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Abstract

We study the relationship between the masses and the geometric properties of central configurations. We prove that in the planar four-body problem, a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. If these two masses are unequal, then the less massive one is closer to the former diagonal. Finally, we extend these results to the case of non-planar central configurations of five particles.

1. Introduction. Let $q_1, \dots, q_n$ represent the positions in a Euclidean space E of n particles with respective positive masses $m_1, \dots, m_n$. Let

$$\gamma_i = \sum_{k \neq i} m_k S_{ik}(q_i - q_k), \qquad S_{ik} = S_{ki} = \|q_i - q_k\|^{2a}, \qquad a = -3/2. \tag{1}$$

Newton's equations

$$\frac{d^2 q_i}{dt^2} + \gamma_i = 0 \tag{2}$$

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define the classical n-body motions of the system of particles. In the Newtonian n-body problem, the simplest possible motions are such that the configuration is constant up to rotation and scaling, and that each body describes a Keplerian orbit. Only some special configurations of particles are allowed in such motions. Wintner called them "central configurations". Famous authors as Euler [10], Lagrange [13], Laplace [14], Liouville [17], Maxwell [20] initiated the study of central configurations, while Chazy [8], Wintner [29], Smale [27] and Atiyah&Sutcliffe [6] called attention to hard unsolved questions. Chazy, Wintner and Smale conjectured that the number of central configurations of n particles with given masses is finite. This was proved recently in the case $n = 4$ by Hampton and Moeckel [11]. This reference also provides an excellent review, which describes how central configurations appear in several applications. Let us mention that central configurations are also needed in the application of Ziglin method or the computation of Kowalevski exponents, in order to get statements of non-existence of additional first integrals in the n-body problem (see e.g. Tsygvintsev[28]).

In terms of

$$M = m_1 + \cdots + m_n, \qquad q_G = \frac{1}{M}(m_1 q_1 + \cdots + m_n q_n), \tag{3}$$

a configuration $(q_1, \ldots, q_n)$ is called a central configuration if and only if there exists a $\lambda \in \mathbb{R}$ such that

$$\gamma_i = \lambda(q_i - q_G). \tag{4}$$

The results in this note are not restricted to the Newtonian case $a = -3/2$. They are true for any $a < 0$. In particular, they apply to relative equilibria of Helmholtz vortices in the plane with positive vorticities $m_1, \ldots, m_n$ (see e.g. O'Neil [23]), for which $a = -1$. While vorticities may be negative, we always assume $m_i > 0$ for all i, $1 \leq i \leq n$. Note that it implies $\lambda > 0$. For results with some negative masses see for example Celli [7]. The main result we prove in this note is:

Theorem 1. Let four particles (q_1, q_2, q_3, q_4) form a two-dimensional central configuration, which is a convex quadrilateral having $[q_1, q_2]$ and $[q_3, q_4]$ as diagonals (see Fig. 1). This configuration is symmetric with respect to the axis $[q_3, q_4]$ if and only if $m_1 = m_2$. It is symmetric with respect to the axis $[q_1, q_2]$ if and only if $m_3 = m_4$. Also $m_1 < m_2$ if and only if $|\Delta_{134}| < |\Delta_{234}|$, where Δ_{ijk} is the oriented area of the triangle $[q_i, q_j, q_k]$. Similarly, $m_3 < m_4$ if and only if $|\Delta_{123}| < |\Delta_{124}|$.

Remark 1. In Theorem 1, each geometric relation between areas has equivalents in terms of distances. For example, $|\Delta_{134}| < |\Delta_{234}|$ is obviously

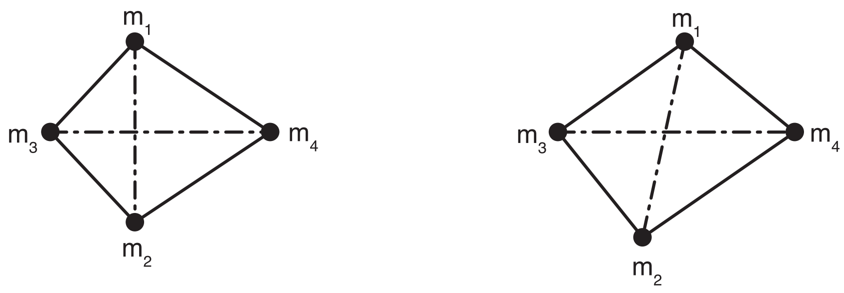

Figure 1: Planar 4-body convex configurations
Left: with an axis of symmetry. Right: asymmetric.

equivalent to that q_1 is closer to the line $[q_3, q_4]$ than q_2. By Lemma 1 and (8), it is also equivalent to $\|q_1 - q_3\| < \|q_2 - q_3\|$ or to $\|q_1 - q_4\| < \|q_2 - q_4\|$.

Remark 2. Our hypotheses are not far from optimal. We assume $a < 0$ in (1) being aware of counter-examples if $a > 0$, and of course if $a = 0$. If the equal masses are m_1 and m_3 there is no symmetry except if also $m_2 = m_4$ (but no proof is known of the symmetry in this latter case). The convexity of the configuration is also a necessary hypothesis: there are asymmetric central configurations where a particle q_1 is inside the triangle formed by q_2, q_3 and q_4, and where $m_3 = m_4$. This can be seen immediately by perturbing the four equal mass solution.

For any choice of four positive masses there is always a convex central configuration with the particles ordered as in Figure 1, according to [19, 30]. Theorem 1 associated to Leandro's [16] shows the uniqueness (up to trivial transformations) in the case where two particles with equal masses are on a diagonal. We conjecture that the uniqueness of the convex central configuration is true for any choice of four positive masses (see also [4] which finishes with a selection of open questions).

Given the presence of some equal masses, it is natural to attack this conjecture by proving some symmetry of the central configuration at first. In this direction, the first named author took the first step [1, 2]. He studied the equal mass case and gave a complete understanding on the geometric properties and the enumeration of the central configurations convex or not. Then Long and the third named author [18] studied the convex central configurations with $m_1 = m_2$ and $m_3 = m_4$, and they proved symmetry and uniqueness under some constraints which are dropped out by Perez-Chavela and Santoprete. In [24], they also generalise further and get symmetry on convex central configurations with $m_1 = m_2$ only, and the uniqueness follows

Leandro [16]. Unfortunately, in this situation, Perez-Chavela and Santoprete also need that $m_1 = m_2$ are not the smallest masses. Our Theorem 1 gives the expected symmetry assuming simply $0 < m_1 = m_2$, $0 < m_3$, $0 < m_4$, and choosing any $a < 0$ in Equation (1). The uniqueness in the Newtonian case $a = -3/2$ follows Leandro [16].

2. Dziobek's equations. The main known results on the planar central configurations of four bodies were obtained or simplified using Dziobek's coordinates [9]. In particular, Meyer and Schmidt [21] remarked their effectiveness, and extended their use to the non-planar central configurations of five bodies.

Given a $(n-2)$-dimensional configuration $(q_1, \dots, q_n)$ of n particles, we consider the linear equations

$$\Delta_1 + \cdots + \Delta_n = 0, \qquad \Delta_1 q_1 + \cdots + \Delta_n q_n = 0. \tag{5}$$

A non-zero $(\Delta_1, \dots, \Delta_n) \in \mathbb{R}^n$ satisfying (5) will be called a system of *homogeneous barycentric coordinates* of the configuration. Such a system is not unique but defined up to a non-zero real factor. There is a $\kappa \in \mathbb{R}$ such that $\pm\kappa\Delta_i$ is the $(n-2)$-dimensional volume of the simplex $(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$. This remark is useful for the geometrical intuition, but at the technical level it only introduces complications. We rewrite the equations for central configuration (4) as

$$0 = \sum_{k \neq i} m_k \check{S}_{ik}(q_k - q_i), \qquad \text{with} \qquad \check{S}_{ik} = S_{ik} - \frac{\lambda}{M}. \tag{6}$$

and compare them with the second equation of (5) written as

$$\sum_{k \neq i} \Delta_k (q_i - q_k) = 0. \tag{7}$$

By the uniqueness, up to a factor, of $(\Delta_1, \dots, \Delta_n)$, there exists a real number θ_i such that $\theta_i \Delta_k = m_k \check{S}_{ik}$. We can write $\check{S}_{ik} = \check{S}_{ki} = \theta_i \Delta_k / m_k = \theta_k \Delta_i / m_i$ for any (i,k), $1 \leq i < k \leq n$. Thus $(\theta_1, \dots, \theta_n)$ is proportional to $(\Delta_1/m_1, \dots, \Delta_n/m_n)$. Calling μ the proportionality factor, we obtain Dziobek's equations

$$S_{ij} - \frac{\lambda}{M} = \mu \frac{\Delta_i \Delta_j}{m_i m_j}. \tag{8}$$

3. Routh versus Dziobek. Here we only consider the case $n = 4$. Equation (8) was proved by Dziobek by considering Cayley's determinant H, differentiating H on $H = 0$ and characterising the central configurations as critical points of the Newtonian potential restricted to some submanifold of $H = 0$. The expression of the derivatives of H is quite nice, but the computation giving it is not so easy. Dziobek and many authors after him chose to skip it. An elegant presentation is given by Moeckel [22].

The "vectorial" deduction of (8) we just gave is simpler than Dziobek's approach. It was also published in [22]. Previously many authors presented similar vectorial computations and deduced relations which are easy consequences of Dziobek's equations. A formula as

$$m_3\Delta_{123}(S_{13} - S_{23}) + m_4\Delta_{124}(S_{14} - S_{24}) = 0 \tag{9}$$

appeared in Routh [25] and Krediet [12] before Dziobek's work, in Laura [15] and Andoyer [5] just after. The symbol Δ_{ijk} represents the oriented area of the triangle (i, j, k). Setting $\Delta_4 = \Delta_{123}$ and $\Delta_3 = -\Delta_{124}$, we recognise (9) as an easy consequence of (8). But (8) is not an obvious corollary of (9), and none of these four authors wrote it.

In the last edition [26] of his treatise, 28 years after [25], Routh considered again the central configurations. Apparently he wanted to claim priority against someone, and it is very likely that this someone is Dziobek. Routh uses Dziobek's notation Δ, which suggests strongly that he read Dziobek's paper. Routh's priority claim concerns (9) and another important formula. We reproduce the text in [25], which is not easy to find in the libraries.

"Ex. 2. If four particles be placed at the corners of a quadrilateral whose sides taken in order are a, b, c, d and diagonals ρ, ρ', then the particles could not move under their mutual attractions so as to remain always at the corners of a similar quadrilateral unless

$$(\rho^n\rho'^n - b^n d^n)(c^n + a^n) + (a^n c^n - \rho^n\rho'^n)(b^n + d^n) + (b^n d^n - a^n c^n)(\rho^n + \rho'^n) = 0,$$

where the law of attraction is the inverse $(n-1)^{\mathrm{th}}$ power of the distance.

Show also that the mass at the intersection of b, c divided by the mass at intersection of c, d, is equal to the product of the area formed by a, ρ', d divided by the area formed by a, b, ρ and the difference $\frac{1}{\rho'^n} - \frac{1}{d^n}$ divided by the difference $\frac{1}{\rho^n} - \frac{1}{b^n}$.

These results may be conveniently arrived at by reducing one angular point as A of the quadrilateral to rest. The resolved part of all the forces which act on each particle perpendicular to the straight line joining it to A will then be zero. The case of three particles may be treated in the same manner."

Routh's priority claim does not concern Dziobek's formula (8), which he does not write. But the formulas he numbered (1), (2), (3), (4) in [26] express the θ_i's of our vectorial proof. Routh could easily prove (8) with his vectorial methods.

4. More equations for mutual distances. It is clear from Equations (5) that the quantity

$$\Delta_1 \|q - q_1\|^2 + \cdots + \Delta_n \|q - q_n\|^2$$

does not depend on the point q. So if we set $s_{ij} = \|q_i - q_j\|^2$ and $t_i = \sum_{j \neq i} \Delta_j s_{ij}$ we get immediately

$$t_1 = t_2 = \cdots = t_n, \tag{10}$$

i.e. for any i, j, $1 \leq i < j \leq n$,

$$t_i - t_j = (\Delta_j - \Delta_i) s_{ij} + \sum_{k \neq i,j} \Delta_k (s_{ik} - s_{jk}) = 0. \tag{11}$$

Using this expression we can get a simpler proof of the following lemma proved in [3].

Lemma 1. For a n body central configuration in dimension $n-2$, the inequality

$$\Big(\frac{\Delta_i}{m_i} - \frac{\Delta_j}{m_j}\Big)(\Delta_i - \Delta_j) \geq 0$$

holds for any i and j, $1 \leq i < j \leq n$.

Proof. As we assume $a < 0$, $s_{ik} - s_{jk}$ has the sign of $s_{jk}^a - s_{ik}^a$ which is the sign of $\mu \Delta_k (\Delta_j/m_j - \Delta_i/m_i)$ by (8). The term with "$\sum$" in (11) has the sign of $\mu(\Delta_j/m_j - \Delta_i/m_i)$. We deduce that $\Delta_j - \Delta_i$ has the sign of $\mu(\Delta_i/m_i - \Delta_j/m_j)$. As there must be a pair of Δ_i's having opposite signs, we deduce that $\mu < 0$. The required inequality then follows immediately. QED

We can restrict the study to *normalised central configurations*, choosing the normalisation $\lambda = M$. As $a \neq 0$, Equation (4) shows that one can obtain a normalised central configuration from any central configuration by changing its scale. Normalised central configurations satisfy Equations (10) and, for some $\mu \in \mathbb{R}$,

$$s_{ij}^a - 1 = \mu \frac{\Delta_i \Delta_j}{m_i m_j}. \tag{12}$$

We said that $(\Delta_1, \ldots, \Delta_n)$ is only defined up to a factor. We can fix this factor and remove the parameter μ, which is negative according to the previous proof. To an $(n-2)$-dimensional normalised central configuration we associate the unique $(\Delta_1,\dots,\Delta_n)\in\mathbb{R}^n$ satisfying (5), $\Delta_i\Delta_j<m_im_j$ and

$$s_{ij}=\Big(1-\frac{\Delta_i\Delta_j}{m_im_j}\Big)^{\alpha},\quad \text{where } \alpha=1/a. \tag{13}$$

These coordinates in turn determine the central configuration up to an isometry: from the Δ_i's we find the mutual distances through Expression (13).

5. The main lemmas

Lemma 2. Set $\rho_1<\rho_2\le 0$, $\rho_1\rho_2<1$, $\rho\ge 0$. Choose any $\alpha<0$, set $s_{12}=(1-\rho_1\rho_2)^\alpha$, $s_{13}=(1-\rho_1\rho)^\alpha$, $s_{23}=(1-\rho_2\rho)^\alpha$ and $A(\rho_1,\rho_2,\rho)=(\rho_2-\rho_1)s_{12}-(\rho_1+\rho_2)(s_{13}-s_{23})$. Then $A(\rho_1,\rho_2,\rho)>0$.

Proof. We first minimise A with respect to ρ. It is enough to minimise $g(\rho)=s_{13}-s_{23}=(1-\rho_1\rho)^\alpha-(1-\rho_2\rho)^\alpha$. We write $g'(\rho)=0$, i.e. $\rho_1(1-\rho_1\rho)^{\alpha-1}=\rho_2(1-\rho_2\rho)^{\alpha-1}$. Taking the power $1/(\alpha-1)$, the equation becomes linear in ρ. There is at most one root, given by

$$\rho_0=\frac{(-\rho_1)^{1/(\alpha-1)}-(-\rho_2)^{1/(\alpha-1)}}{\rho_1(-\rho_1)^{1/(\alpha-1)}-\rho_2(-\rho_2)^{1/(\alpha-1)}}.$$

To simplify this expression we set $u=1/(\alpha-1)$, i.e. $u+1{=}\alpha/(\alpha-1)$. Since $\alpha<0$, we have $-1<u<0$. We suppose also that $q=\rho_2/\rho_1$. It is easy to see that $0<q<1$. So

$$\rho_1\rho_0=\frac{1-q^u}{1-q^{u+1}}<0.$$

By plugging ρ_0 into $g(\rho)$ and using the definition of q, we obtain the minimal value of $s_{13}-s_{23}$:

$$g(\rho_0)=\Big(\frac{q^u-q^{u+1}}{1-q^{u+1}}\Big)^\alpha-\Big(\frac{1-q}{1-q^{u+1}}\Big)^\alpha=(q^{1+u}-1)\Big(\frac{1-q}{1-q^{u+1}}\Big)^\alpha.$$

In the introduced new variables, the problem of minimising A can be converted into that of minimising

$$B=\frac{A}{(-\rho_1)}=(1-q)(1-\rho_1^2q)^\alpha+(1+q)g(\rho).$$

The variables are q, ρ_1 and ρ. Taking minimiser of B in ρ is the same as replacing $g(\rho)$ by $g(\rho_0)$ above. Note that the second term in this expression of B does not depend on ρ_1. So, to minimise B in ρ_1 it suffices to set $\rho_1=0$, as one can see immediately. Then we have

$$B>C=(1-q)+(1+q)g(\rho_0)=(1-q)\Big(1-\frac{(1+q)(1-q)^{\alpha-1}}{(1-q^{1+u})^{\alpha-1}}\Big).$$

To prove $C > 0$, we only need to prove that $(1-q^{1+u})^{1-\alpha}(1+q)(1-q)^{\alpha-1} < 1$. By raising to the power $u = 1/(\alpha-1)$, the formula changes to

$$\frac{(1+q)^u(1-q)}{1-q^{1+u}} > 1.$$

So it suffices to conclude that this inequality holds for $q \in]0,1[$ and $u \in]-1,0[$. We will prove

$$f(q) = (1+q)^u(1-q) + q^{1+u} - 1 > 0.$$

One writes $1-q = -(1+q)+2$ and calculates the derivative $f'(q) = -(1+u)(1+q)^u + 2u(1+q)^{u-1} + (1+u)q^u$. By factoring out $(1+q)^u$, one gets the factor $-(1+u) + 2u(1-x) + (1+u)x^u$. Here we used the new variable $x = q/(1+q) \in]0,1/2[$, which satisfies $(1-x)(1+q) = 1$. This factor is a Laguerre trinomial. It has at most two positive roots. This implies the same for $f'(q)$ and excludes the possibility of a root of f in $]0,1[$. In fact, since $f(0^+) = 0^+$, $f(1) = 0$ and $f'(1) = -2^u + 1 + u < 0$, f having another root between 0 and 1 would imply f' having at least three positive roots. QED

Lemma 3. Substituting the s_{ik} using (13), we consider $t_i = \sum_{k\neq i} \Delta_k s_{ik}$ as a function of $\Delta_1, \ldots, \Delta_n$ satisfying $\Delta_1 + \ldots + \Delta_n = 0$ and of the parameters $a < 0$, $m_1 > 0, \ldots, m_n > 0$. Let $\Delta_1/m_1 < \Delta_2/m_2 \leq 0$, $\Delta_1\Delta_2 < m_1 m_2$, $\Delta_i \geq 0$ for $i \geq 3$. Then $m_1 \geq m_2 \Longrightarrow t_1 > t_2$.

Proof. We consider (11) and renumber the particles in such a way that $s_{13} - s_{23} \leq s_{14} - s_{24} \leq \cdots \leq s_{1n} - s_{2n}$. We get $t_1 - t_2 \geq Z$ where $Z = (\Delta_2 - \Delta_1)s_{12} + (\Delta_3 + \cdots + \Delta_n)(s_{13} - s_{23}) = (\Delta_2 - \Delta_1)s_{12} - (\Delta_1 + \Delta_2)(s_{13} - s_{23})$. We set $s_1 = -s_{12} - s_{13} + s_{23}$ and $s_2 = s_{12} - s_{13} + s_{23}$. By (13) $s_{12} > 1$ and $s_{23} < 1$. Then $s_1 < 0$ and $Z/m_2 = (\Delta_1/m_2)s_1 + (\Delta_2/m_2)s_2 \geq (\Delta_1/m_1)s_1 + (\Delta_2/m_2)s_2 = A(\Delta_1/m_1, \Delta_2/m_2, \Delta_3/m_3) > 0$ by Lemma 2. QED

6. Proof of Theorem 1. Consider the first claim. The "only if" part is easy and does not use the convexity of the configuration. Under the symmetry hypothesis $s_{13} = s_{23}$ and $s_{14} = s_{24}$. Using (13) this gives $\Delta_1/m_1 = \Delta_2/m_2$. But the symmetry also implies $\Delta_1 = \Delta_2$ so $m_1 = m_2$. We pass to the "if" part. We assume $m_1 = m_2$. The convexity with particles 1 and 2 on a diagonal means, without loss of generality, $\Delta_1 \leq 0$, $\Delta_2 \leq 0$, $\Delta_3 \geq 0$ and $\Delta_4 \geq 0$. Assuming $\Delta_1 < \Delta_2$, Lemma 3 applies and gives $t_1 > t_2$. This contradicts (10): there is no planar central configuration with these Δ_i's. The inequality $\Delta_2 < \Delta_1$ is excluded in the same way. So $\Delta_1 = \Delta_2$ and the configuration is symmetric according to (8) or (13).

Concerning the other claims, what remains to prove reduces to $m_1 < m_2 \iff \Delta_1 < \Delta_2 < 0$. This is obtained from Lemma 3, which gives $\Delta_1 < \Delta_2 < 0 \implies m_1 < m_2$ and $\Delta_2 < \Delta_1 < 0 \implies m_2 < m_1$. Note that here we have used the first part of the theorem and Lemma 1. QED

7. Higher dimensional results

Theorem 2. Let five particles $(q_1, q_2, q_3, q_4, q_5)$ form a central configuration, which is a convex three-dimensional configuration having $[q_1, q_2]$ as the diagonal. This configuration is symmetric with respect to the plane $[q_3, q_4, q_5]$ if and only if $m_1 = m_2$. Also $m_1 < m_2$ if and only if $|\Delta_{1345}| < |\Delta_{2345}|$, where Δ_{ijkl} is the oriented volume of the tetrahedron $[q_i, q_j, q_k, q_l]$.

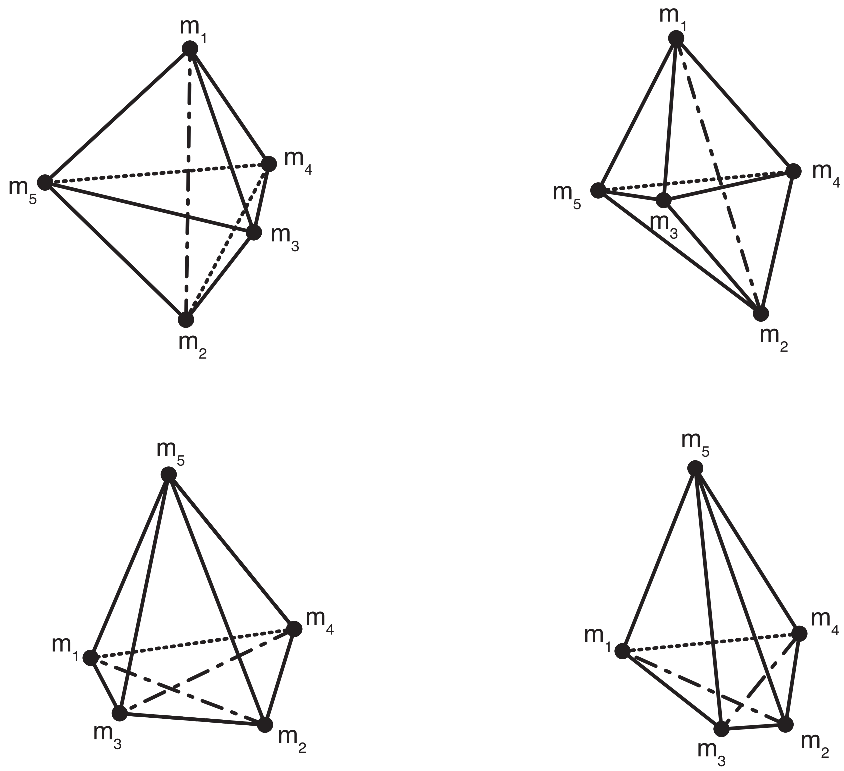

Figure 2: Spatial 5-body convex configurations
Left: with a plane of symmetry. Right: asymmetric.

We say "the diagonal" because a generic convex 3D configuration of five points has a unique diagonal. But if four particles are coplanar there are two diagonals in this plane (see Fig. 2). The above theorem is valid for this kind of diagonals also. The proof is similar to that of Theorem 1.

We obtained the symmetry but do not know the existence and the uniqueness of the central configuration. We do not know if the number of symmetric solutions is finite. About the existence, the nice arguments in Xia's paper [30] show that there exists a convex central configuration, but they don't show there exits one having $[q_1, q_2]$ as the diagonal. In contrast with four body case, there are paths in the space of convex configurations going from configurations having $[q_1, q_2]$ as the diagonal to, let us say, configurations having $[q_3, q_4]$ as the diagonal. The limiting configuration is such that (q_1, q_2, q_3, q_4) form a convex planar quadrilateral configuration. This shape is possible for a central configuration.

In term of the Δ_i's, such a path starts with, for example, a hexahedron with $\Delta_1 < \Delta_2 < 0 < \Delta_5 < \Delta_3 < \Delta_4$, passes through a pyramidal configuration where $\Delta_5 = 0$ and finishes with a hexahedron with $\Delta_1 < \Delta_2 < \Delta_5 < 0 < \Delta_3 < \Delta_4$.

The higher dimensional version, with n particles in dimension $n-2$, is also true, if the convex configuration is obtained by gluing two simplices by a common hyperface. We have to be careful that from $n = 6$ this is not the only way to get a convex configuration. We can think of this in term of the signs of the Δ_i's. If only one of them is negative, we are in a non-convex case. If exactly two of them are negative, we are in the convex case where our Lemma 3 may apply. If there are exactly three negative Δ_i's, the configuration is convex but our Lemma does not apply. Another way to get some geometrical intuition is to think of the configuration as a simplex plus a point, this point being added nearby but outside a hyperface, or nearby but outside a lower dimensional face.

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