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# THE OSCILLATION STABILITY PROBLEM FOR THE URYSOHN SPHERE: A COMBINATORIAL APPROACH.

J. LOPEZ-ABAD AND L. NGUYEN VAN THÉ

ABSTRACT. We study the oscillation stability problem for the Urysohn sphere, an analog of the distortion problem for  $\ell_2$  in the context of the Urysohn space  $\mathbf{U}$ . In particular, we show that this problem reduces to a purely combinatorial problem involving a family of countable ultrahomogeneous metric spaces with finitely many distances.

## 1. INTRODUCTION.

The purpose of this note is to present several partial results related to the *oscillation stability problem for the Urysohn sphere*, a problem about to the geometry of the Urysohn space  $\mathbf{U}$  which can, in some sense, be seen as an analog for  $\mathbf{U}$  of the well-known *distortion problem for  $\ell_2$* . This latter problem appeared after the following central result in geometric functional analysis established by Milman: For  $N \in \omega$  strictly positive, let  $\mathbb{S}^N$  denote the unit sphere of the  $(N + 1)$ -dimensional Euclidean space and let  $\mathbb{S}^\infty$  denote the unit sphere of the Hilbert space  $\ell_2$ . If  $\mathbf{X} = (X, d^{\mathbf{X}})$  is a metric space,  $Y \subset X$  and  $\varepsilon > 0$ , let also

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y \ d^{\mathbf{X}}(x, y) \leq \varepsilon\}.$$

Then:

**Theorem** (Milman [10]). *Let  $\gamma$  be a finite partition of  $\mathbb{S}^\infty$ . Then for every  $\varepsilon > 0$  and every  $N \in \omega$ , there is  $A \in \gamma$  and an isometric copy  $\tilde{\mathbb{S}}^N$  of  $\mathbb{S}^N$  in  $\mathbb{S}^\infty$  such that  $\tilde{\mathbb{S}}^N \subset (A)_\varepsilon$ .*

Whether Milman's theorem still holds when  $N$  is replaced by  $\infty$  is the content of the distortion problem for  $\ell_2$ . Equivalently, if  $\varepsilon > 0$  and  $f : \mathbb{S}^\infty \rightarrow \mathbb{R}$  is bounded and uniformly continuous, is there a closed infinite-dimensional subspace  $V$  of  $\ell_2$  such that  $\sup\{|f(x) - f(y)| : x, y \in V \cap \mathbb{S}^\infty\} < \varepsilon$ ? This question remained unanswered for about 30 years, until the solution of Odell and Schlumprecht in [12]:

**Theorem** (Odell-Schlumprecht [12]). *There is a finite partition  $\gamma$  of  $\mathbb{S}^\infty$  and  $\varepsilon > 0$  such that no  $(A)_\varepsilon$  for  $A \in \gamma$  includes an isometric copy of  $\mathbb{S}^\infty$ .*

This result is traditionally stated in terms of *oscillation stability*, a concept coming from Banach space theory. However, it turns out that it can also be stated thanks to a new concept of oscillation stability due to Kechris, Pestov and Todorcevic introduced in [8] and more fully developed in [13]. The global formulation of

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this notion takes place at a very general level and allows to capture various phenomena coming from combinatorics and functional analysis. Nevertheless, it can be presented quite simply in the realm of complete separable ultrahomogeneous metric spaces, where it coincides with the Ramsey-theoretic concept of approximate indivisibility. Recall that a metric space  $\mathbf{X}$  is *ultrahomogeneous* when every isometry between finite metric subspaces of  $\mathbf{X}$  can be extended to an isometry of  $\mathbf{X}$  onto itself. Now, for  $\varepsilon \geq 0$ , call a metric space  $\mathbf{X}$   $\varepsilon$ -*indivisible* when for every strictly positive  $k \in \omega$  and every  $\chi : \mathbf{X} \rightarrow k$ , there is  $i < k$  and  $\tilde{\mathbf{X}} \subset \mathbf{X}$  isometric to  $\mathbf{X}$  such that

$$\tilde{\mathbf{X}} \subset (\overleftarrow{\chi}\{i\})_\varepsilon.$$

Then  $\mathbf{X}$  is *approximately indivisible* when  $\mathbf{X}$  is  $\varepsilon$ -indivisible for every  $\varepsilon > 0$ , and  $\mathbf{X}$  is *indivisible* when  $\mathbf{X}$  is 0-indivisible. For example, in this terminology, the aforementioned theorem of Odell and Schlumprecht asserts that  $\mathbb{S}^\infty$  is not approximately indivisible. However, in spite of this solution, it is sometimes felt that something essential is still to be discovered about the metric structure of  $\mathbb{S}^\infty$ . Indeed, quite surprisingly, the proof leading to the solution is not based on an analysis of the intrinsic geometry of  $\ell_2$ . This fact is one of the motivations for the present note: In this article, hoping that a better understanding of  $\mathbb{S}^\infty$  might be hidden behind a general approach of approximate indivisibility, we study the approximate indivisibility problem for another complete, separable ultrahomogeneous metric space, the *Urysohn sphere*  $\mathbf{S}$ , defined as follows: Up to isometry, it is the unique metric space to which every sphere of radius  $1/2$  in the Urysohn space  $\mathbf{U}$  is isometric. Equivalently, it is, up to isometry, the unique complete separable ultrahomogeneous metric space with diameter 1 into which every separable metric space with diameter less or equal to 1 embeds isometrically. In this note, we try to answer the following question implicitly present in [8] and explicitly stated in [7] and [13]:

**Question.** Is the Urysohn sphere  $\mathbf{S}$  oscillation stable? That is, given a finite partition  $\gamma$  of  $\mathbf{S}$  and  $\varepsilon > 0$ , is there  $A \in \gamma$  such that  $(A)_\varepsilon$  includes an isometric copy of  $\mathbf{S}$ ?

Our approach here is combinatorial and follows the general intuition according to which the structure of complete separable ultrahomogeneous metric spaces can be approached via combinatorial means. This intuition is based on two ideas. The first one is that the combinatorial point of view is relevant for the study of countable ultrahomogeneous metric spaces in general. This idea is already central in the work of Fraïssé completed in the fifties, even though Fraïssé theory takes place at the level of relational structures and includes much more than metric spaces (for a reference on Fraïssé theory, see [5]). More recently, it was also rediscovered by Bogatyĭ in a purely metric context, see [1] and [2]. The second idea is that the complete separable ultrahomogeneous metric spaces are closely linked to the countable ultrahomogeneous metric spaces. This connection also appears in Bogatyĭ's work but is on the other hand supported by the following result (which answers a question posed in [2]):

**Theorem 1.** *Every complete separable ultrahomogeneous metric space  $\mathbf{Y}$  includes a countable ultrahomogeneous dense metric subspace.*

For example, consider the *rational Urysohn space*  $\mathbf{U}_\mathbb{Q}$  which can be defined up to isometry as the unique countable ultrahomogeneous metric space with rational distances for which every countable metric space with rational distances embeds

isometrically. The Urysohn space  $\mathbf{U}$  arises then as the completion of  $\mathbf{U}_{\mathbb{Q}}$ , a fact which is actually essential as it is at the heart of several important contributions about  $\mathbf{U}$ . In particular, in the original article [15] of Urysohn, the space  $\mathbf{U}$  is precisely constructed as the completion of  $\mathbf{U}_{\mathbb{Q}}$  which is in turn constructed by hand.

Similarly, the Urysohn sphere  $\mathbf{S}$  arises as the completion of the so-called *rational Urysohn sphere*  $\mathbf{S}_{\mathbb{Q}}$ , defined up to isometry as the unique countable ultrahomogeneous metric space with distances in  $\mathbb{Q} \cap [0, 1]$  into which every at most countable metric space with distances in  $\mathbb{Q} \cap [0, 1]$  embeds isometrically.

At first glance, such a representation is relevant with respect to the oscillation stability problem for complete separable ultrahomogeneous metric spaces because it provides a direct way to transfer an approximate indivisibility problem to an exact indivisibility problem. For example, in the present case, it naturally leads to the question (explicitly stated in [11] and in [13]) of knowing whether  $\mathbf{S}_{\mathbb{Q}}$  is indivisible, a question which was answered recently by Delhommé, Laflamme, Pouzet and Sauer in [3], where a detailed analysis of metric indivisibility is provided and several obstructions to indivisibility are isolated. Cardinality is such an obstruction: A classical result in topology asserts that as soon as a metric space  $\mathbf{X}$  is uncountable, there is a partition of  $\mathbf{X}$  into two pieces such that none of the pieces includes a copy of the space via a continuous 1 – 1 map. Unboundedness is another example: If a metric space  $\mathbf{X}$  is indivisible, then its distance set is bounded. Now, it turns out that  $\mathbf{S}_{\mathbb{Q}}$  avoids these obstacles but encounters a third one: For a metric space  $\mathbf{X}$ ,  $x \in \mathbf{X}$ , and  $\varepsilon > 0$ , let  $\lambda_{\varepsilon}(x)$  be the supremum of all reals  $l \leq 1$  such that there is an  $\varepsilon$ -chain  $(x_i)_{i \leq n}$  containing  $x$  and such that  $d^{\mathbf{X}}(x_0, x_n) \geq l$ . Then, define

$$\lambda(x) = \inf\{\lambda_{\varepsilon}(x) : \varepsilon > 0\}.$$

**Theorem** (Delhommé-Laflamme-Pouzet-Sauer [3]). *Let  $\mathbf{X}$  be a countable metric space. Assume that there is  $x_0 \in \mathbf{X}$  such that  $\lambda(x_0) > 0$ . Then  $\mathbf{X}$  is not indivisible.*

Now, for  $\mathbf{S}_{\mathbb{Q}}$ , it is easy to see that ultrahomogeneity together with the fact that the distance set contains 0 as an accumulation point imply that every point  $x$  in  $\mathbf{S}_{\mathbb{Q}}$  is such that  $\lambda(x) = 1$ . It follows that:

**Corollary** (Delhommé-Laflamme-Pouzet-Sauer [3]).  *$\mathbf{S}_{\mathbb{Q}}$  is divisible.*

This result put an end to the first attempt to solve the oscillation stability problem for  $\mathbf{S}$ . Indeed, had  $\mathbf{S}_{\mathbb{Q}}$  been indivisible,  $\mathbf{S}$  would have been oscillation stable. But in the present case, the coloring which is used to divide  $\mathbf{S}_{\mathbb{Q}}$  does not lead to any conclusion and the oscillation stability problem for  $\mathbf{S}$  has to be attacked from another direction.

Here, following with the intuition that approximate indivisibility for  $\mathbf{S}$  can be attacked via the study of the exact indivisibility of simpler spaces, we can show:

**Theorem 2.**  *$\mathbf{S}$  is 1/6-indivisible.*

This result is obtained after having shown that the problem of approximate indivisibility for  $\mathbf{S}$  can be reduced to a purely combinatorial problem involving a family  $(\mathbf{S}_m)_{m \geq 1}$  of countable metric spaces which in some sense approximate the space  $\mathbf{S}$ . For  $m \in \omega$  strictly positive, set

$$[0, 1]_m := \left\{ \frac{k}{m} : k \in \{0, \dots, m\} \right\}.$$

Then  $\mathbf{S}_m$  is defined as follows: Up to isometry it is the unique countable ultrahomogeneous metric space with distances in  $[0, 1]_m$  into which every countable metric space with distances in  $[0, 1]_m$  embeds isometrically. Then:

**Theorem 3.** *The following are equivalent:*

- (i)  $\mathbf{S}$  is oscillation stable (equivalently, approximately indivisible).
- (ii) For every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is  $1/m$ -indivisible.
- (iii) For every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible.

The paper is organized as follows: In section 2, we introduce the spaces  $\mathbf{S}_m$  and study their relationship with  $\mathbf{S}$ . In particular, this leads us to a stronger version of Theorem 3. In section 3, we follow the different directions suggested by Theorem 3 and study the indivisibility as well as the  $1/m$ -indivisibility properties of the spaces  $\mathbf{S}_m$ . We then show how these results can be used to derive Theorem 2. Finally, we close with a short section including some remarks about possible further studies while an Appendix provides a proof of Theorem 1.

*Note:* Item (iii) of Theorem 3 was recently proved by the N. W. Sauer and the second author. The Urysohn sphere is therefore oscillation stable.

## 2. DISCRETIZATION.

The purpose of this section is to prove Theorem 3 and therefore to show that despite the unsuccessful attempt realized with  $\mathbf{S}_\mathbb{Q}$ , the oscillation stability problem for  $\mathbf{S}$  can indeed be understood via the study of the exact indivisibility of simpler spaces. The starting point of our construction consists in the observation that  $\mathbf{S}_\mathbb{Q}$  is the first natural candidate because it is a very good countable approximation of  $\mathbf{S}$ , but this good approximation is paradoxically responsible for the divisibility of  $\mathbf{S}_\mathbb{Q}$ . In particular, the distance set of  $\mathbf{S}_\mathbb{Q}$  is too rich and allows to create a dividing coloring. A natural attempt at that point is consequently to replace  $\mathbf{S}_\mathbb{Q}$  by another space with a simpler distance set but still allowing to approximate  $\mathbf{S}$  in a reasonable sense. In this perspective, general Fraïssé theory provides a whole family of candidates. Indeed, recall that for a strictly positive  $m \in \omega$ ,  $[0, 1]_m$  denotes the set  $\{k/m : k \in \{0, \dots, m\}\}$ . Then one can prove that there is a countable ultrahomogeneous metric space  $\mathbf{S}_m$  with distances in  $[0, 1]_m$  into which every countable metric space with distances in  $[0, 1]_m$  embeds isometrically and that those properties actually characterize  $\mathbf{S}_m$  up to isometry. In other words, the spaces  $\mathbf{S}_m$  are really the analogs of  $\mathbf{S}_\mathbb{Q}$  after having discretized the distance set  $\mathbb{Q} \cap [0, 1]$  with  $[0, 1]_m$ . The intuition is then that in some sense, this should allow them to approximate  $\mathbf{S}$ . This intuition turns out to be right, as shown by the following proposition whose proof is postponed to subsection 2.1:

**Proposition 1.** *For every strictly positive  $m \in \omega$ , there is an isometric copy  $\widetilde{\mathbf{S}}_m$  of  $\mathbf{S}_m$  inside  $\mathbf{S}$  such that  $(\widetilde{\mathbf{S}}_m)_{1/m} = \mathbf{S}$ .*

The spaces  $\mathbf{S}_m$  consequently appear as good candidates towards a discretization of the oscillation stability problem for  $\mathbf{S}$ . However, it turns out that Proposition 1 is not of any help towards a proof of Theorem 3. For example, Proposition 1 does not imply alone that if for some strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible, then  $\mathbf{S}$  is  $1/m$ -indivisible: Assume that  $\chi : \mathbf{S} \rightarrow k$ .  $\chi$  induces a coloring of  $\mathbf{S}_m$  so by indivisibility of  $\mathbf{S}_m$  there is  $\widetilde{\mathbf{S}}_m \subset \mathbf{S}_m$  isometric to  $\mathbf{S}_m$  on which  $\chi$  is constant. But

how does that allow to obtain a copy of  $\mathbf{S}$ ? For example, are we sure that  $(\tilde{\mathbf{S}}_m)_{1/m}$  includes a copy of  $\mathbf{S}$ ? We are not able to answer this question, but recent results of J. Melleray in [9] strongly suggest that  $(\tilde{\mathbf{S}}_m)_{1/m}$  really depends on the copy  $\tilde{\mathbf{S}}_m$  and can be extremely small. In particular, it may not include a copy of  $\mathbf{S}$ . Thus, to our knowledge, Proposition 1 does not say anything about the oscillation stability of  $\mathbf{S}$ , except maybe that the spaces  $\mathbf{S}_m$ 's are not totally irrelevant for our purposes.

Fortunately, the spaces  $\mathbf{S}_m$  do allow to go much further than Proposition 1 and are indeed relevant objects. In particular, they allow to reach the following equivalence, extending Theorem 3:

**Theorem 4.** *The following are equivalent:*

- (i)  $\mathbf{S}$  is oscillation stable.
- (ii)  $\mathbf{S}_{\mathbb{Q}}$  is approximately indivisible.
- (iii) For every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is  $1/m$ -indivisible.
- (iv) For every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible.

Subsections 2.2 to 2.5 are devoted to the proof of this result. But before going deeper into the technical details, let us mention here that part of our hope towards the discretization strategy comes from the proof of a famous result in Banach space theory, namely Gowers' stabilization theorem for  $c_0$  [6], where combinatorial Ramsey-type theorems for the spaces  $\text{FIN}_k^+$  and  $\text{FIN}_k$  imply that the unit sphere  $\mathbb{S}_{c_0}$  of  $c_0$  and its positive part  $\mathbb{S}_{c_0}^+$  are approximately indivisible.

**2.1. Proof of proposition 1.** We start with a definition: Given a metric space  $\mathbf{X} = (X, d^{\mathbf{X}})$ , a map  $f : X \rightarrow ]0, +\infty[$  is *Katětov over  $\mathbf{X}$*  when:

$$\forall x, y \in X, \quad |f(x) - f(y)| \leq d^{\mathbf{X}}(x, y) \leq f(x) + f(y).$$

Equivalently, one can extend the metric  $d^{\mathbf{X}}$  on  $X \dot{\cup} \{f\}$  by defining, for every  $x, y$  in  $X$ ,  $\widehat{d}^{\mathbf{X}}(x, f) = f(x)$  and  $\widehat{d}^{\mathbf{X}}(x, y) = d^{\mathbf{X}}(x, y)$ . The corresponding metric space is then written  $\mathbf{X} \dot{\cup} \{f\}$ . Here, the concept of Katětov map is relevant because of the following standard reformulation of the notion of ultrahomogeneity:

**Lemma 1.** *Let  $\mathbf{X}$  be a countable metric space. Then  $\mathbf{X}$  is ultrahomogeneous iff for every finite subspace  $\mathbf{F} \subset \mathbf{X}$  and every Katětov map  $f$  over  $\mathbf{F}$ , if  $\mathbf{F} \dot{\cup} \{f\}$  embeds into  $\mathbf{X}$ , then there is  $y \in \mathbf{X}$  such that for every  $x \in \mathbf{F}$ ,  $d^{\mathbf{X}}(x, y) = f(x)$ .*

This result will be used constantly throughout the proof. Now, some notation: For  $m \in \omega$  strictly positive, recall that  $[0, 1]_m = \{k/m : k \in \{0, \dots, m\}\}$ . For  $\alpha \in [0, 1]$ , set also

$$[\alpha]_m = \min([\alpha, 1] \cap [0, 1]_m) = \frac{[m\alpha]}{m},$$

where  $[x] = \min([x, \infty[ \cap \mathbb{Z})$  is the ceiling function. Since  $\mathbf{S}$  is the metric completion of  $\mathbf{S}_{\mathbb{Q}}$ , it is enough to show that for every strictly positive  $m \in \omega$ , there is an isometric copy  $\tilde{\mathbf{S}}_m$  of  $\mathbf{S}_m$  inside  $\mathbf{S}_{\mathbb{Q}}$  such that  $(\tilde{\mathbf{S}}_m)_{1/m} = \mathbf{S}_{\mathbb{Q}}$ . This is achieved thanks to a back and forth argument. The following is the main idea.

**Claim.** *Suppose that  $X \subset \mathbf{S}_{\mathbb{Q}}$  is finite and embeddable in  $\mathbf{S}_m$ , and let  $y \in \mathbf{S}_{\mathbb{Q}} \setminus X$ . Then the mapping  $f = f_{X, y, m} : X \cup \{y\} \rightarrow ]0, \infty[$  defined by  $f(x) = [d^{\mathbf{S}_{\mathbb{Q}}}(x, y)]_m$  if  $x \in X$  and  $f(y) = \max\{[d^{\mathbf{S}_{\mathbb{Q}}}(x, y)]_m - d^{\mathbf{S}_{\mathbb{Q}}}(x, y) : x \in X\}$  is Katětov.*

Assume this claim is true. Fix  $(x_n)_{n \in \omega}$  an enumeration of  $\mathbf{S}_m$  and  $(y_n)_{n \in \omega}$  an enumeration of  $\mathbf{S}_{\mathbb{Q}}$ . We are going to construct  $\sigma : \omega \rightarrow \omega$  together with a set  $\tilde{\mathbf{S}}_m = \{\tilde{x}_{\sigma(n)} : n \in \omega\} \subset \mathbf{S}_{\mathbb{Q}}$  so that:

- (i)  $\sigma$  is a bijection.
- (ii)  $\tilde{x}_{\sigma(n)} \mapsto x_{\sigma(n)}$  defines an isometry.
- (iii) For every  $n \in \omega$ ,  $\{y_i : i \leq n\} \subset (\{\tilde{x}_i : i \leq 2n+1\})_{1/m}$ .

Observe that, since  $\sigma$  is a permutation, (i) and (ii) guarantee that  $\tilde{x}_n \mapsto x_n$  defines a surjective isometry between  $\tilde{\mathbf{S}}_m$  and  $\mathbf{S}_m$ . On the other hand, (iii) guarantees that  $(\tilde{\mathbf{S}}_m)_{1/m} = \mathbf{S}_{\mathbb{Q}}$ .

Let  $\sigma(0) = 0$ ,  $\tilde{x}_0 = y_0$ . Suppose now all data up to  $2n$  already defined in the appropriate way, i.e. fulfilling the obvious partial versions of (i), (ii) and (iii). Let

$$\sigma(2n+1) = \min(\omega \setminus \{\sigma(i) : 0 \leq i \leq 2n\}).$$

Set also  $\tilde{x}_{\sigma(2n+1)} \in \mathbf{S}_{\mathbb{Q}}$  such that:

$$\forall i \in \{0, \dots, 2n\}, d^{\mathbf{S}_{\mathbb{Q}}}(\tilde{x}_{\sigma(i)}, \tilde{x}_{\sigma(2n+1)}) = d^{\mathbf{S}_m}(x_{\sigma(i)}, x_{\sigma(2n+1)}).$$

Next, if  $y_n \in (\{\tilde{x}_{\sigma(i)} : i \leq 2n+1\})_{1/m}$ , then we define  $\sigma(2n+2)$  and  $\tilde{x}_{\sigma(2n+2)}$  as we did for  $2n+1$ . Otherwise, let  $f$  be the Katětov map given by the previous claim when applied to  $X = \{\tilde{x}_{\sigma(i)} : 0 \leq i \leq 2n+1\}$  and  $y_n$ . Let  $\tilde{x} \in \mathbf{S}_{\mathbb{Q}}$  realizing  $f$ . Now observe that the map  $g$  defined on  $\{x_{\sigma(i)} : 0 \leq i \leq 2n+1\}$  by  $g(x_{\sigma(i)}) = f(\tilde{x}_{\sigma(i)})$  is Katětov with values in  $[0, 1]_m$ , so

$$\sigma(2n+2) = \min\{k \in \omega : \forall i \in \{0, \dots, 2n+1\}, d^{\mathbf{S}_m}(x_{\sigma(i)}, x_k) = g(x_{\sigma(i)})\}$$

is well defined and we set  $\tilde{x}_{\sigma(2n+2)} = \tilde{x}$ .

We now turn to the proof of the claim. Fix  $x, x' \in X$ . We have to prove:

$$|f(x) - f(x')| \leq d^{\mathbf{S}_{\mathbb{Q}}}(x, x') \leq f(x) + f(x') \quad (1)$$

$$|f(x) - f(y)| \leq d^{\mathbf{S}_{\mathbb{Q}}}(x, y) \leq f(x) + f(y) \quad (2)$$

For (1): The right inequality is not a problem:

$$d^{\mathbf{S}_{\mathbb{Q}}}(x, x') \leq d^{\mathbf{S}_{\mathbb{Q}}}(x, y) + d^{\mathbf{S}_{\mathbb{Q}}}(y, x') \leq f(x) + f(x').$$

For the left inequality, we use the following simple fact:

$$\forall \alpha, \beta \in \mathbb{R}, \forall p \in \omega, |\beta - \alpha| \leq \frac{p}{m} \longrightarrow |[\beta]_m - [\alpha]_m| \leq \frac{p}{m}.$$

Indeed, assume that  $|\beta - \alpha| \leq p/m$ . We want  $|[m\beta] - [m\alpha]| \leq p$ . Without loss of generality,  $\alpha \leq \beta$ . Then  $0 \leq [m\beta] - [m\alpha] < m\beta + 1 - m\alpha \leq p + 1$ , so  $|[m\beta] - [m\alpha]| \leq p$  and we are done. In our case, that property is useful because then the left inequality directly follows from

$$|d^{\mathbf{S}_{\mathbb{Q}}}(x, y) - d^{\mathbf{S}_{\mathbb{Q}}}(y, x')| \leq d^{\mathbf{S}_{\mathbb{Q}}}(x, x') \in [0, 1]_m,$$

because  $X$  is embeddable in  $\mathbf{S}_m$ . For (2):

$$|f(x) - f(y)| = f(x) - f(y).$$

This is because  $f(x) \geq 1/m$  and  $0 \leq f(y) < 1/m$ . Furthermore, by definition of  $f$ ,

$$f(y) \geq f(x) - d^{\mathbf{S}_{\mathbb{Q}}}(x, y).$$

So the left inequality is satisfied. For the right inequality, simply observe that

$$d^{\mathbf{S}_{\mathbb{Q}}}(x, y) \leq f(x). \quad \square$$

## 2.2. From oscillation stability of $\mathbf{S}$ to approximate indivisibility of $\mathbf{S}_{\mathbb{Q}}$ .

The purpose of what follows is to prove the implication (i)  $\rightarrow$  (ii) of Theorem 4 stating that if  $\mathbf{S}$  is oscillation stable, then  $\mathbf{S}_{\mathbb{Q}}$  is approximately indivisible. This is done thanks to the following result:

**Proposition 2.** *Suppose that  $\mathbf{S}_{\mathbb{Q}}^0$  and  $\mathbf{S}_{\mathbb{Q}}^1$  are two copies of  $\mathbf{S}_{\mathbb{Q}}$  in  $\mathbf{S}$  such that  $\mathbf{S}_{\mathbb{Q}}^0$  is dense in  $\mathbf{S}$ . Then for every  $\varepsilon > 0$  the subspace  $\mathbf{S}_{\mathbb{Q}}^0 \cap (\mathbf{S}_{\mathbb{Q}}^1)_{\varepsilon}$  includes a copy of  $\mathbf{S}_{\mathbb{Q}}$ .*

*Proof.* We construct the required copy of  $\mathbf{S}_{\mathbb{Q}}$  inductively. Let  $\{y_n : n \in \omega\}$  enumerate  $\mathbf{S}_{\mathbb{Q}}^1$ . For  $k \in \omega$ , set

$$\delta_k = \frac{\varepsilon}{2} \sum_{i=0}^k \frac{1}{2^i}.$$

Set also

$$\eta_k = \frac{\varepsilon}{3} \frac{1}{2^{k+1}}.$$

$\mathbf{S}_{\mathbb{Q}}^0$  being dense in  $\mathbf{S}$ , choose  $z_0 \in \mathbf{S}_{\mathbb{Q}}^0$  such that  $d^{\mathbf{S}}(y_0, z_0) < \delta_0$ . Assume now that  $z_0, \dots, z_n \in \mathbf{S}_{\mathbb{Q}}^0$  were constructed such that for every  $k, l \leq n$

$$\begin{cases} d^{\mathbf{S}}(z_k, z_l) = d^{\mathbf{S}}(y_k, y_l) \\ d^{\mathbf{S}}(z_k, y_k) < \delta_k. \end{cases}$$

Again by denseness of  $\mathbf{S}_{\mathbb{Q}}^0$  in  $\mathbf{S}$ , fix  $z \in \mathbf{S}_{\mathbb{Q}}^0$  such that

$$d^{\mathbf{S}}(z, y_{n+1}) < \eta_{n+1}.$$

Then for every  $k \leq n$ ,

$$\begin{aligned} |d^{\mathbf{S}}(z, z_k) - d^{\mathbf{S}}(y_{n+1}, y_k)| &= |d^{\mathbf{S}}(z, z_k) - d^{\mathbf{S}}(z_k, y_{n+1}) + d^{\mathbf{S}}(z_k, y_{n+1}) - d^{\mathbf{S}}(y_{n+1}, y_k)| \\ &\leq d^{\mathbf{S}}(z, y_{n+1}) + d^{\mathbf{S}}(z_k, y_k) \\ &< \eta_{n+1} + \delta_k \\ &< \eta_{n+1} + \delta_n. \end{aligned}$$

It follows that there is  $z_{n+1} \in \mathbf{S}_{\mathbb{Q}}^0$  such that

$$\begin{cases} \forall k \leq n \quad d^{\mathbf{S}}(z_{n+1}, z_k) = d^{\mathbf{S}}(y_{n+1}, y_k) \\ d^{\mathbf{S}}(z_{n+1}, z) < \eta_{n+1} + \delta_n. \end{cases}$$

Indeed, consider the map  $f$  defined on  $\{z_k : k \leq n\} \cup \{z\}$  by:

$$\begin{cases} \forall k \leq n \quad f(z_k) = d^{\mathbf{S}}(y_{n+1}, y_k) \\ f(z) = |d^{\mathbf{S}}(z, z_k) - d^{\mathbf{S}}(y_{n+1}, y_k)|. \end{cases}$$

Then  $f$  is Katětov over the subspace of  $\mathbf{S}_{\mathbb{Q}}^0$  supported by  $\{z_k : k \leq n\} \cup \{z\}$ , so simply take  $z_{n+1} \in \mathbf{S}_{\mathbb{Q}}^0$  realizing it. Observe then that

$$\begin{aligned} d^{\mathbf{S}}(z_{n+1}, y_{n+1}) &\leq d^{\mathbf{S}}(z_{n+1}, z) + d^{\mathbf{S}}(z, y_{n+1}) \\ &< \eta_{n+1} + \delta_n + \eta_{n+1} \\ &< \delta_{n+1}. \end{aligned}$$

After  $\omega$  steps, we are left with  $\{z_n : n \in \omega\} \subset \mathbf{S}_{\mathbb{Q}}^0 \cap (\mathbf{S}_{\mathbb{Q}}^1)_{\varepsilon}$  isometric to  $\mathbf{S}_{\mathbb{Q}}$ .  $\square$



We now show how to deduce (i)  $\rightarrow$  (ii) of Theorem 4 from Proposition 2: Let  $\varepsilon > 0$ ,  $k \in \omega$  strictly positive and  $\chi : \mathbf{S}_{\mathbb{Q}} \rightarrow k$ . Then in  $\mathbf{S}$ , seeing  $\mathbf{S}_{\mathbb{Q}}$  as a dense subspace:

$$\mathbf{S} = \bigcup_{i < k} (\overleftarrow{\chi}\{i\})_{\varepsilon/2}.$$

By oscillation stability of  $\mathbf{S}$ , there is  $i < k$  and a copy  $\tilde{\mathbf{S}}$  of  $\mathbf{S}$  included in  $\mathbf{S}$  such that

$$\tilde{\mathbf{S}} \subset ((\overleftarrow{\chi}\{i\})_{\varepsilon/2})_{\varepsilon/4}.$$

Since  $\tilde{\mathbf{S}}$  includes copies of  $\mathbf{S}_{\mathbb{Q}}$ , and since  $\mathbf{S}_{\mathbb{Q}}$  is dense in  $\mathbf{S}$ , it follows by Proposition 2 that there is a copy  $\tilde{\mathbf{S}}_{\mathbb{Q}}$  of  $\mathbf{S}_{\mathbb{Q}}$  in  $\mathbf{S}_{\mathbb{Q}} \cap (\tilde{\mathbf{S}})_{\varepsilon/4}$ . Then in  $\mathbf{S}_{\mathbb{Q}}$

$$\tilde{\mathbf{S}}_{\mathbb{Q}} \subset (\overleftarrow{\chi}\{i\})_{\varepsilon}. \quad \square$$

**2.3. From approximate indivisibility of  $\mathbf{S}_{\mathbb{Q}}$  to  $1/m$ -indivisibility of  $\mathbf{S}_m$ .** Here, we provide a proof for the implication (ii)  $\rightarrow$  (iii) of Theorem 4 according to which if  $\mathbf{S}_{\mathbb{Q}}$  is approximately indivisible, then  $\mathbf{S}_m$  is  $1/m$ -indivisible for every strictly positive  $m \in \omega$ . This is obtained as the consequence of the following proposition:

**Proposition 3.** *Let  $\varepsilon > 0$  and assume that  $\mathbf{S}_{\mathbb{Q}}$  is  $\varepsilon$ -indivisible. Then  $\mathbf{S}_m$  is  $1/m$ -indivisible whenever  $m \leq 1/\varepsilon$ .*

*Proof.* Let  $\varepsilon > 0$ , assume that  $\mathbf{S}_{\mathbb{Q}}$  is  $\varepsilon$ -indivisible and fix  $m \in \omega$  strictly positive such that  $\varepsilon \leq 1/m$ . Define  $[d^{\mathbf{S}_{\mathbb{Q}}}]_m$  by

$$\forall x, y \in X \quad [d^{\mathbf{S}_{\mathbb{Q}}}]_m(x, y) = [d^{\mathbf{S}_{\mathbb{Q}}}(x, y)]_m.$$

**Claim.**  $[d^{\mathbf{S}_{\mathbb{Q}}}]_m$  is a metric on  $\mathbf{S}_{\mathbb{Q}}$ .

*Proof.* Since the function  $[\cdot]_m$  is subadditive and increasing, it easily follows that the composition  $[d^{\mathbf{S}_{\mathbb{Q}}}]_m = [\cdot]_m \circ d^{\mathbf{S}_{\mathbb{Q}}}$  is a metric.  $\square$

Let  $\mathbf{X}_m$  be the metric space

$$\mathbf{X}_m = (\mathbf{S}_{\mathbb{Q}}, [d^{\mathbf{S}_{\mathbb{Q}}}]_m),$$

and let  $\pi_m$  denote the identity map from  $\mathbf{S}_{\mathbb{Q}}$  to  $\mathbf{X}_m$ . Observe that  $\mathbf{X}_m$  and  $\mathbf{S}_m$  embed into each other, and that consequently,  $1/m$ -indivisibility of  $\mathbf{S}_m$  is equivalent to  $1/m$ -indivisibility of  $\mathbf{X}_m$ . So let  $k \in \omega$  be strictly positive and  $\chi : \mathbf{X}_m \rightarrow k$ . Then  $\chi$  induces a coloring  $\chi \circ \pi_m : \mathbf{S}_{\mathbb{Q}} \rightarrow k$ . Since  $\mathbf{S}_{\mathbb{Q}}$  is  $\varepsilon$ -indivisible, there is  $i < k$  and a copy  $\tilde{\mathbf{S}}_{\mathbb{Q}}$  of  $\mathbf{S}_{\mathbb{Q}}$  inside  $\mathbf{S}_{\mathbb{Q}}$  such that

$$\tilde{\mathbf{S}}_{\mathbb{Q}} \subset (\overleftarrow{\chi \circ \pi_m}\{i\})_{\varepsilon}.$$

Now, observe that  $\pi_m''\tilde{\mathbf{S}}_{\mathbb{Q}}$  is a copy of  $\mathbf{X}_m$  inside  $\mathbf{X}_m$ . Furthermore, note that

$$\forall x \neq y \in \mathbf{S}_{\mathbb{Q}} \text{ if } d^{\mathbf{S}_{\mathbb{Q}}}(x, y) \leq \frac{1}{m} \text{ then } d^{\mathbf{X}_m}(\pi_m(x), \pi_m(y)) = \frac{1}{m}.$$

Since  $\varepsilon \leq 1/m$ , it follows that

$$\pi_m''(\overleftarrow{\chi \circ \pi_m}\{i\})_{\varepsilon} \subset (\overleftarrow{\chi}\{i\})_{1/m}.$$

And so

$$\pi_m''\tilde{\mathbf{S}}_{\mathbb{Q}} \subset (\overleftarrow{\chi}\{i\})_{1/m}. \quad \square$$

2.4. **From  $1/2(m^2 + m)$ -indivisibility of  $\mathbf{S}_{2(m^2+m)}$  to indivisibility of  $\mathbf{S}_m$ .** We now turn to the proof of the implication (iii)  $\rightarrow$  (iv) of Theorem 4 stating that if for every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is  $1/m$ -indivisible, then for every strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible. This is done via the following proposition:

**Proposition 4.** *Suppose that for some strictly positive integer  $m$ ,  $\mathbf{S}_{2(m^2+m)}$  is  $1/2(m^2 + m)$ -indivisible. Then  $\mathbf{S}_m$  is indivisible.*

*Proof.* Let  $m \in \omega$  be strictly positive and such that  $\mathbf{S}_{2(m^2+m)}$  is  $1/2(m^2 + m)$ -indivisible. We are going to create a metric space  $\mathbf{W}$  with distances in  $[0, 1]_m$  and a bijection  $\pi : \mathbf{S}_{2(m^2+m)} \rightarrow \mathbf{W}$  such that for every subspace  $\mathbf{Y}$  of  $\mathbf{S}_{2(m^2+m)}$ , if  $(\mathbf{Y})_{1/2(m^2+m)}$  includes a copy of  $\mathbf{S}_m$ , then so does  $\pi''\mathbf{Y}$ .

Assuming that such a space  $\mathbf{W}$  is constructed, the result is proved as follows: Observe first that  $\mathbf{W}$  and  $\mathbf{S}_m$  embed into each other. Indivisibility of  $\mathbf{W}$  is consequently equivalent to indivisibility of  $\mathbf{S}_m$  and it is enough to show that  $\mathbf{W}$  is indivisible. Let  $k \in \omega$  be strictly positive and  $\chi : \mathbf{W} \rightarrow k$ . Then  $\chi \circ \pi : \mathbf{S}_{2(m^2+m)} \rightarrow k$  and by  $1/2(m^2 + m)$ -indivisibility of  $\mathbf{S}_{2(m^2+m)}$ , there is  $i < k$  such that  $(\overleftarrow{\chi \circ \pi}\{i\})_{1/2(m^2+m)}$  includes a copy of  $\mathbf{S}_{2(m^2+m)}$ . Since  $\mathbf{S}_m$  embeds into  $\mathbf{S}_{2(m^2+m)}$ ,  $(\overleftarrow{\chi \circ \pi}\{i\})_{1/2(m^2+m)}$  also includes a copy of  $\mathbf{S}_m$ . Thus,  $\overleftarrow{\chi}\{i\} = \pi''\overleftarrow{\chi \circ \pi}\{i\}$  includes a copy of  $\mathbf{S}_m$ , and therefore a copy of  $\mathbf{W}$ .

We now turn to the construction of  $\mathbf{W}$ . This space is obtained by modifying the metric on  $\mathbf{S}_{2(m^2+m)}$  to a metric  $d$ , so that  $\mathbf{W} = (S_{2(m^2+m)}, d)$  and  $\pi$  is simply the identity map from  $\mathbf{S}_{2(m^2+m)}$  to  $\mathbf{W}$ . The metric  $d$  is defined as follows: consider the map  $f : [0, 1]_{2(m^2+m)} \rightarrow [0, 1]_m$  defined by  $f(x) = \frac{l}{m}$  where  $l$  is the least integer such that

$$x \leq l \left( \frac{1}{m} + \frac{1}{m^2 + m} \right).$$

Observe that  $f$  is increasing, that  $f(0) = 0$ , and that

$$\forall \alpha \in [0, 1]_m \quad \forall \varepsilon \in \{-2, -1, 0, 1, 2\} \quad f \left( \alpha + \frac{\varepsilon}{2(m^2 + m)} \right) = \alpha.$$

Note also that  $f$  is subadditive: Let  $x, y, \in [0, 1]_{2(m^2+m)}$ . Assume that  $f(x) = l/m$ . Then there is  $n \in \{1, \dots, 2m + 4\}$  such that

$$x = \frac{l-1}{m} + \frac{l-1}{m^2+m} + \frac{n}{2(m^2+m)}.$$

Similarly, there are  $l' \in \{0, \dots, m\}$  and  $n' \in \{1, \dots, 2m + 4\}$  such that

$$y = \frac{l'-1}{m} + \frac{l'-1}{m^2+m} + \frac{n'}{2(m^2+m)}.$$

So

$$\begin{aligned} x + y &= (l + l') \left( \frac{1}{m} + \frac{1}{m^2 + m} \right) - 2 \left( \frac{1}{m} + \frac{1}{m^2 + m} \right) + \frac{n + n'}{2(m^2 + m)} \\ &= (l + l') \left( \frac{1}{m} + \frac{1}{m^2 + m} \right) + \frac{n - (2m + 4) + n' - (2m + 4)}{2(m^2 + m)} \\ &\leq (l + l') \left( \frac{1}{m} + \frac{1}{m^2 + m} \right). \end{aligned}$$

Therefore,

$$f(x + y) \leq \frac{l + l'}{m} = \frac{l}{m} + \frac{l'}{m} = f(x) + f(y).$$

It follows that the map  $d := f \circ d^{\mathbf{S}_{2(m^2+m)}}$  is a metric taking values in  $[0, 1]_m$ . Now to show that  $d$  is as required, it suffices to prove that for every subspace  $\mathbf{Y}$  of  $\mathbf{S}_{2(m^2+m)}$ , if  $(\mathbf{Y})_{1/2(m^2+m)}$  includes a copy of  $\mathbf{S}_m$ , then  $\pi''\mathbf{Y}$  includes a copy of  $\mathbf{S}_m$ . So let  $\mathbf{Y}$  be a subspace of  $\mathbf{S}_{2(m^2+m)}$  such that  $(\mathbf{Y})_{1/2(m^2+m)}$  includes a copy  $\tilde{\mathbf{S}}_m$  of  $\mathbf{S}_m$ . Then for every  $x \in \tilde{\mathbf{S}}_m$ , there is an element  $\varphi(x) \in \mathbf{Y}$  such that  $d^{\mathbf{S}_{2(m^2+m)}}(x, \varphi(x)) \leq 1/2(m^2 + m)$ . Thus,

$$\forall x \neq y \in \tilde{\mathbf{S}}_m \quad \left| d^{\mathbf{S}_{2(m^2+m)}}(\varphi(x), \varphi(y)) - d^{\mathbf{S}_{2(m^2+m)}}(x, y) \right| \leq \frac{1}{m^2 + m}.$$

Since  $d^{\mathbf{S}_{2(m^2+m)}}(x, y) \in [0, 1]_m$ ,

$$f \left( d^{\mathbf{S}_{2(m^2+m)}}(\varphi(x), \varphi(y)) \right) = d^{\mathbf{S}_{2(m^2+m)}}(x, y).$$

That is

$$d(\pi(\varphi(x)), \pi(\varphi(y))) = d^{\mathbf{S}_{2(m^2+m)}}(x, y).$$

Thus,  $\pi''\text{ran}(\varphi) \subset \pi''\mathbf{Y}$  is isometric to  $\mathbf{S}_m$ .  $\square$

**2.5. From indivisibility of  $\mathbf{S}_m$  to oscillation stability of  $\mathbf{S}$ .** We are now ready to close the loop of implications of Theorem 4. In what follows, we show that if  $\mathbf{S}_m$  is indivisible for every strictly positive  $m \in \omega$ , then  $\mathbf{S}$  is oscillation stable. This is achieved thanks to the following result:

**Proposition 5.** *Assume that for some strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible. Then  $\mathbf{S}$  is  $1/2m$ -indivisible.*

*Proof.* This is obtained by showing that for every strictly positive  $m \in \omega$ , there is an isometric copy  $\mathbf{S}_m^*$  of  $\mathbf{S}_m$  inside  $\mathbf{S}$  such that for every  $\tilde{\mathbf{S}}_m \subset \mathbf{S}_m^*$  isometric to  $\mathbf{S}_m$ ,  $(\tilde{\mathbf{S}}_m)_{1/2m}$  includes an isometric copy of  $\mathbf{S}$ . This property indeed suffices to prove Proposition 5: Let  $\chi : \mathbf{S} \rightarrow k$  for some strictly positive  $k \in \omega$ .  $\chi$  induces a  $k$ -coloring of the copy  $\mathbf{S}_m^*$ . By indivisibility of  $\mathbf{S}_m$ , find  $i < k$  and  $\tilde{\mathbf{S}}_m \subset \mathbf{S}_m^*$  such that  $\chi$  is constant on  $\tilde{\mathbf{S}}_m$  with value  $i$ . But then, in  $\mathbf{S}$ ,  $(\tilde{\mathbf{S}}_m)_{1/2m}$  includes a copy of  $\mathbf{S}$ . So  $(\overline{\chi}\{i\})_{1/2m}$  includes a copy of  $\mathbf{S}$ .

We now turn to the construction of  $\mathbf{S}_m^*$ . The core of the proof is contained in Lemma 2 which we present now. Fix an enumeration  $\{y_n : n \in \omega\}$  of  $\mathbf{S}_{\mathbb{Q}}$ . Also, keeping the notation introduced in the proof of Proposition 3, let  $\mathbf{X}_m$  be the metric space  $(\mathbf{S}_{\mathbb{Q}}, [d^{\mathbf{S}_{\mathbb{Q}}}]_m)$ . The underlying set of  $\mathbf{X}_m$  is really  $\{y_n : n \in \omega\}$  but to avoid confusion, we write it  $\{x_n : n \in \omega\}$ , being understood that for every  $n \in \omega$ ,  $x_n = y_n$ . On the other hand, remember that  $\mathbf{S}_m$  and  $\mathbf{X}_m$  embed isometrically into each other.

**Lemma 2.** *There is a countable metric space  $\mathbf{Z}$  with distances in  $[0, 1]$  and including  $\mathbf{X}_m$  such that for every strictly increasing  $\sigma : \omega \rightarrow \omega$  such that  $x_n \mapsto x_{\sigma(n)}$  is an isometry,  $(\{x_{\sigma(n)} : n \in \omega\})_{1/2m}$  includes an isometric copy of  $\mathbf{S}_{\mathbb{Q}}$ .*

Assuming Lemma 2, we now show how we can construct  $\mathbf{S}_m^*$ .  $\mathbf{Z}$  is countable with distances in  $[0, 1]$  so we may assume that it is a subspace of  $\mathbf{S}$ . Now, take  $\mathbf{S}_m^*$  a subspace of  $\mathbf{X}_m$  and isometric to  $\mathbf{S}_m$ . We claim that  $\mathbf{S}_m^*$  works: Let  $\tilde{\mathbf{S}}_m \subset \mathbf{S}_m^*$  be isometric to  $\mathbf{S}_m$ . We first show that  $(\tilde{\mathbf{S}}_m)_{1/2m}$  includes a copy of  $\mathbf{S}_{\mathbb{Q}}$ . The enumeration  $\{x_n : n \in \omega\}$  induces a linear ordering  $<$  of  $\tilde{\mathbf{S}}_m$  in type  $\omega$ . According to lemma 2, it suffices to show that  $(\tilde{\mathbf{S}}_m, <)$  includes a copy of  $\{x_n : n \in \omega\}_<$  seen as an ordered metric space. To do that, observe that since  $\mathbf{X}_m$  embeds isometrically

into  $\mathbf{S}_m$ , there is a linear ordering  $<^*$  of  $\mathbf{S}_m$  in type  $\omega$  such that  $\{x_n : n \in \omega\}_{<}$  embeds into  $(\mathbf{S}_m, <^*)$  as ordered metric space. Therefore, it is enough to show:

**Claim.**  $(\tilde{\mathbf{S}}_m, <)$  includes a copy of  $(\mathbf{S}_m, <^*)$ .

*Proof.* Write

$$\begin{aligned}(\mathbf{S}_m, <^*) &= \{s_n : n \in \omega\}_{<^*} \\ (\tilde{\mathbf{S}}_m, <) &= \{t_n : n \in \omega\}_{<}.\end{aligned}$$

Let  $\sigma(0) = 0$ . If  $\sigma(0) < \dots < \sigma(n)$  are chosen such that  $s_k \mapsto t_{\sigma(k)}$  is a finite isometry, observe that the following set is infinite

$$\{i \in \omega : \forall k \leq n \ d^{\mathbf{S}_m}(t_{\sigma(k)}, t_i) = d^{\mathbf{S}_m}(s_k, s_{n+1})\}.$$

Therefore, simply take  $\sigma(n+1)$  in that set and larger than  $\sigma(n)$ .  $\square$

Observe that since the metric completion of  $\mathbf{S}_{\mathbb{Q}}$  is  $\mathbf{S}$ , the closure of  $(\tilde{\mathbf{S}}_m)_{1/2m}$  in  $\mathbf{S}$  includes a copy of  $\mathbf{S}$ . Hence we are done since  $(\tilde{\mathbf{S}}_m)_{1/2m}$  is closed in  $\mathbf{S}$ .  $\square$

We now turn to the proof of lemma 2. The strategy is first to provide the set  $Z$  where the required metric space  $\mathbf{Z}$  is supposed to be based on, and then to argue that the distance  $d^{\mathbf{Z}}$  can be obtained (lemmas 3 to 7). To construct  $Z$ , proceed as follows: For  $t \subset \omega$ , write  $t$  as the strictly increasing enumeration of its elements:

$$t = \{t_i : i \in |t|\}_{<}.$$

Now, let  $T$  be the set of all finite nonempty subsets  $t$  of  $\omega$  such that  $x_n \mapsto x_{t_n}$  is an isometry between  $\{x_n : n \in |t|\}$  and  $\{x_{t_n} : n \in |t|\}$ . This set  $T$  is a tree when ordered by end-extension. Let

$$Z = X_m \dot{\cup} T.$$

For  $z \in Z$ , define

$$\pi(z) = \begin{cases} z & \text{if } z \in X_m. \\ x_{\max z} & \text{if } z \in T. \end{cases}$$

Now, consider an edge-labelled graph structure on  $Z$  by defining  $\delta$  with domain  $\text{dom}(\delta) \subset Z \times Z$  and range  $\text{ran}(\delta) \subset [0, 1]$  as follows:

- If  $s, t \in T$ , then  $(s, t) \in \text{dom}(\delta)$  iff  $s$  and  $t$  are  $<_T$  comparable. In this case,

$$\delta(s, t) = d^{\mathbf{S}_{\mathbb{Q}}}(y_{|s|-1}, y_{|t|-1}).$$

- If  $x, y \in X_m$ , then  $(x, y)$  is always in  $\text{dom}(\delta)$  and

$$\delta(x, y) = d^{\mathbf{X}_m}(x, y).$$

- If  $t \in T$  and  $x \in X_m$ , then  $(x, s)$  and  $(s, x)$  are in  $\text{dom}(\delta)$  iff  $x = \pi(t)$ . In this case

$$\delta(x, s) = \delta(s, x) = \frac{1}{2m}.$$

For a branch  $b$  of  $T$  and  $i \in \omega$ , let  $b(i)$  be the unique element of  $b$  with height  $i$  in  $T$ . Observe that  $b(i)$  is a  $i+1$ -element subset of  $\omega$ . Observe also that for every  $i, j \in \omega$ ,  $b(i)$  is connected to  $\pi(b(i))$  and  $b(j)$ , and

- $\delta(b(i), \pi(b(i))) = 1/2m$ ,
- $\delta(b(i), b(j)) = d^{\mathbf{S}_{\mathbb{Q}}}(y_i, y_j)$ ,
- $\delta(\pi(b(i)), \pi(b(j)))$  is equal to any of the following quantities:  
 $d^{\mathbf{X}_m}(x_{\max b(i)}, x_{\max b(j)}) = d^{\mathbf{X}_m}(x_i, x_j) = \lceil d^{\mathbf{S}_{\mathbb{Q}}}(y_i, y_j) \rceil_m$ .

In particular, if  $b$  is a branch of  $T$ , then  $\delta$  induces a metric on  $b$  and the map from  $\mathbf{S}_{\mathbb{Q}}$  to  $b$  mapping  $y_i$  to  $b(i)$  is a surjective isometry. We claim that if we can show that  $\delta$  can be extended to a metric  $d^{\mathbf{Z}}$  on  $Z$  with distances in  $[0, 1]$ , then lemma 2 will be proved. Indeed, let

$$\tilde{\mathbf{X}}_m = \{x_{\sigma(n)} : n \in \omega\} \subset \mathbf{X}_m,$$

with  $\sigma : \omega \rightarrow \omega$  strictly increasing and  $x_n \mapsto x_{\sigma(n)}$  distance preserving. See  $\text{ran}(\sigma)$  as a branch  $b$  of  $T$ . Then  $(b, d^{\mathbf{Z}}) = (b, \delta)$  is isometric to  $\mathbf{S}_{\mathbb{Q}}$  and

$$b \subset (\pi''b)_{1/2m} = (\tilde{\mathbf{X}}_m)_{1/2m}.$$

Our goal now is consequently to show that  $\delta$  can be extended to a metric on  $Z$  with values in  $[0, 1]$ . Recall that for  $x, y \in Z$ , and  $n \in \omega$  strictly positive, a path from  $x$  to  $y$  of size  $n$  as is a finite sequence  $\gamma = (z_i)_{i < n}$  such that  $z_0 = x$ ,  $z_{n-1} = y$  and for every  $i < n - 1$ ,

$$(z_i, z_{i+1}) \in \text{dom}(\delta).$$

For  $x, y$  in  $Z$ ,  $P(x, y)$  is the set of all paths from  $x$  to  $y$ . If  $\gamma = (z_i)_{i < n}$  is in  $P(x, y)$ ,  $\|\gamma\|$  is defined as:

$$\|\gamma\| = \sum_{i=0}^{n-1} \delta(z_i, z_{i+1}).$$

On the other hand,  $\|\gamma\|_{\leq 1}$  is defined as:

$$\|\gamma\|_{\leq 1} = \min(\|\gamma\|, 1).$$

We are going to see that the required metric can be obtained with  $d^{\mathbf{Z}}$  defined by

$$d^{\mathbf{Z}}(x, y) = \inf\{\|\gamma\|_{\leq 1} : \gamma \in P(x, y)\}.$$

Equivalently, we are going to show that for every  $(x, y) \in \text{dom}(\delta)$ , every path  $\gamma$  from  $x$  to  $y$  is metric, that is:

$$\delta(x, y) \leq \|\gamma\|_{\leq 1} \tag{3}$$

Let  $x, y \in Z$ . Call a path  $\gamma$  from  $x$  to  $y$  *trivial* when  $\gamma = (x, y)$  and *irreducible* when no proper subsequence of  $\gamma$  is a non-trivial path from  $x$  to  $y$ . Finally, say that  $\gamma$  is a *cycle* when  $(x, y) \in \text{dom}(\delta)$ . It should be clear that to prove that  $d^{\mathbf{Z}}$  works, it is enough to show that the previous inequality (3) is true for every irreducible cycle. Note that even though  $\delta$  takes only rational values, it might not be the case for  $d^{\mathbf{Z}}$ . We now turn to the study of the irreducible cycles in  $Z$ .

**Lemma 3.** *Let  $x, y \in T$ . Assume that  $x$  and  $y$  are not  $<_T$ -comparable. Let  $\gamma$  be an irreducible path from  $x$  to  $y$  in  $T$ . Then there is  $z \in T$  such that  $z <_T x$ ,  $z <_T y$  and  $\gamma = (x, z, y)$ .*

*Proof.* Write  $\gamma = (z_i)_{i < n+1}$ .  $z_1$  is connected to  $x$  so  $z_1$  is  $<_T$ -comparable with  $x$ . We claim that  $z_1 <_T x$ : Otherwise,  $x <_T z_1$  and every element of  $T$  which is  $<_T$ -comparable with  $z_1$  is also  $<_T$ -comparable with  $x$ . In particular,  $z_2$  is  $<_T$ -comparable with  $x$ , a contradiction since  $z_2$  and  $x$  are not connected. We now claim that  $z_1 <_T y$ . Indeed, observe that  $z_1 <_T z_2$ : Otherwise,  $z_2 <_T z_1 <_T x$  so  $z_2 <_T x$  contradicting irreducibility. Now, every element of  $T$  which is  $<_T$ -comparable with  $z_2$  is also  $<_T$ -comparable with  $z_1$ , so no further element can be added to the path. Hence  $z_2 = y$  and we can take  $z_1 = z$ .  $\square$

**Lemma 4.** *Every non-trivial irreducible cycle in  $X_m$  has size 3.*

*Proof.* Obvious since  $\delta$  induces the metric  $d^{X_m}$  on  $X_m$ .  $\square$

**Lemma 5.** *Every non-trivial irreducible cycle in  $T$  has size 3 and is included in a branch.*

*Proof.* Let  $c = (z_i)_{i < n}$  be a non-trivial irreducible cycle in  $T$ . We may assume that  $z_0 <_T z_{n-1}$ . Now, observe that every element of  $T$  comparable with  $z_0$  is also comparable with  $z_{n-1}$ . In particular,  $z_1$  is such an element. It follows that  $n = 3$  and that  $z_0, z_1, z_2$  are in a same branch.  $\square$

**Lemma 6.** *Every irreducible cycle in  $Z$  intersecting both  $X_m$  and  $T$  is supported by a set whose form is one of the following ones.*

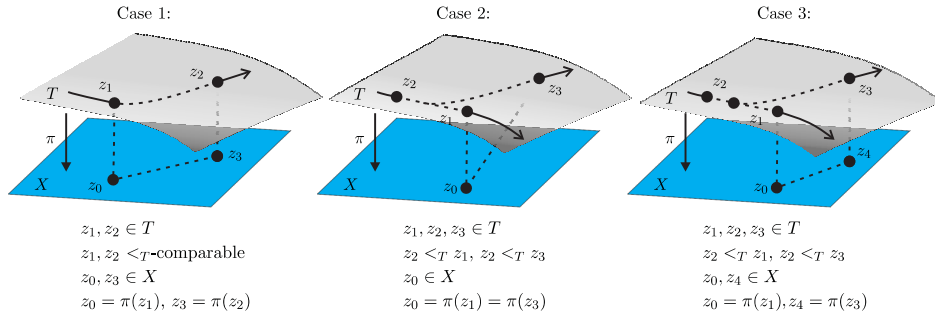


FIGURE 1. Irreducible cycles

*Proof.* Let  $C$  be a set supporting an irreducible cycle  $c$  intersecting both  $X_m$  and  $T$ . It should be clear that  $|C \cap X_m| \leq 2$ : Otherwise since any two points in  $X_m$  are connected,  $c$  would admit a strict subcycle, contradicting irreducibility.

If  $C \cap X_m$  has size 1, let  $z_0$  be its unique element. In  $c$ ,  $z_0$  is connected to two elements which we denote  $z_1$  and  $z_3$ . Note that  $z_1, z_3 \in T$  so  $\pi(z_1) = \pi(z_3) = z_0$ . Since elements in  $T$  which are connected never project on a same point, it follows that  $z_1, z_3$  are  $<_T$ -incomparable. Now,  $c$  induces an irreducible path from  $z_1$  to  $z_3$  in  $T$  so from lemma 3, there is  $z_2 \in C$  such that  $z_2 <_T z_1, z_2 <_T z_3$ , and we are in case 2.

Assume now that  $C \cap X_m = \{z_0, z_4\}$ . Then there are  $z_1, z_3 \in C \cap T$  such that  $\pi(z_1) = z_0$  and  $\pi(z_3) = z_4$ . Note that since  $z_0 \neq z_4$ , we must have  $z_1 \neq z_3$ . Now,  $C \cap T$  induces an irreducible path from  $z_1$  to  $z_3$  in  $T$ . By lemma 3, either  $z_1$  and  $z_3$  are compatible and in this case, we are in case 1, or  $z_1$  and  $z_3$  are  $<_T$ -incomparable and there is  $z_2 \in C \cap T$  such that  $z_2 <_T z_1, z_2 <_T z_3$  and we are in case 3.  $\square$

**Lemma 7.** *Every non-trivial irreducible cycle in  $Z$  is metric.*

*Proof.* Let  $c$  be an irreducible cycle in  $Z$ . If  $c$  is supported by  $X_m$ , then by lemma 4  $c$  has size 3 and is metric since  $\delta$  induces a metric on  $X_m$ . If  $c$  is supported by  $T$ , then by lemma 5  $c$  also has size 3 and is included in a branch  $b$  of  $T$ . Since  $\delta$  induces a metric on  $b$ ,  $c$  is metric. We consequently assume that  $c$  intersects both  $X_m$  and  $T$ . According to lemma 6,  $c$  is supported by a set  $C$  whose form is covered by one of the cases 1, 2 or 3. So to prove the present lemma, it is enough to show every cycle obtained from a re-indexing of the cycles described in those cases is metric.

Case 1: The required inequalities are obvious after having observed that

$$\delta(z_0, z_3) = \lceil \delta(z_1, z_2) \rceil_m \text{ and } \delta(z_0, z_1) = \delta(z_2, z_3) = \frac{1}{2m}.$$

Case 2: Observe that since  $\pi(z_1) = \pi(z_3) = z_0$ , we must have  $\delta(z_1, z_2) = \delta(z_2, z_3)$ . Notice also that  $\delta(z_0, z_1) = \delta(z_0, z_3) = 1/2m$ . The required inequalities follow.

Case 3: Observe that  $\delta(z_0, z_1) = \delta(z_3, z_4) = 1/2m$ , so the inequalities we need to check are

$$\delta(z_1, z_2) \leq \delta(z_2, z_3) + \delta(z_0, z_4) + \frac{1}{m} \quad (4)$$

$$\delta(z_0, z_4) \leq \delta(z_1, z_2) + \delta(z_2, z_3) + \frac{1}{m}. \quad (5)$$

For (4):

$$\begin{aligned} \delta(z_1, z_2) &\leq \lceil \delta(z_1, z_2) \rceil_m \\ &= \delta(\pi(z_1), \pi(z_2)) \\ &= \delta(z_0, \pi(z_2)) \\ &\leq \delta(z_0, z_4) + \delta(z_4, \pi(z_2)) \\ &= \delta(z_0, z_4) + \lceil \delta(z_3, z_2) \rceil_m \\ &\leq \delta(z_0, z_4) + \delta(z_2, z_3) + \frac{1}{m}. \end{aligned}$$

For (5): Write  $z_1 = b(j)$ ,  $z_3 = b'(k)$ ,  $z_2 = b(i) = b'(i)$ . Then  $z_0 = \pi(z_1) = x_{\max b(j)}$  and  $z_4 = \pi(z_3) = x_{\max b'(k)}$ . Observe also that  $\delta(z_1, z_2) = d^{\mathbf{S}^{\mathbb{Q}}}(y_j, y_i)$  and that  $\delta(z_2, z_3) = d^{\mathbf{S}^{\mathbb{Q}}}(y_i, y_k)$ . So

$$\begin{aligned} \delta(z_0, z_4) &= d^{\mathbf{X}^m}(x_{\max b(j)}, x_{\max b'(k)}) \\ &= \lceil d^{\mathbf{S}^{\mathbb{Q}}}(y_{\max b(j)}, y_{\max b'(k)}) \rceil_m \\ &\leq \lceil d^{\mathbf{S}^{\mathbb{Q}}}(y_{\max b(j)}, y_{\max b(i)}) + d^{\mathbf{S}^{\mathbb{Q}}}(y_{\max b'(i)}, y_{\max b'(k)}) \rceil_m \\ &= \lceil d^{\mathbf{S}^{\mathbb{Q}}}(y_j, y_i) + d^{\mathbf{S}^{\mathbb{Q}}}(y_i, y_k) \rceil_m \\ &= \lceil \delta(z_1, z_2) + \delta(z_2, z_3) \rceil_m \\ &\leq \delta(z_1, z_2) + \delta(z_2, z_3) + \frac{1}{m}. \quad \square \end{aligned}$$

### 3. RESULTS AND BOUNDS.

Ideally, the title of this section would have been ‘‘The Urysohn sphere is oscillation stable’’ and we would have ended this article with the proof of one of the different formulations of oscillation stability for  $\mathbf{S}$  presented in Theorem 4. Unfortunately, so far, our numerous attempts to reach this goal did not succeed. This is why this part is entitled ‘‘bounds’’. Instead, what we will be presenting now will show how far we were able to push in the different directions suggested by Theorem 4. We start with a summary about the indivisibility properties of the spaces  $\mathbf{S}_m$ .

3.1. **Are the  $\mathbf{S}_m$ 's indivisible?** Of course, when  $m = 1$ , the space  $\mathbf{S}_m$  is indivisible in virtue of the most elementary pigeonhole principle on  $\omega$ . The first non-trivial case is consequently for  $m = 2$ . However, this case is also easy to solve after having noticed that  $\mathbf{S}_2$  is really the Rado graph  $\mathcal{R}$  where the distance is  $1/2$  between connected points and  $1$  between non-connected distinct points. Therefore, indivisibility for  $\mathbf{S}_2$  is equivalent to indivisibility of  $\mathcal{R}$ , a problem whose solution is well-known:

**Proposition 6.** *The Rado graph  $\mathcal{R}$  is indivisible.*

The following case to consider is  $\mathbf{S}_3$ , which turns out to be another particular case thanks to an observation made in [3]. Indeed,  $\mathbf{S}_3$  can be encoded by the countable ultrahomogeneous edge-labelled graph with edges in  $\{1/3, 1\}$  and forbidding the complete triangle with labels  $1/3, 1/3, 1$ . The distance between two points connected by an edge is the label of the edge while the distance between two points which are not connected is  $2/3$ . This fact allows to show:

**Theorem** (Delhommé-Laflamme-Pouzet-Sauer [3]).  *$\mathbf{S}_3$  is indivisible.*

Indeed, the proof of this theorem can be deduced from the proof of the indivisibility of the  $\mathbf{K}_n$ -free ultrahomogeneous graph by El-Zahar and Sauer in [4]. We do not write more here but the interested reader is referred to [3], section on the indivisibility of Urysohn spaces, for more details.

The very first substantial case consequently shows up for  $m = 4$ . Unfortunately, it appears to be so substantial that so far, we still do not know whether this space is indivisible or not. Nevertheless, we are able to establish that if this space is not indivisible, then  $\mathbf{S}_4$  is quite exceptional, in a sense that we precise now. We already mentioned that [3] contains an analysis of indivisibility in the realm of countable metric spaces. It turns out that this study also led its authors to examine the conditions under which a set of strictly positive reals can be interpreted as the distance set of a countable universal and ultrahomogeneous metric space:

**Definition** (4-values condition). *Let  $S \subset ]0, +\infty[$ .  $S$  satisfies the 4-values condition when for every  $s_0, s_1, s'_0, s'_1 \in S$ , if there is  $t \in S$  such that:*

$$|s_0 - s_1| \leq t \leq s_0 + s_1 \text{ and } |s'_0 - s'_1| \leq t \leq s'_0 + s'_1,$$

*then there is  $u \in S$  such that:*

$$|s_0 - s'_0| \leq u \leq s_0 + s'_0 \text{ and } |s_1 - s'_1| \leq u \leq s_1 + s'_1.$$

**Theorem** (Delhommé-Laflamme-Pouzet-Sauer [3]). *Let  $S \subset ]0, +\infty[$ . TFAE:*

- (i) *There is a countable ultrahomogeneous metric space  $\mathbf{U}_S$  with distances in  $S$  into which every countable metric space with distances in  $S$  embeds isometrically.*
- (ii)  *$S$  satisfies the 4-values condition.*

As detailed in [3], the 4-values condition covers a wide variety of examples. For our purposes, the 4-values condition is relevant because it allows to establish a list of spaces such that any space  $\mathbf{U}_S$  with  $S$  finite is in some sense isomorphic to some space in the list. In particular, it allows to set up a finite list of spaces exhausting all the spaces  $\mathbf{U}_S$  with  $S \leq 4$ . More precisely, for finite subsets  $S = \{s_0, \dots, s_m\}_{<}$ ,  $T = \{t_0, \dots, t_n\}_{<}$  of  $]0, +\infty[$ , define  $S \sim T$  when  $m = n$  and:

$$\forall i, j, k < m, \quad s_i \leq s_j + s_k \leftrightarrow t_i \leq t_j + t_k.$$



Observe that when  $S \sim T$ ,  $S$  satisfies the 4-value condition iff  $T$  does and in this case,  $S$  and  $T$  essentially provide the same metric spaces as it is possible to have  $\mathbf{U}_S$  and  $\mathbf{U}_T$  supported by  $\omega$  with the metrics  $d^{\mathbf{U}_S}$  and  $d^{\mathbf{U}_T}$  being defined such that:

$$\forall x, y \in \omega, \quad d^{\mathbf{U}_S}(x, y) = s_i \leftrightarrow d^{\mathbf{U}_T}(x, y) = t_i.$$

Now, clearly, for a given cardinality  $m$  there are only finitely many  $\sim$ -classes, so we can find a finite collection  $\mathcal{S}_m$  of finite subsets of  $]0, \infty[$  of size  $m$  such that for every  $T$  of size  $m$  satisfying the 4-value condition, there is  $S \in \mathcal{S}_m$  such that  $T \sim S$ . For  $m \leq 3$ , examples of such lists can be easily provided. For instance, one may take:

$$\begin{aligned} \mathcal{S}_1 &= \{\{1\}\} \\ \mathcal{S}_2 &= \{\{1, 2\}, \{1, 3\}\} \\ \mathcal{S}_3 &= \{\{2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 6\}, \{1, 3, 7\}\}. \end{aligned}$$

Notice that in those lists, the set  $[0, 1]_m$  is represented by the set  $\{1, 2, \dots, m\}$ . For  $m = 4$ , a long and tedious checking procedure of the 4-values condition allows to find  $\mathcal{S}_m$  but it then contains more than 20 elements so there is no point writing them all here. Still, it turns out that in most of the cases, we are able to solve the indivisibility problem for the space  $\mathbf{U}_S$ . Our result can be stated as follows:

**Theorem 5.** *Let  $S$  be finite subset of  $]0, +\infty[$  of size  $|S| \leq 4$  and satisfying the 4-values condition. Assume that  $S \approx \{1, 2, 3, 4\}$ . Then  $\mathbf{U}_S$  is indivisible.*

Due to the number of cases to consider, we do not prove this theorem here but simply mention that when the proofs are not elementary, three essential ingredients come into play. The first one is the usual infinite Ramsey theorem, due to Ramsey. The second one is due to El-Zahar and Sauer and was already mentioned when dealing with  $\mathbf{S}_3$ . As for the last one, it is due to Milliken (For more on this theorem and its applications, see [14]).

The case  $S = \{1, 2, 3, 4\}$  is consequently the only case with  $S = 4$  for which the indivisibility problem remains unsolved. In the present case, it is a bit ironical as  $\{1, 2, 3, 4\}$  is precisely the distance set in which we were interested. So far, the reason for which  $\mathbf{S}_4$  stands apart is still unclear. However, it might be that it is actually the very first case where metricity comes into play. Indeed, for all the other sets  $S$  with  $|S| \leq 4$ , the space  $\mathbf{U}_S$  can be coded as an object where the metric aspect does not appear and this is what makes Ramsey, Milliken and El-Zahar - Sauer theorems helpful. Our feeling is consequently that solving the indivisibility problem for  $\mathbf{S}_4$  requires a new approach. Still, we have to admit that what we are hoping for is a positive answer and that Theorem 5 is undoubtedly responsible for that.

**3.2.  $1/m$ -indivisibility of the  $\mathbf{S}_m$ 's.** We now turn to  $1/m$ -indivisibility of the spaces  $\mathbf{S}_m$ . In Theorem 5, we showed how an exact indivisibility result transfers to an approximate one. It turns out that a slight modification of the proof allows to show:

**Proposition 7.** *Assume that for some strictly positive  $m \in \omega$ ,  $\mathbf{S}_m$  is indivisible. Then  $\mathbf{S}_{3m}$  is  $1/3m$ -indivisible.*

*Proof.* To prove this theorem, it suffices to show that there is an isometric copy  $\mathbf{S}_m^{**}$  of  $\mathbf{S}_m$  inside  $\mathbf{S}_{3m}$  such that for every  $\tilde{\mathbf{S}}_m \subset \mathbf{S}_m^{**}$  isometric to  $\mathbf{S}_m$ ,  $(\tilde{\mathbf{S}}_m)_{1/3m}$

includes an isometric copy of  $\mathbf{S}_{3m}$ . The proof is essentially the same as the proof of Proposition 5 where  $\mathbf{S}_m^*$  is constructed except that instead of the metric space  $\mathbf{X}_m = (\mathbf{S}_{\mathbb{Q}}, [d^{\mathbf{S}_{\mathbb{Q}}}]_m)$ , one works with  $(\mathbf{S}_{3m}, [d^{\mathbf{S}_{\mathbb{Q}}}]_m)$ . The fact that the approximation can be made up to  $1/3m$  and not  $1/2m$  comes from the fact that for  $\alpha \in [0, 1]_{3m}$ ,  $\alpha \leq [\alpha]_m \leq \alpha + 2/3m$  whereas if  $\alpha \in [0, 1] \cap \mathbb{Q}$ , one only has  $\alpha \leq [\alpha]_m < \alpha + 1/m$ .  $\square$

Thus:

**Theorem 6.** *For every  $m \leq 9$ ,  $\mathbf{S}_m$  is  $1/m$ -indivisible.*

It follows that as far as  $1/m$ -indivisibility is concerned, the first open case corresponds to  $m = 10$ . On the one hand, this result is quite encouraging as it shows that one of the discretized equivalent formulations of oscillation stability for  $\mathbf{S}$  goes even further than  $m = 3$ . However, it also makes  $1/m$ -indivisibility of the  $\mathbf{S}_m$ 's a difficult track to follow: While it is already difficult to have a feeling about the structure of  $\mathbf{S}_4$ ,  $\mathbf{S}_{10}$  is in turn completely out of reach.

**3.3. Bounds.** We now turn to the computation of values  $\varepsilon$  with respect to which  $\mathbf{S}$  is  $\varepsilon$ -indivisible. At that point, there are two alternatives: Either use the indivisibility results of the spaces  $\mathbf{S}_m$ , or use their  $1/m$ -indivisibility properties. As far as indivisibility is concerned, the best current  $\varepsilon$  with respect to which  $\mathbf{S}$  is  $\varepsilon$ -indivisible is provided by the indivisibility of  $\mathbf{S}_3$  together with Theorem 5, namely:

**Theorem** (Theorem 2).  *$\mathbf{S}$  is  $1/6$ -indivisible.*

Note also that if at some point an approximate indivisibility result for  $\mathbf{S}_m$  showed up independently of an exact one, we would still be able to compute a bound for  $\mathbf{S}$ :

**Proposition 8.** *Suppose that for some strictly positive integer  $m$ ,  $\mathbf{S}_m$  is  $1/m$ -indivisible. Then  $\mathbf{S}$  is  $\varepsilon$ -indivisible for every  $\varepsilon \geq 3/2m$ .*

*Proof.* Let  $\varepsilon \geq 3/2m$ . Consider  $\mathbf{S}_m^*$  constructed in Proposition 5. Now, let  $k \in \omega$  be strictly positive and  $\chi : \mathbf{S} \rightarrow k$ .  $\chi$  induces a coloring of  $\mathbf{S}_m^*$  and  $\mathbf{S}_m$  being  $1/m$ -indivisible, there are  $i < k$  and  $\tilde{\mathbf{S}}_m \subset \mathbf{S}_m^*$  isometric to  $\mathbf{S}_m$  such that  $\tilde{\mathbf{S}}_m \subset (\overleftarrow{\chi}\{i\})_{1/m}$ . By construction,  $(\tilde{\mathbf{S}}_m)_{1/2m}$  includes an isometric copy of  $\mathbf{S}$ . Now,

$$((\overleftarrow{\chi}\{i\})_{1/m})_{1/2m} \subset (\overleftarrow{\chi}\{i\})_{3/2m} \subset (\overleftarrow{\chi}\{i\})_{\varepsilon}.$$

It follows that  $(\overleftarrow{\chi}\{i\})_{\varepsilon}$  includes an isometric copy of  $\mathbf{S}$ .  $\square$

If  $\mathbf{S}_m$  is indivisible, there are consequently two ways to compute bounds on  $\mathbf{S}$ . The first way is provided by Theorem 5 and gives  $1/2m$ . On the other hand, one may also apply Theorem 7 first, and then Theorem 8. The bound is then  $3/2 \cdot 1/3m = 1/2m$ . Thus, the two approaches are equivalent.

#### 4. CONCLUDING REMARKS AND OPEN PROBLEMS.

The equivalence provided by Theorem 4 suggests several lines of future investigation.  $1/m$ -indivisibility of the spaces  $\mathbf{S}_m$  is one of them but we saw earlier - see comment after Theorem 6 - that the partial result we obtained so far, namely that  $1/m$ -indivisibility holds up to  $m = 9$ , makes it difficult to follow this direction by working on a practical case. So apparently, here is the first and most reasonable question to consider:

**Question.** Is  $\mathbf{S}_4$  indivisible? More generally, is  $\mathbf{S}_m$  indivisible for every strictly positive integer  $m$ ?

We finish with two results which might be useful for that purpose. The first one makes a reference to the space  $\mathbf{S}_{\mathbb{Q}}$ :

**Proposition 9.** *Let  $m \in \omega$  be strictly positive. Assume that for every strictly positive  $k \in \omega$  and  $\chi : \mathbf{S}_{\mathbb{Q}} \rightarrow k$ , there is a copy  $\tilde{\mathbf{S}}_m$  of  $\mathbf{S}_m$  in  $\mathbf{S}_{\mathbb{Q}}$  on which  $\chi$  is constant. Then  $\mathbf{S}_m$  is indivisible.*

*Proof.* Once again, we work with  $\mathbf{X}_m = (\mathbf{S}_{\mathbb{Q}}, [d^{\mathbf{S}_{\mathbb{Q}}}]_m)$  and the identity map  $\pi_m : \mathbf{S}_{\mathbb{Q}} \rightarrow \mathbf{S}_m$ . Think of  $\mathbf{X}_m$  as a subspace of  $\mathbf{S}_m$ . Now, let  $k \in \omega$  be strictly positive and  $\chi : \mathbf{S}_m \rightarrow k$ . Then  $\chi$  induces a coloring of  $\mathbf{X}_m$ , and therefore a coloring  $\chi \circ \pi$  of  $\mathbf{S}_{\mathbb{Q}}$ . By hypothesis, there is a copy  $\tilde{\mathbf{S}}_m$  of  $\mathbf{S}_m$  in  $\mathbf{S}_{\mathbb{Q}}$  on which  $\chi \circ \pi$  is constant with value  $i < k$ . Then  $\pi''\tilde{\mathbf{S}}_m \subset \overset{\sim}{\chi}\{i\}$ . The result follows since  $\pi''\tilde{\mathbf{S}}_m$  is isometric to  $\mathbf{S}_m$ .  $\square$

The second result provides a space whose indivisibility is equivalent to the indivisibility of  $\mathbf{S}_m$ . Let  $P$  denote the Cantor space, that is the topological product space  $2^\omega$ . Let  $\mathcal{C}(P)$  denote the set of all continuous maps from  $P$  to  $\mathbb{R}$  equipped with the  $\|\cdot\|_\infty$  norm. Since the work of Banach and Mazur, it is known that  $\mathcal{C}(P)$  is a universal separable metric space. Actually, Sierpinski's proof of that fact allows to show the following result. For  $m \in \omega$  strictly positive, let  $\mathbf{C}_m$  denote the space of all continuous maps from  $P$  to  $[0, 1]_m$  equipped with the distance induced by  $\|\cdot\|_\infty$ .

**Proposition 10.**  *$\mathbf{C}_m$  is a countable metric space and is universal for the class of all countable metric spaces with distances in  $[0, 1]_m$ .*

It follows that  $\mathbf{S}_m$  is indivisible iff  $\mathbf{C}_m$  is.  $\mathbf{C}_m$  being a much more concrete object than  $\mathbf{S}_m$ , studying its indivisibility might be an alternative to solve the indivisibility problem for  $\mathbf{S}_m$ .

## 5. APPENDIX - PROOF OF THEOREM 1.

Unlike the rest of this paper, this section does not specifically deal with the oscillation stability for  $\mathbf{S}$  and is simply included here for the sake of completeness. Our purpose is to prove Theorem 1 by constructing the required subspace of  $\mathbf{Y}$ . Let  $\mathbf{X}_0 \subset \mathbf{Y}$  be countable and dense. Then, assuming that  $\mathbf{X}_n \subset \mathbf{Y}$  countable has been constructed, get  $\mathbf{X}_{n+1}$  as follows: Consider  $\mathcal{F}$  the set of all finite subspaces of  $\mathbf{X}_n$ . For  $\mathbf{F} \in \mathcal{F}$ , consider the set  $E_n(\mathbf{F})$  of all Katětov maps  $f$  over  $\mathbf{F}$  with values in the set  $\{d^{\mathbf{Y}}(x, y) : x, y \in \mathbf{X}_n\}$  and such that  $\mathbf{F} \cup \{f\}$  embeds into  $\mathbf{Y}$ . Observe that  $\mathbf{X}_n$  being countable, so are  $\{d^{\mathbf{Y}}(x, y) : x, y \in \mathbf{X}_n\}$  and  $E_n(\mathbf{F})$ . Then, for  $\mathbf{F} \in \mathcal{F}, f \in E_n(\mathbf{F})$ , fix  $y_{\mathbf{F}}^f \in \mathbf{Y}$  realizing  $f$  over  $\mathbf{F}$ . Finally, let  $\mathbf{X}_{n+1}$  be the subspace of  $\mathbf{Y}$  with underlying set  $\mathbf{X}_n \cup \{y_{\mathbf{F}}^f : \mathbf{F} \in \mathcal{F}, f \in E_n(\mathbf{F})\}$ . After  $\omega$  steps, set  $\mathbf{X} = \bigcup_{n \in \omega} \mathbf{X}_n$ .  $\mathbf{X}$  is clearly a countable dense subspace of  $\mathbf{Y}$ , and it is ultrahomogeneous thanks to the equivalent formulation of ultrahomogeneity provided in lemma 1.

A second proof involves logical methods. Fix a countable elementary submodel  $M \prec H_\theta$  for some large enough  $\theta$  and such that  $Y, d^{\mathbf{Y}} \in M$ . Let  $\mathbf{X} = M \cap \mathbf{Y}$ .

We claim that  $\mathbf{X}$  has the required property. First, observe that  $\mathbf{X}$  is dense inside  $\mathbf{Y}$  since by the elementarity of  $M$ , there is a countable  $D \in M$  (and therefore  $D \subset M$ ) which is a dense subset of  $\mathbf{Y}$ . For ultrahomogeneity, let  $\mathbf{F} \subset \mathbf{X}$  be finite and let  $f$  be a Katětov map over  $\mathbf{F}$  such that  $\mathbf{F} \cup \{f\}$  embeds into  $\mathbf{X}$ . Observe that  $f \in M$ . Indeed,  $\text{dom}(f) \in M$ . On the other hand, let  $\tilde{\mathbf{F}} \cup \{y\} \subset \mathbf{X}$  be isometric to  $\mathbf{F} \cup \{f\}$  via an isometry  $\varphi$ . Then for every  $x \in \mathbf{F}$ ,  $d^{\mathbf{Y}}(\varphi(x), y) \in M$ . But  $d^{\mathbf{Y}}(\varphi(x), y) = f(x)$ . Thus,  $\text{ran}(f) \in M$ . It follows that  $f$  is an element of  $M$ . Now, by ultrahomogeneity of  $\mathbf{Y}$ , there is  $y$  in  $\mathbf{Y}$  realizing  $f$  over  $\mathbf{F}$ . So by elementarity, there is  $x$  in  $\mathbf{X}$  realizing  $f$  over  $\mathbf{F}$ .

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