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# Thurston obstructions and Ahlfors regular conformal dimension

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## Abstract

Let  $f : S^2 \rightarrow S^2$  be an expanding branched covering map of the sphere to itself with finite postcritical set  $P_f$ . Associated to  $f$  is a canonical quasisymmetry class  $\mathcal{G}(f)$  of Ahlfors regular metrics on the sphere in which the dynamics is (non-classically) conformal. We show

$$\inf_{X \in \mathcal{G}(f)} \text{H.dim}(X) \geq \max \left\{ 2, \sup_{\Gamma} \{Q | \lambda(f_{\Gamma, Q}) = 1\} \right\}.$$

The supremum is taken over all multicurves  $\Gamma \subset S^2 - P_f$ . The map  $f_{\Gamma, Q} : \mathbb{R}^{\Gamma} \rightarrow \mathbb{R}^{\Gamma}$  is defined by

$$f_{\Gamma, Q}(\gamma) = \sum_{[\gamma'] \in \Gamma} \sum_{\delta \sim \gamma'} \deg(f : \delta \rightarrow \gamma)^{1-Q} [\gamma'],$$

where the second sum is over all preimages  $\delta$  of  $\gamma$  freely homotopic to  $\gamma'$  in  $S^2 - P_f$ , and  $\lambda(f_{\Gamma, Q})$  is its Perron-Frobenius leading eigenvalue. This refines Thurston's observation that if  $Q(f) > 2$ , then there is no  $f$ -invariant classical conformal structure.

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*Keywords:* postcritically finite, conformal dimension, finite subdivision rule, combinatorial moduli.

# 1 Introduction

A fundamental principle of dynamical systems is that in the presence of sufficient expansion, topology determines a preferred class  $\mathcal{G}$  of geometric structures. For example, suppose  $G \curvearrowright X$  is an action of a group  $G$  on a perfect metrizable compactum  $X$  by homeomorphisms. Bowditch [Bow] showed that if the induced diagonal action on the space of ordered triples of pairwise distinct points of  $X$  is properly discontinuous and cocompact, then  $G$  is hyperbolic, and there is a  $G$ -equivariant homeomorphism  $\phi$  of  $X$  onto  $\partial G$ . The boundary  $\partial G$  carries a preferred (quasisymmetry) class of metrics in which the group elements act by quasiconformal maps. Elements of this class of metrics can be transported via  $\phi$  to  $X$ , yielding a class of metrics  $\mathcal{G}(G \curvearrowright X)$  canonically associated to the dynamics in which the elements act in a geometrically special way.

*Cannon's Conjecture* is equivalent to the assertion that under the hypotheses of Bowditch's theorem, whenever  $X$  is homeomorphic to  $S^2$ , then the standard Euclidean metric two-sphere  $\mathbb{S}^2$  belongs to  $\mathcal{G}(G \curvearrowright X)$  [BK]. Thus, conjecturally, no "exotic" metrics on the sphere arise from such group actions. In contrast, the dynamics of certain iterated maps  $f : S^2 \rightarrow S^2$  provide a rich source of examples of metrics on the sphere in which the dynamics is (non-classically) conformal.

We now explain this precisely. The results summarized below are consequences of the general theory developed in [HP].

## Topologically coarse expanding conformal (cxc) dynamics.

**Definition 1.1 (Topologically cxc)** *A continuous, orientation-preserving, branched covering  $f : S^2 \rightarrow S^2$  is called topologically cxc provided there exists a finite open covering  $\mathcal{U}_0$  of  $S^2$  by connected sets satisfying the following properties:*

**[Expansion]** *The mesh of the coverings  $\mathcal{U}_n$  tends to zero as  $n \rightarrow \infty$ , where  $\mathcal{U}_n$  denotes the set of connected components of  $f^{-n}(U)$  as  $U$  ranges over  $\mathcal{U}_0$ . That is, for any finite open cover  $\mathcal{Y}$  of  $S^2$  by open sets, there exists  $N$  such that for all  $n \geq N$  and all  $U \in \mathcal{U}_n$ , there exists  $Y \in \mathcal{Y}$  with  $U \subset Y$ .*

**[Irreducibility]** *The map  $f$  is locally eventually onto: for any  $x \in S^2$ , and any neighborhood  $W$  of  $x$ , there is some  $n$  with  $f^n(W) = S^2$ .*

**[Degree]** *The set of degrees of maps of the form  $f^k|_{\tilde{U}} : \tilde{U} \rightarrow U$ , where  $U \in \mathcal{U}_n$ ,  $\tilde{U} \in \mathcal{U}_{n+k}$ , and  $n$  and  $k$  are arbitrary, has a finite maximum.*

We denote by  $\mathbf{U} = \cup_{n \geq 0} \mathcal{U}_n$ .

Note that the definition prohibits recurrent or periodic branch points.

Rational maps  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  are dynamical systems which are conformal in the classical sense. A rational map  $f$  is called *semihyperbolic* if it has no parabolic cycles and no recurrent critical points in its Julia set. A rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  which is chaotic on all of  $\widehat{\mathbb{C}}$  (that is, has Julia set the whole sphere) is topologically cxc if and only if it is semihyperbolic [HP, Corollary 4.4.2].

**Metric cxc.** Now suppose  $S^2$  is equipped with a metric  $d$  (that is, a distance function) compatible with its topology.

Let us first recall the notion of *roundness*.

**Roundness.** Let  $(Z, d)$  be a metric space and let  $A$  be a bounded, proper subset of  $Z$  with non-empty interior. Given  $a \in \text{int}(A)$ , let

$$L(A, a) = \sup\{d(a, b) : b \in A\}$$

and

$$l(A, a) = \sup\{r : r \leq L(A, a) \text{ and } B(a, r) \subset A\}$$

denote, respectively, the *outradius* and *inradius* of  $A$  about  $a$ . While the outradius is intrinsic, the inradius depends on how  $A$  sits in  $Z$ . The condition  $r \leq L(A, a)$  is necessary to guarantee that the outradius is at least the inradius. The *roundness of  $A$  about  $a$*  is defined as

$$\text{Round}(A, a) = L(A, a)/l(A, a) \in [1, \infty).$$

One says  $A$  is  *$K$ -almost-round* if  $\text{Round}(A, a) \leq K$  for some  $a \in A$ , and this implies that for some  $s > 0$ ,

$$B(a, s) \subset A \subset B(a, Ks).$$

**Definition 1.2 (Metric cxc)** *A continuous, orientation-preserving branched covering  $f : (S^2, d) \rightarrow (S^2, d)$  is called metric cxc provided it is topologically cxc with respect to some covering  $\mathcal{U}_0$  such that there exist*

- *continuous, increasing embeddings  $\rho_{\pm} : [1, \infty) \rightarrow [1, \infty)$ , the forward and backward roundness distortion functions, and*
- *increasing homeomorphisms  $\delta_{\pm} : [0, 1] \rightarrow [0, 1]$ , the forward and backward relative diameter distortion functions*

*satisfying the following axioms:*

**[Roundness distortion]**  $(\forall n, k)$  *and for all*

$$U \in \mathcal{U}_n, \tilde{U} \in \mathcal{U}_{n+k}, \tilde{y} \in \tilde{U}, y \in U$$

*if*

$$f^{\circ k}(\tilde{U}) = U, f^{\circ k}(\tilde{y}) = y$$

*then the backward roundness bound*

$$\text{Round}(\tilde{U}, \tilde{y}) < \rho_{-}(\text{Round}(U, y)) \tag{1}$$

*and the forward roundness bound*

$$\text{Round}(U, y) < \rho_{+}(\text{Round}(\tilde{U}, \tilde{y})). \tag{2}$$

*hold.*

*In other words: for a given element of  $\mathbf{U}$ , iterates of  $f$  both forward and backward distorts its roundness by an amount independent of the iterate.*

**[Diameter distortion]** ( $\forall n_0, n_1, k$ ) and for all

$$U \in \mathcal{U}_{n_0}, U' \in \mathcal{U}_{n_1}, \tilde{U} \in \mathcal{U}_{n_0+k}, \tilde{U}' \in \mathcal{U}_{n_1+k}, \tilde{U}' \subset \tilde{U}, U' \subset U$$

if

$$f^k(\tilde{U}) = U, f^k(\tilde{U}') = U'$$

then

$$\frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} < \delta_- \left( \frac{\text{diam}U'}{\text{diam}U} \right)$$

and

$$\frac{\text{diam}U'}{\text{diam}U} < \delta_+ \left( \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} \right)$$

*In other words: given two nested elements of  $\mathbf{U}$ , iterates of  $f$  both forward and backward distort their relative sizes by an amount independent of the iterate.*

A homeomorphism  $h : X \rightarrow Y$  between metric spaces is called *quasisymmetric* provided there exists a homeomorphism  $\eta : [0, \infty) \rightarrow [0, \infty)$  such that  $d_X(x, a) \leq t d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \leq \eta(t) d_Y(f(x), f(b))$  for all triples of points  $x, a, b \in X$  and all  $t \geq 0$ . More succinctly:  $h$  distorts ratios of distances, and the roundness of balls, by controlled amounts.

A branched covering map  $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  from the standard Euclidean sphere to itself is metric cxc if and only if it is quasisymmetrically conjugate to a semihyperbolic rational map with Julia set the whole sphere [HP, Theorems 4.2.4 and 4.2.7]. The class of metric cxc dynamical systems is closed under quasisymmetric conjugation, and a topological conjugacy between metric cxc maps is quasisymmetric [HP, Theorem 2.8.2]

**Conformal gauges.** A metric space  $X$  is *Ahlfors regular* of dimension  $Q$  provided there is a Radon measure  $\mu$  such that for any  $x \in X$  and  $r \in (0, \text{diam}X]$ ,

$$\mu(B(x, r)) \asymp r^Q.$$

If this is the case, this estimate also holds for the  $Q$ -dimensional Hausdorff measure.

Suppose now that  $f : S^2 \rightarrow S^2$  is topologically cxc.

**Theorem 1.3 (Canonical Gauge)** *Given a topologically cxc dynamical system  $f : S^2 \rightarrow S^2$ , there exists an Ahlfors regular metric  $d$  on  $S^2$ , unique up to quasisymmetry, such that  $f : (S^2, d) \rightarrow (S^2, d)$  is metrically cxc.*

It follows that the set  $\mathcal{G}(f)$  of all Ahlfors regular metric spaces  $Y$  quasisymmetrically equivalent to  $(S^2, d)$  is an invariant, called the *Ahlfors regular conformal gauge*, of the topological conjugacy class of  $f$ . Therefore, the *Ahlfors regular conformal dimension*

$$\text{confdim}_{AR}(f) := \inf_{Y \in \mathcal{G}(f)} \text{H.dim}(Y)$$

is a numerical topological dynamical invariant as well; it is distinct from the entropy. Moreover, this invariant almost characterizes rational maps among topologically cxc maps on the sphere. In [HP, Theorem 4.2.11] the following theorem is proved.

**Theorem 1.4 (Characterization of rational maps)** *A topologically cxc map  $f : S^2 \rightarrow S^2$  is topologically conjugate to a semihyperbolic rational map if and only if  $\text{confdim}_{AR}(f) = 2$  and is realized.*

There are many examples of topologically cxc maps which are not topologically conjugate to rational maps. The following are well-known combinatorial obstructions.

**Thurston obstructions.** The Riemann-Hürwitz formula implies that the set  $C_f$  of branch points at which  $f$  fails to be locally injective is equal to  $2 \deg(f) - 2$ , counted with multiplicity. The *postcritical set* is defined as  $P_f = \overline{\cup_{n>0} f^{\circ n} C_f}$ . When  $\#P_f < \infty$ , Thurston characterized when such branched coverings are homotopic to rational maps [DH]. The obstructions which arise are of the following nature.

A *multicurve*  $\Gamma \subset S^2 - P_f$  is a set of simple, closed, essential, pairwise disjoint, pairwise non-freely homotopic, unoriented curves, no element of which is freely homotopic into an arbitrarily small neighborhood of an element of  $P_f$ . A multicurve  $\Gamma$  is *invariant* if every preimage of an element of  $\Gamma$  under  $f$  regarded as an element of  $S^2 - P_f$  is either inessential, homotopic into a neighborhood of an element of  $P_f$ , or homotopic to an element of  $\Gamma$ .

Let  $\Gamma$  be an arbitrary multicurve and  $Q \geq 2$ . Define

$$f_{\Gamma,Q} : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$$

on basis elements  $\gamma \in \Gamma$  by

$$f_{\Gamma,Q}(\gamma) = \sum_{[\gamma'] \in \Gamma} \sum_{\delta \sim \gamma'} \deg(f : \delta \rightarrow \gamma)^{1-Q} [\gamma'].$$

Note that we do not require  $f$ -invariance.

Since  $f_{\Gamma,Q}$  is represented by a nonnegative matrix, it has a real nonnegative Perron-Frobenius eigenvalue  $\lambda(f_{\Gamma,Q})$  equal to its spectral radius and a corresponding nonnegative eigenvector  $v(f_{\Gamma,Q})$ . Note that  $\lambda(f_{\Gamma,Q})$  decreases as  $Q$  increases, so given  $\Gamma$ , there is a unique value  $Q(\Gamma)$  such that  $\lambda(f_{\Gamma,Q(\Gamma)}) = 1$ .

A *Thurston obstruction* is a multicurve  $\Gamma$  for which  $\lambda(f_{\Gamma,2}) \geq 1$ . Equivalently,  $\Gamma$  is an obstruction if and only if  $Q(\Gamma) \geq 2$ . By a theorem of McMullen [McM], a semihyperbolic rational map has no Thurston obstructions unless it is very special. The reason these form obstructions to (classical Riemannian) conformality is roughly the following. If  $f$  were rational and semihyperbolic, we could find a collection of annuli  $A(\gamma), \gamma \in \Gamma$  such that the vector of classical moduli  $\text{mod}(A(\gamma))$  is a scalar multiple of a Perron-Frobenius eigenvector  $v$ . The sharp Grötzsch inequality asserts that if  $A(\delta_i)$  are disjoint essential subannuli of  $A(\gamma')$  then  $\sum_i \text{mod}(A(\delta_i)) \leq \text{mod}(A(\gamma'))$  with equality if and only if each  $A(\delta_i)$  is a right subannulus in  $A(\gamma')$  equipped with its extremal Euclidean metric. The transformation  $f_{\Gamma,2}$  encodes lower bounds for the maximum moduli of annuli homotopic to the preimages of the annuli  $A(\gamma)$  under  $f$ . It follows that such a rational map cannot have a Thurston obstruction unless it is a so-called *integral Lattès example* [DH]. In this case,  $\#P_f = 4$ ,  $(f_{\Gamma,2}) = (1)$ , and  $f$  lifts under a twofold covering ramified at  $P_f$  to an unbranched covering map of the torus that preserves two transverse foliations by closed curves.

Since the existence of Thurston obstructions is preserved under topological (indeed, under combinatorial) equivalence, it follows that

$$f \text{ obstructed} \Rightarrow \text{confdim}_{AR}(f) \text{ is either } \begin{cases} 2, & \text{but not realized, or} \\ > 2. \end{cases}$$

Our main result quantifies the influence of Thurston obstructions for  $f$  on the conformal gauge  $\mathcal{G}(f)$ . Inspired by discussions with M. Bonk and L. Geyer, set

$$Q(f) = \max \left\{ 2, \sup_{\Gamma} Q(\Gamma) \right\}.$$

where the inner supremum is over all multicurves  $\Gamma$ . Thus

$$f \text{ obstructed} \Rightarrow Q(f) = \sup_{\Gamma} Q(\Gamma)$$

where  $\Gamma$  ranges over all obstructions. We prove:

**Theorem 1.5** *Suppose  $f : S^2 \rightarrow S^2$  is topologically cxc. Then*

$$\text{confdim}_{AR}(f) \geq Q(f).$$

The finite subdivision rules of Cannon, Floyd, and Parry [CFP] provide a wealth of examples of topologically cxc maps on the sphere [HP, §4.3]. As a special case of the above theorem, we have the following.

**Corollary 1.6** *Suppose  $\mathcal{R}$  is a finite subdivision rule with bounded valence, mesh going to zero, underlying surface the two-sphere, and whose subdivision map  $f : S^2 \rightarrow S^2$  is orientation-preserving. Then  $\text{confdim}_{AR}(f) \geq Q(f)$ .*

In [Bon, Conjecture 6.4] it is guessed that for obstructed maps induced by such finite subdivision rules, equality actually holds. The preceding Corollary establishes one direction of this conjecture. Our methods are in spirit similar to those sketched above for the classical case  $Q = 2$ . Instead of classical analytic moduli, combinatorial moduli are used. The outline of our argument is the same as the brief sketch in [Bon]. However, Theorem 1.5 applies to maps which need not be postcritically finite and hence need not arise from finite subdivision rules. A key ingredient is the construction, given in [HP], of a suitable metric on  $S^2$  in which the coverings  $\mathcal{U}_n$  have the geometric regularity property of being a family of *uniform quasipackings*. Also, our proof makes use of a succinct comparison relation (Proposition 3.2 below) between combinatorial and analytic moduli articulated by the first author in [Ha].

Unfortunately our proof is somewhat indirect: we do not prove the existence of an Ahlfors  $Q$ -regular metric in which some curve family has positive  $Q$ -modulus, where  $Q \geq 2$ .

### Outline of paper.

In §2 we develop the machinery of combinatorial  $Q$ -moduli of path families associated to sequences  $\mathcal{S}_n$  of coverings of surfaces. Much of this material is now standard.

In §3 we state known results which relate combinatorial and analytic moduli in Ahlfors regular metric spaces. These results apply to covering sequences  $\mathcal{S}_n$  which are *quasipackings with mesh tending to zero*.

In §4, we briefly recall the construction in [HP] of a canonical metric and its properties. We also prove that the sequence of coverings  $(\mathcal{U}_n)_n$  defines a uniform sequence of quasipackings.

In §5, we complete the proof of Theorem 1.5.

**Remarks.** By [HP, Proposition 3.3.8], if  $f : S^2 \rightarrow S^2$  has periodic branch points, the canonical gauge  $\mathcal{G}(f)$  contains a non-doubling metric. Since Ahlfors regularity implies doubling and doubling is preserved under quasimetric maps, in this case  $\mathcal{G}(f)$  is empty.

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## 2 Combinatorial moduli

**Definitions.** Let  $\mathcal{S}$  be a covering of a topological space  $X$ , and let  $Q > 1$ . Denote by  $\mathcal{M}_Q(\mathcal{S})$  the set of functions  $\rho : \mathcal{S} \rightarrow \mathbb{R}_+$  such that  $0 < \sum \rho(s)^Q < \infty$ ; elements of  $\mathcal{M}_Q(\mathcal{S})$  we call *admissible metrics*. For  $K \subset X$  we denote by  $\mathcal{S}(K)$  the set of elements of  $\mathcal{S}$  which intersect  $K$ . The  $\rho$ -length of  $K$  is by definition

$$\ell_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)$$

and its  $\rho$ -volume is

$$V_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)^Q.$$

If  $\Gamma$  is a family of curves in  $X$  and if  $\rho \in \mathcal{M}_Q(\mathcal{S})$ , we define

$$L_\rho(\Gamma, \mathcal{S}) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

$$\text{mod}_Q(\Gamma, \rho, \mathcal{S}) = \frac{V_\rho(X)}{L_\rho(\Gamma, \mathcal{S})^Q},$$

and the *combinatorial modulus* by

$$\text{mod}_Q(\Gamma, \mathcal{S}) = \inf_{\rho \in \mathcal{M}_Q(\mathcal{S})} \text{mod}_Q(\Gamma, \rho, \mathcal{S}).$$

A metric  $\rho$  for which  $\text{mod}_Q(\Gamma, \rho, \mathcal{S}) = \text{mod}_Q(\Gamma, \mathcal{S})$  will be called *optimal*. We will consider here only finite coverings; in this case the proof of the existence of optimal metrics is a straightforward argument in linear algebra. The following result is the analog of the classical Beurling's criterion which characterises optimal metrics.

**Proposition 2.1** *Let  $\mathcal{S}$  be a finite cover of a space  $X$ ,  $\Gamma$  a family of curves and  $Q > 1$ . A metric  $\rho$  is optimal if and only if there is a nonempty subfamily  $\Gamma_0 \subset \Gamma$  and non-negative scalars  $\lambda_\gamma$ ,  $\gamma \in \Gamma_0$ , such that*

1. for all  $\gamma \in \Gamma_0$ ,  $\ell_\rho(\gamma) = L_\rho(\Gamma, \mathcal{S})$ ;



2. for any  $s \in \mathcal{S}$ ,

$$Q\rho(s)^{Q-1} = \sum \lambda_\gamma$$

where the sum is taken over curves in  $\Gamma_0$  which go through  $s$ .

Moreover, this metric is unique up to a normalization.

For a proof, see Proposition 2.1 and Lemma 2.2 in [Ha].

**Composition law: subadditivity.**

**Proposition 2.2** *Let  $\mathcal{S}$  be a locally finite cover of a topological space  $X$  and  $Q \geq 1$ .*

1. *If  $\Gamma_1 \subset \Gamma_2$  then  $\text{mod}_Q(\Gamma_1, \mathcal{S}) \leq \text{mod}_Q(\Gamma_2, \mathcal{S})$ .*
2. *Let  $\Gamma_1, \dots, \Gamma_n$  be a set of curve families in  $X$  and  $Q \geq 1$ . Then*

$$\text{mod}_Q(\cup \Gamma_j, \mathcal{S}) \leq \sum \text{mod}_Q(\Gamma_j, \mathcal{S}).$$

*If, furthermore, the supports of each family of curves are pairwise disjoint i.e., a piece of  $\mathcal{S}$  intersects at most one curve family, then*

$$\text{mod}_Q(\cup \Gamma_j, \mathcal{S}) = \sum \text{mod}_Q(\Gamma_j, \mathcal{S}).$$

**Proof:** The first conclusion is obvious from the definition. To prove the second, let  $\rho_j$  be an extremal metric for  $\Gamma_j$  normalised so that  $L_{\rho_j}(\Gamma_j, \mathcal{S}) = 1$ . Set

$$\rho = \left( \sum_j \rho_j^Q \right)^{1/Q}.$$

If  $\gamma \in \Gamma_j$ , then

$$\ell_\rho(\gamma) = \sum_{\mathcal{S}(\gamma)} \rho(s) \geq \sum_{\mathcal{S}(\gamma)} (\rho_j(s)^Q)^{1/Q} \geq 1.$$

Furthermore,

$$\sum_{\mathcal{S}} \rho^Q(s) = \sum_{\mathcal{S}} \sum_j \rho_j(s)^Q = \sum_j \text{mod}_Q(\Gamma_j, \mathcal{S}).$$

Therefore,

$$\text{mod}_Q(\cup \Gamma_j, \mathcal{S}) \leq \text{mod}_Q(\cup \Gamma_j, \rho, \mathcal{S}) \leq \sum_j \text{mod}_Q(\Gamma_j, \mathcal{S}).$$

Assuming the disjointness assumption, consider an optimal metric  $\rho$  for  $\cup \Gamma_j$  such that  $L_\rho(\cup_j \Gamma_j, \mathcal{S}) = 1$ . It follows that we may define  $\rho_j$  as the restriction of  $\rho$  to the support of  $\Gamma_j$ . Hence, for any  $p \geq 1$ ,

$$\sum \rho_j^p \leq \rho^p.$$

For any curve  $\gamma \in \Gamma_j$ ,

$$\ell_{\rho_j}(\gamma, \mathcal{S}) = \sum_{\mathcal{S}(\gamma)} \rho_j(s) = \sum_{\mathcal{S}(\gamma)} \rho(s) \geq 1.$$

Therefore,

$$\text{mod}_Q(\Gamma_j, \mathcal{S}) \leq \sum_j \rho_j^Q.$$

Since the supports of the  $\rho_j$ 's are pairwise disjoint, it follows that

$$\sum_j \text{mod}_Q(\Gamma_j, \mathcal{S}) \leq \sum_j \sum_{\mathcal{S}} \rho_j^Q \leq \sum_j \rho^Q = \text{mod}_Q(\cup \Gamma_j, \mathcal{S}).$$

■

### Naturality under coverings.

A (closed) *annulus* in a surface  $X$  is an subset homeomorphic to  $[0, 1] \times S^1$ . Suppose  $A$  is an annulus in a surface  $X$  and  $\mathcal{S}$  is a finite covering of  $A$  by subsets of  $X$ . For  $Q \geq 1$  we define

$$\text{mod}_Q(A, \mathcal{S}) = \text{mod}_Q(\Gamma, \mathcal{S})$$

where  $\Gamma$  is the set of curves which are contained in  $A$  and which separate the boundary components of  $A$ .

Note that  $\text{mod}_Q(A, \mathcal{S})$  is an invariant of the triple  $(X, A, \mathcal{S})$  and is not purely intrinsic to  $A$ . The following result describes how combinatorial moduli of annuli change under coverings. Since the elements of  $\mathcal{S}$  meeting  $A$  need not be contained in  $A$ , it is necessary to have some additional space surrounding  $A$  on which the covering map is defined.

**Proposition 2.3** *Suppose  $A, B, A', B'$  are open annuli such that  $\overline{A} \subset B$ ,  $\overline{A'} \subset B'$ ,  $A$  is essential in  $B$ , and  $A'$  is essential in  $B'$ . Let  $f : B' \rightarrow B$  be a covering map of degree  $d$  such that  $f|_{A'} : A' \rightarrow A$  is also a covering map of degree  $d$ . Let  $\mathcal{S}$  be a finite cover of  $A$  by Jordan domains  $s \subset B$  and  $\mathcal{S}'$  be the induced covering of  $A'$ , i.e. the covering whose pieces  $s'$  are the components of  $f^{-1}(\{s\})$ ,  $s \in \mathcal{S}$ . Then*

$$\text{mod}_Q(A', \mathcal{S}') = \frac{1}{d^{Q-1}} \text{mod}_Q(A, \mathcal{S}).$$

**Proof:** Let  $\Gamma, \Gamma'$  denote respectively the curve families in  $A, A'$  separating the boundary components. We note that since  $f$  is a covering and each piece of  $\mathcal{S}$  is a Jordan domain,  $f^{-1}(s)$  has  $d$  components each of which is also a Jordan domain.

Let  $\rho$  be an optimal metric for  $\text{mod}_Q(\Gamma, \mathcal{S})$ . Consider the subfamily  $\Gamma_0$  and the scalars  $\lambda_\gamma$  given by Proposition 2.1. Set  $\Gamma'_0 = f^{-1}(\Gamma_0)$ ,  $\rho' = \rho \circ f$ , and for  $\lambda' \in \Gamma'$  define  $\lambda_{\gamma'} = \lambda_{f(\gamma')}$ .

The preimage  $\gamma'$  of a curve  $\gamma$  in  $\Gamma$  is connected and all the preimages of  $s \in \mathcal{S}(\gamma)$  belong to  $\mathcal{S}'(\gamma')$ . Therefore, for  $\gamma' \in \Gamma'_0$ , one has  $\ell_{\rho'}(\gamma') = dL_\rho(\Gamma)$ , and for any other curve,  $\ell_{\rho'}(\gamma') \geq dL_\rho(\Gamma)$ .

Clearly, for any  $s' \in \mathcal{S}'$ ,

$$Q\rho'(s')^{Q-1} = \sum_{\gamma' \in \Gamma'_0} \lambda_{\gamma'}$$

so that Proposition 2.1 implies that  $\rho'$  is optimal.

It follows that

$$\text{mod}_Q(\Gamma', \mathcal{S}') = \frac{dV_Q(\rho)}{(dL_\rho(\Gamma))^Q} = \frac{d}{d^Q} \text{mod}_Q(\Gamma, \mathcal{S}).$$

■

### 3 Combinatorial moduli and Ahlfors regular conformal dimension

Under suitable conditions, the combinatorial moduli obtained from a sequence  $\{\mathcal{S}_n\}_n$  of coverings can be used to approximate analytic moduli on metric measure spaces. Suppose  $(X, d, \mu)$  is a metric measure space,  $\Gamma$  is a family of curves in  $X$ , and  $p > 0$ . The (*analytic*)  $Q$ -*modulus* of  $\Gamma$  is defined by

$$\text{mod}_Q(\Gamma) = \inf \int_X \rho^Q d\mu$$

where the infimum is taken over all measurable functions  $\rho : X \rightarrow \mathbb{R}_+$  such that

$$\int_\gamma \rho ds \geq 1$$

for all  $\gamma \in \Gamma$  which are rectifiable. If  $\Gamma$  contains no rectifiable curves,  $\text{mod}_Q(\Gamma)$  is defined to be zero. When  $X \subset \mathbb{C}$  and  $Q = 2$ , this definition coincides with the classical one.

The approximation result we use requires the sequence of coverings  $\{\mathcal{S}_n\}$  to be a *uniform family of quasipackings*.

#### Quasipackings.

**Definition 3.1 (Quasipacking)** *A quasipacking of a metric space is a locally finite cover  $\mathcal{S}$  such that there is some constant  $K \geq 1$  which satisfies the following property. For any  $s \in \mathcal{S}$ , there are two balls  $B(x_s, r_s) \subset s \subset B(x_s, K \cdot r_s)$  such that the family  $\{B(x_s, r_s)\}_{s \in \mathcal{S}}$  consists of pairwise disjoint balls. A family  $\{\mathcal{S}_n\}_n$  of quasipackings is called *uniform* if the mesh of  $\mathcal{S}_n$  tends to zero as  $n \rightarrow \infty$  and the constant  $K$  defined above can be chosen independent of  $n$ .*

**Proposition 3.2** *Suppose  $Q \geq 1$ ,  $X$  is a  $Q$ -regular compact metric space, and  $(\mathcal{S}_n)_n$  is a sequence of uniform quasipackings. Then for any curve family  $\Gamma$  which is closed in the Hausdorff topology, there exist  $C \geq 1$  and  $N \in \mathbb{N}$  such that for any  $n > N$ ,*

$$\frac{1}{C} \text{mod}_Q(\Gamma, \mathcal{S}_n) \leq \text{mod}_Q(\Gamma) \leq C \text{mod}_Q(\Gamma, \mathcal{S}_n)$$

if  $\text{mod}_Q(\Gamma) > 0$  and otherwise,  $\lim \text{mod}_Q(\Gamma, \mathcal{S}_n) = 0$ .

See Proposition B.2 in [Ha].

**Corollary 3.3** *Let  $X$  be a compact metric space, and  $(\mathcal{S}_n)_n$  a sequence of uniform quasipackings. If  $Q > \text{confdim}_{AR}X$ , then, for any family of curves which is closed in the Hausdorff topology,*

$$\lim_{n \rightarrow \infty} \text{mod}_Q(\Gamma, \mathcal{S}_n) = 0.$$

In other words, if there is a curve family  $\Gamma$  and some  $Q > 1$  such that

$$\text{mod}_Q(\Gamma, \mathcal{S}_n) \gtrsim 1$$

for all  $n$ , then the AR-conformal dimension is at least  $Q$ .

**Proof:** There is some Ahlfors-regular metric  $d$  in the conformal gauge of  $X$  of dimension  $p \in (\text{confdim}_{AR}X, Q)$ . Therefore,  $(X, d^{p/Q})$  is regular of dimension  $Q$ , but has no rectifiable curves: no  $Q$ -modulus is positive for this metric. Furthermore, the property of being a quasipacking is preserved under quasimetric maps, so that Proposition 3.2 implies that  $\lim \text{mod}_Q(\Gamma, \mathcal{S}_n) = 0$  for any family of curves. ■

**Remark.** Corollary 3.3 takes its origin in the work of Pansu [Pan, Prop.3.2] where a similar statement is proved for his *modules grossiers*. It is also closely related to a theorem of Bonk and Tyson which asserts that if the  $Q$ -modulus of curves in a  $Q$ -Ahlfors-regular space is positive, then the Ahlfors-regular conformal dimension of that space is  $Q$  [Hei, Thm 15.10]. In particular, if a metric is  $Q$ -regular for some  $Q$  strictly larger than the AR-conformal dimension, then the  $Q$ -modulus of any non-trivial family of curves is zero.

## 4 The conformal gauge of a topological cxc map

In this section, we recall from [HP] the construction of the natural metrics associated to topologically cxc maps, specialized to the case of maps  $f : S^2 \rightarrow S^2$ . After summarizing their properties, we prove that with respect to these metrics, the induced coverings  $\mathcal{U}_n$  obtained by pulling back an initial covering  $\mathcal{U}_0$  under iteration form a sequence of uniform quasipackings.

**Associated graph  $\Gamma$ .** Suppose  $f : S^2 \rightarrow S^2$  is topologically cxc with respect to an open covering  $\mathcal{U}_0$ . Let  $\Gamma$  be the graph whose vertices are elements of  $\mathcal{U}_n$ , together with a distinguished root vertex  $o = S^2 = \mathcal{U}_{-1}$ . The set of edges is defined as a disjoint union of two types of edges: horizontal edges join elements  $U_1, U_2 \in \mathcal{U}_n$  if and only if  $U_1 \cap U_2 \neq \emptyset$ , while vertical edges join elements  $U \in \mathcal{U}_n, V \in \mathcal{U}_{n\pm 1}$  at consecutive levels if and only if  $U \cap V \neq \emptyset$ . Note that there is a natural map  $F : \Gamma \rightarrow \Gamma$  which is cellular on the complement of the set of closed edges meeting  $\mathcal{U}_0$ .

**Associated metrics.** Equip  $\Gamma$  temporarily with the length metric  $d(\cdot, \cdot)$  in which edges are isometric to unit intervals.

Axiom [Expansion] implies that the metric space  $\Gamma$  is hyperbolic. One may define its compactification in the following way [CDP]. Fix  $\varepsilon > 0$ . For  $x \in \Gamma$  let  $\varrho_\varepsilon(x) = \exp(-\varepsilon d(o, x))$ . Define a new metric  $d_\varepsilon$  on  $\Gamma$  by

$$d_\varepsilon(x, y) = \inf l_\varepsilon(\gamma)$$

where

$$l_\varepsilon(\gamma) = \int_\gamma \varrho_\varepsilon ds$$

and where as usual the infimum is over all rectifiable curves in  $\Gamma$  joining  $x$  to  $y$ . The resulting metric space  $\Gamma_\varepsilon$  is incomplete. Its complement in its completion defines the boundary  $\partial_\varepsilon \Gamma$ .

If  $\varepsilon$  is sufficiently small, then the boundary  $\partial_\varepsilon \Gamma$  coincides with the set of the classes of asymptotic geodesic rays (in the metric  $d$ ), and is homeomorphic to  $S^2$ . More precisely, the natural map  $S^2 \rightarrow \partial_\varepsilon \Gamma$  given by  $\phi(z) = \lim U_n(z)$  where  $z \in U_n \in \mathcal{U}_n \subset \Gamma_\varepsilon$  is well-defined and a homeomorphism conjugating  $f$  on  $S^2$  to the map on  $\partial_\varepsilon \Gamma$  induced by the cellular map  $F$ .

The quasisymmetry type of  $d_\varepsilon$  is independent of choice of initial covering  $\mathcal{U}_0$  so long as Axiom [Expansion] is satisfied; it is clearly independent of the choice of  $\varepsilon$  as well (as soon as  $\varepsilon$  is small enough). Axiom [Degree] is satisfied if and only if  $d_\varepsilon$  is Ahlfors regular.

**Natural metric on  $S^2$ .** The conformal gauge  $\mathcal{G}(f)$  is then defined as the set of all metrics on  $S^2$  quasisymmetrically equivalent to the metric  $\phi^*(d_\varepsilon)$ , which we (abusing terminology) refer to as the *natural metric* on  $S^2$  obtained by pulling back  $d_\varepsilon$  under the conjugacy  $\phi$ . We remark that balls for the natural metric need not be connected.

It what follows, for convenience we denote by  $d_\varepsilon$  the pulled-back metric  $\phi^*(d_\varepsilon)$  on  $S^2$ .

**Theorem 4.1** *Let  $f : S^2 \rightarrow S^2$  be a topological cxc dynamical system with respect to an open covering  $\mathcal{V}_0$ . Then there exists a finite cover  $\mathcal{U}_0$  of  $S^2$  by Jordan domains such that the sequence  $\mathcal{U}_n$  of pulled-back covers, in the natural metric  $d_\varepsilon$  defined above, defines a sequence of uniform quasipackings.*

**Proof:** For convenience, equip  $S^2$  with the standard Euclidean spherical metric and denote the resulting metric space by  $\mathbb{S}^2$ . Then spherical balls  $D(x, r)$  are Jordan domains. For each  $x \in \mathbb{S}^2$ , consider an open ball  $U_x = D(x, r_x)$  centred at  $x$ . Choose  $n_0$  so large that no element of  $\mathcal{V}_n$ ,  $n \geq n_0$ , contains more than one critical value of  $f$ . By choosing  $r_x$  sufficiently small and sufficiently generic, we may arrange so that each  $U_x$  (i) is contained in some element of  $\mathcal{V}_{n_0}$ , and (ii) does not contain a critical value of an iterate of  $f$  on its boundary. It follows that every iterated preimage of  $U_x$  under  $f$  is a Jordan domain.

Let  $\mathcal{U}_0 = \{U_{x_j}\}_j$  be a finite subcover. From the  $5r$ -covering theorem, we may assume that  $\{D(x_j, r_j/5)\}_j$  are pairwise disjoint. Since we assumed that each disk in  $\mathcal{U}_0$  was contained in an element of  $\mathcal{V}_{n_0}$ , it follows that  $\mathcal{U}_0$  satisfies both [Expansion] and [Degree]. Furthermore, there is some  $r_0 > 0$  such that the collection  $\{B_\varepsilon(x_j, r_0)\}$  of balls in the natural metric  $d_\varepsilon$  is a disjointed family. The proof is completed by

appealing to the fact that with respect to the natural metric, iterates of  $f$  distort the roundness of elements of  $\mathbf{U} = \cup_n \mathcal{U}_n$  by controlled amounts. More precisely, it follows from [HP, Prop. 3.3.2] that there is a constant  $C > 0$  for which the following property holds. If  $\tilde{x} \in \tilde{U} \in \mathcal{U}_n$ ,  $f^n(\tilde{x}) = x_j$ ,  $f^n(\tilde{U}) = U_j$ , then

$$B_\varepsilon(\tilde{x}, (r_0/C)e^{-\varepsilon n}) \subset \tilde{U} \subset B_\varepsilon(\tilde{x}, Ce^{-\varepsilon n})$$

and

$$f^n(B_\varepsilon(\tilde{x}, (r_0/C)e^{-\varepsilon n})) \subset B_\varepsilon(x_j, r_0).$$

This implies that  $\{\mathcal{U}_n\}_n$  is a sequence of quasipackings by Jordan domains. ■

We note also that the quasipackings we have just constructed have uniformly bounded overlap by axiom [Degree].

## 5 Ahlfors-regular conformal dimension and multicurves

The original definition of ‘‘Thurston obstruction’’ has no minimality assumption and so  $\Gamma$  could contain curve families which do not comprise obstructions; cf. [Pil]. To clarify the pathology, we introduce the following notion. A multicurve  $\Gamma$  is called *irreducible* if some iterate  $f_{\Gamma, Q}^{o k}$  has all matrix entries positive; this is independent of  $Q$ .

For an irreducible obstruction, the Perron-Frobenius eigenvalue is positive, has geometric multiplicity one, and it is strictly larger than the norm of all other eigenvalues; the corresponding eigenvector is also strictly positive. A Thurston obstruction always contains an irreducible Thurston obstruction with the same Perron-Frobenius eigenvalue [Pil, §2]. When  $\#P_f < \infty$ , a noninvariant irreducible multicurve  $\Gamma$  can always be enlarged to an invariant multicurve  $\Gamma'$  such that  $\lambda(f_{\Gamma, Q}) \leq \lambda(f_{\Gamma', Q})$ .

Suppose  $f : S^2 \rightarrow S^2$  is topologically cxc. We choose  $\mathcal{U}_0$  as in Theorem 4.1 so that the sequence  $\{\mathcal{U}_n\}$  is a sequence of uniform quasipackings by Jordan domains with mesh going to zero. As coverings for the definition of combinatorial moduli, we take  $\mathcal{S}_n = \mathcal{U}_n$ .

**Proposition 5.1** *Let  $Q > 1$ , and let  $\Gamma$  be an invariant multicurve with  $\lambda(f_{\Gamma, Q}) \geq 1$ . Then  $\text{mod}_Q([\Gamma], \mathcal{U}_n) \gtrsim 1$ , where  $[\Gamma]$  denotes the family of curves homotopic to a curve in  $\Gamma$  rel.  $P_f$ .*

**Proof:** Without loss of generality we may assume  $\Gamma$  is irreducible. Write  $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_q\}$  and equip  $\mathbb{R}^\Gamma = \mathbb{R}^q$  with the  $|\cdot|_1$  norm, so that  $|\sum_i a_i \gamma_i|_1 = \sum_i |a_i|$ . For each  $1 \leq i \leq q$  choose an annulus  $B_i$  which is a regular neighborhood of  $\gamma_i$  such that  $B_i \cap B_j = \emptyset$ ,  $i \neq j$ . Within each  $B_i$  choose a smaller such neighborhood  $A_i$  so that  $\gamma \subset A_i \subset \bar{A}_i \subset B_i$  and each inclusion is essential. Since the mesh tends to zero, there exists a level  $n_0$  such that the covering  $\mathcal{U}_{n_0}$  has the following properties:

1.  $s \in \mathcal{U}_{n_0}$ ,  $s \cap A_i \neq \emptyset \Rightarrow s \subset B_i$ .
2.  $\text{mod}_Q(A_i, \mathcal{U}_{n_0}) > 0$  for all  $1 \leq i \leq q$ .

By the monotonicity and additivity of moduli (Proposition 2.2) we have that

$$\text{mod}_Q([\Gamma], \mathcal{U}_{n_0+k}) \geq \sum_i \text{mod}_Q(A_i, \mathcal{U}_{n_0+k}).$$

Let  $v \in \mathbb{R}^\Gamma$  be the vector of combinatorial moduli at level  $n_0$  given by

$$v = (\text{mod}_Q(A_1, \mathcal{U}_{n_0}), \dots, \text{mod}_Q(A_q, \mathcal{U}_{n_0})).$$

The definition of  $f_{\Gamma, Q}$ , Proposition 2.3, and an induction argument then shows that for  $k \geq 0$ ,

$$\text{mod}_Q([\Gamma], \mathcal{U}_{n_0+k}) \geq |f_{\Gamma, Q}^k(v)|_1.$$

We apply the next proposition to conclude the proof.

**Proposition 5.2** *If  $A$  is non-negative and irreducible with  $\lambda(A) \geq 1$  and  $v$  is a vector with positive entries then  $\liminf |A^k v|_1 > 0$ .*

**Proof:** We apply the Perron-Frobenius theorem. There is a unique leading eigenvalue  $\lambda$  which is a simple root of the characteristic polynomial of  $A$  and which is strictly greater in norm than the norm of all other eigenvalues. We study two cases.

1. Suppose  $\lambda(A) = 1$ .

Let  $u$  be the unique Perron-Frobenius eigenvector with eigenvalue  $\lambda(A) = 1$ , normalized so that  $|u|_1 = 1$ ; it has positive entries. Write  $v = au + w$  where  $w \in \oplus_i E_i$  and the  $E_i$ 's are the other generalized eigenspaces. Then  $A^k v = au + w_k$  where  $w_k = A^k w$ . Since all other eigenvalues are less than one in absolute value,  $w_k \rightarrow 0$  and so  $|A^k v|_1 \geq a - |w_k|_1 \geq a/2 > 0$  for all  $k$  sufficiently large.

2. Suppose  $\lambda(A) > 1$ . By case 1 applied to  $\lambda(A)^{-1}A$ ,

$$\liminf |\lambda(A)^{-k} A^k v|_1 > 0 \Rightarrow \liminf |A^k v|_1 > 0.$$

■

We conclude with the proof of Theorem 1.5.

**Proof:** The Ahlfors regular conformal dimension is bounded below by the topological dimension, so  $\text{confdim}_{AR}(f) \geq 2$ .

By Theorem 4.1, there is an Ahlfors regular metric  $d_\varepsilon \in \mathcal{G}(f)$  on  $S^2$  for which the coverings  $\mathcal{U}_n$  form a uniform family of quasipackings.

Let  $\Gamma$  be an irreducible multicurve. By Proposition 5.1,  $\text{mod}_{Q(\Gamma)}([\Gamma], \mathcal{U}_n) \gtrsim 1$ . Thus, Corollary 3.3 implies that  $\text{confdim}_{AR}(S^2, d_\varepsilon) \geq Q(\Gamma)$  and so  $\text{confdim}_{AR}(f) \geq Q(\Gamma)$ . Since this holds for all irreducible multicurves, one gets

$$\text{confdim}_{AR}(f) \geq Q(f).$$

■

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