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THURSTON OBSTRUCTIONS AND AHLFORS REGULAR CONFORMAL DIMENSION

PETER HAÏSSINSKY AND KEVIN M. PILGRIM

Abstract. Let $f : S^2 \to S^2$ be a postcritically finite expanding branched covering map of the sphere to itself. Associated to f is a canonical quasisymmetry class $\mathcal{G}(f)$ of Ahlfors regular metrics on the sphere in which the dynamics is (non-classically) conformal. We find a lower bound on the Hausdorff dimension of metrics in $\mathcal{G}(f)$ in terms of the combinatorics of f.

Soit $f : S^2 \to S^2$ un revêtement ramifié de la sphère topologiquement expansif et à ensemble postcritique fini. On lui associe une famille de métriques Ahlfors-régulières canoniques $\mathcal{G}(f)$ qui rendent f grossièrement conforme. On établit une minoration de la dimension de Hausdorff de ces métriques en termes combinatoires.

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1. Introduction

A fundamental principle of dynamical systems is that in the presence of sufficient expansion, topology determines a preferred class $\mathcal{G}$ of geometric structures. For example, suppose $G \rightsquigarrow X$ is an action of a group G on a perfect metrizable compactum X by homeomorphisms. Bowditch [Bow] showed that if the induced diagonal action on the space of ordered triples of pairwise distinct points of X is properly discontinuous and cocompact, then G is hyperbolic, and there is a G-equivariant homeomorphism ϕ of X onto ∂G. The boundary ∂G carries a preferred (quasisymmetry) class of metrics in which the group elements act by uniformly quasi-Möbius maps. Elements of this class of metrics can be transported via ϕ to X, yielding a class of metrics $\mathcal{G}(G \rightsquigarrow X)$ canonically associated to the dynamics in which the elements act in a geometrically special way.

Cannon's Conjecture is equivalent to the assertion that under the hypotheses of Bowditch's theorem, whenever X is homeomorphic to the two-sphere S^2, then the standard Euclidean metric belongs to $\mathcal{G}(G \rightsquigarrow X)$ [BK]. Thus, conjecturally, no "exotic" metrics on the sphere arise from such group actions. In contrast, the dynamics of certain iterated maps $f : S^2 \to S^2$ provide a rich source of examples of metrics on the sphere in which the dynamics is (non-classically) conformal.

We now explain this precisely. The results summarized below are consequences of the general theory developed in [HP].

Topologically coarse expanding conformal (cxc) dynamics.

Definition 1.1 (Topologically cxc)**.** *A continuous, orientation-preserving, branched covering $f : S^2 \to S^2$ is called* topologically cxc *provided there exists a finite open covering $\mathcal{U}_0$ of S^2 by connected sets satisfying the following properties:*

> **[Expansion]** *The mesh of the covering $\mathcal{U}_n$ tends to zero as $n \to \infty$, where $\mathcal{U}_n$ denotes the set of connected components of $f^{-n}(U)$ as U ranges over $\mathcal{U}_0$. That is, for any finite open cover $\mathcal{Y}$ of S^2 by open sets, there exists N such that for all $n \geq N$ and all $U \in \mathcal{U}_n$, there exists $Y \in \mathcal{Y}$ with $U \subset Y$.*
> **[Irreducibility]** *The map f is* locally eventually onto*: for any $x \in S^2$, and any neighborhood W of x, there is some n with $f^n(W) = S^2$.*
> **[Degree]** *The set of degrees of maps of the form $f^k|\widetilde{U} : \widetilde{U} \to U$, where $U \in \mathcal{U}_n$, $\widetilde{U} \in \mathcal{U}_{n+k}$, and n and k are arbitrary, has a finite maximum.*

We denote by $\mathbf{U} = \cup_{n\geq 0}\mathcal{U}_n$.

Note that the definition prohibits periodic or recurrent branch points, i.e.branch points x for which the orbit $x, f(x), f(f(x)), \ldots$ contains or accumulates on x.

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere. A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a dynamical system which is conformal in the Riemannian sense. It is called *semihyperbolic* if it has no parabolic cycles and no recurrent critical points in its Julia set. A rational map f which is chaotic on all of $\widehat{\mathbb{C}}$ (that is, has Julia set the whole sphere) is topologically cxc if and only if it is semihyperbolic [HP, Corollary 4.4.2].

Metric cxc. Now suppose S^2 is equipped with a metric d (that is, a distance function) compatible with its topology.

We first recall the notion of *roundness*.

Roundness. Let (X, d) be a metric space. We denote by $B(x, r)$ and $\overline{B}(x, r)$ respectively the open and closed ball of radius r about x. Let A be a bounded, proper subset of X with non-empty interior. Given $x \in \text{int}(A)$, let

$$L(A, x) = \sup\{d(x, b) : b \in A\}$$

and

$$l(A, x) = \sup\{r : r \leq L(A, x) \text{ and } B(x, r) \subset A\}$$

denote, respectively, the *outradius* and *inradius* of A about x. While the outradius is intrinsic, the inradius depends on how A sits in X. The condition $r \leq L(A, x)$ is necessary to guarantee that the outradius is at least the inradius. The *roundness of A about x* is defined as

$$\text{Round}(A, x) = L(A, x)/l(A, x) \in [1, \infty).$$

A set A is *K-almost-round* if $\text{Round}(A, x) \leq K$ holds for some $x \in A$, and this implies that for some $s > 0$, there exists a ball $B(x, s)$ satisfying

$$B(x, s) \subset A \subset \overline{B}(x, Ks).$$

Definition 1.2 (Metric cxc). *A continuous, orientation-preserving branched covering $f : (S^2, d) \to (S^2, d)$ is called* metric cxc *provided it is topologically cxc with respect to some covering $\mathcal{U}_0$ such that there exist*

- *continuous, increasing embeddings $\rho_{\pm} : [1, \infty) \to [1, \infty)$, the* forward and backward roundness distortion functions*, and*
- *increasing homeomorphisms $\delta_{\pm} : [0, 1] \to [0, 1]$, the* forward and backward relative diameter distortion functions

satisfying the following axioms:

[Roundness distortion] *For all $n, k \in \mathbb{N}$ and for all*

$$U \in \mathcal{U}_n, \ \widetilde{U} \in \mathcal{U}_{n+k}, \ \tilde{x} \in \widetilde{U}, \ x \in U$$

if

$$f^k(\widetilde{U}) = U, \ f^k(\tilde{x}) = x$$

then the backward roundness bound

(1) $$\text{Round}(\widetilde{U}, \tilde{x}) \leq \rho_-(\text{Round}(U, x))$$

and the forward roundness bound

(2) $$\text{Round}(U, x) \leq \rho_+(\text{Round}(\widetilde{U}, \tilde{x})).$$

hold.

In other words: for a given element of $\mathbf{U}$, iterates of f both forward and backward distorts its roundness by an amount independent of the iterate.

[Diameter distortion] *For all $n_0, n_1, k \in \mathbb{N}$ and for all*

$$U \in \mathcal{U}_{n_0}, \ U' \in \mathcal{U}_{n_1}, \ \widetilde{U} \in \mathcal{U}_{n_0+k}, \ \widetilde{U}' \in \mathcal{U}_{n_1+k}, \ \widetilde{U}' \subset \widetilde{U}, \ U' \subset U$$

if

$$f^k(\widetilde{U}) = U, \ f^k(\widetilde{U}') = U'$$

then

$$\frac{\text{diam}\widetilde{U}'}{\text{diam}\widetilde{U}} \leq \delta_-\left(\frac{\text{diam}U'}{\text{diam}U}\right)$$

and

$$\frac{\text{diam}U'}{\text{diam}U} \leq \delta_+\left(\frac{\text{diam}\widetilde{U}'}{\text{diam}\widetilde{U}}\right)$$

hold.

In other words: given two nested elements of $\mathbf{U}$*, iterates of* f *both forward and backward distort their relative sizes by an amount independent of the iterate.*

A homeomorphism $h : X \to Y$ between metric spaces is called *quasisymmetric* provided there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that $d_X(x,a) \le t d_X(x,b)$ implies $d_Y(f(x), f(a)) \le \eta(t) d_Y(f(x), f(b))$ for all triples of points $x, a, b \in X$ and all $t \ge 0$. Loosely: h distorts ratios of distances, and the roundness of balls, by controlled amounts.

An orientation-preserving branched covering map $f : \mathbb{S}^2 \to \mathbb{S}^2$ from the standard Euclidean sphere to itself is metric cxc if and only if it is quasisymmetrically conjugate to a semihyperbolic rational map with Julia set the whole sphere [HP, Theorems 4.2.4 and 4.2.7]. The class of metric cxc dynamical systems is closed under quasisymmetric conjugation, and a topological conjugacy between metric cxc maps is quasisymmetric [HP, Theorem 2.8.2].

Conformal gauges. The *conformal gauge* of a metric space X is the set of all metric spaces quasisymmetrically equivalent to X. A metric space X is *Ahlfors regular of dimension* Q provided there is a Radon measure μ and a constant $C \ge 1$ such that for any $x \in X$ and $r \in (0, \mathrm{diam} X]$,

$$\frac{1}{C} r^Q \le \mu(B(x,r)) \le C r^Q\,.$$

The Hausdorff dimension $\mathrm{H.dim}(X)$ of an Ahlfors Q-regular metric space X is equal to Q.

Suppose now that $f : S^2 \to S^2$ is topologically cxc. By [HP, Corollary 3.5.4] we have

Theorem 1.3 (Canonical gauge)**.** *Given a topologically cxc dynamical system* $f : S^2 \to S^2$*, there exists an Ahlfors regular metric* d *on* S^2*, unique up to quasisymmetry, such that* $f : (S^2, d) \to (S^2, d)$ *is metrically cxc.*

It follows that the set $\mathcal{G}(f)$ of all Ahlfors regular metric spaces Y quasisymmetrically equivalent to (S^2, d) is an invariant, called the *Ahlfors regular conformal gauge*, of the topological conjugacy class of f. Therefore, the *Ahlfors regular conformal dimension*

$$\mathrm{confdim}_{AR}(f) := \inf_{Y \in \mathcal{G}(f)} \mathrm{H.dim}(Y)$$

is a numerical topological dynamical invariant as well; it is distinct from the entropy. Moreover, this invariant almost characterizes rational maps among topologically cxc maps on the sphere. In [HP, Theorem 4.2.11] the following theorem is proved.

Theorem 1.4 (Characterization of rational maps)**.** *A topologically cxc map* $f : S^2 \to S^2$ *is topologically conjugate to a semihyperbolic rational map if and only if the Ahlfors regular conformal dimension* $\mathrm{confdim}_{AR}(f)$ *is equal to* 2*, and is achieved by an Ahlfors regular metric.*

There are many examples of topologically cxc maps which are not topologically conjugate to rational maps. The following are well-known combinatorial obstructions.

Thurston obstructions. Let $f : S^2 \to S^2$ be an orientation-preserving branched covering. The Riemann-Hurwitz formula implies that the cardinality of the set C_f of branch points at which f fails to be locally injective is equal to $2\deg(f) - 2$, counted with multiplicity, where $\deg(f)$ is the degree of f. The *postcritical set* is defined as $P_f = \overline{\cup_{n>0} f^n(C_f)}$. Under the assumption that the postcritical set is finite, Thurston characterized when f is equivalent to a rational map R in the following sense: $h_0 \circ f = R \circ h_1$ for orientation-preserving homeomorphisms h_0, h_1 which are homotopic through homeomorphisms fixing P_f pointwise; see [DH]. The obstructions which arise are of the following nature.

A *multicurve* $\Gamma \subset S^2 - P_f$ is a finite set of simple, closed, unoriented curves

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$$

in $S^2 - P_f$ satisfying the following properties: (i) they are disjoint and pairwise distinct, up to free homotopy in $S^2 - P_f$, and (ii) each curve γ_j is *non-peripheral*—that is, each component of $S^2 - \gamma_j$ contains at least two elements of P_f. A multicurve Γ is *invariant* if for each $\gamma_j \in \Gamma$, every connected component δ of $f^{-1}(\gamma_j)$ is either homotopic in $S^2 - P_f$ to an element $\gamma_i \in \Gamma$, or else is peripheral. If α and β are unoriented curves in $S^2 - P_f$, we write $\alpha \sim \beta$ if they are freely homotopic in $S^2 - P_f$.

Let Γ be an arbitrary multicurve and $Q \geq 1$. Let $\mathbb{R}^\Gamma$ denote the real vector space with basis Γ; thus γ_j is identified with the jth standard basis vector of $\mathbb{R}^{\#\Gamma}$. Define

$$f_{\Gamma,Q} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

by

$$f_{\Gamma,Q}(\gamma_j) = \sum_{\gamma_i \in \Gamma} \sum_{\delta \sim \gamma_i} |\deg(f : \delta \to \gamma_j)|^{1-Q} \gamma_i.$$

In words: the (i,j)-matrix coefficient $(f_{\Gamma,Q})_{i,j}$ is obtained by considering the connected preimages δ of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 - P_f$; recording the positive degree of the restriction $f|_\delta : \delta \to \gamma$, raised to the power $(1 - Q)$; and summing these numbers together. If there are no such curves δ, the coefficient is defined to be zero. Note that in the definition, we do not require invariance.

Since $f_{\Gamma,Q}$ is represented by a non-negative matrix, one can apply the structure theory for such matrices, summarized at the beginning of Appendix A. This theory implies the following results.

The matrix $(f_{\Gamma,Q})$ has a real non-negative Perron-Frobenius eigenvalue $\lambda(f_{\Gamma,Q})$ equal to its spectral radius and a corresponding non-negative eigenvector $v(f_{\Gamma,Q})$. A multicurve Γ is called *irreducible* if given any $\gamma_i, \gamma_j \in \Gamma$ there exists an iterate $q \geq 1$ such that the corresponding coefficient $(f^q_{\Gamma,Q})_{i,j}$ is positive; this property is independent of Q. For an irreducible multicurve, the Perron-Frobenius eigenvalue is positive, has geometric multiplicity one, and is strictly larger than the norm of all other eigenvalues; the corresponding eigenvector is also strictly positive.

A *Thurston obstruction* is defined as a multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ for which the inequality $\lambda(f_{\Gamma,2}) \geq 1$ holds. An obstruction always contains an irreducible obstruction with the same Perron-Frobenius eigenvalue (cf. Appendix A). By a theorem of McMullen [McM], a semihyperbolic rational map has no obstructions unless it is extremely special (see below).

The reason these form obstructions to (classical Riemannian) conformality is roughly the following; see [DH] for details. Suppose a semihyperbolic rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ had an obstruction. Then one could find a collection of disjoint annular neighborhoods A_j of γ_j such that the vector of classical moduli $(\text{mod}(A_1), \ldots, \text{mod}(A_m))$ is a scalar multiple of a Perron-Frobenius eigenvector v. Classical moduli are subadditive and monotone: the sharp Grötzsch inequality implies that if $A(\delta_k), k = 1, \ldots, l$ are disjoint essential open subannuli of A_i, then $\sum_{k=1}^{l} \text{mod}(A(\delta_k)) \leq \text{mod}(A_i)$; equality holds if and only if each $A(\delta_k)$ is a right Euclidean subannulus in a conformally equivalent Euclidean metric on A_i, and the union of their closures contains A_i. If $f : A(\delta_k) \to A_j$ is a degree d covering, then $\text{mod}(A(\delta_k)) = \text{mod}(A_j)/d$. It follows by induction that for fixed $j \in \{1, \ldots, m\}$ and for all $n \in \mathbb{N}$, the j-th coordinate of the vector $f^n_{\Gamma,2}(v)$ is a lower bound for the maximum modulus of an annulus homotopic in $\mathbb{P}^1 - P_f$ to A_j. It follows that such a rational map cannot have an obstruction unless it is extremely special—a so-called *integral Lattès example* [DH]. In this case, $\#P_f = 4$, $(f_{\Gamma,2}) = (1)$, and f lifts under a twofold covering ramified at P_f to an unbranched covering map of the complex torus given by $z \mapsto dz$ in the group law, where $d = \deg(f)$. Summarizing, we say that branched covering f is *obstructed* if (i) it is not topologically conjugate to an integral Lattès example, and (ii) it has an obstruction.

Suppose that the map $f : S^2 \to S^2$ is topologically cxc. Since the property of being obstructed is invariant under topological conjugacy, Theorem 1.4 yields

$$f \text{ obstructed} \Rightarrow \mathrm{confdim}_{AR}(f) \text{ is either } \begin{cases} 2, & \text{but not realized, or} \\ > 2. & \end{cases}$$

Our main result, which was inspired by discussions with M. Bonk and L. Geyer, quantifies the influence of obstructions on the Ahlfors regular conformal gauge $\mathcal{G}(f)$.

Let Γ be a multicurve. If Γ contains an irreducible multicurve, then there is a unique value $Q(\Gamma) \geq 1$ such that $\lambda(f_{\Gamma,Q(\Gamma)}) = 1$ (Lemma A.2). Otherwise, we set $Q(\Gamma) = 0$. Define

$$Q(f) = \sup\{Q(\Gamma) : \Gamma \text{ is a multicurve}\}.$$

We note that:

- If f is obstructed, then $Q(f) \geq 2$ and is a rough numerical measurement of the extent to which f is obstructed.
- If $\#P_f < \infty$, then there are only finitely many possible irreducible matrices $f_{\Gamma,2}$, and the supremum is achieved by some multicurve.
- If f is not conjugate to a Lattès example, and if Γ is not an obstruction, then $Q(\Gamma) < 2$, by Lemma A.2.

We prove:

Theorem 1.5. *Suppose $f : S^2 \to S^2$ is topologically cxc. Then*

$$\mathrm{confdim}_{AR}(f) \geq Q(f).$$

The finite subdivision rules of Cannon, Floyd, and Parry [CFP] provide a wealth of examples of topologically cxc maps on the sphere [HP, § 4.3]. As a special case of the above theorem, we have the following.

Corollary 1.6. *Suppose $\mathcal{R}$ is a finite subdivision rule with bounded valence, mesh going to zero, underlying surface the two-sphere, and whose subdivision map $f : S^2 \to S^2$ is orientation-preserving. Then* $\mathrm{confdim}_{AR}(f) \geq Q(f)$.

In [Bon, Conjecture 6.4] it is guessed that for obstructed maps induced by such finite subdivision rules, equality actually holds. The preceding corollary establishes one direction of this conjecture. Our methods are in spirit similar to those sketched above for the classical case $Q = 2$. Instead of classical analytic moduli, combinatorial moduli are used. The outline of our argument is the same as the brief sketch in [Bon]. However, Theorem 1.5 applies to maps which need not be postcritically finite and hence need not arise from finite subdivision rules. A key ingredient is the construction of a suitable metric on S^2 in which the coverings $\mathcal{U}_n$ have the geometric regularity property of being a family of *uniform quasipackings*. Also, our proof makes use of a succinct comparison relation (Proposition 3.2 below) between combinatorial and analytic moduli articulated by the first author in [Haï].

Unfortunately our proof is somewhat indirect: apart from the bound on dimension, our methods shed very little light on the structure of the elements of the gauge $\mathcal{G}(f)$.

Outline of paper.

In §2 we develop the machinery of combinatorial Q-moduli of path families associated to sequences $(\mathcal{S}_n)_n$ of coverings of surfaces. Much of this material is now standard.

In §3 we state results that relate combinatorial and analytic moduli in Ahlfors regular metric spaces. These results apply to covering sequences $(\mathcal{S}_n)_n$ which are *quasipackings with mesh tending to zero.*

In §4, we briefly recall the construction in [HP] of the gauge $\mathcal{G}(f)$ and its properties. We also prove that when S^2 is equipped with any metric in the gauge $\mathcal{G}(f)$, the sequence of coverings $(\mathcal{U}_n)_n$ defines a uniform sequence of quasipackings.

In §5, we complete the proof of Theorem 1.5.

In Appendix A, we summarize facts about non-negative matrices and prove Lemma A.2.

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Notation. For positive quantities a, b, we write $a \lesssim b$ (resp. $a \gtrsim b$) if there is a universal constant $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$). The notation $a \asymp b$ will mean $a \lesssim b$ and $a \gtrsim b$.

If A is a matrix, the notation $A \geq 0$ means the entries of A are non-negative, and $A \geq B$ means $A - B \geq 0$.

The cardinality of a set A is denoted $\#A$.

2. Combinatorial moduli

Definitions. Let $\mathcal{S}$ be a covering of a topological space X, and let $Q \geq 1$. Denote by $\mathcal{M}_Q(\mathcal{S})$ the set of functions $\rho : \mathcal{S} \to \mathbb{R}_+$ such that $0 < \sum \rho(s)^Q < \infty$; elements of $\mathcal{M}_Q(\mathcal{S})$ we call *admissible metrics*. For $K \subset X$ we denote by $\mathcal{S}(K)$ the set of elements of $\mathcal{S}$ which intersect K. The *ρ-length* of K is by definition

$$\ell_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)$$

and its *ρ-volume* is

$$V_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)^Q .$$

If Γ is a family of curves in X and if $\rho \in \mathcal{M}_Q(\mathcal{S})$, we define

$$L_\rho(\Gamma, \mathcal{S}) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

$$\mathrm{mod}_Q(\Gamma, \rho, \mathcal{S}) = \frac{V_\rho(X)}{L_\rho(\Gamma, \mathcal{S})^Q},$$

and the *combinatorial modulus* by

$$\mathrm{mod}_Q(\Gamma, \mathcal{S}) = \inf_{\rho \in \mathcal{M}_Q(\mathcal{S})} \mathrm{mod}_Q(\Gamma, \rho, \mathcal{S}).$$

A metric ρ for which $\mathrm{mod}_Q(\Gamma,\rho,\mathcal{S}) = \mathrm{mod}_Q(\Gamma,\mathcal{S})$ will be called *optimal*. We will consider here only finite coverings; in this case the proof of the existence of optimal metrics is a straightforward argument in linear algebra. The following result is the analog of the classical Beurling's criterion which characterises optimal metrics.

Proposition 2.1. *Let $\mathcal{S}$ be a finite cover of a space X, Γ a family of curves and $Q > 1$. An admissible metric ρ is optimal if and only if there is a non-empty subfamily $\Gamma_0 \subset \Gamma$ and non-negative scalars λ_γ, $\gamma \in \Gamma_0$, such that*

(1) *for all $\gamma \in \Gamma_0$, $\ell_\rho(\gamma) = L_\rho(\Gamma,\mathcal{S})$;*
(2) *for any $s \in \mathcal{S}$,*
$$Q\rho(s)^{Q-1} = \sum \lambda_\gamma$$
where the sum is taken over curves in Γ_0 which go through s.

Moreover, an optimal metric is unique up to scale.

For a proof, see Proposition 2.1 and Lemma 2.2 in [Haï].

Monotonicity and subadditivity.

Proposition 2.2. *Let $\mathcal{S}$ be a locally finite cover of a topological space X and $Q \geq 1$.*

(1) *If $\Gamma_1 \subset \Gamma_2$ then $\mathrm{mod}_Q(\Gamma_1,\mathcal{S}) \leq \mathrm{mod}_Q(\Gamma_2,\mathcal{S})$.*
(2) *Let $\Gamma_1,\ldots,\Gamma_n$ be a set of curve families in X and $Q \geq 1$. Then*
$$\mathrm{mod}_Q(\cup\Gamma_j,\mathcal{S}) \leq \sum \mathrm{mod}_Q(\Gamma_j,\mathcal{S}) .$$
Furthermore, if $\mathcal{S}(\Gamma_i) \cap \mathcal{S}(\Gamma_j) = \emptyset$ for $i \neq j$, then
$$\mathrm{mod}_Q(\cup\Gamma_j,\mathcal{S}) = \sum \mathrm{mod}_Q(\Gamma_j,\mathcal{S}) .$$

The proof is the same as the standard one for classical moduli (see for instance [Väi, Thms 6.2 and 6.7]) and so is omitted.

Naturality under coverings.

A closed (resp. open) *annulus* in a surface X is a subset homeomorphic to $[0,1] \times S^1$ (resp. $(0,1) \times S^1$). Suppose A is an annulus in a surface X and $\mathcal{S}$ is a finite covering of A by subsets of X. For $Q \geq 1$ we define
$$\mathrm{mod}_Q(A,\mathcal{S}) = \mathrm{mod}_Q(\Gamma,\mathcal{S})$$
where Γ is the set of closed curves which are contained in A and which separate the boundary components of A.

Note that $\mathrm{mod}_Q(A,\mathcal{S})$ is an invariant of the triple $(X,A,\mathcal{S})$ and is not purely intrinsic to A. The following result describes how combinatorial moduli of annuli change under coverings. Since the elements of $\mathcal{S}$ meeting A need not be contained in A, it is necessary to have some additional space surrounding A on which the covering map is defined.

Proposition 2.3. *Suppose A, B, A', B' are open annuli such that $\overline{A} \subset B$, $\overline{A'} \subset B'$, A is essential in B, and A' is essential in B'. Let $f : B' \to B$ be a covering map of degree d such that $f|A' : A' \to A$ is also a covering map of degree d. Let $\mathcal{S}$ be a finite cover of A by Jordan domains $s \subset B$ and $\mathcal{S}'$ be the induced covering of A', i.e. the covering whose elements s' are the components of $f^{-1}(\{s\})$, $s \in \mathcal{S}$. Then, for $Q > 1$,*
$$\mathrm{mod}_Q(A',\mathcal{S}') = d^{1-Q} \cdot \mathrm{mod}_Q(A,\mathcal{S}) .$$

Proof: Let Γ, Γ' denote respectively the curve families in A, A' separating the boundary components. We note that since f is a covering and each piece of $\mathcal{S}$ is a Jordan domain, $f^{-1}(s)$ has d components each of which is also a Jordan domain.

Let ρ be an optimal metric for $\mathrm{mod}_Q(\Gamma, \mathcal{S})$. Consider the subfamily Γ_0 and the scalars λ_γ given by Proposition 2.1. Set $\Gamma'_0 = f^{-1}(\Gamma_0)$, $\rho' = \rho \circ f$, and for $\gamma' \in \Gamma'_0$ define $\lambda_{\gamma'} = \lambda_{f(\gamma')}$.

The preimage γ' of a curve γ in Γ is connected and all the preimages of $s \in \mathcal{S}(\gamma)$ belong to $\mathcal{S}'(\gamma')$. Therefore, for $\gamma' \in \Gamma'_0$, one has $\ell_{\rho'}(\gamma') = dL_\rho(\Gamma)$, and for any other curve, $\ell_{\rho'}(\gamma') \geq dL_\rho(\Gamma)$.

Clearly, for any $s' \in \mathcal{S}'$,

$$Q\rho'(s')^{Q-1} = \sum_{\gamma' \in \Gamma'_0} \lambda_{\gamma'}$$

so that Proposition 2.1 implies that ρ' is optimal.

It follows that

$$\mathrm{mod}_Q(\Gamma', \mathcal{S}') = \frac{dV_Q(\rho)}{(dL_\rho(\Gamma))^Q} = d^{1-Q} \cdot \mathrm{mod}_Q(\Gamma, \mathcal{S})\,.$$

■

3. Combinatorial moduli and Ahlfors regular conformal dimension

Under suitable conditions, the combinatorial moduli obtained from a sequence $(\mathcal{S}_n)_n$ of coverings can be used to approximate analytic moduli on metric measure spaces. Suppose (X, d, μ) is a metric measure space, Γ is a family of curves in X, and $Q \geq 1$. The *(analytic) Q-modulus* of Γ is defined by

$$\mathrm{mod}_Q(\Gamma) = \inf \int_X \rho^Q d\mu$$

where the infimum is taken over all measurable functions $\rho : X \to \mathbb{R}_+$ such that ρ is *admissible*, i.e.

$$\int_\gamma \rho ds \geq 1$$

for all $\gamma \in \Gamma$ which are rectifiable. If Γ contains no rectifiable curves, $\mathrm{mod}_Q(\Gamma)$ is defined to be zero. Note that when Γ contains a constant curve, then there are no admissible ρ, so we set $\mathrm{mod}_Q\Gamma = +\infty$. When $X \subset \mathbb{C}$ is a domain, μ is Euclidean area, and $Q = 2$, this definition coincides with the classical one.

The approximation result we use requires the sequence of coverings $(\mathcal{S}_n)_n$ to be a *uniform family of quasipackings.*

Definition 3.1 (Quasipacking)**.** *A* quasipacking *of a metric space is a locally finite cover $\mathcal{S}$ such that there is some constant $K \geq 1$ which satisfies the following property. For any $s \in \mathcal{S}$, there are two balls $B(x_s, r_s) \subset s \subset B(x_s, K \cdot r_s)$ such that the family $\{B(x_s, r_s)\}_{s \in \mathcal{S}}$ consists of pairwise disjoint balls. A family $(\mathcal{S}_n)_n$ of quasipackings is called* uniform *if the mesh of $\mathcal{S}_n$ tends to zero as $n \to \infty$ and the constant K defined above can be chosen independent of n.*

The next result says roughly that for the family consisting of all sufficiently large curves, analytic and combinatorial moduli are comparable.

Proposition 3.2. *Suppose $Q > 1$, X is an Ahlfors Q-regular compact metric space, and $(\mathcal{S}_n)_n$ is a sequence of uniform quasipackings. Fix $L > 0$, and let Γ_L be the family of curves in X of diameter at least L. Then either*

(1) $\mathrm{mod}_Q(\Gamma_L) = 0$ *and* $\lim_{n\to\infty} \mathrm{mod}_Q(\Gamma_L, \mathcal{S}_n) = 0$*, or*
(2) $\mathrm{mod}_Q(\Gamma_L) > 0$*, and there exists constants* $C \geq 1$ *independent of* L *and* $N = N(L) \in \mathbb{N}$ *such that for any* $n > N$*,*

$$\frac{1}{C}\mathrm{mod}_Q(\Gamma_L, \mathcal{S}_n) \leq \mathrm{mod}_Q(\Gamma_L) \leq C\mathrm{mod}_Q(\Gamma_L, \mathcal{S}_n).$$

See Proposition B.2 in [Haï].

Corollary 3.3. *Under the hypotheses of Proposition 3.2, if* $Q > \mathrm{confdim}_{AR}(X) \geq 1$*, and if* Γ *is a curve family contained in some* Γ_L*,* $L > 0$*, then* $\lim_{n\to\infty} \mathrm{mod}_Q(\Gamma, \mathcal{S}_n) = 0$.

In other words, if there is a curve family Γ each of whose elements has diameter at least $L > 0$, and if for some $Q > 1$ one has

$$\mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \gtrsim 1$$

for all n, then the AR-conformal dimension is at least Q.

The lower bound on the diameter is necessary. If Γ is any family of curves such that, for any n, there is some $\gamma \in \Gamma$ contained in an element of $\mathcal{S}_n$, then $\mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \geq 1$ for all n. So, for example, if Γ consists of a countable family of non-constant curves as above, then $\mathrm{mod}_Q(\Gamma) = 0$ while $\mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \geq 1$ for all n.

Proof: By assumption, there is some metric d in the conformal gauge of X which is Ahlfors regular of dimension $p \in (\mathrm{confdim}_{AR} X, Q)$, and $\Gamma \subset \Gamma_L$ for some $L > 0$. Let $d' = d^{p/Q}$. Then the sequence $\{\mathcal{S}_n, n \geq 0\}$ is again a family of uniform quasipackings. Though d' is Ahlfors regular of dimension Q, it has no rectifiable curves. In particular, for the metric d', we have $\mathrm{mod}_Q(\Gamma_L) = 0$. By Proposition 3.2, $\mathrm{mod}_Q(\Gamma_L, \mathcal{S}_n) \to 0$ as $n \to \infty$. Since $\Gamma \subset \Gamma_L$, the monotonicity of moduli (Proposition 2.2) implies that $\mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \leq \mathrm{mod}_Q(\Gamma_L, \mathcal{S}_n)$ so that $\mathrm{mod}_Q(\Gamma, \mathcal{S}_n)$ tends to 0 as well.

■

Remark. Corollary 3.3 takes its origin in the work of Pansu [Pan, Prop. 3.2] where a similar statement is proved for his *modules grossiers*. It is also closely related to a theorem of Bonk and Tyson which asserts that if the Q-modulus of curves in a Q-Ahlfors regular space is positive, then the Ahlfors regular conformal dimension of that space is Q [Hei, Thm 15.10]. In particular, if a metric is Q-regular for some Q strictly larger than the Ahlfors regular conformal dimension, then the Q-modulus of any non-trivial family of curves is zero.

4. The conformal gauge of a topological cxc map

In this section, we recall from [HP] the construction of the metrics associated to topologically cxc maps, specialized to the case of maps $f : S^2 \to S^2$. After summarizing their properties, we prove that with respect to these metrics, the induced coverings $\mathcal{U}_n$ obtained by pulling back an initial covering $\mathcal{U}_0$ under iteration form a sequence of uniform quasipackings.

Associated graph Σ. Suppose $f : S^2 \to S^2$ is topologically cxc with respect to an open covering $\mathcal{U}_0$. Let Σ be the graph whose vertices are elements of $\cup_n \mathcal{U}_n$, together with a distinguished root vertex $o = S^2 = \mathcal{U}_{-1}$. The set of edges is defined as a disjoint union of two types of edges: horizontal edges join elements $U_1, U_2 \in \mathcal{U}_n$ if and only if $U_1 \cap U_2 \neq \emptyset$, while vertical edges join elements $U \in \mathcal{U}_n, V \in \mathcal{U}_{n\pm 1}$ at consecutive levels if and only if $U \cap V \neq \emptyset$. Note that there is a natural map $F : \Sigma \to \Sigma$ which is cellular on the complement of the set of closed edges meeting $\mathcal{U}_0$.

Associated metrics. Equip Σ temporarily with the length metric $d(\cdot,\cdot)$ in which edges are isometric to unit intervals.

Axiom [Expansion] implies that the metric space Σ is hyperbolic in the sense of Gromov [HP, Theorem 3.3.1]; see [GdlH] for background on hyperbolic metric spaces. One may define its compactification in the following way.

Fix $\varepsilon > 0$. For $\xi \in \Sigma$ let $\varrho_\varepsilon(\xi) = \exp(-\varepsilon d(o,\xi))$. Define a new metric d_ε on Σ by

$$d_\varepsilon(\xi,\zeta) = \inf \ell_\varepsilon(\gamma)$$

where

$$\ell_\varepsilon(\gamma) = \int_\gamma \varrho_\varepsilon \, ds$$

and where as usual the infimum is over all rectifiable curves in the metric space (Σ, d) joining ξ to ζ. The resulting metric space $\Sigma_\varepsilon = (\Sigma, d_\varepsilon)$ is incomplete. Its complement in its completion defines the boundary $\partial_\varepsilon\Sigma$ which is an Ahlfors regular metric space of dimension $\frac{1}{\varepsilon}\log\deg(f)$ by axiom [Degree] if ε is small enough. The map F is e^ε-Lipschitz in the d_ε-metric, so it extends to $\partial_\varepsilon\Sigma$.

If ε is sufficiently small, the boundary $\partial_\varepsilon\Sigma$ is homeomorphic to the usual Gromov boundary, and there is a natural homeomorphism $\phi : S^2 \to \partial_\varepsilon\Sigma$ given as follows. For $x \in S^2$ let $U_n(x)$ be any element of $\mathcal{U}_n$ containing x. We may regard $U_n(x)$ as a vertex of Σ and hence as an element of its completion $\overline{\Sigma}_\varepsilon$. The definitions of Σ and of d_ε imply that

$$\phi(x) = \lim_{n\to\infty} U_n(x) \in \partial_\varepsilon\Sigma$$

exists and is independent of the choice of sequence $\{U_n(x)\}_n$. The homeomorphism ϕ conjugates f on S^2 to the map F on $\partial_\varepsilon\Sigma$.

Associated metrics on S^2. *A priori* the boundary $\partial_\varepsilon\Sigma$ depends on the choice of $\mathcal{U}_0$ and of ε. However, by [HP, Proposition 3.3.12], its quasisymmetry class is independent of such choices, provided the covering satisfies axiom [Expansion] and the parameter is small enough to guarantee that ϕ is a homeomorphism. We remark that balls for such metrics need not be connected.

The *Ahlfors regular conformal gauge* $\mathcal{G}(f)$ is then defined as the set of all Ahlfors regular metrics on S^2 quasisymmetrically equivalent to a metric of the form $\phi^*(d_\varepsilon)$. Elements of $\mathcal{G}(f)$ will be referred to as *associated metrics*.

It what follows, for convenience we denote by d_ε the pulled-back metric $\phi^*(d_\varepsilon)$ on S^2.

Theorem 4.1. *Let $f : S^2 \to S^2$ be a topological cxc dynamical system with respect to an open covering $\mathcal{V}_0$. Then there exists a finite cover $\mathcal{U}_0$ of S^2 by Jordan domains such that, for any associated metric, the sequence of coverings $\{\mathcal{U}_n, n \geq 0\}$ is a uniform family of quasipackings.*

Proof: It is easily shown that the property of being a uniform quasipacking is preserved under quasisymmetric changes of metric. Hence, it suffices to show the conclusion for a metric d_ε as constructed above.

For convenience, equip S^2 with the standard Euclidean spherical metric and denote the resulting metric space by $\mathbb{S}^2$. Then small spherical balls $D(x,r)$ are Jordan domains. For each $x \in \mathbb{S}^2$, consider an open ball $U_x = D(x, r_x)$ centred at x. By expansion, there exists n_0 such that no element of $\mathcal{V}_n$, $n \geq n_0$, contains more than one critical value of f. By choosing r_x sufficiently small and sufficiently generic, we may arrange so that each U_x (i) is contained in some element of $\mathcal{V}_{n_0}$, and (ii) does not contain a critical value of an iterate of f on its boundary.

A covering of a disk ramified above at most one point is again a disk, by the Riemann-Hurwitz formula. It follows that every iterated preimage of U_x under f is a Jordan domain.

Let $\mathcal{U}_0 = \{U_{x_j}\}_j$ be a finite subcover. From the $5r$-covering theorem [Hei, Thm 1.2], we may assume that the balls $\{D(x_j, r_j/5)\}_j$ are pairwise disjoint. Since we assumed that each disk in $\mathcal{U}_0$ was contained in an element of $\mathcal{V}_{n_0}$, it follows that $\mathcal{U}_0$ satisfies both axioms [Expansion] and [Degree]. Furthermore, there is some $r_0 > 0$ such that the collection $\{B_\varepsilon(x_j, r_0)\}$ of balls in the metric d_ε is a disjointed family.

The proof is completed by appealing to the axiom [Expansion] and to the fact that with respect to the metric d_ε, iterates of f distort elements of $\mathbf{U} = \cup_n \mathcal{U}_n$ by controlled amounts. More precisely, it follows from [HP, Prop. 3.3.2] that there is a constant $C \geq 1$ for which the following property holds. If $\tilde{x} \in \widetilde{U} \in \mathcal{U}_n$, $f^n(\tilde{x}) = x_j$, $f^n(\widetilde{U}) = U_j$, then

$$B_\varepsilon(\tilde{x}, (r_0/C)e^{-\varepsilon n}) \subset \widetilde{U} \subset B_\varepsilon(\tilde{x}, Ce^{-\varepsilon n})$$

and

$$f^n(B_\varepsilon(\tilde{x}, (r_0/C)e^{-\varepsilon n})) \subset B_\varepsilon(x_j, r_0)\,.$$

This implies that the sequence $\{\mathcal{U}_n, n \geq 0\}$ is a uniform family of quasipackings by Jordan domains.

■

We note also that the quasipackings we have just constructed have uniformly bounded overlap by axiom [Degree].

5. Ahlfors regular conformal dimension and multicurves

Suppose $f : S^2 \to S^2$ is topologically cxc. By Theorem 4.1, there exists an associated metric such that the sequence $\{\mathcal{U}_n, n \geq 0\}$ is a family of uniform quasipackings by Jordan domains. As coverings for the definition of combinatorial moduli, we take $\mathcal{S}_n = \mathcal{U}_n$.

Proposition 5.1. *Let $Q > 1$, and let Γ be a multicurve with $\lambda(f_{\Gamma,Q}) \geq 1$. Then, for any n large enough, $\mathrm{mod}_Q([\Gamma], \mathcal{U}_n) \gtrsim 1$, where $[\Gamma]$ denotes the family of all curves in $S^2 - P_f$ homotopic to a curve in Γ.*

Proof: Without loss of generality we may assume Γ is irreducible. Write $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ and equip $\mathbb{R}^\Gamma$ with the L^1-norm $|\cdot|_1$ norm, so that $|\sum_j a_j\gamma_j|_1 = \sum_j |a_j|$. For each $1 \leq j \leq m$ choose an annulus B_j which is a regular neighborhood of γ_j and such that $B_i \cap B_j = \emptyset, i \neq j$. Within each B_j choose a smaller such neighborhood A_j so that $\gamma_j \subset A_j \subset \overline{A}_j \subset B_j$ and each inclusion is essential. By expansion, there exists a level n_0 such that the covering $\mathcal{U}_{n_0}$ has the following properties:

(1) $s \in \mathcal{U}_{n_0}, s \cap A_j \neq \emptyset \Rightarrow s \subset B_j$.
(2) $\mathrm{mod}_Q(A_j, \mathcal{U}_{n_0}) > 0$ for all $1 \leq j \leq m$.

For $n \geq 1$, let Γ_n denote the finite family of curves $\widetilde{\gamma}$ in $[\Gamma]$ arising as connected components of curves of the form $f^{-n}(\gamma_j), \gamma_j \in \Gamma$. Given such a curve $\widetilde{\gamma} \subset f^{-n}(\gamma_j)$, denote by $A(\widetilde{\gamma})$ the unique component of $f^{-n}(A_j)$ containing $\widetilde{\gamma}$. Note that for fixed n, the resulting collection of annuli $A(\widetilde{\gamma}), \widetilde{\gamma} \in \Gamma_n$, are disjoint.

By the monotonicity and additivity of moduli (Proposition 2.2) we have that

$$\mathrm{mod}_Q([\Gamma], \mathcal{U}_{n_0+n}) \geq \sum_{\widetilde{\gamma} \in \Gamma_n} \mathrm{mod}_Q(A(\widetilde{\gamma}), \mathcal{U}_{n_0+n}).$$

Let $v \in \mathbb{R}^\Gamma$ be the vector of combinatorial moduli at level n_0 given by

$$v = (\mathrm{mod}_Q(A_1, \mathcal{U}_{n_0}), \ldots, \mathrm{mod}_Q(A_m, \mathcal{U}_{n_0})).$$

Proposition 2.3 implies that if $n \geq 1$ and $\widetilde{\gamma} \in \Gamma_n$ then

$$\mathrm{mod}_Q(A(\widetilde{\gamma}), \mathcal{U}_{n_0+n}) = \deg(f : \widetilde{\gamma} \to f(\widetilde{\gamma}))^{1-Q} \cdot \mathrm{mod}_Q(A(f(\widetilde{\gamma})), \mathcal{U}_{n_0+n-1}).$$

By induction and the fact that degrees multiply under compositions of coverings, for each fixed n, the j-th entry of the vector $f^n_{\Gamma,Q}(v)$ is the sum of the moduli $\{\mathrm{mod}_Q(A(\widetilde{\gamma})), \mathcal{U}_{n_0+n})\}$ over all curves $\widetilde{\gamma} \in \Gamma_n$ homotopic to $\gamma_j \in \Gamma$. By the monotonicity and subadditivity of moduli (Proposition 2.2) we conclude that, for any $n \geq 1$,

$$\mathrm{mod}_Q([\Gamma], \mathcal{U}_{n_0+n}) \geq |f^n_{\Gamma,Q}(v)|_1 \,.$$

By the Perron-Frobenius theorem, there is a positive vector w_Q for which $f_{\Gamma,Q}(w_Q) = \lambda(f_{\Gamma,Q}) \cdot w_Q$. By scaling, we may assume $w_Q \leq v$. Since the entries of the matrix for $f_{\Gamma,Q}$ are non-negative, we have

$$f^n_{\Gamma,Q}(v) \geq f^n_{\Gamma,Q}(w_Q) = \lambda(f_{\Gamma,Q})^n w_Q \geq w_Q > 0$$

and so

$$\liminf_{n\to\infty} |f^n_{\Gamma,Q}(v)|_1 > 0$$

which completes the proof.

■

We conclude with the proof of Theorem 1.5.

Proof: By Theorem 4.1, there is an Ahlfors regular metric $d_\varepsilon \in \mathcal{G}(f)$ on S^2 for which the sequence of coverings $\{\mathcal{U}_n, n \geq 0\}$ is a uniform family of quasipackings. Let Γ be a multicurve and $[\Gamma]$ the family of all curves homotopic to an element of Γ. If Γ contains no irreducible multicurve, then $Q(\Gamma) = 0 \leq \mathrm{confdim}_{AR}(f)$. Otherwise, by Lemma A.2 and the definition of $Q(\Gamma)$, we have $\lambda(f_{\Gamma,Q(\Gamma)}) = 1$ for some $Q(\Gamma) \geq 1$. By Proposition 5.1 applied with $Q = Q(\Gamma)$, $\mathrm{mod}_{Q(\Gamma)}([\Gamma], \mathcal{U}_n) \gtrsim 1$ as $n \to \infty$. Since curves in Γ are non-peripheral, there is a positive lower bound for the diameter of any curve in the family $[\Gamma]$. Thus, Corollary 3.3 implies that $\mathrm{confdim}_{AR}(S^2, d_\varepsilon) \geq Q(\Gamma)$ and so $\mathrm{confdim}_{AR}(f) \geq Q(\Gamma)$. Since Γ is an arbitrary multicurve, we conclude

$$\mathrm{confdim}_{AR}(f) \geq Q(f)\,.$$

■

Appendix A. Monotonicity of leading eigenvalues

We first recall some facts concerning non-negative square matrices A; see [BP].

- **Perron-Frobenius theorem, irreducible version.** [BP, Theorem 1.4]. A k-by-k non-negative matrix A is said to be *irreducible* if, for any ordered pair (i, j), $1 \leq i, j \leq k$, there is some power $q > 0$ for which $(A^q)_{i,j} > 0$. If A is irreducible, then there is a simple eigenvalue $\lambda(A)$ of A which is larger than the norm of any other eigenvalue, and up to scale, there is a unique corresponding eigenvector, all of whose entries are positive.
- **Perron-Frobenius theorem, general version.** [BP, Theorem 1.1]. If A is merely non-negative, then there exists a non-negative eigenvalue $\lambda(A)$ equal to its spectral radius, and any corresponding eigenvector is also non-negative.

- **Monotonicity** [BP, Corollary 2.1.5]. The function $A \mapsto \lambda(A)$ satisfies

$$A \geq B \Rightarrow \lambda(A) \geq \lambda(B) \tag{3}$$

- **Irreducible decomposition** [BP, pp. 39-40]. Given any non-negative matrix A, there is a permutation matrix P such that $U = PAP^{-1}$ has block upper triangular form, where the diagonal blocks D of U are square and either irreducible or zero. For some diagonal block D, $\lambda(D) = \lambda(A)$.

Definition. Let $p \geq 1$ be an integer. A *Levy cycle of length p* is a multicurve $\Gamma = \{\gamma_j,\ j \in \mathbb{Z}/p\mathbb{Z}\}$ such that for each $j \in \mathbb{Z}/p\mathbb{Z}$, $f^{-1}(\gamma_j)$ contains a preimage δ which is homotopic to γ_{j+1}, and such that $\deg(f : \delta \to \gamma_j) = 1$.

Lemma A.1. *If $f : S^2 \to S^2$ is a branched covering satisfying Axiom [Expansion] with respect to an open covering $\mathcal{U}_0$, then f has no Levy cycles.*

Proof: Fix a metric on the sphere compatible with its topology. Axiom [Expansion] implies that there are constants $d_n \downarrow 0$ as $n \to \infty$ such that $\max\{\text{diam} U | U \in \mathcal{U}_n\} \leq d_n$.

Suppose f had a Levy cycle Γ of length p, and let $\gamma \in \Gamma$. Then $g = f^p$ also satisfies Axiom [Expansion] with respect to $\mathcal{U}_0$, and $g^{-1}(\gamma)$ has a connected component δ homotopic to γ and satisfying $\deg(g : \delta \to \gamma) = 1$. There is an open annulus $A \subset S^2 - P_f$ containing γ such that the inclusion map $\gamma \hookrightarrow A$ is essential. By compactness and Axiom [Expansion], there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{U}_{n_0}(\gamma) = \{U \in \mathcal{U}_{n_0} | U \cap \gamma \neq \emptyset\} \subset A.$$

Let $N = \#\mathcal{U}_{n_0}(\gamma)$. Since $\deg(g : \delta \to \gamma) = 1$ and δ is homotopic to γ, it follows by induction and the construction of the annulus A that for all $k \in \mathbb{N}$, there exists a component A_k of $g^{-k}(A)$ homotopic to A such that $\deg(g^k : A_k \to A) = 1$, i.e. $g^k|A_k$ is a homeomorphism onto A. Hence, the annulus A_k contains a unique component δ_k of $g^{-k}(\gamma)$ which is homotopic to γ, and δ_k is covered by N elements of $\mathcal{U}_{n_0+pk}$. Thus $\text{diam}\delta_k \leq N \cdot d_{n_0+pk} \to 0$ as $k \to \infty$. But this is impossible: since γ is non-peripheral, there is a positive lower bound on the diameter of any curve homotopic to γ.

■

Lemma A.2. *Let $f : S^2 \to S^2$ be a branched covering satisfying Axiom [Expansion], and let Γ be a multicurve.*

1. *If Γ does not contain an irreducible multicurve, then $f_{\Gamma,Q}$ is nilpotent and $\lambda(f_{\Gamma,Q}) = 0$ for all $Q \geq 1$.*
2. *If Γ contains an irreducible multicurve, then $\lambda(f_{\Gamma,1}) \geq 1$, and the function $Q \mapsto \lambda(f_{\Gamma,Q})$ is strictly decreasing on $[1, \infty)$ and tends to zero as Q tends to ∞.*

Proof: Re-indexing the elements of Γ, we may assume the matrix $(f_{\Gamma,Q})$ has block-upper triangular form for all $Q \geq 1$.

Suppose that Γ contains no irreducible multicurve. Then the matrix $(f_{\Gamma,Q})$ is upper triangular and has zeros on the diagonal. Hence $f_{\Gamma,Q}$ is nilpotent and $\lambda(f_{\Gamma,Q}) = 0$.

Suppose now that Γ contains an irreducible multicurve Γ'. It follows that $\lambda(f_{\Gamma,1}) \geq \lambda(f_{\Gamma',1})$. Since the matrix $(f_{\Gamma',1})$ is irreducible, non-negative and with positive entries at least 1, there is some permutation matrix P' and a re-indexing such that

$$f_{\Gamma',1} \geq \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}.$$

We then have by Equation (3)

$$\lambda(f_{\Gamma,1}) \geq \lambda(f_{\Gamma',1}) \geq \lambda(P') = 1.$$

We now prove the second assertion. For convenience, denote by A_Q the matrix $(f_{\Gamma,Q})$. If $Q_1 > Q_2$ then $A_{Q_1} \leq A_{Q_2}$ entrywise and Equation (3) implies that $\lambda(A_{Q_1}) \leq \lambda(A_{Q_2})$. If equality holds for distinct Q_1, Q_2, then $\lambda(A_Q)$ is constant for all $Q_2 \leq Q \leq Q_1$. Since eigenvalues are algebraic functions, this would imply $\lambda(A_Q)$ is constant for all Q. Since $\lambda(A_1) \geq 1$ by assumption, it suffices to show that $\lambda(A_Q) \to 0$ as $Q \to \infty$.

The definition of $f_{\Gamma,Q}$ implies that

$$A_Q = B_Q + C$$

where B_Q, C are non-negative, $B_Q \to 0$, and the entries of C are of the form $1^{1-Q} + \ldots + 1^{1-Q}$. Hence C is constant in Q.

In this paragraph, we prove that $C^m = 0$ where $m = \#\Gamma$. Suppose D is an irreducible diagonal block in the decomposition of C. Then $(D^q)_{i,i} > 0$ for some index i and some power $q > 0$. But this implies that Γ contains a Levy cycle, which is impossible by Lemma A.1. Hence all diagonal blocks are zero, which implies $C^m = 0$.

Since $C^m = 0$, every term in the expansion of $(B_Q + C)^m$ contains B_Q as a factor. Therefore

$$\lim_{Q\to\infty} A_Q^m = \lim_{Q\to\infty} (B_Q + C)^m = 0$$

entrywise. Hence $\lambda(A_Q^m) = \lambda(A_Q)^m \to 0$ and so $\lambda(A_Q) \to 0$.

■

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