since ${}^t\mathcal{M}_1^{\mathrm{eff,fr}} \hookrightarrow {}^t\mathcal{M}_1^{\mathrm{eff,tf}}$, the claimed equivalence is obtained from the canonical q.i. $M \to M_{\mathrm{fr}}$ for $M \in {}^t\mathcal{M}_1^{\mathrm{eff,tf}}$, see (C.1). Finally, consider the exact sequence (C.6) of 1-motives with torsion such that $M'_{\mathrm{tor}} = M''_{\mathrm{tor}} = 0$. Since M_{tor} is mapped to zero in M'', it injects in M'. Thus also M is torsion-free, *i.e.* $M = M_{\mathrm{tf}}$, and quasi-isomorphic to M_{fr}. □

C.7.2. **Remark.** We also clearly have that the functor $M \mapsto M_{\mathrm{tor}}$ is a right-adjoint to the embedding ${}^t\mathcal{M}_1^{\mathrm{tor}}[1/p] \hookrightarrow {}^t\mathcal{M}_1[1/p]$, *i.e.*

$$\mathrm{Hom}_{\mathrm{eff}}(M, M'_{\mathrm{tor}}) \cong \mathrm{Hom}(M, M')$$

for $M \in {}^t\mathcal{M}_1^{\mathrm{tor}}[1/p]$ and $M' \in {}^t\mathcal{M}_1[1/p]$.

C.7.3. **Corollary.** *We have* ${}^t\mathcal{M}_1^{\mathrm{tor}} \otimes \mathbb{Q} = 0$ *and the full embedding* $\mathcal{M}_1[1/p] \to {}^t\mathcal{M}_1[1/p]$ *induces an equivalence*

$$\mathcal{M}_1 \otimes \mathbb{Q} \xrightarrow{\sim} {}^t\mathcal{M}_1 \otimes \mathbb{Q}.$$

C.8. **Non-connected 1-motives.** We consider a larger category allowing non-connected (reduced) group schemes as a supplement of Proposition 1.1.5.

C.8.1. **Definition.** Let $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ denote the following category. The objects are $N = [L \to G]$ complexes of étale sheaves over the field k where L is discrete and G is a reduced group scheme locally of finite type over k such that

(i) the connected component of the identity G^0 is semiabelian, and
(ii) $\pi_0(G)$ is finitely generated.

The morphisms are just maps of complexes. We call $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ the category of *effective non-connected* 1-motives.

We denote $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ the full subcategory of $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ whose objects are $N = [L \to G]$ as above such that G is of finite type over k (then condition (ii) is automatically granted and $\pi_0(G)$ is a finite group scheme). We call $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ the category of *algebraic effective non-connected* 1-motives.

Note that a representable presheaf on the category of schemes over k can be characterised by axiomatic methods, including the condition (i) above, *cf.* the Appendix in [5].

Associated to $N = [L \to G]$ we have the following diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
L^0 \arrow[r] \arrow[d] & G^0 \arrow[d] \\
L \arrow[r] \arrow[d] & G \arrow[d] \\
L/L^0 \arrow[r, "\subseteq"] \arrow[d] & \pi_0(G) \arrow[d] \\
0 & 0
\end{tikzcd}$$

here L^0 denote the pull-back of G^0 along $L \to G$. Let

$$N^0 := [L^0 \to G^0]$$

denote the effective 1-motive associated to N and denote

$$\pi_0(N) := [L/L^0 \hookrightarrow \pi_0(G)].$$

We say that $D = [L \hookrightarrow L']$ is *discrete* if L' is a discrete sheaf and L injects into L'. Denote $\mathcal{M}_{\mathrm{nc}}^{\mathrm{dis}}$ the full subcategory of $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ given by discrete objects. Note that N is discrete if and only if $\pi_0(N) = N$ (if and only if $N^0 = 0$).

C.8.2. **Proposition.** *The operation $N \mapsto N^0$ defines a functor*

$$c^{\mathrm{eff}} : \mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}} \to {}^t\mathcal{M}_1^{\mathrm{eff}}$$

which is right adjoint to the embedding $i_1 : {}^t\mathcal{M}_1^{\mathrm{eff}} \hookrightarrow \mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ and $c^{\mathrm{eff}} i_1 = 1$. Moreover, we have a functor

$$\pi_0 : \mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}} \to \mathcal{M}_{\mathrm{nc}}^{\mathrm{dis}}$$

which is left adjoint to $i_{\mathrm{dis}} : \mathcal{M}_{\mathrm{nc}}^{\mathrm{dis}} \hookrightarrow {}^t\mathcal{M}_1^{\mathrm{eff}}$ and $\pi_0 i_{\mathrm{dis}} = 1$.

Proof. Straightforward. □

C.8.3. **Remarks.** 1) The same results as in Proposition C.8.2 above refine to $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ and $\mathcal{M}_{\mathrm{anc}}^{\mathrm{dis}}$.
2) Note that $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ has kernels. Let $\varphi = (f, g) : N \to N'$ be a map in $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$. Let $g^0 : G^0 \to G'^0$ and $\pi_0(g) : \pi_0(G) \to \pi_0(G')$ be the induced maps. Then $\mathrm{Ker}(\varphi) = [\mathrm{Ker}(f) \to \mathrm{Ker}(g)]$ as a complex of sheaves; in fact $\mathrm{Ker}(g)$ is representable, $\mathrm{Ker}^0(g^0) = (\mathrm{Ker}(g))^0$ and $\pi_0(\mathrm{Ker}(g))$ maps to $\mathrm{Ker}(\pi_0(g))$ with finite kernel. However, it is easy to see that $\mathcal{M}_{\mathrm{nc}}^{\mathrm{eff}}$ is not abelian.

C.8.4. **Proposition.** *The category $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}[1/p]$ is abelian.*

Proof. Regard $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}[1/p]$ as a full subcategory of $C^{[-1,0]}(\mathrm{Shv}(k_{\text{ét}}))[1/p]$ of the abelian category of complexes of sheaves concentrated in degree -1 and 0. For a map $\varphi = (f, g) : N \to N'$, $\mathrm{Ker}(\varphi) = [\mathrm{Ker}(f) \to \mathrm{Ker}(g)] \in \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ and $\mathrm{Coker}(\varphi) = [\mathrm{Coker}(f) \to \mathrm{Coker}(g)] \in \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$. For an extension $0 \to N \to N' \to N'' \to 0$ in $C^{[-1,0]}(\mathrm{Shv}(k_{\text{ét}}))$ such that N and N'' belongs to $\mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$ then also $N'' \in \mathcal{M}_{\mathrm{anc}}^{\mathrm{eff}}$. □

C.9. **Homs and Extensions.** We will provide a characterisation of the Yoneda Ext in the abelian category ${}^t\mathcal{M}_1[1/p]$.

C.9.1. **Proposition.** *We have*

(a) $\mathrm{Hom}_{{}^t\mathcal{M}_1}(L[1], L'[1]) = \mathrm{Hom}_k(L, L')$,
(b) $\mathrm{Hom}_{{}^t\mathcal{M}_1}(L[1], G') = 0$,
(c) $\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, G') \subseteq \mathrm{Hom}_k(A, A') \times \mathrm{Hom}_k(T, T')$ *if G (resp. G') is an extension of an abelian variety A by a torus T (resp. of A' by T'),*

(d) $\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_k({}_nG, L'_{\mathrm{tor}})$ *if* $nL'_{\mathrm{tor}} = 0$.

In particular, the group $\mathrm{Hom}_{{}^t\mathcal{M}_1}(M, M')$ *is finitely generated for all* 1*-motives* $M, M' \in {}^t\mathcal{M}_1[1/p]$.

Proof. Since there are no q.i. to $L[1]$, we have $\mathrm{Hom}_{{}^t\mathcal{M}_1}(L[1], L'[1]) = \mathrm{Hom}_{\mathrm{eff}}(L[1], L'[1])$ and the latter is clearly isomorphic to $\mathrm{Hom}_k(L, L')$. By Proposition C.7.1, we have $\mathrm{Hom}_{{}^t\mathcal{M}_1}(L[1], G') = \mathrm{Hom}_{\mathrm{eff}}(L[1], G')$ and $\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, G') = \mathrm{Hom}_{\mathrm{eff}}(G, G')$. The former is clearly 0 while $\mathrm{Hom}_{\mathrm{eff}}(G, G') = \mathrm{Hom}_k(G, G') \subseteq \mathrm{Hom}_k(A, A') \times \mathrm{Hom}_k(T, T')$ since $\mathrm{Hom}_k(T, A') = \mathrm{Hom}_k(A, T') = 0$. For (d), let $[F \to \widetilde{G}] \to [0 \to G]$ be a q.i. and $[F \to \widetilde{G}] \to [L' \to 0]$ be an effective map providing an element of $\mathrm{Hom}(G, L'[1])$. If L' is free then it yields the zero map, as F is torsion. Thus $\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_{{}^t\mathcal{M}_1}(G, L'_{\mathrm{tor}}[1])$. For $n \in \mathbb{N}$ consider the short exact sequence (C.8) in ${}^t\mathcal{M}_1$. If n is such that $nL'_{\mathrm{tor}} = 0$ taking $\mathrm{Hom}_{{}^t\mathcal{M}_1}(-, L'_{\mathrm{tor}}[1])$ we further obtain $\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_k({}_nG, L'_{\mathrm{tor}})$.

The last statement follows from these computations and an easy dévissage from (C.9). □

C.9.2. **Remark.** If we want to get rid of the integer n in (d), we may equally write

$$\mathrm{Hom}_{{}^t\mathcal{M}_1}(G, L'[1]) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{T}(G), L'_{\mathrm{tor}}) = \mathrm{Hom}_{\mathrm{cont}}(\widehat{T}(G), L')$$

where $\widehat{T}(G) = \prod_\ell T_\ell(G)$ is the complete Tate module of G.

C.9.3. **Proposition.** *We have isomorphisms (for* Ext *in* ${}^t\mathcal{M}_1[1/p]$*):*

(a) $\mathrm{Ext}^1_k(L, L') \xrightarrow{\sim} \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(L[1], L'[1])$,

(b) $\mathrm{Hom}_k(L, G') \xrightarrow{\sim} \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(L[1], G')$,

(c) $\mathrm{Ext}^1_k(G, G') \xrightarrow{\sim} \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, G')$ *and*

(d) $\varinjlim_n \mathrm{Ext}^1_k({}_nG, L') \xrightarrow{\sim} \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, L'[1])$*; these two groups are* 0 *if* L' *is torsion.*

Proof. By Corollary C.5.5, any short exact sequence of 1-motives can be represented up to isomorphism by a short exact sequence of complexes in which each term is an effective 1-motive.

For (a), just observe that there are no nontrivial q.i. of 1-motives with zero semiabelian part. For (b), note that an extension of $L[1]$ by G' is given by a diagram

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d, "v"'] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{G}' \arrow[r] & G'' \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{C.11}$$

where $\widetilde{M}' = [F' \to \widetilde{G}']$ is q.i. to $[0 \to G']$. When $F' = 0$, this diagram is equivalent to the datum of v: this provides a linear map $\mathrm{Hom}_k(L, G') \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(L[1], G')$. This map is surjective since we may always mod out by F' in (C.11) and get an quasi-isomorphic exact sequence with $F' = 0$. It is also injective: if (C.11) (with $F' = 0$) splits in ${}^t\mathcal{M}_1[1/p]$, it already splits in ${}^t\mathcal{M}_1^{\mathrm{eff}}[1/p]$ and then $v = 0$.

For (c) we see that an extension of G by G' in ${}^t\mathcal{M}_1[1/p]$ can be represented by a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & L'' & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{G}' & \longrightarrow & G'' & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \end{array}$$

with $\widetilde{M}'$ as in (b) and $\widetilde{M} = [F \to \widetilde{G}]$ q.i. to $[0 \to G]$. Since the top line is exact, L'' is finite. For $F = F' = 0$ we just get a group scheme extension of G by G', hence a homomorphism $\mathrm{Ext}^1_k(G, G') \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, G')$. This homomorphism is surjective: dividing by F' we get a quasi-isomorphic exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & L''/F' & \xrightarrow{\sim} & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & G''/F' & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \end{array}$$

and further dividing by F' we then obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & G''/L'' & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Injectivity is seen as in (b).

For (d) we first construct a map $\Phi_n : \mathrm{Ext}^1_k({}_nG, L') \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, L'[1])$ for all n. Let $[L''] \in \mathrm{Ext}_k({}_nG, L')$ and consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L'' & \longrightarrow & {}_nG & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & G & = & G & \longrightarrow & 0. \end{array} \tag{C.12}$$

Since $[{}_nG \to G]$ is q.i. to $[0 \to G]$, this provides an extension of G by $L'[1]$ in ${}^t\mathcal{M}_1[1/p]$. For n variable $\{\mathrm{Ext}^1_k({}_nG, L')\}_n$ is a direct system and one checks easily that the maps Φ_n are compatible (by pull-back), yielding a well-defined linear map

$$\Phi : \varinjlim \mathrm{Ext}^1_k({}_nG, L') \to \mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, L'[1]).$$

This map is surjective since any extension of G by $L'[1]$ can be represented by a diagram (C.12) for some n (as multiplication by n is cofinal in the direct sytem of isogenies). We now show that Φ is also injective.

Let $n \mid m$, *e.g.* $rn = m$, so that the following sequence is exact

$$0 \to {}_rG \to {}_mG \xrightarrow{r} {}_nG \to 0$$

and yields a long exact sequence

$$\mathrm{Hom}_k({}_mG, L') \to \mathrm{Hom}_k({}_rG, L') \to \mathrm{Ext}^1_k({}_nG, L') \xrightarrow{r} \mathrm{Ext}^1_k({}_mG, L').$$

If L' is torsion, we have $rL' = 0$ for some r, hence $\varinjlim \mathrm{Ext}^1_k({}_nG, L') = 0$. This shows in particular that $\mathrm{Ext}^1_{{}^t\mathcal{M}_1}(G, L'[1]) = 0$ in this case.

Suppose now that L' is free. Then we have $\mathrm{Hom}_k({}_rG, L') = 0$, hence the transition maps are injective. Therefore, to check that Φ is injective it suffices to check that Φ_n is injective for all n.

Let $\sigma : G \to [L'' \to G]$ be a section of (C.12) in ${}^t\mathcal{M}_1[1/p]$. Then σ can be represented by a diagram of effective maps

$$\begin{tikzcd}
 & & {}_{rn}G \arrow[dl] \arrow[d] \arrow[dr, "r"] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & {}_nG \arrow[d] \\
 & & G \arrow[dl] \arrow[dr, "r"] & \\
0 \arrow[r] & 0 \arrow[r] & G \arrow[r, equal] & G
\end{tikzcd}$$

for some r, where the southwest map is a q.i. To say that σ is a section is to say that this diagram commutes. Hence the image of ${}_{rn}G$ in L'' surjects onto ${}_nG$, and it also injects since L' is torsion-free. This means that the projection $L'' \to {}_nG$ has a section, hence $[L''] = 0$ in $\mathrm{Ext}^1_k({}_nG, L'')$.

For a general L', we reduce to these two special cases through an easy diagram chase. □

C.10. **Projective objects in ${}^t\mathcal{M}_1[1/p]$.** We show that there are not enough projective objects in ${}^t\mathcal{M}_1[1/p]$, at least when k is algebraically closed:

C.10.1. **Proposition.** *Suppose that $k = \overline{k}$. Then the only projective object of ${}^t\mathcal{M}_1[1/p]$ is* 0.

Proof. Suppose that $M = [L \to G] \in {}^t\mathcal{M}_1[1/p]$ is such that $\mathrm{Ext}(M, N) = 0$ for any $N \in {}^t\mathcal{M}_1[1/p]$. From (C.9) we then get a long exact

sequence

$$\begin{aligned}\mathrm{Hom}(G,\mathbb{G}_m) \to \mathrm{Ext}(L[1],\mathbb{G}_m) \to \mathrm{Ext}(M,\mathbb{G}_m)\\ \to \mathrm{Ext}(G,\mathbb{G}_m) \to \mathrm{Ext}^2(L[1],\mathbb{G}_m)\end{aligned}$$

where $\mathrm{Ext}(M,\mathbb{G}_m)=0$, thus *i)* $\mathrm{Ext}(G,\mathbb{G}_m)$ is finite, and *ii)* $\mathrm{Hom}(L,\mathbb{G}_m)$ is finitely generated. We also have an exact sequence

$$\mathrm{Hom}(T,\mathbb{G}_m) \to \mathrm{Ext}(A,\mathbb{G}_m) \to \mathrm{Ext}(G,\mathbb{G}_m)$$

where $\mathrm{Hom}(T,\mathbb{G}_m)$ is the character group of the torus T and $\mathrm{Ext}(A,\mathbb{G}_m)$ is the group of k-points of the dual abelian variety A, the abelian quotient of G. From *i)* we get $A=0$. Since $\mathrm{Hom}(L,\mathbb{G}_m)$ is an extension of a finite group by a divisible group, from *ii)* we get that L is a finite group. Now consider the exact sequence, for $l \neq p$

$$\begin{aligned}0 \to \mathrm{Hom}(L[1],\mathbb{Z}/l[1]) \to \mathrm{Hom}(M,\mathbb{Z}/l[1])\\ \to \mathrm{Hom}(T,\mathbb{Z}/l[1]) \to \mathrm{Ext}(L[1],\mathbb{Z}/l[1]) \to 0\end{aligned}$$

where the right-end vanishing is $\mathrm{Ext}(M,\mathbb{Z}/l[1])=0$ by assumption. Now $\mathrm{Hom}(T,\mathbb{Z}/l[1]) = \mathrm{Ext}(T,\mathbb{Z}/l)$ and any extension of the torus T is lifted to an extension of M by $\mathbb{Z}/l$, therefore to an element of $\mathrm{Hom}(M,\mathbb{Z}/l[1]) = \mathrm{Ext}(M,\mathbb{Z}/l)$. This yields $\mathrm{Ext}(L,\mathbb{Z}/l)=0$ for any prime $l \neq p$, thus we see that $L=0$.

Finally, $[0 \to \mathbb{G}_m]$ is not projective since, for $n>1$, the epimorphism

$$[0 \to \mathbb{G}_m] \xrightarrow{n} [0 \to \mathbb{G}_m]$$

is not split. □

C.11. **Weights.** If $M=[L \xrightarrow{u} G] \in {}^t\mathcal{M}_1[1/p]$ is free then Deligne [16] equipped $M=M_{\mathrm{fr}}$ with an increasing filtration by sub-1-motives as follows:

$$W_{-2}(M) := [0 \to T] \subseteq W_{-1}(M) := [0 \to G] \subseteq W_0(M) := M$$

If M is torsion-free we then pull-back the weight filtration along the effective map $M \to M_{\mathrm{fr}}$ as follows:

$$W_i(M) := \begin{cases} M & i \geq 0 \\ [L_{\mathrm{tor}} \hookrightarrow G] & i=-1 \\ [L_{\mathrm{tor}} \cap T \hookrightarrow T] & i=-2 \\ 0 & i \leq -3 \end{cases}$$

Note that $W_i(M)$ is q.i. to $W_i(M_{\mathrm{fr}})$.

If M has torsion we then further pull-back the weight filtration along the effective map $M \to M_{\mathrm{tf}}$.

C.11.1. **Definition.** Let $M = [L \xrightarrow{u} G]$ be an effective 1-motive. Let $u_A : L \to A$ denote the induced map where $A = G/T$. Define

$$W_i(M) := \begin{cases} M & i \geq 0 \\ [L_{\mathrm{tor}} \to G] & i = -1 \\ [L_{\mathrm{tor}} \cap \mathrm{Ker}(u_A) \to T] & i = -2 \\ M_{\mathrm{tor}} = L_{\mathrm{tor}} \cap \mathrm{Ker}(u)[1] & i = -3 \\ 0 & i \leq -4 \end{cases}$$

C.11.2. **Remark.** It is easy to see that $M \mapsto W_i(M)$ yields a functor from ${}^t\mathcal{M}_1[1/p]$ to ${}^t\mathcal{M}_1[1/p]$.

Appendix D. Homotopy invariance for étale sheaves with transfers

One of the main results of Voevodsky concerning presheaves with transfers is that, over a perfect field k, a Nisnevich sheaf with transfers F is homotopy invariant (that is, $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for any smooth X) if and only if it is strongly homotopy invariant, that is, $H^i_{\mathrm{Nis}}(X, F) \xrightarrow{\sim} H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, F)$ for any smooth X and any $i \geq 0$. This allows him to define the *homotopy t-structure* on $\mathrm{DM}^{\mathrm{eff}}_-$.

These results remain "as true as can be" in the étale topology, at least if k has finite étale cohomological dimension. According to an established tradition, this result is probably well-known to experts but we haven't been able to find it in the literature: it could have been formulated and proven for example in [39]. The aim of this appendix is to provide proofs, for which our main source of results will be [39].

D.1. **Homotopy invariance and strict homotopy invariance.**

D.1.1. **Definition.** We denote as in [39, Def. 2.1] by $\mathrm{PST}(k) = \mathrm{PST}$ the category of presheaves with transfers on smooth k-varieties. We also denote by $\mathrm{EST}(k)$, or simply EST, the catégory of étale sheaves with transfers over k.

According to [39, Def. 2.15 and 9.22]:

D.1.2. **Definition.** a) An object F of PST or EST is *homotopy invariant* if $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$ for any smooth k-variety X.
b) Let $F \in \mathrm{EST}$. Then F is *strictly homotopy invariant* if $H^i_{\text{ét}}(X, F) \xrightarrow{\sim} H^i_{\text{ét}}(X \times \mathbb{A}^1, F)$ for any smooth k-variety X and any $i \geq 0$.
We denote by $\mathrm{HI}_{\text{ét}}(k) = \mathrm{HI}_{\text{ét}}$ the full subcategory of EST consisting of homotopy invariant sheaves, and by $\mathrm{HI}^s_{\text{ét}}(k) = \mathrm{HI}^s_{\text{ét}}$ the full subcategory of $\mathrm{HI}_{\text{ét}}$ consisting of strictly homotopy invariant sheaves.

(Strict homotopy invariance for F simply means that F is $\mathbb{A}^1$-local in $D^-(\mathrm{EST})$, see [39, Lemma 9.24].)

Note that $\mathrm{HI}_{\text{ét}}$ is a thick abelian subcategory of EST: if $0 \to F' \to F \to F'' \to 0$ is an exact sequence in EST, then $F \in \mathrm{HI}_{\text{ét}}$ if and only if $F', F'' \in \mathrm{HI}_{\text{ét}}$. We shall see below that the same is true for $\mathrm{HI}^s_{\text{ét}}$.

The main example of a sheaf F which is in $\mathrm{HI}_{\text{ét}}$ but not in $\mathrm{HI}^s_{\text{ét}}$ is $F = \mathbb{Z}/p$ in characteristic p: because of the Artin-Schreier exact sequence we have

$$k[t]/\mathcal{P}(k[t]) \xrightarrow{\sim} H^1_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/p)$$

where $\mathcal{P}(x) = x^p - x$.

We are going to show that this captures entirely the obstruction for a sheaf in $\mathrm{HI}_{\text{ét}}$ not to be in $\mathrm{HI}^s_{\text{ét}}$.

The following is an étale analogue of [39, Th. 13.8]:

D.1.3. **Lemma.** *Let F be a homotopy invariant presheaf with transfers. Suppose moreover that F is a presheaf of $\mathbb{Z}[1/p]$-modules, where p is the exponential characteristic of k. Then the associated étale sheaf with transfers* [39, Th. 6.17] *$F_{\text{ét}}$ is strictly homotopy invariant.*

Proof. The following method is classical: let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of homotopy invariant presheaves with transfers, and consider the corresponding exact sequence $0 \to F'_{\text{ét}} \to F_{\text{ét}} \to F''_{\text{ét}} \to 0$. If, among $F'_{\text{ét}}, F_{\text{ét}}$ and $F''_{\text{ét}}$, two are in $\mathrm{HI}^s_{\text{ét}}$, then clearly so is the third. Using the exact sequence

$$0 \to F_{tors} \to F \to F \otimes \mathbb{Q} \to F \otimes \mathbb{Q}/\mathbb{Z} \to 0$$

for the sheaf F of Lemma D.1.3, this reduces us to the following cases:

- F is a presheaf of $\mathbb{Q}$-vector spaces. Then the result is true by [39, Lemma 14.25] (reduction to [39, Th. 13.8] by the comparison theorem [39, Prop. 14.23]).
- F is a presheaf of torsion abelian groups. Since, by assumption, this torsion is prime to p, $F_{\text{ét}}$ is locally constant by Suslin-Voevodsky rigidity [39, Th. 7.20]. Then the result follows from [SGA4, XV 2.2] (compare [39, Lemma 9.23]).

□

D.1.4. **Proposition.** *The inclusion $\mathrm{HI}^s_{\text{ét}} \to \mathrm{HI}_{\text{ét}}$ has an exact left adjoint/left inverse given by $F \mapsto F \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. In particular, $\mathrm{HI}^s_{\text{ét}} = \mathrm{HI}_{\text{ét}}[1/p]$.*

Proof. The fact that $\mathrm{HI}^s_{\text{ét}}$ is $\mathbb{Z}[1/p]$-linear follows from the Artin-Schreier exact sequence plus the contractibility of $\mathbb{G}_a$ (compare [63, Prop. 3.3.3

2)]). Conversely, Lemma D.1.3 implies that any $\mathbb{Z}[1/p]$-linear sheaf of $\mathrm{HI}_{\text{ét}}$ belongs to $\mathrm{HI}^s_{\text{ét}}$. The rest of the proposition follows. □

As a complement, let us mention the following proposition, which extends Proposition D.1.4:

D.1.5. **Proposition.** *Let $F \in \mathrm{HI}_{\text{ét}}$. Then the complex $C_*(F)$ is canonically isomorphic to $F[1/p]$ in $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$.*

Proof. The map $F \to F[1/p]$ induces a map $C_*(F) \to C_*(F[1/p])$. The latter complex is tautologically equal to $C_*(F)[1/p]$. Since $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ is $\mathbb{Z}[1/p]$-linear, the map $C_*(F) \to C_*(F)[1/p]$ is a quasi-isomorphism. Finally, since $F[1/p] \in \mathrm{HI}^s_{\text{ét}}$ (Prop. D.1.4), the augmentation $F[1/p] \to C_*(F[1/p])$ is a quasi-isomorphism by [39, Lemma 9.15]. □

D.1.6. **Corollary.** *Let F be a homotopy invariant Nisnevich sheaf with transfers. Then, the natural functor $\alpha^* : \mathrm{DM}^{\mathrm{eff}}_- \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ sends F to $F_{\text{ét}}[1/p]$.*

Proof. According to [39, Remark 14.3], α^* may be described as the composition

$$\mathrm{DM}^{\mathrm{eff}}_- \hookrightarrow D^-(\mathrm{Shv}_{\mathrm{Nis}}(Sm(k))) \xrightarrow{\alpha^*} D^-(\mathrm{Shv}_{\text{ét}}(Sm(k))) \xrightarrow{RC} \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$$

where the middle functor is induced by the inverse image functor (change of topology) on sheaves and RC is induced by $K \mapsto C_*(K)$. The result then follows from Proposition D.1.5. □

D.2. **Friendly complexes.**

D.2.1. **Definition.** A object $C \in D^-(\mathrm{EST})$ is *friendly* if there exists an integer $N = N(C)$ such that, for any prime number $l \neq p$, $H_q(C/l) = 0$ for $q > N$ (in other terms, C/l is uniformly bounded below). We denote by $D^-_{\mathrm{fr}}(\mathrm{EST})$ the full subcategory of friendly objects and by $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{fr},\text{ét}}$ the intersection $D^-_{\mathrm{fr}}(\mathrm{EST}) \cap \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$.

D.2.2. **Theorem.** $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},\text{ét}} \subset \mathrm{DM}^{\mathrm{eff}}_{\mathrm{fr},\text{ét}}$.

Proof. If is clear that $D^-_{\mathrm{fr}}(\mathrm{EST})$ is a thick triangulated subcategory of $D^-(\mathrm{EST})$; hence it suffices to prove that $C_*(L_{\text{ét}}(X))$ is friendly for any smooth scheme X. By [39, Lemmas 6.23 and 9.15], we have for any smooth U, any prime $l \neq p$ and any $q \in \mathbb{Z}$

$$\mathrm{Hom}(C_*(L_{\text{ét}}(X \times U)), \mathbb{Z}/l[q]) \simeq H^q_{\text{ét}}(X \times U, \mathbb{Z}/l)$$

hence

$$\mathcal{E}xt^{\,q}(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l) \simeq R^q_{\text{ét}}\pi_*\mathbb{Z}/l$$

where $\pi : X \to \operatorname{Spec} k$ is the structural morphism. By the cohomological dimension results of [SGA4, Exp. X] and the finiteness results of [SGA4 1/2, Th. finitude], this shows that $\underline{\operatorname{Hom}}_{\text{ét}}(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l)$ is a bounded complex of constructible $\mathbb{Z}/l$-sheaves. It follows that the biduality morphism

$$C_*(L_{\text{ét}}(X))/l \to \underline{\operatorname{Hom}}_{\text{ét}}(\underline{\operatorname{Hom}}_{\text{ét}}(C_*(L_{\text{ét}}(X)), \mathbb{Z}/l), \mathbb{Z}/l)$$

is an isomorphism of bounded complexes of constructible $\mathbb{Z}/l$-sheaves. Moreover, the lower bound is at most $2 \dim X$, hence is independent of l. □

D.3. **The étale homotopy t-structure.** The following is an étale analogue of [39, Prop. 14.8]:

D.3.1. **Proposition.** *Let $K \in D^-(\mathrm{EST})$ be a bounded above complex of étale sheaves with transfers. Suppose either that the étale cohomological dimension of k is finite, or that K is friendly. Then K is $\mathbb{A}^1$-local if and only if all its cohomology sheaves are strictly homotopy invariant.*

Proof. In the finite cohomological dimension case, "if" is trivial (*cf.* [39, Prop. 9.30]). For "only if", same proof as that of [39, Prop. 14.8], by replacing the reference to [39, Th. 13.8] in *loc. cit.* by a reference to Lemma D.1.3 (note that if K is $\mathbb{A}^1$-local, then it is $\mathbb{Z}[1/p]$-linear by [63, Prop. 3.3.3 2)] and thus so are its cohomology sheaves).

In the friendly case, note that the two conditions

- $\mathbb{A}^1$-local
- having strictly homotopy invariant cohomology sheaves

are stable under triangles: for the first it is obvious and for the second it is because $\mathrm{HI}^s_{\text{ét}}$ is thick in EST by Proposition D.1.4. Considering the exact triangle

$$K \to K \otimes \mathbb{Q} \to K \otimes (\mathbb{Q}/\mathbb{Z})' \xrightarrow{+1}$$

we are reduced to show the statement separately for $K \otimes \mathbb{Q}$ and for $K \otimes (\mathbb{Q}/\mathbb{Z})'$. In the first case this works by reduction to Nisnevich cohomology, while in the second case the spectral sequence of [39, Prop. 9.30] also converges, this time because $K \otimes (\mathbb{Q}/\mathbb{Z})'$ is bounded below. □

D.3.2. **Remark.** The finite cohomological dimension hypothesis appears in the spectral sequence arguments of the proofs of [39, Prop. 9.30 and 14.8]. We don't know if it is really necessary. Nevertheless, Joël Riou pointed out that this argument trivially extends to fields of virtually finite cohomological dimension: the only issue is for the "if" part, but if we know that an object K is $\mathbb{A}^1$-local étale-locally, then

it is clearly $\mathbb{A}^1$-local. (For example, this covers all fields of arithmetic origin.) Therefore:

D.3.3. **Corollary.** *If the virtual étale cohomological dimension of k is finite, then $\mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ has a homotopy t-structure, with heart $\mathrm{HI}^{s}_{\text{ét}}$, and the functor $\alpha^* : \mathrm{DM}^{\mathrm{eff}}_{-} \to \mathrm{DM}^{\mathrm{eff}}_{-,\text{ét}}$ is t-exact. Without any cohomological dimension assumption, $\mathrm{DM}_{\mathrm{fr},\text{ét}}$ has a homotopy t-structure.* $\square$

Index

$D^{\text{ét}}_{\leq 1}$, $D^{\text{Nis}}_{\leq 1}$, 52, 54
$D^b(\mathcal{M}_1[1/p])$, 16
$H^{(1)}_j$, 101
$M_{\text{ét}}(X)$, 23
M_{fr}, M_{tor}, M_{tf}, 118
S_1, S_1^{eff}, 38
$Sch(k)$, $Sm(k)$, 8
$W_i(M)$, 11
Biext, 44
$\mathrm{DM}^{\text{eff}}_{-,\text{ét}}(k)$, 3
$\mathrm{DM}^{\text{eff}}_{\text{gm}}(k)$, 2
$\mathrm{DM}^{\text{eff}}_{\text{gm,ét}}$, 23
$\mathrm{EST}(k)$, 12
$\mathrm{HI}_{\text{ét}}$, 12
$\mathrm{HI}^s_{\text{ét}}$, 41, 135
LAlb, 55
LAlb^*, 68
LAlb^c, 66
$\mathcal{M}_1$, $\mathcal{M}_1 \otimes \mathbb{Q}$, 10
$\mathrm{Pic}_{X/k}$, $\mathrm{NS}_{X/k}$, 34, 35
RPic, 56
RPic^c, RPic^*, 69
SAb, 31
Shv_1, Shv_0, 30
Shv^s_1, 41
Tot, 27
$\mathcal{M}^{\text{eff}}_{\text{anc}}$, 10, 129
$\mathcal{A} \boxtimes R$, 10
$\mathcal{A} \otimes R$, 10, 112
$\mathcal{A}_{X/k}$, 69
$\mathcal{A}^{\text{éh}}_{X/k}$, 89
$\mathcal{M}_0$, ${}^t\mathcal{M}_0$, 9
$\underline{\mathrm{Hom}}_{\text{ét}}$, 25
Tot, 49
$\pi_0(X)$, 25
$\mathcal{H}^n$, $\mathcal{H}_n$, 42
tot, 42
$\underline{G}$, 28
$\underline{\mathrm{Biext}}$, 49
$d_{\leq 1}$, 54
$d_{\leq 1}\,\mathrm{DM}^{\text{eff}}_{\text{gm,ét}}$, 23
${}^tH^n$, ${}_tH^n$, 22
${}^t\,\mathrm{AbS}$, AbS, 37, 38
${}^t\mathcal{M}_1$, ${}^t\mathcal{M}_1^{\text{eff}}$, 14, 117, 122
${}_t\mathcal{M}_1$, ${}_t\mathcal{M}_1^{\text{eff}}$, 19

References

[1] M. Artin: Néron models, in "Arithmetic geometry" (G. Cornell, J.H. Silverman, eds), Springer, 1986, 213–230.

[2] L. Barbieri-Viale: On the theory of 1-motives, in "Algebraic Cycles and Motives" (J. Nagel, C. Peters, ed.), London Math. Soc. Lect. Notes Series **343** (Vol. 1), 2007.

[3] L. Barbieri-Viale and B. Kahn: A note on relative duality for Voevodsky's motives, preprint http://arxiv.org/abs/math.AG/0701782

[4] L. Barbieri-Viale, A. Rosenschon and M. Saito: Deligne's conjecture on 1-motives, Ann. of Math. **158** (2003), 593–633.

[5] L. Barbieri-Viale and V. Srinivas: Albanese and Picard 1-motives, Mémoire SMF **87**, Paris, 2001.

[6] G. Laumon: Transformation de Fourier généralisée, preprint http://arxiv.org/abs/alg-geom/9603004 - Preprint IHÉS (Transformation de Fourier géométrique, IHÉS/85/M/52) 47 pages.

[7] A.A. Beilinson, J. Bernstein, P. Deligne: Faisceaux pervers, Astérisque **100**, 1982.

[8] A. Bertapelle: Remarks on 1-motivic sheaves, preprint, 2007.

[9] C. Bertolin: Biextensions and 1-motives, C. R. Math. Acad. Sci. Soc. R. Can. **27** (2005), 65–71.

[10] J. Biswas and V. Srinivas: A Lefschetz $(1,1)$-theorem for normal projective complex varieties, Duke Math. J. **101** (2000), 427–458.

[11] S. Bloch and H. Esnault: Künneth projectors, in "Algebraic cycles and motives" (J. Nagel, C. Peters, ed.), London Math. Soc. Lect. Notes Series **344** (Vol. 2), 2007, 54–72.

[12] S. Bloch and A. Ogus: Gersten's conjecture and the homology of schemes, Ann. Sci. Éc. Norm. Sup. **7** (1974), 181–202.

[13] M. Bondarko: Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura, preprint, 2007, http://arxiv.org/abs/math/0601713.

[14] N. Bourbaki: Algèbre commutative, Masson, 1985.

[15] L. Breen: Extensions of abelian sheaves and Eilenberg-Mac Lane algebras, Invent. Math. **9** (1969), 15–44.

[16] P. Deligne: Théorie de Hodge III, Publ. Math. IHÉS **44** (1974) 5–78.

[17] M. Demazure: Motifs de variétés algébriques, Sém. Bourbaki 1969/70, Exp. **365**, Lect. Notes in Math. **180**, Springer, 1971.

[18] C. Deninger: A proper base change theorem for nontorsion sheaves in étale cohomology, J. pure appl. Algebra **50** (1988), 231–235.

[19] H. Esnault, B. Kahn, M. Levine and E. Viehweg: The Arason invariant and mod 2 algebraic cycles, J. AMS **11** (1998), 73–118.

[20] P. Gabriel and M. Zisman: Calculus of fractions and homotopy theory, Erg. in Math., Springer, 1967.

[21] T. Geisser: Arithmetic Cohomology over finite fields and special values of zeta-functions, Duke Math. J. **133** (2006), 27–57.

[22] T. Geisser: Duality via cycle complexes, preprint http://arxiv.org/abs/math/0608456

[23] H. Gillet and C.Soulé: Descent, motives and K-theory, J. reine angew. Math. **478** (1996), 127–176.
[24] A. Grothendieck: Théorèmes de descente et théorèmes d'existence en géométrie algébrique VI. Les schémas de Picard: propriétés générales, Sém. Bourbaki **236** (1962), Vol. 7, 221–243, Soc. Math. France, 1995.
[25] R. Hartshorne: Residues and Duality, Lecture Notes **20**, Springer, 1966.
[26] A. Huber and B. Kahn: The slice filtration and mixed Tate motives, Compositio Math. **142** (2006), 907–936.
[27] A.J. de Jong: Smoothness, semistability and alterations, Publ. Math. IHÉS **83** (1996) 51–93.
[28] B. Kahn: Motivic cohomology of smooth geometrically cellular varieties, in "Algebraic K-theory", Proc. Sym. Pure Math. **67**, AMS, 1999.
[29] B. Kahn: Sur le groupe des classes d'un schéma arithmétique (avec un appendice de Marc Hindry), Bull. SMF **134** (2006), 395-415.
[30] M. Kashiwara and P. Schapira Categories and sheaves, Springer, 2005.
[31] B. Keller: Chain complexes and exact categories, Manuscripta Math. **67** (1990), 379–417.
[32] S. Kleiman: Motives, *in* Algebraic geometry (Oslo, 1970), Walters-Noordhoff, 1972, 53–82.
[33] A. Krishna and V. Srinivas: Zero Cycles on Singular Varieties, in "Algebraic cycles and motives" (J. Nagel, C. Peters, ed.), London Math. Soc. Lect. Notes Series **343** (Vol. 1), 2007.
[34] S. Lichtenbaum: Suslin homology and Deligne 1-motives, in "Algebraic K-theory and algebraic topology" NATO ASI Series, 1993, 189–197.
[35] S. Lichtenbaum: Quasi-motives of curves, Algebraic K-theory (Toronto, ON, 1996), 185–197, Fields Inst. Commun., **16**, AMS, 1997.
[36] V. M. Mallick: Roĭtman's for singular projective varieties in arbitrary characteristic, preprint, 2006.
[37] S. Mac Lane: Homology, Springer, 1963.
[38] J. I. Manin: Correspondences, motifs and monoidal transformations, Math. USSR Sbornik **6** (1968), 439–470.
[39] C. Mazza, V. Voevodsky and C. Weibel: Lecture Notes on Motivic Cohomology, Clay Mathematics Monographs, Vol. **2**, 2006 (Based upon lectures given by V. Voevodsky at IAS in 1999-2000)
[40] J.S. Milne: Étale cohomology, Princeton University Press, 1980.
[41] J.S. Milne: Abelian varieties, in "Arithmetic geometry" (G. Cornell, J.H. Silverman, eds), Springer, 1986, 103–150.
[42] D. Mumford: "Abelian varieties" (2nd ed.), TIFR Studies in Mathematics, Oxford University Press, 1974.
[43] J.-P. Murre: On contravariant functors from the category of preschemes over a field into the category of abelian groups (with an application to the Picard functor), Publ. Math. IHÉS **23** (1964), 581–619.
[44] A. Neeman: The derived category of an exact category, J. Alg. **135** (1990), 388–394.
[45] F. Orgogozo: Isomotifs de dimension inférieure ou égale à 1, Manuscripta Math. **115** (2004), 339–360.
[46] C. Pedrini and C. Weibel: Invariants of real curves, Rend. Sem. Mat. Univ. Politec. Torino **49** (1993), 139–173.

[47] D. Quillen: Higher algebraic K-theory, I, Lect. Notes in Math. **341**, Springer, 1973.

[48] N. Ramachandran: Albanese and Picard motives of schemes, preprint, 1998.

[49] N. Ramachandran: Duality of Albanese and Picard 1-motives, K-Theory **22** (2001), 271–301.

[50] N. Ramachandran: One-motives and a conjecture of Deligne, J. Algebraic Geom. **13** (2004), 29–80.

[51] J. Riou: "Opérations sur la K-théorie algébrique et régulateurs *via* la théorie homotopique des schémas", PhD thesis, Paris 7, 2006.

[52] A. Scholl: Classical motives, *in* Motives, Proc. Symp. pure Math. **55** (I), 163–187.

[53] J.-P. Serre: "Groupes algébriques et corps de classes", Hermann, 1959.

[54] J.-P. Serre: Morphismes universels et variétés d'Albanese, *in* Exposés de séminaires, 1950–1989, Doc. mathématiques, SMF, 2001, 141–160.

[55] J.-P. Serre: Groupes proalgébriques, Publ. Math. IHÉS **7** (1960), 5–68.

[56] J.-P. Serre: "Représentations linéaires des groupes finis", Hermann, 1970.

[57] M. Spiess and T. Szamuely: On the Albanese map for smooth quasiprojective varieties, Math. Ann. **325** (2003), 1–17.

[58] A. Suslin: Higher Chow groups and etale cohomology, in "Cycles, transfers, and motivic homology theories", Princeton Univ. Press, Annals of Math. Studies **143**, 2000, 239–254.

[59] A. Suslin and V. Voevodsky: Singular homology of abstract algebraic varieties, Invent. Math. **123** (1996), 61–94.

[60] V. Voevodsky: Letter to A.A. Beilinson, June 12, 1992.

[61] V. Voevodsky: Letter to the first author, October 14, 1996.

[62] V. Voevodsky: Cohomological theory of presheaves with transfers, in "Cycles, Transfers, and Motivic Cohomology Theories" Princeton Univ. Press, Annals of Math. Studies **143** 2000.

[63] V. Voevodsky: Triangulated categories of motives over a field, in "Cycles, Transfers, and Motivic Cohomology Theories" Princeton Univ. Press, Annals of Math. Studies **143** 2000, 188–238.

[64] V. Voevodsky: Cancellation theorem, Preprint http://www.math.uiuc.edu/K-theory/0541/

[65] C. Weibel: Pic is a contracted functor, Invent. Math. **103** (1991), 351–377.

[SGA1] A. Grothendieck: "Revêtements étales et groupe fondamental", Séminaire de géométrie algébrique du Bois-Marie (SGA 1), Documents Mathématiques **3**, SMF, 2003.

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier: "Théorie des topos et cohomologie étale des schémas", Séminaire de géométrie algébrique du Bois-Marie (SGA 4), Vol. 3, Lect. Notes in Math. **305**, Springer, 1973.

[SGA4 1/2] P. Deligne: "Cohomologie étale", Séminaire de géométrie algébrique du Bois-Marie (SGA 4 1/2), Lect. Notes in Math. **569**, Springer, 1977.

[SGA7] P. Deligne, A. Grothendieck, N. Katz, M. Raynaud: "Groupes de monodromie en géométrie algébrique" (SGA 7, I), Lect. Notes in Math. **288**, Springer, 1972, 133–217.

Dipartimento di Matematica Pura e Applicata, Università degli Studi di Padova, Via Trieste, 63, 35121 Padova, Italy
E-mail address: barbieri@math.unipd.it

Institut de Mathématiques de Jussieu, 175–179 rue du Chevaleret, 75013 Paris, France
E-mail address: kahn@math.jussieu.fr