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# RATE OF CONVERGENCE OF SPACE TIME APPROXIMATIONS FOR STOCHASTIC EVOLUTION EQUATIONS

ISTVÁN GYÖNGY AND ANNIE MILLET

ABSTRACT. Stochastic evolution equations in Banach spaces with unbounded nonlinear drift and diffusion operators are considered. Under some regularity condition assumed for the solution, the rate of convergence of various numerical approximations are estimated under strong monotonicity and Lipschitz conditions. The abstract setting involves general consistency conditions and is then applied to a class of quasilinear stochastic PDEs of parabolic type.

## 1. INTRODUCTION

Let  $V \hookrightarrow H \hookrightarrow V^*$  be a *normal triple* of spaces with dense and continuous embeddings, where  $V$  is a separable and reflexive Banach space,  $H$  is a Hilbert space, identified with its dual by means of the inner product in  $H$ , and  $V^*$  is the dual of  $V$ . Thus  $\langle v, h \rangle = (v, h)$  for all  $v \in V$  and  $h \in H^* = H$ , where  $\langle v, v^* \rangle = \langle v^*, v \rangle$  denotes the duality product of  $v \in V$ ,  $v^* \in V^*$ , and  $(h_1, h_2)$  denotes the inner product of  $h_1, h_2 \in H$ . We assume, without loss of generality, that  $|v|_H \leq |v|_V$  for all  $v \in V$ . Let  $W = \{W(t) : t \geq 0\}$  be a  $d_1$ -dimensional Brownian motion carried by a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ . Consider the stochastic evolution equation

$$u(t) = u_0 + \int_0^t A(s, u(s)) ds + \sum_{k=1}^{d_1} \int_0^t B_k(s, u(s)) dW^k(s), \quad (1.1)$$

where  $u_0$  is a  $H$ -valued  $\mathcal{F}_0$ -measurable random variable such that  $E|u_0|_H^2 < \infty$ ,  $A$  and  $B$  are  $\mathcal{P} \otimes \mathcal{B}(V)$ -measurable mappings from  $[0, \infty[ \times \Omega \times V$  into  $V^*$  and  $H^{d_1} := H \times \dots \times H$ , respectively, where  $\mathcal{P}$  denotes the  $\sigma$ -algebra of predictable subset of  $[0, \infty[ \times \Omega$  and  $\mathcal{B}(V)$  is the Borel  $\sigma$ -algebra on  $V$ .

It is well-known, see [10], [13] and [16], that this equation admits a unique solution on any interval  $[0, T]$ , if the following conditions are met: There exist constants  $\lambda > 0$ ,  $K \geq 0$ ,  $K_1 \geq 0$ ,  $K_2 \geq 0$  and  $(\mathcal{F}_t)$ -adapted non-negative locally integrable stochastic processes  $f = \{f(t) : t \geq 0\}$  and  $g = \{g(t) : t \geq 0\}$  such that

$$E \int_0^T f(t) dt + E \int_0^T g(t) dt < \infty,$$

and almost surely

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**(R0)** (i) (Strong monotonicity) Almost surely for all  $t \in [0, T]$ ,  $u, v \in V$ ,

$$2\langle u - v, A(t, u) - A(t, v) \rangle + \sum_k |B_k(t, u) - B_k(t, v)|_H^2 + \lambda \|u - v\|_V^2 \leq K \|u - v\|_H^2, \quad (1.2)$$

(ii) (Growth conditions on  $A$  and  $B$ ) Almost surely for all  $t \in [0, T]$ ,  $u \in V$ ,

$$|A(t, u)|_{V^*}^2 \leq K_1 \|u\|_V^2 + f(t), \quad \sum_k |B_k(t, u)|_H^2 \leq K_2 \|u\|_V^2 + g(t). \quad (1.3)$$

(iii) (Lipschitz conditions on  $A$  and  $B$ ) Almost surely for all  $t \in [0, T]$ ,  $u, v \in V$ ,

$$|A(t, u) - A(t, v)|_{V^*}^2 \leq K_1 \|u - v\|_V^2, \quad \sum_k |B_k(t, u) - B_k(t, v)|_H^2 \leq K_2 \|u - v\|_V^2. \quad (1.4)$$

In [7] it is shown that under these conditions the solutions of various implicit and explicit schemes converge to  $u$ . In [8] the rate of convergence of implicit Euler approximations is estimated under more restrictive hypotheses:  $A$  and  $B$  are Lipschitz continuous in  $v \in V$ , satisfy a strong monotonicity condition, and the solution  $u$  satisfies some regularity condition. Then Theorem 3.2 from [7] in the case of time independent operators  $A$  and  $B$  reads as follows. For the implicit Euler approximation  $u^\tau$  corresponding to the mesh size  $\tau = T/m$  of the partition of  $[0, T]$  one has

$$E \max_{i \leq m} |u(i\tau) - u^\tau(i\tau)|_H^2 + \tau E \sum_{i \leq m} \|u(i\tau) - u^\tau(i\tau)\|_V^2 \leq C\tau^{2\nu}$$

with some constant  $C$  independent of  $\tau$ .

In this paper, we study space and space-time approximations schemes in a general framework. In order to formulate the space discretizations we consider a normal triple  $V_n \hookrightarrow H_n \hookrightarrow V_n^*$  and a bounded linear operator  $\Pi_n : H \rightarrow H_n$  for each  $n \geq 0$ , such that  $\Pi_n : V \rightarrow V_n$  is also continuous. We have in mind wavelets, finite elements and discrete Sobolev spaces as examples for the spaces  $V_n$ .

The space discretization scheme is defined by equation (1.1) with some operators

$$A^n : [0, \infty[ \times \Omega \times V_n \rightarrow V_n^*, \quad B^n : [0, \infty[ \times \Omega \times V_n \rightarrow H_n^{d_1}$$

in place of  $A$  and  $B$ , respectively, such that  $A^n, B^n$  are Lipschitz in  $v \in V_n$ , satisfy a strong monotonicity condition, and are related to  $A$  and  $B$  via a consistency condition, **(Cn)** below. Then assuming also a regularity condition, **(R1)**, on the solution  $u$  of (1.1) we prove the existence of a constant  $C$  such that for all  $n \geq 1$

$$E \sup_{0 \leq t \leq T} |\Pi_n u(t) - u_n(t)|_{H_n}^2 + E \int_0^T \|\Pi_n u(t) - u_n(t)\|_{V_n}^2 dt \leq C\varepsilon_n^2,$$

where  $u_n$  is the solution of (1.1) with  $A^n, B^n$  in place of  $A, B$ , and the sequence  $\varepsilon_n \rightarrow 0$  is from the consistency condition.

We define the space-time implicit and the space-time explicit approximations,  $u_i^{n,\tau}$  and  $u_{\tau,i}^n$ , on the time grid  $\{t_i = i\tau : 0 \leq i \leq m\}$  by  $u_0^{n,\tau} = u_{\tau,0}^n = \Pi_n u_0$ , and respectively

$$u_{i+1}^{n,\tau} = u_i^{n,\tau} + \tau A_{i+1}^{n,\tau}(u_{i+1}^{n,\tau}) + \sum_k B_{k,i}^{n,\tau}(u_i^{n,\tau})(W^k(t_{i+1}) - W^k(t_i)),$$

$$u_{\tau,i+1}^n = u_{\tau,i}^n + \tau A_i^{n,\tau}(u_{\tau,i}^n) + \sum_k B_{k,i}^{n,\tau}(u_{\tau,i}^n)(W^k(t_{i+1}) - W_k(t_i)),$$

for  $i = 0, \dots, m-1$  with some operators

$$A_i^{n,\tau} : \Omega \times V_n \rightarrow V_n^*, \quad B_{k,i}^{n,\tau} : \Omega \times V_n \rightarrow H_n^{d_1}$$

which are Lipschitz continuous in  $v \in V_n$ , satisfy a strong monotonicity condition and are related to  $A$  and  $B$  by a consistency condition. Then if  $u$  is sufficiently regular, we prove that

$$E \sup_{0 \leq i \leq m} |\Pi_n u(t_i) - u_i^{n,\tau}|_{H_n}^2 + E \sum_{0 \leq i \leq m} \|\Pi_n u(t_i) - u_i^{n,\tau}\|_{V_n}^2 \tau \leq C(\tau^{2\nu} + \varepsilon_n^2).$$

We have the same estimate for the explicit space-time approximations if as in [7], a *stability relation* between the time mesh  $\tau$  and a space approximation parameter is satisfied.

Finally, we present as examples a class of quasi-linear stochastic partial differential equations (SPDEs) and linear SPDEs of parabolic type. We show that they satisfy the conditions of the abstract results, Theorems 3.1, 4.3 and 5.2, when we use wavelets, or finite differences. In particular, we obtain rate of convergence results for space and space-time approximations of linear parabolic SPDEs, in particular, for the Zakai equation of nonlinear filtering. We would like to mention that as far as we know, discrete Sobolev spaces are applied first in [18] to space discretizations and explicit space-time discretizations of linear SPDEs, and it inspired our approach to finite difference schemes. Our abstract results can also be applied to finite elements approximations. To keep down the size of the paper we will consider such applications elsewhere.

We denote by  $C$  and  $M$  some constants which, as usual, can change from line to line. Given  $a \in \mathbb{R}^k$ , let  $C(a)$  denote a constant which depends on  $a$ .

## 2. CONDITIONS ON EQUATION (1.1) AND ON THE APPROXIMATION SPACES

**2.1. Conditions on equation (1.1).** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  be a stochastic basis, satisfying the usual conditions, i.e.,  $(\mathcal{F}_t)_{t \geq 0}$  is an increasing right-continuous family of sub- $\sigma$ -algebras of  $\mathcal{F}$  such that  $\mathcal{F}_0$  contains every  $P$ -null set. Let  $W = \{W(t) : t \geq 0\}$  be a  $d_1$ -dimensional Wiener martingale with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , i.e.,  $W$  is an  $\mathcal{F}_t$ -adapted Wiener process with values in  $\mathbb{R}^{d_1}$  such that  $W(t) - W(s)$  is independent of  $\mathcal{F}_s$  for all  $0 \leq s \leq t$ . We use the notation  $\mathcal{P}$  for the sigma-algebra of predictable subsets of  $[0, \infty) \times \Omega$ . If  $V$  is a Banach space then  $\mathcal{B}(V)$  denotes the sigma-algebra generated by the (closed) balls in  $V$ .

Let  $V$  be a separable Hilbert space embedded densely and continuously into a Hilbert space  $H$ , which is denoted by  $V \hookrightarrow H$ . Let  $V^*$  denote the dual space of  $V$  and suppose that  $H$  is embedded continuously in  $V^*$ . The triplet  $V \hookrightarrow H \hookrightarrow V^*$  is called a normal triplet, where  $H$  is identified with its dual  $H^*$  by the help of the inner product in  $H$ , and  $|v|_H \leq |v|_V$  holds for  $v \in V$ .

Consider the initial value problem

$$du(t) = A(t, u(t)) dt + \sum_k B_k(t, u(t)) dW^k(t), \quad u(0) = u_0 \quad (2.1)$$

on a fixed time interval  $[0, T]$ , where  $u_0$  is a  $H$ -valued  $\mathcal{F}_0$ -measurable random variable,  $A$  and  $B = (B_k)$  are  $\mathcal{P} \otimes \mathcal{B}(V)$ -measurable mappings from  $[0, T] \times \Omega \times V$  into  $V^*$  and  $H^{d_1}$ , respectively such that the conditions (i)-(iii) from **(R0)** hold:

**Definition 2.1.** *An  $V$ -valued progressively measurable process  $u = (u(t))_{t \in [0, T]}$  is a solution to equation (2.1) on  $[0, T]$  if almost surely*

$$u(t) = u_0 + \int_0^t A(s, u(s)) ds + \sum_k \int_0^t B_k(s, u(s)) dW^k(s)$$

holds for  $dt$ -almost every  $t \in [0, T]$ .

Note that if  $V \hookrightarrow H \hookrightarrow V^*$  is a normal triplet, then the following theorem is well-known (see [10], [13] and [16]).

**Theorem 2.2.** *Let  $A$  and  $B$  satisfy **(R0)** from the Introduction. Then for every  $H$ -valued  $\mathcal{F}_0$ -measurable random variable  $u_0$ , (2.1) has a unique solution  $u$ . Moreover,  $u$  has an  $H$ -valued continuous stochastic modification, identified with  $u$ , and if  $E|u(0)|_H^2 < \infty$  then*

$$\begin{aligned} E \sup_{t \in [0, T]} |u(t)|_H^2 + E \int_0^T \|u(s)\|_V^2 ds \\ \leq CE|u_0|_H^2 + CE \int_0^T (f(t) + g(t)) dt < \infty, \end{aligned} \quad (2.2)$$

where  $C$  is a constant depending only on the constants  $\lambda$ , and  $K$  from the conditions (i)-(iii).

Instead of borrowing the conditions of this theorem, we simply assume that problem (2.1) has a solution  $u$ , and that it satisfies the regularity properties formulated below.

Let  $K$  and  $M$  be some constants,  $\mathcal{V}$  and  $\mathcal{H}$  be separable Hilbert spaces such that  $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow V$ , and let us make the following assumptions.

**(R1)** The solution  $u$  takes values in  $\mathcal{V}$  for  $dt \times P$ -almost every  $(t, \omega) \in [0, T] \times \Omega$  such that

(i)

$$E \int_0^T |u(t)|_{\mathcal{V}}^2 dt =: r_1 < \infty.$$

(ii)  $u$  has an  $\mathcal{H}$ -valued stochastic modification such that

$$\sup_{t \in [0, T]} E|u(t)|_{\mathcal{H}}^2 =: r_2 < \infty.$$

We assume the following regularity conditions on the operators  $A$  and  $B$ .

**(R2)** (*Linear growth in the triplet  $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow V$* ) Almost surely

$$\|A(t, v)\|_V^2 \leq K|v|_{\mathcal{V}}^2 + \xi(t), \quad \sum_k \|B_k(t, v)\|_V^2 \leq K|v|_{\mathcal{H}}^2 + \eta(t)$$

for all  $t \in [0, T]$  and  $v \in \mathcal{V}$ , where  $\xi$  and  $\eta$  are non-negative processes such that

$$E \int_0^T \xi(t) dt \leq M, \quad \sup_{t \in [0, T]} E\eta(t) \leq M.$$

When we discretise problem (2.1) in time we use the following Hölder condition required from  $A$  and  $B$  in  $t$ .

**(R3)** There exists a constant  $\nu \in ]0, \frac{1}{2}]$  and a non-negative random variable  $\eta$  such that  $E\eta \leq M$ , and almost surely

(i) (*Time regularity of A*)

$$|A(s, v) - A(t, v)|_V^2 \leq (K |v|_V^2 + \eta) |t - s|^{2\nu} \quad \text{for } v \in \mathcal{V},$$

(ii) (*Time regularity of B*)

$$\sum_k |B_k(s, v) - B_k(t, v)|_V^2 \leq (K |v|_{\mathcal{H}}^2 + \eta) |t - s|^{2\nu} \quad \text{for } v \in \mathcal{H},$$

for all  $0 \leq s < t \leq T$ .

**Remark 2.3.** Let  $u$  be a solution of (2.1) on  $[0, T]$  such that  $E\|u_0\|_V^2 < \infty$  and condition (i) from **(R1)** and condition **(R2)** are met. Then the following statements hold.

(i)  $u$  has a  $V$ -valued continuous stochastic modification, denoted also by  $u$ , such that

$$E \sup_{t \in [0, T]} \|u(t)\|_V^2 \leq 3E\|u_0\|_V^2 + C(r_1 + M).$$

(ii) If (ii) from **(R1)** also holds, then for  $s, t \in [0, T]$ ,

$$E\|u(t) - u(s)\|_V^2 \leq C|t - s|(r_1 + r_2 + M), \quad (2.3)$$

where  $r_1, r_2$  and  $M$  are from **(R1)** and **(R2)**, and  $C$  is a constant depending only on  $T$  and on the constant  $K$  from **(R2)**.

*Proof.* Define

$$F(t) = \int_0^t A(s, u(s)) ds \quad \text{and} \quad G(t) = \sum_k \int_0^t B_k(s, u(s)) dW^k(s).$$

Notice that

$$\begin{aligned} E \int_0^T \|A(s, u(s))\|_V^2 ds &\leq KE \int_0^T |u(s)|_V^2 ds + E \int_0^T \xi(s) ds =: M_1 < \infty, \\ \sum_k E \int_0^T \|B_k(s, u(s))\|_V^2 ds &\leq KE \int_0^T |u(s)|_{\mathcal{H}}^2 ds + E \int_0^T \eta(s) ds =: M_2 < \infty. \end{aligned}$$

Hence  $F$  and  $G$  are  $V$ -valued continuous processes, and by Jensen's and Doob's inequalities

$$E \sup_{t \leq T} \|F(t)\|_V^2 \leq TM_1, \quad E \sup_{t \leq T} \|G(t)\|_V^2 \leq 4 \sum_k E \int_0^T \|B_k(s, u(s))\|_V^2 ds \leq 4M_2.$$

Consequently, the process  $u_0 + F(t) + G(t)$  is a  $V$ -valued continuous modification of  $u$ , and statement (i) holds. Moreover, if **(R1)** (ii) holds, then

$$\sup_{t \in [0, T]} \sum_k E \|B_k(s, u(s))\|_V^2 \leq K \sup_{t \in [0, T]} E |u(t)|_{\mathcal{H}}^2 + \sup_{t \in [0, T]} E \eta(t) := M_3 < +\infty,$$

and

$$E\|F(t) - F(s)\|_V^2 \leq |t - s|M_1,$$

$$E\|G(t) - G(s)\|_V^2 = \sum_k \int_s^t E\|B_k(r, u((r)))\|_V^2 dr \leq |t - s|M_3$$

for any  $0 \leq s \leq t \leq T$ , which proves (ii).  $\square$

**2.2. Approximation spaces and operators  $\Pi_n$ .** Let  $V_n \hookrightarrow H_n \hookrightarrow V_n^*$  be a normal triple for each  $n \geq 0$  and let  $\Pi_n : V \rightarrow V_n$  denote a linear continuous operator. Suppose that

$$\sup_{\|v\|_V=1} \|\Pi_n v\|_{V_n} \leq p \quad (2.4)$$

for some constant  $p$  independent of  $n$ . Note that we do not require that the maps  $\Pi_n$  be orthogonal projections on the Hilbert space  $H$ .

We denote by  $\langle v, w \rangle_n$  the duality between  $v \in V_n$  and  $w \in V_n^*$  and similarly by  $(h, k)_n$  the inner product of  $h, k \in H_n$ . To lighten the notations, we will denote by  $\|v\| := \|v\|_V$  the  $V$  norm of  $v \in V$ ,  $\|v\|_n := \|v\|_{V_n}$  the  $V_n$  norm of  $v \in V_n$ ,  $|u| := |u|_H$  the  $H$  norm of  $u \in H$ ,  $|u|_n = |u|_{H_n}$  the  $H_n$  norm of  $u \in H_n$  and finally  $|y|_* := |y|_{V^*}$  the  $V^*$  norm of  $y \in V^*$  and  $|y|_{n^*} := |y|_{V_n^*}$  the  $V_n^*$  norm of  $y \in V_n^*$ .

For  $r \geq 0$  let  $W_2^r$  denote the space of Borel functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  whose derivatives up to order  $r$  are square integrable functions. The norm  $|\varphi|_r$  of  $\varphi$  in  $W_2^r$  is defined by

$$|\varphi|_r^2 = \sum_{|\gamma| \leq r} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^2 dx.$$

In particular,  $W_2^0 = L_2(\mathbb{R}^d)$  and  $|\varphi|_0 := |\varphi|_{L_2(\mathbb{R}^d)}$ .

The following basic example using wavelets will be used in the sequel. It describes spaces  $V_n, H_n$  and  $V_n^*$  and operators  $\Pi_n$  such that condition (2.4) is satisfied.

**Example 2.4. Wavelet approximation** We refer to [2] for a more detailed account on the subject; all the references in this example are taken from this book which will not be mentioned to lighten the statements.

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be an orthonormal scaling function, i.e., a real-valued, compactly supported function, such that:

(i) there exists a sequence  $(h_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$  for which  $\varphi(x) = \sum_k h_k \varphi(2x - k)$  in  $L^2(\mathbb{R})$ ,

(ii)  $\int \varphi(x - k) \varphi(x - l) dx = \delta_{k,l}$  for any  $k, l \in \mathbb{Z}$ .

We also suppose that there exist constants  $0 < C_1 < C_2$  such that for any finitely supported sequence  $(x_n)_{n \in \mathbb{Z}}$ ,  $C_1 \sum_n x_n^2 \leq \|\sum_n x_n \varphi(\cdot - n)\|_{L^2(\mathbb{R})}^2 \leq C_2 \sum_n x_n^2$ . We assume that the scaling function  $\varphi$  belongs to the Sobolev space  $H^s(\mathbb{R}) := W_2^s(\mathbb{R})$  for sufficiently large integer  $s > 0$ .

For  $d > 1$ ,  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , set  $\phi(x) = \varphi(x_1) \cdots \varphi(x_d)$  and for  $j \geq 0$  and  $k \in \mathbb{Z}^d$ , set  $\phi_{j,k}(x) = 2^{\frac{jd}{2}} \phi(2^j x - k) \in H^s := W_2^s(\mathbb{R}^d)$ . For any integer  $j \geq 0$ , let  $V_j$  denote the closure in  $L^2(\mathbb{R}^d)$  of the vector space generated by  $(\phi_{j,k}, k \in \mathbb{Z}^d)$  and for any  $f \in L^2(\mathbb{R}^d)$ , let  $\Pi_j f$  denote the projection defined by

$$\Pi_j f = \sum_{k \in \mathbb{Z}^d} (f, \phi_{j,k}) \phi_{j,k},$$

where  $(\cdot, \cdot)$  denotes the  $L^2(\mathbb{R}^d)$  scalar product. We also assume that  $(\phi_{j,k}, j \geq 0, k \in \mathbb{Z}^d)$  is a complete orthonormal system of  $L^2(\mathbb{R}^d)$ , so that  $\lim_j \|f - \Pi_j f\|_{L^2(\mathbb{R}^d)} = 0$ .

Set  $H := L^2(\mathbb{R}^d)$ ,  $V := H^s = W_2^s(\mathbb{R}^d)$  for some  $s \geq 0$ , and identify  $H$  with its dual  $H^*$  by the help of the inner product in  $H$ . Then  $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$  is a normal triple, where  $H^* \hookrightarrow V^*$  is the adjoint of the embedding  $V \hookrightarrow H$ . In particular,

$$\langle \Pi_n f, g \rangle = (\Pi_n f, g) = (f, \Pi_n g) = \langle f, \Pi_n g \rangle$$

for  $f \in V$  and  $g \in H$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $H^s$  and  $H^{-s}$ . This equality extends to the case  $f \in V$  and  $g \in V^*$ . Fix  $s \geq r \geq 0$ , let  $\varphi \in H^s$  be an orthonormal scaling function defined as above. The following inequalities give approximation properties:

$$(Direct) \quad \|f - \Pi_j f\|_{H^r} \leq C 2^{-j(s-r)} \|f\|_{H^s} \quad \forall f \in H^s, \quad (2.5)$$

$$(Converse) \quad \|\Pi_j f\|_{H^s} \leq C 2^{j(s-r)} \|f\|_{H^r}, \quad \forall f \in H^r. \quad (2.6)$$

Furthermore, for any  $r \in [0, s]$ ,  $\Pi_j$  is a bounded operator on  $W_2^r$  with norm uniformly bounded in  $j$ . The operators  $\Pi_j$  are clearly contractions of  $L_2 = W_2^0(\mathbb{R}^d)$ .

**Example 2.5. Finite differences – Discrete Sobolev spaces.** Consider for fixed  $h \in (0, 1)$  the grid

$$\mathbb{G} = h\mathbb{Z}^d = \{(k_1 h, k_2 h, \dots, k_d h) : k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d\},$$

where  $\mathbb{Z}$  denotes the set of integers. Use the notation  $\{e_1, e_2, \dots, e_d\}$  for the standard basis in  $\mathbb{R}^d$ . For any integer  $m \geq 0$ , let  $W_{h,2}^m$  be the set of real valued functions  $v$  on  $\mathbb{G}$  with

$$|v|_{h,m}^2 := \sum_{|\alpha| \leq m} \sum_{z \in \mathbb{G}} |\delta_+^\alpha v(z)|^2 h^d < \infty,$$

where  $\delta_{\pm i}^0$  is the identity and  $\delta_{\pm}^\alpha = \delta_{\pm 1}^{\alpha_1} \delta_{\pm 2}^{\alpha_2} \dots \delta_{\pm d}^{\alpha_d}$  for multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \{0, 1, 2, \dots\}^d$  of length  $|\alpha| := \alpha_1 + \dots + \alpha_d \geq 1$  is defined for by

$$\delta_{\pm i} v(z) := \pm \frac{1}{h} (v(z \pm h e_i) - v(z)).$$

We write also  $\delta^\alpha$  and  $\delta_i$  in place of  $\delta_+^\alpha$  and  $\delta_{+i}$ , respectively. Then  $W_{h,2}^m$  with the norm  $|\cdot|_{h,m}$  is a separable Hilbert space. It is the discrete counterpart of the Sobolev space  $W_2^m(\mathbb{R}^d)$ . Set  $W_{h,2}^{-1} = (W_{h,2}^1)^*$ , the adjoint of  $W_{h,2}^1$ , with its norm denoted by  $|\cdot|_{h,-1}$ . It is easy to see that  $W_{h,2}^m \hookrightarrow W_{h,2}^{m-1}$  is a dense and continuous embedding,

$$\begin{aligned} |v|_{h,m-1} &\leq |v|_{h,m}, \\ |v|_{h,m} &\leq \frac{\kappa}{h} |v|_{h,m-1}, \end{aligned} \quad (2.7)$$

for all  $v \in W_{h,2}^m$ ,  $m \geq 0$  and  $h \in (0, 1)$ , where  $\kappa$  is a constant depending only on  $d$ . Notice that for  $m \geq 1$

$$\langle v, u \rangle := \sum_{|\alpha| \leq m} \sum_{z \in \mathbb{G}} \delta^\alpha v \delta^\alpha u \leq C |v|_{h,m-1} |u|_{h,m+1} \quad \text{for all } v, u \in W_{h,2}^{m+1}$$

extends to a duality product between  $W_{h,2}^{m-1}$  and  $W_{h,2}^{m+1}$ , which makes it possible to identify  $W_{h,2}^{m-1}$  with  $(W_{h,2}^{m+1})^*$ .

Assume that  $m > \frac{d}{2}$ . Then by Sobolev's theorem on embedding  $W_2^m := W_2^m(\mathbb{R}^d)$  into  $\mathcal{C}(\mathbb{R}^d)$ , there is a bounded linear operator  $I : W_2^m(\mathbb{R}^d) \rightarrow \mathcal{C}(\mathbb{R}^d)$ , such that  $Iu = u$  almost everywhere on  $\mathbb{R}^d$ . Thus, identifying  $u$  with  $Iu$ , we can define the

operator  $R_h : W_2^m(\mathbb{R}^d) \rightarrow W_{h,2}^m$  by restricting the functions  $u \in W_2^m$  onto  $\mathbb{G} \subset \mathbb{R}^d$ . Moreover, due to Sobolev's theorem,

$$\sum_{z \in \mathbb{G}} \sup_{x \in \mathcal{I}(z)} |u(x)|^2 h^d \leq p^2 |u|_{W_2^m}^2,$$

where  $\mathcal{I}(z) := \{x \in \mathbb{R}^d : z_k \leq x_k \leq z_k + h, \quad k = 1, 2, \dots, d\}$  and  $p$  is a constant depending only on  $m$  and  $d$ . Hence obviously

$$|R_h u|_{h,0}^2 \leq p |u|_{W_2^m}^2 \quad \text{for all } u \in W_2^m. \quad (2.8)$$

Moreover, for every integer  $l \geq 0$

$$|R_h u|_{h,l} \leq p |u|_{W_2^{m+l}} \quad \text{for all } u \in W_2^{m+l}, \quad (2.9)$$

with a constant  $p$  depending only on  $m$ ,  $l$  and  $d$ . Thus setting

$$V_n := W_{h_n,2}^{m+l}, \quad H_n := W_{h_n,2}^{m+l-1}, \quad V_n^* \equiv W_{h_n,2}^{m+l-2}, \\ \Pi_n := R_{h_n}$$

for any sequence  $\{h_n\}_{n=0}^\infty \subset (0, 1)$  and any integers  $m > \frac{d}{2}$ ,  $l \geq 0$  we get examples of approximation spaces.

When approximating differential operators by finite differences we need to estimate  $D_i u - \delta_{\pm i} u$  in discrete Sobolev norms. For  $d = 1$  we can estimate this as follows. Let  $l \geq 0$  be an integer and set  $z_k := kh$  for  $k \in \mathbb{Z}$ . By the mean value theorem there exist  $z'_k$  and  $z''_k$  in  $[z_k, z_k + lh]$  such that  $\delta^l Du(z_k) = D^{l+1}u(z'_k)$  and  $\delta^l \delta u(z_k) = D^{l+1}u(z''_k)$ , where  $D := \frac{d}{dx}$ . Hence

$$|\delta^l (Du(z_k) - \delta u(z_k))|^2 = |D^{l+1}u(z'_k) - D^{l+1}u(z''_k)|^2 = \left| \int_{z'_k}^{z''_k} D^{l+2}u(y) dy \right|^2 \\ \leq lh \int_{z_k}^{z_k+lh} |D^{l+2}u(y)|^2 dy$$

for  $u \in C_0^\infty(\mathbb{R})$ . Consequently,

$$|Du - \delta_{\pm}^l u|_{h,l} \leq lh |u|_{W_2^{l+2}(\mathbb{R})} \quad (2.10)$$

for  $u \in C_0^\infty(\mathbb{R})$ , and hence for all  $u \in W_2^{l+2}(\mathbb{R})$ . For  $d > 1$  by similar calculation combined with Sobolev's embedding, we get that for  $m > l + 2 + \frac{d-1}{2}$

$$|D_i u - \delta_{\pm i} u|_{h,l} \leq Ch |u|_{W_2^m} \quad (2.11)$$

for all  $u \in W_2^m$ ,  $h \in (0, 1)$ , where  $C$  is a constant depending on  $l$ ,  $m$  and  $d$ .

### 3. SPACE DISCRETIZATION

**3.1. Description of the scheme.** Consider for each integer  $n \geq 0$  the problem

$$du^n(t) = A^n(t, u^n(t)) dt + \sum_k B^n(t, u^n(t)) dW^k(t), \quad u^n(0) = u_0^n, \quad (3.1)$$

where  $u_0^n$  is an  $H_n$ -valued  $\mathcal{F}_0$ -measurable random variable, and

$$A^n : [0, T] \times \Omega \times V_n \rightarrow V_n^*, \quad B^n : [0, T] \times \Omega \times V_n \rightarrow H_n^{d_1}$$

are  $\mathcal{P} \otimes \mathcal{B}(V_n)$ -measurable mappings such that the following conditions hold.

**(C1n)** (*Strong monotonicity*) There exist constants  $\lambda > 0$  and  $L$  such that for all  $n \geq 0$  almost surely

$$\begin{aligned} 2\langle u - v, A^n(t, u) - A^n(t, v) \rangle_n + \sum_k |B_k^n(t, u) - B_k^n(t, v)|_{H_n}^2 + \lambda \|u - v\|_{V_n}^2 \\ \leq L \|u - v\|_{H_n}^2 \quad \text{for all } t \in [0, T], u, v \in V_n. \end{aligned}$$

**(C2n)** (*Growth condition on  $A^n, B^n$* ) Almost surely

$$|A^n(t, v)|_{V_n^*}^2 \leq K \|v\|_{V_n}^2 + f^n(t), \quad |B^n(t, v)|_{H_n}^2 \leq K \|v\|_{V_n}^2 + g^n(t)$$

for all  $t \in [0, T]$ ,  $v \in V_n$  and  $n \geq 0$ , where  $K$  is a constant, independent of  $n$ , and  $f^n$  and  $g^n$  are non-negative stochastic processes such that

$$\sup_n E \int_0^T f^n(t) dt =: M_1 < \infty, \quad \sup_n E \int_0^T g^n(t) dt =: M_2 < \infty.$$

**(C3n)** (*Lipschitz condition on  $B^n$* ) Almost surely

$$\sum_k |B_k^n(t, u) - B_k^n(t, v)|_{H_n}^2 \leq L_B \|u - v\|_{V_n}^2$$

for all  $t \in [0, T]$  and  $u, v \in V_n$ .

**(C4n)**  $\sup_n E |u_0^n|_{H_n}^2 < \infty$ .

**(C5n)** (*Hemicontinuity of  $A^n$* ) For every  $n \geq 0$ , the operators  $A^n$  are hemicontinuous in  $v \in V_n$ , i.e., almost surely

$$\lim_{\varepsilon \rightarrow 0} \langle A^n(t, v + \varepsilon u), w \rangle_n = \langle A^n(t, v), w \rangle_n$$

for all  $t \in [0, T]$ ,  $v, u, w \in V_n$ .

Notice that **(C1n)** - **(C3n)** imply the coercivity condition

$$2\langle v, A^n(t, v) \rangle_n + \sum_k |B_k^n(t, v)|_{H_n}^2 + \frac{\lambda}{2} \|v\|_{V_n}^2 \leq C (\|v\|_{H_n}^2 + f^n(t) + g^n(t))$$

with a constant  $C$  depending on  $\lambda$  and  $L_B$ . It is classical (see e.g. [10], [13] and [16]) that **(C1n)**–**(C5n)** ensure the existence of a unique solution  $u^n$  to (3.1). It has an  $H_n$ -valued continuous stochastic modification and

$$\begin{aligned} E \sup_{0 \leq t \leq T} |u^n(t)|_{H_n}^2 + E \int_0^T \|u^n(t)\|_{V_n}^2 dt \\ \leq C \sup_n \left( E |u_0^n|_{H_n}^2 + E \int_0^T (f^n(t) + g^n(t)) dt \right) < \infty, \end{aligned} \quad (3.2)$$

where  $C$  is a constant depending only on  $\lambda$ ,  $K$  and  $L_B$ .

**3.2. Rate of convergence of the space discretization.** Recall that if **(R1)(i)** and **(R2)** are satisfied, for every  $t \in [0, T]$  and  $v \in \mathcal{V}$ ,  $A(t, v) \in V$  and  $B_k(t, v) \in V$ ; hence  $\Pi_n A(t, v) \in V_n$  and  $\Pi_n B_k(t, v) \in V_n$ . In order to estimate the norm of the difference  $\Pi_n u - u^n$ , we need to relate the various operators  $A, A^n, B$  and  $B^n$ , which is done with the following consistency condition.

**(Cn)** (*Consistency condition*) There exist a sequence  $(\varepsilon_n)_{n \geq 0}$  of positive numbers and a sequence  $(\xi^n)_{n \geq 0}$  of non-negative adapted processes such that

$$\sup_n E \int_0^T \xi^n(t) dt \leq M < +\infty,$$

and almost surely

$$\begin{aligned} |\Pi_n A(t, v) - A^n(t, \Pi_n v)|_{V_n^*}^2 + \sum_k |\Pi_n B_k(t, v) - B_k^n(t, \Pi_n v)|_{H_n}^2 \\ \leq \varepsilon_n^2 (\|v\|_{\mathcal{V}}^2 + \xi^n(t)) \end{aligned} \quad (3.3)$$

for all  $t \in [0, T]$  and  $v \in \mathcal{V}$ .

**Theorem 3.1.** *Let (R1) (i), (R2), (C1n)–(C5n) and (Cn) hold. Then for  $e^n(t) := \Pi_n u(t) - u^n(t)$ ,*

$$E \sup_{0 \leq t \leq T} |e^n(t)|_{H_n}^2 + E \int_0^T \|e^n(t)\|_{V_n}^2 dt \leq C_1 E |e^n(0)|_{H_n}^2 + C_2 (r_1 + M) \varepsilon_n^2 \quad (3.4)$$

holds for all  $n \geq 0$ , where  $C_1 = C_1(\lambda, L, T)$  and  $C_2 = C_2(\lambda, L, L_B, T)$  are constants.

*Proof.* From equation (1.1) we deduce that for every  $n \geq 0$ ,

$$\Pi_n u(t) = \Pi_n u_0 + \int_0^t \Pi_n A(s, u(s)) ds + \sum_k \int_0^t \Pi_n B_k(s, u(s)) dW^k(s).$$

Using Itô's formula

$$|e^n(t)|_n^2 = |e^n(0)|_n^2 + \sum_{i \leq 3} I_i(t), \quad (3.5)$$

where

$$\begin{aligned} I_1(t) &= 2 \int_0^t \langle e^n(s), \Pi_n A(s, u(s)) - A^n(s, u^n(s)) \rangle_n ds, \\ I_2(t) &= 2 \sum_k \int_0^t \langle e^n(s), \Pi_n B_k(s, u(s)) - B_k^n(s, u^n(s)) \rangle_n dW^k(s), \\ I_3(t) &= \sum_k \int_0^t |\Pi_n B_k(s, u(s)) - B_k^n(s, u^n(s))|_n^2 ds. \end{aligned}$$

We first prove

$$\sup_{0 \leq t \leq T} E |e^n(t)|_n^2 + E \int_0^T \|e^n(t)\|_n^2 dt \leq C_1 E |e^n(0)|_n^2 + C_2 (r_1 + M) \varepsilon_n^2, \quad (3.6)$$

where  $C_1 = C_1(\lambda, L, T)$  and  $C_2 = C_2(\lambda, L, L_B, T)$  are constants. The strong monotonicity condition **(C1n)** implies

$$I_1(t) + I_3(t) \leq -\lambda \int_0^t \|e^n(s)\|_n^2 ds + L \int_0^t |e^n(s)|_n^2 ds + \sum_{i=1,2} R_i(t), \quad (3.7)$$

where

$$R_1(t) = \int_0^t 2 \langle e^n(s), \Pi_n A(s, u(s)) - A^n(s, \Pi_n u(s)) \rangle_n ds,$$

$$R_2(t) = \sum_k \int_0^t \left[ |\Pi_n B_k(s, u(s)) - B_k^n(s, u^n(s))|_n^2 - |B_k^n(s, \Pi_n u(s)) - B_k^n(s, u^n(s))|_n^2 \right] ds.$$

Schwarz's inequality, then the consistency condition **(Cn)** and **(R1)** (i) imply that for every  $n \geq 0$  and  $t \in [0, T]$ ,

$$\begin{aligned} |R_1(t)| &\leq \frac{\lambda}{3} \int_0^t \|e^n(s)\|_n^2 ds + \frac{3}{\lambda} \int_0^t |\Pi_n A(s, u(s)) - A^n(s, \Pi_n u(s))|_{n^*}^2 ds \\ &\leq \frac{\lambda}{3} \int_0^t \|e^n(s)\|_n^2 ds + \frac{3}{\lambda} \varepsilon_n^2 (r_1 + M). \end{aligned}$$

Schwarz's inequality, the consistency condition **(Cn)**, **(R1)** (i) and the Lipschitz condition **(C3n)** yield that for every  $\alpha > 0$ ,

$$\begin{aligned} |R_2(t)| &\leq \sum_k \int_0^t \left[ |\Pi_n B_k(s, u(s)) - B_k^n(s, \Pi_n u(s))|_n^2 + 2 \left( \Pi_n B_k(s, u(s)) - B_k^n(s, \Pi_n u(s)), B_k^n(s, \Pi_n u(s)) - B_k^n(s, u^n(s)) \right)_n \right] ds \\ &\leq (1 + \frac{1}{\alpha}) \int_0^t \sum_k |\Pi_n B_k(s, u(s)) - B_k^n(s, \Pi_n u(s))|_n^2 ds \\ &\quad + \alpha \int_0^t \sum_k |B_k^n(s, \Pi_n u(s)) - B_k^n(s, u^n(s))|_n^2 ds \\ &\leq (1 + \frac{1}{\alpha})(r_1 + M)\varepsilon_n^2 + \alpha L_B \int_0^t \|e^n(s)\|_n^2 ds. \end{aligned} \tag{3.8}$$

Thus, for  $\alpha L_B \leq \frac{\lambda}{3}$ , taking expectations in (3.5) and (3.7)-(3.8), we deduce that

$$E|e^n(t)|_n^2 + \frac{\lambda}{3} E \int_0^t \|e^n(s)\|_n^2 ds \leq LE \int_0^t |e^n(s)|_n^2 ds + E|e^n(0)|_n^2 + C(r_1 + M)\varepsilon_n^2,$$

where  $C = C(\lambda, L_B)$  is a constant. Since by (2.2) and (3.2)

$$\sup_{0 \leq t \leq T} E|e^n(t)|_n^2 < +\infty,$$

Gronwall's lemma gives

$$\sup_{0 \leq t \leq T} E|e^n(t)|_n^2 \leq e^{LT} (C(r_1 + M)\varepsilon_n^2 + E|e^n(0)|_n^2),$$

which in turn yields (3.6). We now prove (3.4). From (3.6)-(3.8) we deduce

$$\begin{aligned} E \sup_{0 \leq t \leq T} (I_1(t) + I_3(t)) &\leq LE \int_0^T |e^n(s)|_n^2 ds + \frac{2\lambda}{3} E \int_0^T \|e^n(s)\|_n^2 ds + C_2(cr_1 + M)\varepsilon_n^2 \\ &\leq C_1 E|e^n(0)|_n^2 + C_2(r_1 + M)\varepsilon_n^2. \end{aligned} \tag{3.9}$$

By Davies' inequality, (2.2), **(C3n)**, **(Cn)** and **(R1)**(i),

$$E \sup_{0 \leq t \leq T} |I_2(t)| \leq 6E \left( \int_0^T \sum_k \left| \left( e^n, \Pi_n B_k(u) - B_k^n(u^n) \right)_n \right|^2 ds \right)^{\frac{1}{2}}$$

$$\begin{aligned}
&\leq 6E \left\{ \sup_{0 \leq t \leq T} |e^n(t)|_n \left( \int_0^T \sum_k |\Pi_n B_k(u) - B_k^n(u^n)|_n^2 ds \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{1}{2} E \sup_{0 \leq t \leq T} |e^n(t)|_n^2 \\
&\quad + 36E \sum_k \int_0^T \left[ |\Pi_n B_k(u) - B_k^n(\Pi_n u)|_n^2 + |B_k^n(\Pi_n u) - B_k^n(u^n)|_n^2 \right] ds \\
&\leq \frac{1}{2} E \sup_{0 \leq t \leq T} |e^n(t)|_n^2 + 36L_B E \int_0^T \|e^n(s)\|_n^2 ds + 36(r_1 + M)\varepsilon_n^2, \tag{3.10}
\end{aligned}$$

where the argument  $s$  is omitted from most integrands. Thus, inequalities (3.5), (3.9), (3.10) and (3.6) yield

$$\frac{1}{2} E \sup_{0 \leq t \leq T} |e^n(t)|_n^2 \leq C_1 E |e^n(0)|_n^2 + C_2 (r_1 + M) \varepsilon_n^2,$$

with some constants  $C_1 = C_1(L, T)$  and  $C_2 = (\lambda, L, L_B, T)$ , which completes the proof of (3.4).  $\square$

**3.3. Examples.** Let  $r > 0$  be an integer,  $V = \mathcal{H} = W_2^r(\mathbb{R}^d)$ ,  $H = L^2(\mathbb{R}^d)$ ,  $V^* = W_2^{-r}(\mathbb{R}^d)$  denote its dual with respect to the  $L^2$ -inner product, and for  $l > 0$  let  $\mathcal{V} = W_2^{r+l}(\mathbb{R}^d)$ . Let  $A$  and  $B$  be operators which satisfy the conditions **(R0)**, **(R2)** and **(R3)**. Let  $\varphi$  be an orthonormal scaling function in  $W_2^{r+l}(\mathbb{R})$ ; for any  $n \geq 0$  let  $V_n$  and  $\Pi_n$  be defined as in Example 2.4. We denote by  $H_n$  and  $\Pi_n$  the subspaces of  $H$  and the orthogonal projections defined in Example 2.4,  $V_n = \Pi_n V$  the Banach space endowed with the topology induced by that of  $V$ , and  $V_n^*$  denotes its dual. Then the operators  $\Pi_n$  are bounded from  $V$  to  $V_n$ ,  $H$  to  $H_n$  and from  $V^*$  to  $V_n^*$ ; furthermore, given  $u, v \in V$ , and resp.  $u, v \in H$ ,

$$\langle \Pi_n u, v \rangle = \langle u, \Pi_n v \rangle, \quad \text{and resp.} \quad (\Pi_n u, v) = (u, \Pi_n v). \tag{3.11}$$

For  $t \in [0, T]$ ,  $k = 1, \dots, d_1$ ,  $n \geq 0$  and  $u \in V_n$ , set

$$A^n(t, u, \omega) = \Pi_n A(t, u, \omega), \quad B_k^n(t, u, \omega) = \Pi_n B_k(t, u, \omega). \tag{3.12}$$

Then it is easy to see that the operators  $A^n$  and  $B^n$  satisfy the condition **(C1n)**. Indeed, for  $u, v \in V_n \subset V$ ,  $t \in [0, T]$ , **(R0)** and (2.4) yield

$$\begin{aligned}
&2 \langle u - v, A^n(s, u) - A^n(s, v) \rangle_n + \sum_k |B_k^n(s, u) - B_k^n(s, v)|_{H_n}^2 \\
&\leq -\lambda \|u - v\|_{V_n}^2 + K \|u - v\|_{H_n}^2.
\end{aligned}$$

Furthermore, conditions **(R0)** (ii)-(iii) and (2.4) imply **(C2n)**-**(C3n)** and **(C5n)**. Finally, if  $p$  is the constant defined in (2.4), (2.5), **(R0)** (ii) and (iii) imply for  $u \in \mathcal{V}$ ,

$$\begin{aligned}
&|\Pi_n A(t, u) - A^n(t, \Pi_n u)|_{V_n^*}^2 + \sum_k |\Pi_n B_k(t, u) - B_k^n(t, \Pi_n u)|_{H_n}^2 \\
&\leq p^2 (K_1 + K_2) \|u - \Pi_n u\|_V^2 \leq C p^2 (K_1 + K_2) 2^{-2nl} \|u\|_{\mathcal{V}}^2,
\end{aligned}$$

which yields **(Cn)** with  $\varepsilon_n = 2^{-nl}$  and  $h_n = 0$ . In the last section, we will give examples of operators such that condition **(R1)**(i) holds.

## 4. IMPLICIT SPACE-TIME DISCRETIZATIONS

**4.1. Description of the scheme.** For a fixed integer  $m \geq 1$  set  $\tau := T/m$  and  $t_i = i\tau$  for  $i = 0, \dots, m$ . Let  $V_n \hookrightarrow H_n \hookrightarrow V_n^*$  satisfy the conditions in section 2.2. Consider for each  $n$  the system of equations

$$u_{i+1}^{n,\tau} = u_i^{n,\tau} + \tau A_{i+1}^{n,\tau}(u_{i+1}^{n,\tau}) + \sum_k B_{k,i}^{n,\tau}(u_i^{n,\tau}) (W^k(t_{i+1}) - W^k(t_i)) \quad (4.1)$$

for  $V_n$ -valued  $\mathcal{F}_{t_i}$ -measurable random variables  $u_i^{n,\tau}$  for  $i = 1, 2, \dots, m$ .

Here  $u_0^{n,\tau}$  is a given  $V_n$ -valued  $\mathcal{F}_0$ -measurable random variable, while

$$A_i^{n,\tau} : \Omega \times V_n \rightarrow V_n^* \quad \text{and} \quad B_{k,i}^{n,\tau} : \Omega \times V_n \rightarrow H_n, \text{ for } k = 1, \dots, d_1,$$

are given  $\mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)$ -measurable mappings for  $i = 0, 1, \dots, m$ , such that the following conditions hold.

**(C1n $\tau$ )** (*Strong monotonicity*) There exist constants  $\lambda > 0$  and  $L \geq 0$  such that almost surely

$$\begin{aligned} 2\langle u - v, A_i^{n,\tau}(u) - A_i^{n,\tau}(v) \rangle_n + \sum_k |B_{k,i}^{n,\tau}(u) - B_{k,i}^{n,\tau}(v)|_{H_n}^2 \\ \leq -\lambda \|u - v\|_{V_n}^2 + L \|u - v\|_{H_n}^2 \end{aligned}$$

for all  $v, u \in V_n$ ,  $i = 1, 2, \dots, m$ ,  $m \geq 1$  and  $n \geq 0$ .

**(C2n $\tau$ )** (*Growth condition on  $A_i^{n,\tau}$  and  $B_i^{n,\tau}$* ) There is a constant  $K$  such that almost surely

$$|A_i^{n,\tau}(u)|_{V_n^*}^2 \leq K \|u\|_{V_n}^2 + f_i^{n,\tau}, \quad |B_i^{n,\tau}(u)|_{H_n}^2 \leq K \|u\|_{V_n}^2 + g_i^{n,\tau},$$

for all  $u \in V_n$ ,  $n \geq 0$ ,  $m \geq 1$  and  $i = 0, 1, \dots, m$ , where  $f_i^{n,\tau}$  and  $g_i^{n,\tau}$  are non-negative random variables, such that

$$\sup_n \sup_m \sum_{1 \leq i \leq m} \tau E f_i^{n,\tau} \leq M < +\infty, \quad \sup_n \sup_m \max_{0 \leq i < m} E g_i^{n,\tau} \leq M < +\infty.$$

**(C3n $\tau$ )** (*Lipschitz condition on  $A_i^{n,\tau}$  and  $B_i^{n,\tau}$* ) There exist constants  $L_1$  and  $L_2$  such that almost surely

$$\begin{aligned} |A_i^{n,\tau}(u) - A_i^{n,\tau}(v)|_{V_n^*}^2 &\leq L_1 \|u - v\|_{V_n}^2, \\ \sum_k |B_{k,i}^{n,\tau}(u) - B_{k,i}^{n,\tau}(v)|_{H_n}^2 &\leq L_2 \|u - v\|_{V_n}^2 \end{aligned} \quad (4.2)$$

for all  $u, v \in V_n$ ,  $n \geq 0$ ,  $m \geq 1$  and  $i = 0, 1, \dots, m$ .

**(C4n $\tau$ )**  $\sup_{n,m} E \|u_0^{n,\tau}\|_{V_n}^2 \leq M$ .

**Remark 4.1.** Conditions **(C1n $\tau$ )**–**(C2n $\tau$ )** imply that almost surely

$$2\langle u, A_i^{n,\tau}(u) \rangle_n + \sum_k |B_{k,i}^{n,\tau}(u)|_{H_n}^2 \leq -\frac{\lambda}{2} \|u\|_{V_n}^2 + L \|u\|_{H_n}^2 + C(f_i^{n,\tau} + g_i^{n,\tau})$$

for all  $u \in V_n$ ,  $n, i$  and  $m$ , where  $C = C(\lambda, K)$  is a constant. The Lipschitz condition on  $A_i^{n,\tau}$  from **(C3n $\tau$ )** obviously implies that  $A_i^{n,\tau}$  is hemicontinuous.

**Proposition 4.2.** Let **(C1n $\tau$ )**–**(C4n $\tau$ )** hold. Then for  $\tau < 1/L$  equation (4.1) has a unique  $V_n$ -valued solution  $(u_i^{n,\tau})_{i=0}^m$ , such that  $u_i^{n,\tau}$  is  $\mathcal{F}_{t_i}$ -measurable and  $E \|u_i^{n,\tau}\|_{V_n}^2$  is finite for each  $i, n$ . (Here  $1/L := \infty$  if  $L = 0$ .)

*Proof.* Equation (4.1) can be rewritten as

$$D_{i+1}(u_{i+1}^{n,\tau}) = u_i^{n,\tau} + \sum_k B_{k,i}^{n,\tau}(u_i^{n,\tau}) (W^k(t_{i+1}) - W^k(t_i)), \quad (4.3)$$

where  $D_i : V_n \rightarrow V_n^*$  is defined by  $D_i(v) = v - \tau A_i^{n,\tau}(v)$  for each  $i = 1, 2, \dots, m$ . Due to **(C1n $\tau$ )**-**(C3n $\tau$ )** and Remark 4.1 the operator  $D_i$  satisfies the assumptions (monotonicity, coercivity, linear growth and hemicontinuity) of Proposition 3.4 in [7] with  $p = 2$ . By virtue of this proposition, for  $\tau < 1/L$ , equation (4.3) has a unique  $V_n$ -valued  $\mathcal{F}_{t_{i+1}}$ -measurable solution  $u_{i+1}^{n,\tau}$  for every given  $V$ -valued  $\mathcal{F}_{t_i}$ -measurable random variable  $u_i^{n,\tau}$ , and

$$\begin{aligned} E \|u_{i+1}^{n,\tau}\|_{V_n}^2 &\leq C E \left( 1 + f_i^{n,\tau} + g_i^{n,\tau} + \left| \sum_k B_{k,i}^{n,\tau}(u_i^{n,\tau}) (W^k(t_{i+1}) - W^k(t_i)) \right|^2 \right) \\ &\leq C \left( 1 + E f_i^{n,\tau} + E g_i^{n,\tau} + \sum_k \tau E |B_{k,i}^{n,\tau}(u_i^{n,\tau})|_n^2 \right) \\ &\leq C (1 + E f_i^{n,\tau} + E g_i^{n,\tau} + K \tau E \|u_i^{n,\tau}\|_{V_n}^2 + \tau E g_i^{n,\tau}), \end{aligned}$$

where  $C = C(\lambda, \tau)$  is a constant. Hence induction on  $i$  concludes the proof.  $\square$

**4.2. Rate of convergence of the implicit scheme.** Assume that **(R1)** and **(R2)** hold. To approximate  $\Pi_n u$  by  $u^{n,\tau}$  we have to relate the operators  $\Pi_n A(t_i, \cdot)$  and  $A_i^{n,\tau} \circ \Pi_n$  as well as the operators  $\Pi_n B_k(t_i, \cdot)$  and  $B_{k,i}^{n,\tau} \circ \Pi_n$  on  $\mathcal{V}$ .

**(Cn $\tau$ )** (*Consistency condition*) There exist constants  $\nu \in ]0, \frac{1}{2}]$ ,  $c \geq 0$ , a sequence of numbers  $\varepsilon_n \geq 0$ , such that almost surely

$$\begin{aligned} |\Pi_n A(t_i, u) - A_i^{n,\tau}(\Pi_n u)|_{V_n^*}^2 &\leq c (\|u\|_{\mathcal{V}}^2 + \xi_i^{n,\tau}) (\tau^{2\nu} + \varepsilon_n^2), \\ \sum_k |\Pi_n B_k(t_i, u) - B_{k,i}^{n,\tau}(\Pi_n u)|_{H_n}^2 &\leq c (\|u\|_{\mathcal{V}}^2 + \eta_i^{n,\tau}) (\tau^{2\nu} + \varepsilon_n^2) \end{aligned}$$

for all  $i = 0, \dots, m-1$  and  $u \in \mathcal{V}$ , where  $\xi_i^{n,\tau}$  and  $\eta_i^{n,\tau}$  are non-negative random variables such that

$$\sup_{n,m} \sum_i \tau E \xi_i^{n,\tau} \leq M, \quad \sup_{n,m} \sum_i \tau E \eta_i^{n,\tau} \leq M.$$

**Theorem 4.3.** *Let **(R1)**-**(R3)** hold. Assume **(C1n $\tau$ )**-**(C4n $\tau$ )** and **(Cn $\tau$ )**. Set  $e_i^{n,\tau} = \Pi_n u(t_i) - u_i^{n,\tau}$ . Then for  $\tau < 1/L$  and  $n \geq 1$*

$$E \max_{0 \leq i \leq m} |e_i^{n,\tau}|_{H_n}^2 + \sum_{1 \leq i \leq m} \tau E \|e_i^{n,\tau}\|_{V_n}^2 \leq C_1 E |e_0^{n,\tau}|_{H_n}^2 + C_2 (\tau^{2\nu} + \varepsilon_n^2) (r_1 + r_2 + M), \quad (4.4)$$

where  $C_1 = C_1(\lambda, L, T)$  and  $C_2 = C_2(\lambda, L, K, T, p, c, L_1, L_2)$  are constants.

*Proof.* We fix  $n, \tau$ , and to ease notation we write  $e_i, A_i$  and  $B_{k,i}$  in place of  $e_i^{n,\tau}, A_i^{n,\tau}$  and  $B_{k,i}^{n,\tau}$ , respectively. Similarly, we often use  $u_i$  in place of  $u_i^{n,\tau}$  for  $i = 1, 2, \dots, m$ . Then for any  $i = 0, \dots, m-1$ ,

$$\begin{aligned} |e_{i+1}|_n^2 - |e_i|_n^2 &= 2 \int_{t_i}^{t_{i+1}} \langle e_{i+1}, \Pi_n A(s, u(s)) - A_{i+1}(u_{i+1}) \rangle_n ds \\ &\quad + 2 \sum_k \int_{t_i}^{t_{i+1}} (e_{i+1}, F_k(s))_n dW^k(s) \end{aligned}$$

$$\begin{aligned}
& - \left| \int_{t_i}^{t_{i+1}} [\Pi_n A(s, u(s)) - A_{i+1}(u_{i+1})] ds + \sum_k \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right|_n^2 \\
& = 2 \int_{t_i}^{t_{i+1}} \langle e_{i+1}, \Pi_n A(s, u(s)) - A_{i+1}(u_{i+1}) \rangle_n ds \\
& \quad + \left| \sum_k \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right|_n^2 + 2 \sum_k \int_{t_i}^{t_{i+1}} (e_i, F_k(s))_n dW^k(s) \\
& \quad - \left| \int_{t_i}^{t_{i+1}} [\Pi_n A(s, u(s)) - A_{i+1}(u_{i+1})] ds \right|_n^2,
\end{aligned}$$

where for  $k = 1, \dots, d_1$  one sets

$$F_k(s) = \Pi_n B_k(s, u(s)) - B_{k,i}(u_i^{n,\tau}), \quad s \in [t_i, t_{i+1}], \quad i = 0, 1, \dots, m-1.$$

Summing up for  $i = 0, \dots, l-1$ , we obtain

$$|e_l|_n^2 \leq |e_0|_n^2 + 2 \sum_{0 \leq i < l} \int_{t_i}^{t_{i+1}} \langle e_{i+1}, \Pi_n A(s, u(s)) - A_{i+1}(u_{i+1}) \rangle_n ds + Q(t_l) + I(t_l), \quad (4.5)$$

where

$$\begin{aligned}
Q(t_l) &= \sum_{0 \leq i < l} \left| \sum_k \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right|_n^2, \\
I(t_l) &= 2 \sum_k \int_0^{t_l} (e(s), F_k(s))_n dW^k(s), \quad e(s) := e_i \text{ for } s \in [t_i, t_{i+1}], \quad i = 0, \dots, m.
\end{aligned}$$

First we show

$$\sup_{0 \leq l \leq m} E|e_l|_n^2 + E \sum_{1 \leq i \leq m} \tau \|e_i\|_n^2 \leq C_1 E|e_0|_n^2 + C_2(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \quad (4.6)$$

where  $C_1 = C_1(\lambda, L, T)$  and  $C_2 = C_2(\lambda, L, K, T, p, c, L_1, L_2)$  are constants. To this end we take expectation in both sides of (4.5) and use the strong monotonicity condition **(C1n $\tau$ )**, to get

$$\begin{aligned}
E|e_l|_n^2 &\leq E|e_0|_n^2 + 2E \sum_{0 \leq i < l} \tau \langle e_{i+1}, A_{i+1}(\Pi_n u(t_{i+1})) - A_{i+1}(u_{i+1}) \rangle_n \\
&\quad + E \sum_{0 \leq i < l-1} \sum_k \tau |B_{k,i+1}(\Pi_n u(t_{i+1})) - B_{k,i+1}(u_{i+1})|_n^2 + \sum_{1 \leq j \leq 3} S_j \\
&\leq E|e_0|_n^2 - \lambda \sum_{1 \leq i \leq l} \tau E \|e_i\|_n^2 + L \sum_{1 \leq i \leq l} \tau E |e_i|_n^2 + \sum_{1 \leq j \leq 3} S_j \quad (4.7)
\end{aligned}$$

for  $l = 1, \dots, m$ , where

$$\begin{aligned}
S_1 &= 2 \sum_{1 \leq i \leq l} E \int_{t_{i-1}}^{t_i} \langle e_i, \Pi_n A(s, u(s)) - A_i(\Pi_n u(t_i)) \rangle_n ds, \\
S_2 &= \sum_k \sum_{1 \leq i < l} E \int_{t_i}^{t_{i+1}} [|F_k(s)|_n^2 - |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2] ds,
\end{aligned}$$

$$S_3 = \sum_k E \int_0^\tau |F_k(s)|_n^2 ds.$$

For any  $\varepsilon > 0$

$$S_1 \leq \varepsilon \sum_{1 \leq i \leq l} \tau E \|e_i\|_n^2 + \frac{1}{\varepsilon} R,$$

where

$$\begin{aligned} R &= \sum_{1 \leq i \leq l} E \int_{t_{i-1}}^{t_i} |\Pi_n A(s, u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2 ds \leq 3 \sum_{1 \leq j \leq 3} R_j, \quad (4.8) \\ R_1 &= \sum_{1 \leq i \leq l} E \int_{t_{i-1}}^{t_i} |\Pi_n A(s, u(s)) - \Pi_n A(t_i, u(s))|_{n^*}^2 ds, \\ R_2 &= \sum_{1 \leq i \leq l} E \int_{t_{i-1}}^{t_i} |\Pi_n A(t_i, u(s)) - A_i(\Pi_n u(s))|_{n^*}^2 ds, \\ R_3 &= \sum_{1 \leq i \leq l} E \int_{t_{i-1}}^{t_i} |A_i(\Pi_n u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2 ds. \end{aligned}$$

Due to **(R3)** (i), (2.4), **(Cn $\tau$ )**, **(C3n $\tau$ )** and (2.3)

$$R_1 \leq \tau^{2\nu} p^2 E \int_0^T (K|u(s)|_{\mathcal{V}}^2 + \eta) ds, \quad (4.9)$$

$$R_2 \leq (\tau^{2\nu} + \varepsilon_n^2) \left( cE \int_0^T |u(s)|_{\mathcal{V}}^2 ds + \sum_{1 \leq i \leq m} \tau E \xi_i^{n,\tau} \right), \quad (4.10)$$

$$R_3 \leq L_1 p^2 \sum_{1 \leq i \leq l} \int_{t_{i-1}}^{t_i} E \|u(s) - u(t_i)\|^2 ds \leq T L_1 p^2 M_1 \tau. \quad (4.11)$$

with  $M_1 := C(r_1 + r_2 + M)$ . By (2.4), the growth conditions **(R2)**, **(C2n $\tau$ )** and by condition **(C4n $\tau$ )** on the initial values

$$\begin{aligned} S_3 &\leq 2 \sum_k \int_0^\tau E |\Pi_n B_k(s, u(s))|_n^2 ds + 2 \sum_k \tau E |B_{k,0}(u_0^{n,\tau})|_n^2 \\ &\leq 2\tau p^2 \left( K \sup_{t \in [0, T]} E \|u(t)\|_{\mathcal{H}}^2 + \sup_{t \in [0, T]} E \eta(t) \right) \\ &\quad + 2\tau \left( K \sup_{n,m} E \|u_0^{n,\tau}\|_n^2 + \sup_{n,m} E g_0^{n,\tau} \right). \end{aligned} \quad (4.12)$$

Using the simple inequality  $|b|_n^2 - |a|_n^2 \leq \varepsilon |a|_n^2 + (1 + \frac{1}{\varepsilon}) |b - a|_n^2$  with

$$a := B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i), \quad b := F_k(s),$$

for any  $\varepsilon > 0$  we have  $S_2 \leq \varepsilon P_1 + (1 + \frac{1}{\varepsilon}) P_2$  with

$$\begin{aligned} P_1 &:= \sum_{1 \leq i \leq l} E \sum_k \tau |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2, \\ P_2 &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s, u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2 ds. \end{aligned} \quad (4.13)$$

By the Lipschitz condition **(C3n $\tau$ )** on  $B_{k,i}$  we get  $P_1 \leq L_2 E \sum_{1 \leq i \leq l} \tau \|e_i\|_n^2$ . Clearly,  $P_2 \leq 3(Q_1 + Q_2 + Q_3)$  with

$$\begin{aligned} Q_1 &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s, u(s)) - \Pi_n B_k(t_i, u(s))|_n^2 ds, \\ Q_2 &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(t_i, u(s)) - B_{k,i}(\Pi_n u(s))|_n^2 ds, \\ Q_3 &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \sum_k |B_{k,i}(\Pi_n u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2 ds. \end{aligned}$$

Due to (ii) from **(R3)**, consistency **(Cn $\tau$ )**, Lipschitz condition **(C3n $\tau$ )**, (2.3) and (2.4)

$$\begin{aligned} Q_1 &\leq \tau^{2\nu} p^2 \left( K E \int_0^T |u(s)|_{\mathcal{H}}^2 ds + TE\eta \right), \\ Q_2 &\leq c(\tau^{2\nu} + \varepsilon_n^2) \left( E \int_0^T |u(s)|_{\mathcal{V}}^2 ds + \sup_{n,m} \sum_{0 \leq i < l} \tau E \eta_i^{n,\tau} \right), \\ Q_3 &\leq L_2 p^2 T \sup_{|t-s| \leq \tau} E \|u(t) - u(s)\|^2 \leq \tau L_2 p^2 T M_1. \end{aligned}$$

Hence

$$\begin{aligned} S_2 &\leq \varepsilon L_2 E \sum_{1 \leq i \leq l} \tau \|e_i\|_n^2 + C \left( 1 + \frac{1}{\varepsilon} \right) (\tau^{2\nu} + \varepsilon_n^2) \\ &\quad \times \left( E \int_0^T |u(s)|_{\mathcal{V}}^2 ds + TE\eta + \sup_{n,m} \sum_i \tau E \eta_i^{n,\tau} + M_1 \right), \end{aligned} \quad (4.14)$$

where  $C = C(p, K, L_2, c)$ . Choosing  $\varepsilon > 0$  sufficiently small, from (4.7) and (4.9)–(4.14) we obtain for  $l = 1, \dots, m$ ,

$$\begin{aligned} E|e_l|_n^2 + \frac{\lambda}{2} E \sum_{1 \leq i \leq l} \tau \|e_i\|_n^2 \\ \leq E|e_0|_n^2 + L \sum_{1 \leq i \leq l} \tau E|e_i|_n^2 + C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \end{aligned} \quad (4.15)$$

where  $C = C(K, \lambda, p, T, c, L_1, L_2)$  is a constant. Since  $\sup_m \sum_{i=1}^m \tau = T < +\infty$ , if  $L\tau < 1$  a discrete version of Gronwall's lemma yields the existence of constants  $C_1 = C_1(L, \lambda, T)$  and  $C_2 = C_2(L, K, \lambda, p, T, c, L_1, L_2)$  such that for sufficiently large  $m$

$$\max_{1 \leq l \leq m} E|e_l|_n^2 \leq C_1 E|e_0|_n^2 + C_2 (r_1 + r_2 + M) (\tau^{2\nu} + \varepsilon_n^2)$$

holds for all  $n$ . This together with (4.15) concludes the proof of (4.6). To prove (4.4) notice that (4.5) and Schwarz's inequality yield

$$E \max_{1 \leq i \leq m} |e_i|_n^2 \leq E|e_0|_n^2 + E \sum_{1 \leq i \leq m} \tau \|e_i\|_n^2 + 2R + 2R_0 + E \max_{0 \leq i \leq m} I(t_i) + EQ(T), \quad (4.16)$$

where  $R$  is defined by (4.8),

$$R_0 := E \sum_{0 \leq i < m} \tau |A_{i+1}(\Pi_n(u(t_{i+1}))) - A_{i+1}(u_{i+1})|_{n^*}^2 \leq L_1 E \sum_{1 \leq i \leq m} \tau \|e_i\|_n^2, \quad (4.17)$$

by the Lipschitz condition **(C3n $\tau$ )**, and

$$EQ(T) := E \int_0^T \sum_k |F_k(s)|_n^2 ds \leq 2P_1 + 2P_2,$$

with  $P_1$  and  $P_2$  from (4.13) with  $l = m$ . By Davis' inequality

$$\begin{aligned} E \max_{0 \leq i \leq m} I(t_i) &\leq 6E \left\{ \int_0^T \sum_k |(e(s), F_k(s))_n|^2 ds \right\}^{1/2} \\ &\leq \frac{1}{2} E \max_{0 \leq i < m} |e_i|_n^2 + 18E \int_0^T |F_k(s)|_n^2 ds. \end{aligned} \quad (4.18)$$

Thus from (4.16), by (4.6), (4.14) with  $\varepsilon = 1$  and (4.17) we obtain (4.4).  $\square$

**Remark 4.4.** *One can show, like it is observed in [8] that if we drop the Lipschitz condition (4.2) on  $A_i^{n,\tau}$  from **(C3n $\tau$ )**, then the order of the speed of convergence is divided by two.*

**4.3. Examples.** (i) Let  $r > 0$ ,  $V = \mathcal{H} = W_2^r(\mathbb{R}^d)$ ,  $H = L^2(\mathbb{R}^d)$ ,  $V^* = W_2^{-r}(\mathbb{R}^d)$ ,  $\mathcal{V} = W_2^{r+l}(\mathbb{R}^d)$  for some  $l > 0$ , let  $\varphi$  be an orthonormal scaling function in  $W_2^{r+l}(\mathbb{R})$ . Let  $V_n$ ,  $H_n$  and  $\Pi_n$  be defined as in examples 2.4 and 3.3. Let  $A$  and  $B$  satisfy the conditions **(R0)**, **(R2)** and **(R3)**. For every  $u \in V_n \subset V$  and  $k = 1, \dots, d_1$ , set

$$A_i^{n,\tau}(u) = \Pi_n A(t_i, u) \quad \text{and} \quad B_{k,i}^{n,\tau}(u) = \Pi_n B_k(t_i, u). \quad (4.19)$$

The arguments used in Example 3.3 show that the conditions **(C1n $\tau$ )**-**(C3n $\tau$ )** are satisfied. Let us check that the consistency assumption **(Cn $\tau$ )** holds. For  $u \in \mathcal{V} \subset V$ , (2.4), (1.4) and (2.5) imply

$$|A_i^{n,\tau}(\Pi_n u) - \Pi_n A(t_i, u)|_{n^*}^2 \leq p^2 K_1 \|\Pi_n u - u\|_V^2 \leq Cp^2 K_1 2^{-2nl} \|u\|_V^2,$$

and similarly,

$$\sum_k |B_{k,i}^{n,\tau}(\Pi_n u) - \Pi_n B_k(t_i, u)|_n^2 \leq Cp^2 K_2 2^{-2nl} \|u\|_V^2.$$

Hence, **(Cn $\tau$ )** holds with  $\varepsilon_n = 2^{-nl}$ ,  $\xi_i^{n,\tau} = \eta_i^{n,\tau} = 0$  and any  $\nu > 0$ .

(ii) Another choice consists in setting  $A_0^{n,\tau}(u) = \Pi_n A(0, u)$ ,  $B_{k,0}^{n,\tau}(u) = \Pi_n B_k(0, u)$ , and for  $i = 1, \dots, m$ ,  $u \in V_n \subset V$  and  $k = 1, \dots, d_1$

$$A_{t_i}^{n,\tau}(u) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \Pi_n A(s, u) ds, \quad B_{k,t_i}^{n,\tau}(u) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \Pi_n B_k(s, u) ds.$$

A similar computation based on Jensen's inequality, (2.4), (1.3), (1.4), **(R3)** and (2.5) proves that relating  $\lambda$ ,  $p$  and  $K_2$ , **(C1n $\tau$ )**-**(C3n $\tau$ )** hold with  $f_i^{n,\tau} = g_i^{n,\tau} = \eta + 1$  and that **(Cn $\tau$ )** holds with  $\varepsilon_n = 2^{-nl}$ ,  $\zeta_i^{n,\tau} = \eta_i^{n,\tau} = \eta + 1$ .

(iii) Finally, let  $V_n = V$ ,  $H_n = H$  and let  $\Pi_n$  be the identity operator for every  $n$ . Assume that **(R0)**, **(R1)**, **(R2)** and **(R3)** hold. Then one recovers the conclusions

of Theorems 3.2 and 3.4 in [8] concerning the rate of convergence of the implicit time discretization scheme with  $\varepsilon_n = 0$ .

## 5. EXPLICIT SPACE-TIME DISCRETIZATION SCHEME

**5.1. Description of the scheme.** Let  $V_n$ ,  $H_n$  and  $V_n^*$  be a normal triple and  $\Pi_n$  be continuous linear operators which satisfy the condition (2.4). Assume moreover that for each  $n \geq 1$  as sets

$$V_n = H_n = V_n^*,$$

and there is a constant  $\vartheta(n)$  such that

$$\|u\|_{V_n}^2 \leq \vartheta(n) |u|_{H_n}^2, \quad \forall u \in H_n. \quad (5.1)$$

Then by duality we also have

$$|u|_{H_n}^2 \leq \vartheta(n) \|u\|_{V_n^*}^2, \quad \forall u \in V_n^*.$$

Consider for each  $n$  and  $i = 0, 1, \dots, m-1$  the equations

$$u_{\tau,i+1}^n = u_{\tau,i}^n + \tau A_i^{n,\tau}(u_{\tau,i}^n) + \sum_k B_{k,i}^{n,\tau}(u_{\tau,i}^n) (W^k(t_{i+1}) - W^k(t_i)), \quad (5.2)$$

for  $V_n$ -valued  $\mathcal{F}_{t_i}$ -measurable random variables  $u_{\tau,i}^n$  for  $i = 1, \dots, m$ , where  $u_{\tau,0}^n$  is a given  $V_n$ -valued  $\mathcal{F}_0$ -measurable random variable, and

$$A_i^{n,\tau} : \Omega \times V_n \rightarrow V_n^* \quad \text{and} \quad B_{k,i}^{n,\tau} : \Omega \times V_n \rightarrow H_n$$

are given  $\mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)$ -measurable mappings satisfying the conditions **(Cn1 $\tau$ )**–**(Cn4 $\tau$ )** for  $i = 0, 1, \dots, m-1$  with  $u_{\tau,0}^n$  in place of  $u_0^{n,\tau}$ .

**Proposition 5.1.** *Let **(C1n $\tau$ )**–**(C3n $\tau$ )** hold. Then for any  $V$ -valued  $\mathcal{F}_0$ -measurable random variable  $u_{\tau,0}^n$  such that  $E\|u_{\tau,0}^n\|_{V_n}^2 < \infty$ , the system of equations (5.2) has a unique solution  $(u_{\tau,i}^n)_{i=0}^m$  such that  $u_{\tau,i}^n$  is  $\mathcal{F}_{t_i}$ -measurable and  $E\|u_{\tau,i}^n\|_{V_n}^2 < \infty$  for all  $i, m$  and  $n$ .*

*Proof.* By (5.1) we have  $\|u_{\tau,i+1}^n\|_n^2 \leq \vartheta(n) |u_{\tau,i+1}^n|_n^2$ , and by (5.2)

$$\begin{aligned} E|u_{\tau,i+1}^n|_n^2 &\leq 3E|u_{\tau,i}^n|_n^2 + 3\tau E|A_i^{n,\tau}(u_{\tau,i}^n)|_n^2 + 3\tau \sum_k E|B_{k,i}^{n,\tau}(u_{\tau,i}^n)|_n^2 \\ &\leq 3(\vartheta(n) + \vartheta(n)\tau K + \tau K) E\|u_{\tau,i}^n\|_n^2 + 3\tau(\vartheta(n) + 1)M. \end{aligned}$$

Hence we get the proposition by induction on  $i$ .  $\square$

**5.2. Rate of convergence of the scheme.** The following theorem gives the rate of convergence of  $e_{\tau,i}^n := \Pi_n u(t_i) - u_{\tau,i}^n$ .

**Theorem 5.2.** *Let **(R1)** - **(R3)** hold. Assume **(C1n $\tau$ )**–**(C3n $\tau$ )**, **(C4n $\tau$ )** formulated with  $u_{\tau,0}^n$  instead of  $u_0^{n,\tau}$  and **(Cn $\tau$ )**. Let  $n$  and  $\tau$  satisfy*

$$L_1\tau\vartheta(n) + 2\sqrt{L_1L_2\tau\vartheta(n)} \leq q \quad (5.3)$$

for some constant  $q < \lambda$ . Then

$$E \max_{0 \leq i \leq m} |e_{\tau,i}^n|_{H_n}^2 + \sum_{0 \leq i < m} \tau E \|e_{\tau,i}^n\|_{V_n}^2 \leq C_1 E |e_{\tau,0}^n|_{H_n}^2 + C_2 (\tau^{2\nu} + \varepsilon_n^2) (r_1 + r_2 + M), \quad (5.4)$$

where  $C_1 = C_1(\lambda, q, L, T)$  and  $C_2 = C_2(\lambda, q, L, K, T, p, c, L_1, L_2)$  are constants.

*Proof.* To ease notation we omit the indices  $n$  and  $\tau$  from  $e_{\tau,i}^n$ ,  $u_{\tau,i}^n$ ,  $A_i^{n,\tau}$  and  $B_i^{n,\tau}$  when this does not cause ambiguity. For any  $i = 0, \dots, m-1$

$$\begin{aligned} |e_{i+1}|_n^2 - |e_i|_n^2 &= 2 \int_{t_i}^{t_{i+1}} \langle e_i, \Pi_n A(s, u(s)) - A_i(u_i) \rangle_n ds \\ &+ 2 \sum_k \int_{t_i}^{t_{i+1}} (e_i, F_k(s))_n dW^k(s) + \sum_k \left| \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right|_n^2 \\ &+ \left| \int_{t_i}^{t_{i+1}} [\Pi_n A(s, u(s)) - A_i(u_i)] ds \right|_n^2 \\ &+ 2 \sum_k \left( \int_{t_i}^{t_{i+1}} [\Pi_n A(s, u(s)) - A_i(u_i)] ds, \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right)_n, \end{aligned}$$

where

$$F_k(s) = \Pi_n B_k(s, u(s)) - B_{k,i}(u_i), \quad s \in ]t_i, t_{i+1}], \quad i = 0, 1, \dots, m-1.$$

Hence for  $l = 1, \dots, m$  and every  $\delta > 0$ ,

$$\begin{aligned} |e_l|_n^2 &\leq |e_0|_n^2 + 2 \sum_{0 \leq i < l} \int_{t_i}^{t_{i+1}} \langle e_i, \Pi_n A(s, u(s)) - A_i(u_i) \rangle_n ds + 2I(t_l) + Q(t_l) \\ &+ (1 + \frac{1}{\delta})S(t_l) + \delta Q(t_l), \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} I(t_l) &= \int_0^{t_l} (e(s), F_k(s)) dW^k(s), \quad e(s) := e_i \quad \text{for } s \in ]t_i, t_{i+1}], \quad i \geq 0, \\ S(t_l) &= \sum_{0 \leq i < l} \left| \int_{t_i}^{t_{i+1}} [\Pi_n A(s, u(s)) - A_i(u_i)] ds \right|_n^2, \\ Q(t_l) &= \sum_{0 \leq i < l} \sum_k \left| \int_{t_i}^{t_{i+1}} F_k(s) dW^k(s) \right|_n^2. \end{aligned}$$

First we prove

$$\max_{1 \leq i \leq m} E|e_i|_n^2 + \sum_{0 \leq i < m} \tau E \|e_i\|_n^2 \leq C_1 E|e_0|_n^2 + C_2 (\tau^{2\nu} + \varepsilon_n^2) (r_1 + r_2 + M), \quad (5.6)$$

with some constants  $C_1 = C_1(\lambda, q, L, T)$  and  $C_2 = C_2(\lambda, q, L, K, T, p, c, L_1, L_2)$ . To this end we take expectation in both sides of (5.5) and use the strong monotonicity condition **(C1n $\tau$ )**, to get

$$\begin{aligned} E|e_l|_n^2 &\leq E|e_0|_n^2 + 2E \sum_{0 \leq i < l} \tau \langle e_i, A_i(\Pi_n(t_i)) - A_i(u_i) \rangle_n \\ &+ E \sum_{0 \leq i < l} \tau |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2 + \sum_{i=1,2} S_i + (1 + \frac{1}{\delta})ES(t_l) + \delta EQ(t_l) \\ &\leq E|e_0|_n^2 - \lambda E \sum_{0 \leq i < l} \tau \|e_i\|_n^2 + LE \sum_{0 \leq i < l} \tau |e_i|_n^2 \\ &+ \sum_{i=1,2} S_i + (1 + \frac{1}{\delta})ES(t_l) + \delta EQ(t_l), \end{aligned}$$

for any  $\delta > 0$ , where

$$\begin{aligned} S_1 &= 2 \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \langle e_i, \Pi_n A(s, u(s)) - A_i(\Pi_n u(t_i)) \rangle_n ds, \\ S_2 &= \sum_k \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} [|F_k(s)|_n^2 - |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2] ds. \end{aligned}$$

As in the proof of Theorem 4.3 we get for any  $\varepsilon > 0$ ,

$$\begin{aligned} S_1 &\leq \varepsilon \sum_{0 \leq i < l} \tau E \|e_i\|_n^2 + \frac{1}{\varepsilon} C(r_1 + r_2 + M)(\tau^{2\nu} + \varepsilon_n^2), \\ S_2 &\leq L_2 \varepsilon \sum_{0 \leq i < l} \tau E \|e_i\|_n^2 + \frac{1}{\varepsilon} C(r_1 + r_2 + M)(\tau^{2\nu} + \varepsilon_n^2) \end{aligned}$$

with a constant  $C = C(K, p, T, L_1, L_2, c)$ . Notice that for any  $\varepsilon > 0$ ,

$$\begin{aligned} ES(t_l) &\leq \tau \vartheta(n) J(t_l), \\ J(t_l) &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} |\Pi_n A(s, u(s)) - A_i(u_i)|_{n^*}^2 ds \leq (1 + \varepsilon) R_0 + (1 + \frac{1}{\varepsilon}) R, \end{aligned} \quad (5.7)$$

$$EQ(t_l) \leq (1 + \varepsilon) P_1 + (1 + \frac{1}{\varepsilon}) P_2, \quad (5.8)$$

where

$$\begin{aligned} P_1 &:= \sum_{0 \leq i < l} E \sum_k |B_{k,i}(\Pi_n u(t_i)) - B_{k,i}(u_i)|_n^2 \tau \leq L_2 E \sum_{0 \leq i < l} \tau \|e_i\|_n^2 \\ P_2 &:= \sum_{0 \leq i < l} E \int_{t_i}^{t_{i+1}} \sum_k |\Pi_n B_k(s, u(s)) - B_{k,i}(\Pi_n u(t_i))|_n^2 ds, \\ R_0 &:= E \sum_{0 \leq i < l} \tau |A_i(\Pi_n u(t_i)) - A_i(u_i)|_{n^*}^2 \leq L_1 E \sum_{0 \leq i < l} \tau \|e_i\|_n^2, \\ R &:= E \sum_{0 \leq i < l} \int_{t_i}^{t_{i+1}} |\Pi_n A(s, u(s)) - A_i(\Pi_n u(t_i))|_{n^*}^2 ds, \end{aligned}$$

for any  $l = 1, 2, \dots, m$ . In the same way as in the proof of Theorem 4.3 we obtain

$$R \leq C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \quad (5.9)$$

and that

$$P_2 \leq C'(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \quad (5.10)$$

where  $C = C(K, p, c, L_1, T)$  and  $C' = C'(K, p, c, L_2, T)$  are constants. Consequently,

$$\begin{aligned} E|e_l|_n^2 &\leq E|e_0|_n^2 + (\mu - \lambda) E \sum_{0 \leq i < l} \tau \|e_i\|_n^2 + LE \sum_{0 \leq i < l} \tau |e_i|_n^2 \\ &\quad + (1 + \tau \vartheta(n)) (1 + \frac{1}{\delta} + \frac{1}{\varepsilon}) C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M) \end{aligned} \quad (5.11)$$

for any  $\delta > 0$  and  $\varepsilon > 0$ , where

$$\mu = (1 + \varepsilon) \left[ (1 + \frac{1}{\delta}) \tau \vartheta(n) L_1 + \delta L_2 \right] + \varepsilon(1 + L_2),$$

and  $C = C(K, p, c, T, L_1, L_2)$  is a constant. It is easy to see that due to (5.3)

$$\inf_{\delta > 0} (1 + \frac{1}{\delta})\tau\vartheta(n)L_1 + \delta L_2 = \tau\vartheta(n)L_1 + 2\sqrt{\tau\vartheta(n)L_1L_2} \leq q.$$

Therefore we can take  $\delta > 0$  and  $\varepsilon > 0$  such that  $\mu \leq (q + \lambda)/2$ . Thus from (5.11) we can get

$$\begin{aligned} E|e_l|_n^2 &\leq E|e_0|_n^2 - \frac{1}{2}(\lambda - q)E \sum_{0 \leq i < l} \tau \|e_i\|_n^2 + LE \sum_{0 \leq i < l} \tau |e_i|_n^2 \\ &\quad + C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M), \end{aligned}$$

with a constant  $C = C(K, \lambda, q, p, c, T, L_1, L_2)$ . Hence by a discrete version of Gronwall's lemma we obtain (5.6). To prove (5.4) note that (5.5) yields

$$E \max_{1 \leq l \leq m} |e_l|_n^2 \leq |e_0|_n^2 + E \sum_{0 \leq i < l} \tau \|e_i\|_n^2 + 2ES(T) + 2E \max_{1 \leq l \leq m} I(t_l) + 2EQ(T), \quad (5.12)$$

where by (5.7)–(5.9)  $ES(T) \leq \tau\vartheta(n)J(T)$ , and

$$J(T) \leq 2L_1E \sum_{0 \leq i < m} \tau \|e_i\|_n^2 + 2C(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M).$$

By (5.8), (5.9) and (5.10)

$$EQ(T) \leq 2L_2 \sum_{0 \leq i < m} \tau \|e_i\|_n^2 + 2C'(\tau^{2\nu} + \varepsilon_n^2)(r_1 + r_2 + M).$$

Finally, in the same way as equation (4.18) is obtained, we get

$$\begin{aligned} E \max_{1 \leq i \leq m} I(t_i) &\leq 6E \left\{ \int_0^T \sum_k |(e(s), F_k(s))_n|^2 ds \right\}^{1/2} \\ &\leq \frac{1}{2}E \max_{0 \leq i < m} |e_i|_n^2 + 18E \int_0^T F_k^2(s) ds \leq \frac{1}{2}E \max_{0 \leq i < m} |e_i|_n^2 + 18EQ(T). \end{aligned}$$

Consequently, from (5.12) we obtain (5.4) by (5.6).  $\square$

**5.3. Examples.** Let  $r > 0$ ,  $V = \mathcal{H} = W_2^r(\mathbb{R}^d)$ ,  $H = L^2(\mathbb{R}^d)$ ,  $V^* = W_2^{-r}(\mathbb{R}^d)$ ,  $\mathcal{V} = W_2^{r+l}(\mathbb{R}^d)$  for some  $l > 0$ , let  $\varphi$  be an orthonormal scaling function in  $W_2^{r+l}(\mathbb{R})$ . Let  $V_n$ ,  $H_n$  and  $\Pi_n$  be defined as in examples 2.4, 3.3 and 4.3. Then the converse inequality (2.6) yields  $\vartheta(n) = 2^{nr}$ . Let  $A$  and  $B$  satisfy the conditions **(R0)**, **(R2)** and **(R3)**. For every  $u \in V_n \subset \mathcal{V}$  and  $k = 1, \dots, d_1$ , let  $A^{n,\tau}$  and  $B^{n,\tau}$  be defined by (4.19). Then, as shown in section 4.3,  $A^{n,\tau}$  and  $B^{n,\tau}$  satisfy the conditions **(C1n $\tau$ )**–**(C3n $\tau$ )** as well as **(Cn $\tau$ )** with  $\varepsilon_n = 2^{-nl}$  and any  $\nu > 0$ . Hence, if the solution  $u$  satisfies **(R1)**, and  $L_1\tau\vartheta_n + 2\sqrt{L_1L_2\tau\vartheta_n} \leq q < \lambda$ , the conditions of Theorem 5.2 hold.

## 6. EXAMPLES OF APPROXIMATIONS OF STOCHASTIC PDES

In this section we present some examples of stochastic PDEs for which the previous theorems provide rates of convergence for the above space and space-time discretization schemes. We refer to section 5 in [8] for more details.

**6.1. Quasilinear equations.** Let us consider the stochastic partial differential equation

$$\begin{aligned} du(t, x) = & (Lu(t) + F(t, x, \nabla u(t, x), u(t, x))) dt \\ & + \sum_k (M_k u(t, x) + g_k(t, x)) dW^k(t), \quad t \in (0, T], x \in \mathbb{R}^d, \end{aligned} \quad (6.1)$$

with initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^d, \quad (6.2)$$

where  $F$  and  $g_k$  are Borel functions of  $(\omega, t, x, p, r) \in \Omega \times [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$  and of  $(\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}^d$ , respectively, and  $L, M_k$  are differential operators of the form

$$L(t)v(x) = \sum_{|\alpha| \leq 1, |\beta| \leq 1} D^\alpha (a^{\alpha\beta}(t, x) D^\beta v(x)), \quad M_k(t)v(x) = \sum_{|\alpha| \leq 1} b_k^\alpha(t, x) D^\alpha v(x),$$

with functions  $a^{\alpha\beta}$  and  $b_k^\alpha$  of  $(\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R}^d$ , for all multi-indices  $\alpha = (\alpha_1, \dots, \alpha_d), \beta = (\beta_1, \dots, \beta_d)$  of length  $|\alpha| = \sum_i \alpha_i \leq 1, |\beta| \leq 1$ . Here, and later on  $D^\alpha$  denotes  $D_1^{\alpha_1} \dots D_d^{\alpha_d}$  for any multi-indices  $\alpha = (\alpha_1, \dots, \alpha_d) \in \{0, 1, 2, \dots\}^d$ , where  $D_i = \frac{\partial}{\partial x_i}$  and  $D_i^0$  is the identity operator. We use the notation  $\nabla_p := (\partial/\partial p_1, \dots, \partial/\partial p_d)$ .

Let  $K$  and  $M$  denote some non-negative numbers. Fix an integer  $l \geq 0$  and suppose that the following conditions hold:

**Assumption (A1)** (Stochastic parabolicity). *There exists a constant  $\lambda > 0$  such that*

$$\sum_{|\alpha|=1, |\beta|=1} \left( a^{\alpha\beta}(t, x) - \frac{1}{2} \sum_k (b_k^\alpha b_k^\beta)(t, x) \right) z^\alpha z^\beta \geq \lambda \sum_{|\alpha|=1} |z^\alpha|^2$$

for all  $\omega \in \Omega, t \in [0, T], x \in \mathbb{R}^d$  and  $z = (z^1, \dots, z^d) \in \mathbb{R}^d$ , where  $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$  for  $z \in \mathbb{R}^d$  and multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ .

**Assumption (A2)** (Smoothness of the initial condition). *Let  $u_0$  be  $W_l^2$ -valued  $\mathcal{F}_0$ -measurable random variable such that  $E|u_0|_l^2 \leq M$ .*

**Assumption (A3)** (Smoothness of the linear term). *The derivatives of  $a^{\alpha\beta}$  and  $b_k^\alpha$  up to order  $l$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable real functions such that almost surely*

$$|D^\gamma a^{\alpha\beta}(t, x)| + |D^\gamma b_k^\alpha(t, x)| \leq K, \quad \text{for all } |\alpha| \leq 1, |\beta| \leq 1, k = 1, \dots, d_1,$$

$t \in [0, T], x \in \mathbb{R}^d$  and multi-indices  $\gamma$  with  $|\gamma| \leq 2$ .

**Assumption (A4)** (Smoothness of the nonlinear term). *The function  $F$  and their first order partial derivatives in  $p, x$  and  $r$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ -measurable functions. The function  $g_k$  and its derivatives in  $x$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions for every  $k = 1, \dots, d_1$ . There exists a constant  $K$  and a  $\mathcal{P} \otimes \mathcal{B}$ -measurable function  $\xi$  of  $(\omega, t, x)$  such that almost surely*

$$|\nabla_p F(t, x, p, r)| + \left| \frac{\partial}{\partial r} F(t, x, p, r) \right| \leq K,$$

$$|F(t, \cdot, 0, 0)|_0^2 + \sum_k |g_k(t, \cdot)|_2^2 \leq \eta,$$

$$|\nabla_x F(t, x, p, r)| \leq L(|p| + |r|) + \xi(t, x), \quad |\xi(t)|_0^2 \leq \eta$$

for all  $t, x, p, r$ , where  $\eta$  is a random variable such that  $E\eta \leq M$ .

Set  $H = L^2(\mathbb{R}^d) = W_2^0$ ,  $V = W_2^1$ ,  $\mathcal{H} = W_2^2$  and  $\mathcal{V} = W_2^3$  and suppose that the assumptions **(A1)**–**(A4)** hold with  $l = 2$ . Then the operators

$$A(t, \varphi) = L(t)\varphi + F(t, \cdot, \nabla\varphi, \varphi), \quad B_k(t, \varphi) = M_k(t)\varphi + g_k(t, \cdot), \quad \varphi \in V$$

and  $u_0$  satisfy the conditions of Theorem 2.2. Hence (6.1)–(6.2) has a unique solution  $u$  on  $[0, T]$ . Furthermore,  $u$  has a  $W_2^2$ -valued continuous modification such that

$$E \sup_{0 \leq t \leq T} |u(t)|_2^2 + E \int_0^T |u(t)|_3^2 dt < \infty.$$

Consequently **(R1)** holds. It is easy to check that  $A$  and  $B_k$  verify condition **(R2)**.

**Assumption (A5)** (Time regularity of  $A$  and  $B$ ) *Almost surely*

(i)

$$\begin{aligned} \sum_k |D^\gamma(b_k^\alpha(t, x) - b_k^\alpha(s, x))|^2 &\leq K|t - s|, \\ \sum_k |g_k(s, \cdot) - g_k(t, \cdot)|_1^2 &\leq \eta|t - s|. \end{aligned}$$

(ii)

$$\begin{aligned} |D^\gamma(a^{\alpha, \beta}(t, x) - a^{\alpha, \beta}(s, x))|^2 &\leq K|t - s|, \\ |F(t, x, p, r) - F(s, x, p, r)|^2 &\leq K|t - s|(|p|^2 + |r|^2), \\ |\nabla_x F(t, x, p, r) - \nabla_x F(s, x, p, r)|^2 &\leq K|t - s|(|p|^2 + |r|^2), \\ |\nabla_p F(t, x, p, r) - \nabla_p F(s, x, p, r)|^2 &\leq K|t - s|, \\ \left| \frac{\partial}{\partial r} F(t, x, p, r) - \frac{\partial}{\partial r} F(s, x, p, r) \right|^2 &\leq K|t - s|. \end{aligned}$$

for all  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$ ,  $|\gamma| \leq 1$ ,  $s, t \in [0, T]$  and  $x \in \mathbb{R}^d$ , where  $K$  is a constant and  $\eta$  is a random variable such that  $E\eta \leq M$ .

Clearly, Assumptions (i) and (ii) of **(A5)** imply conditions (i) and (ii) of **(R3)**, respectively with  $\nu = 1/2$ .

Let  $H_n, V_n$  and  $\Pi_n$  be defined as in Example 2.4 for a scaling function  $\varphi \in W_2^3$ . For every  $n$  let  $A^n(t, u) = \Pi_n A(t, u)$  and  $B_k^n(t, u) = \Pi_n B_k(t, u)$  be defined as in (3.12). Then, if  $u_0^n = \Pi_n u_0$  and  $u_0 \in W_2^2 = \mathcal{H}$ , we can apply Theorem 3.1 with  $\varepsilon_n = 2^{-n}$ , and using (2.5) to estimate  $|\Pi_n u(t) - u(t)|_r$  for  $r = 0, 1$ , we deduce

$$E \left( \sup_{0 \leq t \leq T} |u^n(t) - u(t)|_0^2 + \int_0^T |u^n(t) - u(t)|_1^2 dt \right) \leq C 2^{-2n}.$$

If one requires also **(A5)**, then for  $A^{n, \tau}$  and  $B^{n, \tau}$  defined by (4.19), the conclusions of Theorem 4.3 hold with  $\nu = \frac{1}{2}$  and  $\alpha_n = 2^{-n}$ . Hence if  $u_0^{n, \tau} = \Pi_n u(0)$ , then using Theorem 4.3, (2.5) and (2.2), we deduce that

$$E \left( \max_{0 \leq i \leq m} |u_i^{n, \tau} - u(i\tau)|_0^2 + \tau \sum_{0 \leq i \leq m} |u_i^{n, \tau} - u(i\tau)|_1^2 \right) \leq C \left( \tau + 2^{-2n} \right).$$

Finally, if one requires that the condition **(A5)** holds, then the inequality (2.6) shows that  $\vartheta(n) = 2^n$ . Hence by Theorems 3.1 and 5.2, if  $u_{\tau, 0}^n = \Pi_n u(0)$  and  $\frac{T}{m} 2^{2n} \leq \gamma$  for

some constant  $\gamma < c\lambda$ , then

$$E \left( \max_{0 \leq i \leq m} |u_{\tau,i}^n - u(i\tau)|_0^2 + \tau \sum_{0 \leq i < m} |u_{\tau,i}^n - u(i\tau)|_1^2 \right) \leq C \left( \tau + 2^{-2n} \right).$$

Let us now recall Example 2.5 and approximate (6.1)–(6.2) by finite difference schemes. Consider first the following system of SDEs, corresponding to the space discretization with finite differences for fixed  $h \in (0, 1)$ :

$$\begin{aligned} dv(t) &= (L_h(t)v(t) + F_h(t, \nabla_h v(t), v(t))) dt \\ &\quad + \sum_k (M_{k,h}(t)v(t) + g_{k,h}(t)) dW^k(t), \quad z \in \mathbb{G} = hZ^d, \end{aligned} \quad (6.3)$$

$$v(0) = (u_0(z))_{z \in \mathbb{G}}, \quad (6.4)$$

where  $g_{k,h}(t) = (g_k(t, z))_{z \in \mathbb{G}}$ ,  $F_h(t, p, r) = (F(t, z, p, r))_{z \in \mathbb{G}}$  and

$$L_h(t)\varphi := \sum_{|\alpha| \leq 1, |\beta| \leq 1} \delta_-^\alpha (a^{\alpha\beta}(t, \cdot) \delta_+^\beta \varphi), \quad \nabla_h \varphi := (\delta_1 \varphi, \delta_2 \varphi, \dots, \delta_d \varphi), \quad (6.5)$$

$$M_{k,h}(t)\varphi := \sum_{|\alpha| \leq 1} b_k^\alpha(t) \delta^\alpha \varphi, \quad (6.6)$$

for functions  $\varphi$  defined on  $\mathbb{G}$ . It is not difficult to see that taking the triplet  $V_n := W_{h,2}^1$ ,  $H_n := W_{h,2}^0$ ,  $V_n^* = (W_{h,2}^1)^*$ , problem (6.3)–(6.4) can be cast into equation (3.1), and we can easily check that conditions **(C1n)**–**(C5n)** hold. Thus (6.3)–(6.4) has a unique continuous  $W_{h,2}^0$ -valued solution  $v = v^h$  such that for every  $h \in (0, 1)$ ,

$$E \sup_{t \in [0, T]} |v^h(t)|_{h,0}^2 + E \int_0^T |v^h(t)|_{h,1}^2 dt \leq M < \infty.$$

Assume now that  $d = 1$ . Consider the normal triplet  $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$  with  $V := W_2^1(\mathbb{R})$ ,  $H := W_2^0(\mathbb{R})$  and  $V^* \equiv W_2^{-1}(\mathbb{R})$ . Notice that Using (2.10) we can see that there is a constant  $C$  such that almost surely for all  $t \in [0, T]$

$$\begin{aligned} |D^\alpha (a^{\alpha\beta}(t) D^\beta \varphi) - \delta_-^\alpha (a^{\alpha\beta}(t) \delta_+^\beta \varphi)|_{h,0} &\leq Ch |\varphi|_{W_2^3(\mathbb{R})}, \\ |b_k^\alpha(t) D^\alpha \varphi - b_k^\alpha(t) \delta^\alpha \varphi|_{h,0} &\leq Ch |\varphi|_{W_2^2(\mathbb{R})}, \\ |F_h(t, D\varphi, \varphi) - F_h(t, \delta\varphi, \varphi)|_{h,0} &\leq C|h|_{W_2^2(\mathbb{R})} \end{aligned}$$

for all  $\varphi \in W_2^3(\mathbb{R})$  and  $h \in (0, 1)$ . Hence the consistency condition **(Cn)** holds with  $\mathcal{V} = W_2^3(\mathbb{R})$  and  $\varepsilon_n = h$ . Set  $\mathcal{H} = W_2^2(\mathbb{R})$ . Assume (A1)–(A4) with  $l = 2$ . Then the assumptions of Theorem 3.1 are satisfied. Thus there is a constant  $C$  such that

$$E \sup_{t \in [0, T]} |u(t) - v^h(t)|_{h,0}^2 + E \int_0^T |u(t) - v^h(t)|_{h,1}^2 dt \leq Ch^2$$

for all  $h \in (0, 1)$ . Now we approximate (6.4) by the following Euler approximation schemes:

$$\begin{aligned} w_{i+1} &= w_i + (L_h(t_{i+1})w_{i+1} + F_h(t_{i+1}, \nabla_h w_{i+1}, w_{i+1})) \tau \\ &\quad + \sum_k (M_{k,h}(t_i)w_i + g_{k,h}(t_i))(W^k(t_{i+1}) - W^k(t_i)), \quad w_0 = u_0, \end{aligned} \quad (6.7)$$

$$\begin{aligned} u_{i+1} &= u_i + (L_h(t_i)u_i + F_h(t_{i+1}, \nabla_h u_{i+1}, u_i))\tau \\ &\quad + \sum_k (M_{k,h}(t_i)u_i + g_{k,h}(t_i))(W^k(t_{i+1}) - W^k(t_i)), \quad v_0 = u_0. \end{aligned} \quad (6.8)$$

for  $i = 0, 1, 2, \dots, m-1$ ,  $\tau = T/m$ ,  $t_i = i\tau$ . Then by Proposition 4.2 we get the existence of a unique  $W_{h,2}^1$ -valued solution  $w_i$  of (6.7), such that  $w_i$  is  $\mathcal{F}_{t_i}$ -measurable for  $i = 1, 2, \dots, m$ , if  $\tau$  is sufficiently small. By Theorem 4.3 for  $e_i^{h,\tau} = (u(t_i, z) - w_i(z))_{z \in \mathbb{G}}$ , we get

$$E \max_{0 \leq i \leq m} |e_i^{h,\tau}|_{W_{h,2}^0}^2 + \tau \sum_{1 \leq i \leq m} E |e_i^{h,\tau}|_{W_{h,2}^1}^2 \leq C(\tau + h^2)$$

with a constant  $C$  independent of  $\tau$  and  $h$ . Recall that  $\vartheta(n) = \kappa^2/h_n^2$  for any sequence  $h_n \in (0, 1)$  by (2.7). Set  $e_{i,\tau}^h = (u(t_i, z) - u_i(z))_{z \in \mathbb{G}}$ . Then applying Theorem 5.2 we get

$$E \max_{0 \leq i \leq m} |e_{\tau,i}^h|_{W_{h,2}^0}^2 + \tau \sum_{0 \leq i < m} E |e_{\tau,i}^h|_{W_{h,2}^1}^2 \leq Ch^2,$$

with a constant  $C$  independent of  $\tau$  and  $h$ , provided (5.3) holds with  $\kappa^2/h^2$  in place of  $\vartheta(n)$ . To obtain the corresponding results when  $d > 1$  we need more regularity in the space variable from the solution  $u$  of (6.1)-(6.2). Assuming more regularity on the data, it is possible to get the required regularity of  $u$ . We do not want to prove in this paper further results on regularity of the solutions to (6.1). Instead of that we consider the case of linear equations, i.e., when  $F$  does not depend on  $p$  and  $r$ , since in this case the necessary results on regularity of the solutions are well known in the literature. (See e.g. [9] and [16].)

**6.2. Linear stochastic PDEs.** We consider again equation (6.1)-(6.2) and assume that  $F = F(t, x, p, r)$  does not depend on  $p$  and  $r$ . We fix an integer  $l \geq 0$ . Instead of **(A4)** we assume the following.

**Assumption (A\*4)**  $F(t, x, p, r) = f(t, x)$  and  $g_k(t, x)$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions of  $(t, \omega, x)$ , and their derivatives in  $x$  up to order  $l$  are  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ -measurable functions such that

$$|f(t, \cdot)|_l^2 + \sum_k |g_k(t, \cdot)|_l^2 \leq \eta,$$

where  $\eta$  is a random variable such that  $E\eta \leq M$ .

Instead of **(A5)** we make the following assumption.

**Assumption (A\*5)** *Almost surely*

$$(i) \sum_k |D^\gamma (b_k^\alpha(t) - b_k^\alpha(s))| \leq K|t - s|^{\frac{1}{2}}, \quad \sum_k |g_k(s) - g_k(t)|_l^2 \leq \eta|t - s|.$$

$$(ii) |D^\gamma (a^{\alpha,\beta}(t) - a^{\alpha,\beta}(s))| \leq K|t - s|^{\frac{1}{2}}, \quad |f(t) - f(s)|_l^2 \leq \eta|t - s|$$

for all  $|\gamma| \leq l$ ,  $s, t \in [0, T]$ ,  $x \in \mathbb{R}^d$  and multi-indices  $|\alpha| \leq 1$  and  $|\beta| \leq 1$ , where  $K$  is a constant and  $\eta$  is a random variable such that  $E\eta \leq M$ .

Consider the space-time discretizations with finite differences. The implicit and the explicit approximations,  $v^{h,\tau}$  and  $v_\tau^h$  are given by the systems of equations defined for  $i = 0, \dots, m-1$  by

$$\begin{aligned} v^{h,\tau}(t_{i+1}) &= v^{h,\tau}(t_i) + \tau(L_h(t_{i+1})v^{h,\tau}(t_{i+1}) + f(t_{i+1})) \\ &\quad + \sum_k (M_{k,h}v^{h,\tau}(t_i) + g(t_i))(W^k(t_{i+1}) - W^k(t_i)), \end{aligned} \quad (6.9)$$

$$v^{h,\tau}(0, z) = u(0, z), \quad z \in \mathbb{G}, \quad (6.10)$$

and

$$\begin{aligned} v_\tau^h(t_{i+1}) &= v_\tau^h(t_i) + \tau(L_h(t_i)v_\tau^h(t_i) + f(t_i)) \\ &\quad + \sum_k (M_{k,h}v_\tau^h(t_i) + g(t_i))(W^k(t_{i+1}) - W^k(t_i)), \end{aligned} \quad (6.11)$$

$$v_\tau^h(0, z) = u(0, z), \quad z \in \mathbb{G}, \quad (6.12)$$

respectively, where  $t_i = i\tau = iT/m$ ,  $v^{h,\tau}(t_i)$  and  $v_\tau^h(t_i)$  are functions on  $\mathbb{G}$ ,  $L_h(t)$  and  $M_{k,h}(t)$  are defined by (6.5) and (6.6).

Take  $H_n := W_{h,2}^0$  and the normal triplet  $V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*$  with  $V_n := W_{h,2}^1$ . Then it is easy to see that

$$(L_h(t_i)\varphi, \psi)_n \leq C|\varphi|_{V_n}|\psi|_{V_n}, \quad (M_{k,h}(t_i)\varphi, \psi)_n \leq C|\varphi|_{V_n}|\psi|_{H_n} \quad (6.13)$$

for all  $\varphi, \psi \in V_n$ , where  $(\cdot, \cdot)_n$  denotes the inner product in  $H_n$ , and  $C$  is a constant depending only on  $d$  and the constant  $K$  from Assumption **(A3)**. Thus we can define  $L_h(t_i)$  and  $M_{h,k}(t_i)$  as bounded linear operators from  $V_n$  into  $V_n^*$  and  $H_n$  respectively. Due to (2.8) and (2.9), the restriction of  $u_0$ ,  $f(t_i)$  and  $g_k(t_i)$  onto  $\mathbb{G}$  are  $H_n$ -valued random variables such that

$$\begin{aligned} E|f(t_i)|_{H_n}^2 &\leq p^2 E|f(t_i)|_l^2, \quad E|g_k(t_i)|_{H_n}^2 \leq p^2 E|g(t_i)|_l^2, \\ E|u_0|_{H_n}^2 &\leq p^2 E|u_0|_l^2, \end{aligned}$$

where  $p$  is the constant from (2.8). Moreover,

$$2(L_h(t_i)\varphi, \varphi)_n + \sum_k |M_{h,k}\varphi|_{H_n}^2 \leq -\frac{\lambda}{2}|\varphi|_{V_n}^2 + C|\varphi|_{H_n}^2 \quad (6.14)$$

for all  $\varphi \in V_n$ , where  $C$  is a constant depending only on  $d$  and on the constant  $K$  from Assumption **(A2)**. Thus using the notation  $u_i^{n,\tau} = v^{h,\tau}(t_i)$ ,  $u_{\tau,i}^n = v_\tau^h(t_i)$  and defining

$$A_i^{n,\tau}(\varphi) = L_h(t_i)\varphi + f(t_i), \quad B_{k,i}^{n,\tau}(\varphi) = M_{k,h}(t_i)\varphi + g_k(t_i) \quad (6.15)$$

for  $\varphi \in W_{h,2}^1$ , we can cast (6.9)–(6.10) and (6.11)–(6.12) into (4.1) and into (5.2), respectively, and we can see that Assumptions **(C1n $\tau$ )**–**(C4n $\tau$ )** hold. Consequently, by virtue of Proposition 4.2, for sufficiently small  $\tau$  (6.9)–(6.10) has a unique solution  $\{v^{h,\tau}(t_i)\}_{i=0}^m$ , such that  $v^{h,\tau}(t_i)$  is a  $W_{h,2}^1$ -valued  $\mathcal{F}_{t_i}$ -measurable random variable and  $E|v_i^{h,\tau}|_{h,2}^2 < \infty$ . Furthermore, by virtue of Proposition 5.1, (6.11)–(6.12) has a unique solution  $\{v_\tau^h(t_i)\}_{i=0}^m$ , such that  $v_\tau^h(t_i)$  is a  $W_{h,2}^1$ -valued  $\mathcal{F}_{t_i}$ -measurable random variable and  $E|v_\tau^h(t_i)|_{h,2}^2 < \infty$ .

Let  $r \geq 0$  be an integer, and assume that

$$l > r + 2 + \frac{d}{2}. \quad (6.16)$$

Then Theorem 4.3 gives the following result.

**Theorem 6.1.** *Let Assumptions **(A1)**, **(A2)**, **(A3)**, **(A\*4)** and **(A\*5)** hold with  $l$  satisfying (6.16). Then for sufficiently small  $\tau$*

$$E \max_{1 \leq i \leq m} |v^{h,\tau}(t_i) - u(t_i)|_{h,r}^2 + E \sum_{1 \leq i \leq m} \tau |v^{h,\tau}(t_i) - u(t_i)|_{h,r+1}^2 \leq C(h^2 + \tau)$$

for all  $h \in (0, 1)$ , where  $C = C(r, l, p, \lambda, T, K, M, d, d_1)$  is a constant.

*Proof.* Take  $H_n := W_{h,2}^r$ ,  $H := W_2^{l-2}(\mathbb{R}^d)$ ,  $\mathcal{H} := W_2^l(\mathbb{R}^d)$  and the normal triplets

$$V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*, \quad V \hookrightarrow H \equiv H^* \hookrightarrow V^*, \quad \mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

where  $V_n := W_{h,2}^{r+1}$ ,  $V_n^* \equiv W_{h,2}^{r-1}$ ,  $V := W_2^{l-1}(\mathbb{R}^d)$ ,  $V^* \equiv W_2^{l-3}(\mathbb{R}^d)$ ,  $\mathcal{V} := W_2^{l+1}(\mathbb{R}^d)$  and  $\mathcal{V}^* \equiv W_2^{l-1}(\mathbb{R}^d) = V$ . Then due to (6.16) there is a constant  $p$  such that for  $\Pi_n := R_h$ ,

$$|\Pi_n \varphi|_{V_n} \leq p |\varphi|_V,$$

for all  $\varphi \in V$ , by virtue of (2.11). It is easy to check that (6.13)–(6.14) still hold, and hence (6.9)–(6.10), written as equation (4.1), satisfies **(C1n $\tau$ )**–**(C4n $\tau$ )** in the new triplet as well. Using (2.11) it is easy to show that due to Assumption **(A3)**

$$|L(t_i)\varphi - L_h(t_i)\varphi|_{V_n^*} \leq |L(t_i)\varphi - L_h(t_i)\varphi|_{H_n} \leq Ch|\varphi|_{W_2^{l+1}(\mathbb{R}^d)},$$

$$\sum_k |M_k(t_i)\varphi - M_{k,h}(t_i)\varphi|_{H_n} \leq Ch|\varphi|_{W_2^l(\mathbb{R}^d)}$$

for all  $\varphi \in W_2^{l+1}(\mathbb{R}^d)$ , where  $C$  is a constant depending on  $d, l, r$  and on the constant  $K$  from Assumption **(A3)**. Hence we can see that **(Cn $\tau$ )** holds with  $\alpha_n = h$ . Due to Assumption **(A\*5)** we have

$$\begin{aligned} |L(t)\varphi - L(s)\varphi|_V^2 &\leq C|t-s|, & |f(t) - f(s)|_V^2 &\leq \eta|t-s|, \\ \sum_k |M_k(t)\varphi - M_k(s)\varphi|_V^2 &\leq C|t-s|, & \sum_k |g_k(t) - g_k(s)|_V^2 &\leq \eta|t-s|, \end{aligned}$$

where  $\eta$  is the random variable from Assumption **(A\*5)**, and  $C$  is a constant depending on  $d, d_1, l$  and on the constant  $K$  from Assumption **(A\*5)**. It is an easy exercise to show that due to Assumptions **(A3)** and **(A\*4)** condition **(R2)** holds. From [9] it is known that under the Assumptions **(A1)**–**(A3)** and **(A\*4)** the problem (6.3)–(6.4) has a unique solution  $u$  on  $[0, T]$ , and that  $u$  is a continuous  $W_2^l(\mathbb{R}^d)$ -valued  $(\mathcal{F}_t)$ -adapted stochastic process such that

$$\begin{aligned} E \sup_{t \in [0, T]} |u(t)|_l^2 + E \int_0^T |u(t)|_{l+1}^2 dt &\leq \\ CE|u_0|_l^2 + CE \int_0^T \left( |f(t)|_{l-1}^2 + \sum_k |g_k(t)|_l^2 \right) dt, \end{aligned}$$

where  $C$  is a constant depending on  $d, d_1$  and the constants  $\lambda$  and  $K$  from Assumptions **(A1)**, **(A3)** and **(A\*4)**. Hence condition **(R1)** clearly holds. Now we can conclude the proof by applying Theorem 4.3.  $\square$

Let us now investigate the rate of convergence of the explicit space-time approximations.

Take the normal triple  $V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*$  with  $V_n := W_{h,2}^{r+1}$ ,  $H_n := W_{h,2}^r$ , and notice that due to Assumption **(A3)**

$$(L(t_i)\varphi, \psi)_n \leq C_1 |\varphi|_{V_n} |\psi|_{V_n}, \quad (M_{k,h}(t_i)\varphi, \psi)_n \leq C_{2k} |\varphi|_{V_n} |\psi|_{H_n} \quad (6.17)$$

with some constants  $C_1$  and  $C_{2k}$  depending only on  $d, r$  and the constant  $K$  from Assumption **(A3)**. Set  $L_1 = C_1^2$  and  $L_2 = \sum_k C_{2k}^2$ . Then Theorem 5.2 yields the following theorem, which improves a result from [18].

**Theorem 6.2.** *Let Assumptions (A1), (A2), (A3), (A\*4) and (A\*5) hold with  $l$  satisfying (6.16). Let  $h$  and  $\tau$  satisfy*

$$L_1 \kappa^2 \frac{\tau}{h^2} + 2\kappa(L_1 L_2)^{1/2} \frac{\sqrt{\tau}}{h} \leq q \quad (6.18)$$

for a constant  $q < \lambda$ . Then

$$E \max_{1 \leq i \leq m} |v_\tau^h(t_i) - u(t_i)|_{h,r}^2 + E \sum_{0 \leq i < m} \tau |v_\tau^h(t_i) - u(t_i)|_{h,r+1}^2 \leq C(h^2 + \tau)$$

for all  $h \in (0, 1)$ , where  $C = C(r, l, p, \lambda, q, T, K, M, d, d_1)$  is a constant.

*Proof.* As in the proof of Theorem 6.1 we take  $H_n := W_{h,2}^r$ ,  $H := W_2^{l-2}(\mathbb{R}^d)$ ,  $\mathcal{H} := W_2^l(\mathbb{R}^d)$  and the normal triplets

$$V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*, \quad V \hookrightarrow H \equiv H^* \hookrightarrow V^*, \quad \mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

with  $V_n := W_{h,2}^{r+1}$ ,  $V := W_2^{l-1}(\mathbb{R}^d)$ ,  $\mathcal{V} = W_2^{l+1}(\mathbb{R}^d)$ , we cast (6.11)–(6.12) into (5.2), and see that conditions (R1)–(R3), (C1n $\tau$ )–(C4n $\tau$ ) and (Cn $\tau$ ) of Theorem 5.2 are met. Furthermore,  $\vartheta(n) = \frac{\kappa^2}{h^2}$ . We can easily check that by virtue of (6.17) and (2.7), condition (6.18) yields condition (5.3). Hence applying Theorem 5.2 we finish the proof.  $\square$

**Corollary 6.3.** *Let  $k \geq 0$  be an integer and let Assumptions (A1), (A2), (A3), (A\*4) and (A\*5) hold with  $l$  satisfying  $l > k + 2 + d$ . Then the following statements are valid for all multi-indices  $|\alpha| \leq k$ :*

(i) *For sufficiently small  $\tau$*

$$E \max_{1 \leq i \leq m} \sup_{z \in \mathbb{G}} |\delta^\alpha(v^{h,\tau}(t_i, z) - u(t_i, z))| \leq C(h + \sqrt{\tau})$$

holds for all  $h \in (0, 1)$ , where  $C = C(l, p, \lambda, T, K, M, d, d_1)$  is a constant.

(ii) *Assume also that  $\tau$  and  $h$  satisfy (6.18). Then*

$$E \max_{1 \leq i \leq m} \sup_{z \in \mathbb{G}} |\delta^\alpha(v_\tau^h(t_i, z) - u(t_i, z))| \leq C(h + \sqrt{\tau}) \leq C \left(1 + \kappa^{-1} \sqrt{\lambda/L_1}\right) h$$

for all  $h \in (0, 1)$ , where  $C = C(r, l, p, \lambda, q, T, K, M, d, d_1)$  is a constant.

*Proof.* By the discrete version of Sobolev's theorem on embedding  $W_m^2(\mathbb{R}^d)$  into  $\mathcal{C}^k(\mathbb{R}^d)$  one knows that if  $m \geq k + \frac{d}{2}$ , then

$$\sup_{z \in \mathbb{G}} |\delta^\alpha \varphi(z)| \leq C |\varphi|_{W_{h,2}^m}$$

for all  $h \in (0, 1)$ ,  $\varphi \in W_{h,2}^m$  and  $|\alpha| \leq k$ , where  $C = C(d, m, k)$  is a constant (see, e.g., [18]). Hence the above statements follow immediately from Theorems 6.1 and 6.2.  $\square$

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