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Maximal slope of tensor product of Hermitian vector bundles

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Abstract

We give an upper bound for the maximal slope of the tensor product of several non-zero Hermitian vector bundles on the spectrum of an algebraic integer ring. By Minkowski's theorem, we need to estimate the Arakelov degree of an arbitrary Hermitian line subbundle \overline{M} of the tensor product. In the case where the generic fiber of M is semistable in the sense of geometric invariant theory, the estimation is established by constructing, through the classical invariant theory, a special polynomial which does not vanish on the generic fibre of M . Otherwise we use an explicit version of a result of Ramanan and Ramanathan to reduce the general case to the former one.

1 Introduction

It is well known that on a projective and smooth curve over a field of characteristic 0, the tensor product of two *semistable* vector bundles is still semistable. This result has been firstly proved by Narasimhan and Seshadri [NS65] in the complex algebraic geometry framework by using analytic method. Then it has been reestablished by Ramanan and Ramanathan [RR84] in the purely algebraic context, through the geometric invariant theory. Their method is based on a result of Kempf [Kem78], which has also been independently obtained by Rousseau [Rou78], generalizing the Hilbert-Mumford criterion [MFK94] of semi-stability in the sense of geometric invariant theory. By reformulating the results of Kempf and Ramanan-Ramanathan, Totaro [Tot96] (see also [dS] for a review) has given a new proof of a conjecture due to Fontaine [Fon79], which had been firstly proved by Faltings [Fal89] asserting that the tensor product of two semistable weakly admissible filtered isocrystals is still semistable.

Let us go back to the case of vector bundles. The fact that the tensor product of two semistable vector bundles on a curve C over a field k of characteristic 0 is still semistable is equivalent to the fact that for any non-zero vector bundles E and F on C , we have $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$, where the maximal slope μ_{\max} of a non-zero vector bundle G over C is by definition the maximal value of slopes (that is to say, the quotient of the degree by the rank) of non-zero subbundles of G . When k is of characteristic positive, the above equality is not true in general (see [Gie73] for a counter-example). Nevertheless, it is easy to prove that there exists a constant a which depends only on C such that

$$\mu_{\max}(E) + \mu_{\max}(F) \leq \mu_{\max}(E \otimes F) \leq \mu_{\max}(E) + \mu_{\max}(F) + a \quad (1)$$

for any non-zero vector bundles E and F on C .

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Hermitian vector bundles play in Arakelov geometry the role of vector bundles in algebraic geometry. Let K be a number field and \mathcal{O}_K be its integer ring. A Hermitian vector bundle $\overline{E} = (E, h)$ on $\text{Spec } \mathcal{O}_K$ is by definition a projective \mathcal{O}_K -module of finite type E together with a family of Hermitian metric $h = (\|\cdot\|_\sigma)_{\sigma \in \Sigma_\infty}$, where Σ_∞ is the set of all embedding of K into \mathbb{C} , and for any $\sigma \in \Sigma_\infty$, $\|\cdot\|_\sigma$ is a Hermitian norm on $E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$, subject to the condition that the data $(\|\cdot\|_\sigma)_{\sigma \in \Sigma_\infty}$ is invariant by the complex conjugation, namely for any $e \in E$, any $z \in \mathbb{C}$ and any $\sigma \in \Sigma_\infty$, $\|e \otimes \bar{z}\|_\sigma = \|e \otimes z\|_\sigma$. Similarly to the geometric case, one can define the (normalized) *Arakelov degree* of a Hermitian vector bundle \overline{E} on $\text{Spec } \mathcal{O}_K$ as

$$\widehat{\deg}_n \overline{E} = \frac{1}{[K : \mathbb{Q}]} \left(\log \#(E/\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r) - \frac{1}{2} \sum_{\sigma \in \Sigma_\infty} \log \det(\langle s_i, s_j \rangle_\sigma) \right),$$

where $(s_1, \dots, s_r) \in E^r$ is an arbitrary element in E^r which defines a base of E_K over K . This definition does not depend on the choice of (s_1, \dots, s_r) . For more details, see [Bos96], [Bos01], [CL02], [Bos04] and [BK07]. The *slope* of a non-zero Hermitian vector bundle \overline{E} on $\text{Spec } \mathcal{O}_K$ is then defined to be the quotient $\widehat{\mu}(\overline{E}) := \widehat{\deg}_n(\overline{E})/\text{rk}(E)$. We say that a non-zero Hermitian vector bundle \overline{E} is *semistable* if the *maximal slope* $\widehat{\mu}_{\max}(\overline{E})$ of \overline{E} , defined as the maximal value of slopes of its non-zero Hermitian subbundles, equals to its slope. If \overline{E} is a non-zero Hermitian vector bundle on $\text{Spec } \mathcal{O}_K$, Stuhler [Stu76] and Grayson [Gra76] have proved that there exists a unique Hermitian subbundle $\overline{E}_{\text{des}}$ of \overline{E} having $\widehat{\mu}_{\max}(\overline{E})$ as its slope, and containing all Hermitian subbundle \overline{F} of \overline{E} such that $\widehat{\mu}(\overline{F}) = \widehat{\mu}_{\max}(\overline{E})$. Clearly \overline{E} is semistable if and only if $\overline{E} = \overline{E}_{\text{des}}$. If it is **not** the case, then $\overline{E}_{\text{des}}$ is said to be the Hermitian subbundle which *destabilize* \overline{E} . In a lecture at Oberwolfach, J.-B. Bost [Bos97] conjectured that the tensor product of two semistable Hermitian vector bundles on the spectrum of an algebraic integer ring is semistable. This conjecture is equivalent to the assertion that for any non-zero Hermitian vector bundle \overline{E} and \overline{F} on $\text{Spec } \mathcal{O}_K$,

$$\widehat{\mu}_{\max}(\overline{E} \otimes \overline{F}) = \widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}).$$

It is clear that we always have the inequality $\widehat{\mu}_{\max}(\overline{E} \otimes \overline{F}) \geq \widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F})$ since

$$\widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}) = \widehat{\mu}(\overline{E}_{\text{des}}) + \widehat{\mu}(\overline{F}_{\text{des}}) = \widehat{\mu}(\overline{E}_{\text{des}} \otimes \overline{F}_{\text{des}}) \leq \widehat{\mu}_{\max}(\overline{E} \otimes \overline{F}).$$

But the inverse inequality remains always open. Several special cases of the conjecture have been proved and some estimations of type (1) have been obtained with error terms depending on the ranks of the vector bundles and on the number field K . In the following, we resume some known results on this conjecture.

- 1) De Shalit and Parzanovski [dSP06] have proved that if \overline{E} and \overline{F} are two semistable Hermitian vector bundles on $\text{Spec } \mathbb{Z}$ such that $\text{rk } E + \text{rk } F \leq 5$, then $\overline{E} \otimes \overline{F}$ is semistable.
- 2) In [Bos96] (see also [Gra00]), using the comparison of a Hermitian vector bundle to a direct sum of Hermitian line bundles, J.-B. Bost has proved that

$$\widehat{\mu}_{\max}(\overline{E}_1 \otimes \cdots \otimes \overline{E}_n) \leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(\overline{E}_i) + 3 \text{rk } E_i \log(\text{rk } E_i) \right)$$

for any family of non-zero Hermitian vector bundles $(\overline{E}_i)_{i=1}^n$ on the spectrum of an algebraic integer ring.

3) Recently J.-B. Bost and K. Künnemann [BK07] have proved that, if K is a number field and \overline{E} and \overline{F} are two non-zero Hermitian vector bundles on $\text{Spec } \mathcal{O}_K$, then

$$\widehat{\mu}_{\max}(\overline{E} \otimes \overline{F}) \leq \widehat{\mu}_{\max}(\overline{E}) + \widehat{\mu}_{\max}(\overline{F}) + \frac{1}{2}(\log \text{rk } E + \log \text{rk } F) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]},$$

where Δ_K is the discriminant of K .

We state the main result of this article as follows:

Theorem 1.1 *Let K be a number field and \mathcal{O}_K be its integer ring. If $(\overline{E}_i)_{i=1}^n$ is a family of non-zero Hermitian vector bundles on $\text{Spec } \mathcal{O}_K$, then*

$$\widehat{\mu}_{\max}(\overline{E}_1 \otimes \cdots \otimes \overline{E}_n) \leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(\overline{E}_i) + \log(\text{rk } E_i) \right). \quad (2)$$

The idea goes back to an article of J.-B. Bost [Bos94] inspired by Bogomolov [Ray81], Gieseker [Gie77] and Cornalba-Harris [CH88]. In an article of Gasbarri [Gas00] appears also the similar idea. After the first theorem of Minkowski, we reduce our problem to finding an upper bound for the Arakelov degree of an arbitrary hermitien line subbundle \overline{M} of $\overline{E}_1 \otimes \cdots \otimes \overline{E}_n$. In the case where M_K is semistable (in the sense of geometric invariant theory) for the action of $\text{GL}(E_{1,K}) \times \cdots \times \text{GL}(E_{n,K})$, the classical invariant theory gives invariant polynomials with coefficients in \mathbb{Z} whose Archimedean norms are “small”. The general case can be reduced to the former one using an explicit version of a result of Ramanan-Ramanathan [RR84].

The structure of this article is as follows. In the second section we recall the first principal theorem in classical invariant theory and discuss some generalizations in several vector spaces case. We then establish in the third section an upper bound for the Arakelov degree of a Hermitian line subbundle with semistable hypothesis. In the fourth section, we give a criterion of semi-stability (for Hermitian vector bundles) which is an arithmetic analogue of a result of Bogomolov. We recall in the fifth section some basic notions for filtrations in the category of vector spaces. Then in the sixth section, we state an explicit version of a result of Ramanan-Ramanathan in our context and, following the methode of Totaro, give a proof for it. In the seventh section, we explain how to use the result in previous section to reduce the majoration of the Arakelov degree of an arbitrary Hermitian line subbundle to the case with semistability hypothesis, which has already been discussed in the third section. Finally, in the eighth and the last section, we give the proof of Theorem 1.1.

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2 Recall on invariant theory

In this section we recall some known results in classical invariant theory. We fix K to be a field of characteristic 0.

For any finite dimensional vector space V over K and any strictly positive integer u , we denote by $J_u : \text{End}_K(V)^{\otimes u} \rightarrow \text{End}_K(V^{\otimes u})$ the K -linear homomorphism (of vector spaces)

which sends the tensor product $T_1 \otimes \cdots \otimes T_u$ of u elements in $\text{End}_K(V)$ to their tensor product as endomorphism of $V^{\otimes u}$. The mapping J_u is actually a homomorphism of K -algebras. Furthermore, as homomorphism of vector spaces, J_u can be written as the composition of following natural isomorphisms:

$$\text{End}_K(V)^{\otimes u} \longrightarrow (V^\vee \otimes V)^{\otimes u} \longrightarrow (V^\vee)^{\otimes u} \otimes V^{\otimes u} \longrightarrow (V^{\otimes u})^\vee \otimes V^{\otimes u} \longrightarrow \text{End}_K(V^{\otimes u}),$$

so is itself an isomorphism. Moreover, there exists an action of the symmetric group \mathfrak{S}_u on $V^{\otimes u}$ by permuting the factors. This representation of \mathfrak{S}_u defines a homomorphism from the group algebra $K[\mathfrak{S}_u]$ to $\text{End}_K(V^{\otimes u})$. The elements of \mathfrak{S}_u acts by conjugation on $\text{End}_K(V^{\otimes u})$. If we identify $\text{End}_K(V^{\otimes u})$ to $\text{End}_K(V)^{\otimes u}$ by the isomorphism J_u , then the corresponding \mathfrak{S}_u -action is just the permutation of factors in tensor product. Finally the group $\text{GL}_K(V)$ acts naturally on $V^{\otimes u}$.

Lemma 2.1 *If we denote by $\rho : \text{GL}_K(V) \rightarrow \text{End}_K(V^{\otimes u})$ the natural representation of $\text{GL}_K(V)$ on $V^{\otimes u}$, then the \mathfrak{S}_u -invariant subalgebra $\text{End}_K(V^{\otimes u})^{\mathfrak{S}_u}$ is generated by the image of ρ .*

Proof. The action of an arbitrary element in \mathfrak{S}_u by permuting factors and the action of an arbitrary element in $\text{GL}_K(V)$ on $V^{\otimes u}$ commute. Hence $\text{End}_K(V^{\otimes u})^{\mathfrak{S}_u}$ contains the subalgebra of $\text{End}_K(V^{\otimes u})$ generated by the image of ρ . Therefore it suffices to prove the inverse inclusion. As K is of characteristic 0, $\text{End}_K(V^{\otimes u})^{\mathfrak{S}_u} \cong (\text{End}_K(V)^{\otimes u})^{\mathfrak{S}_u} \cong S^u \text{End}_K(V)$. Since $S^u \text{End}_K(V)$ is generated as vector space over K by the elements of the form $g^{\otimes u}$ ($g \in \text{End}_K(V)$), it is also generated by $g^{\otimes u}$ ($g \in \text{GL}_K(V)$) since $\text{GL}_K(V)$ is Zariski dense in $\text{End}_K(V)$ (K being of characteristic 0). \square

We recall below the “first principal theorem” of classical invariant theory (cf. [Wey97] Chapter III, see also [ABP73] Appendix 1).

Theorem 2.2 *Let V be a finite dimensional vector space over K and let $u, v \geq 1$ be two integers. If T is a non-zero elements in $V^{\otimes u} \otimes V^{\vee \otimes v}$, which is invariant under the action of $\text{GL}_K(V)$, then $u = v$, and T is a linear combination of permutations (here we identify $V^{\otimes u} \otimes V^{\vee \otimes u}$ with $\text{End}_K(V^{\otimes u})$).*

Proof. Let $a \in K^\times$ be an element which is not root of unity (this is always possible since K is of characteristic 0) and let $g = a \text{Id}_V \in \text{GL}_K(V)$. The action of g on $V^{\otimes u} \otimes V^{\vee \otimes v}$ is $a^{u-v} \text{Id}$. If there exists a non-zero vector in $V^{\otimes u} \otimes V^{\vee \otimes v}$ which is invariant by the action of $\text{GL}_K(V)$, then $a^{u-v} = 1$. Therefore $u = v$ since a is not root of unity.

Let $A = \text{End}_K(V^{\otimes u})$ and let B be the image of $K[\mathfrak{S}_u]$ in A by the representation of permutations, which is a subalgebra of A . The K -algebra A is simple, central and of finite rank over K . the sub- K -algebra B is semistable. After Lemma 2.1, the commutant $B' = \text{End}_K(V^{\otimes u})^{\mathfrak{S}_u}$ of B is the subalgebra of A generated by the image of the representation $\rho : \text{GL}_K(V) \rightarrow \text{End}_K(V^{\otimes u})$. By the bicommutant theorem (see for example [Jac89], [Lan02] or [Pie82]), we have $B'' = B$. Therefore, the endomorphisms of $V^{\otimes u}$ which are invariant by the action of $\text{GL}_K(V)$ lie in B , i.e., are linear combination of permutations. \square

Let $(V_i)_{1 \leq i \leq n}$ be a family of finite dimensional non-zero vector space over K . Let $\mathbf{u} = (u_i)_{1 \leq i \leq n}$ be a family of strictly positive integers. We denote by G the group $\text{GL}_K(V_1) \times \cdots \times \text{GL}_K(V_n)$ and by $\mathfrak{S}_{\mathbf{u}}$ the product $\mathfrak{S}_{u_1} \times \cdots \times \mathfrak{S}_{u_n}$ of symmetric groups. The group algebra $K[\mathfrak{S}_{\mathbf{u}}]$ is isomorphic to $K[\mathfrak{S}_{u_1}] \otimes_K \cdots \otimes_K K[\mathfrak{S}_{u_n}]$. On the other hand, we have a natural isomorphism of K -algebras from $\text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})$ to $\text{End}_K(V_1)^{\otimes u_1} \otimes_K \cdots \otimes_K \text{End}_K(V_n)^{\otimes u_n}$. The group G acts naturally on $V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n}$. We denote by

$\rho : G \rightarrow \text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})$ the corresponding representation. Finally the group $\mathfrak{S}_{\mathbf{u}}$ acts on $V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n}$ by permutating tensor factors.

Lemma 2.3 *The algebra $\text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})^{\mathfrak{S}_{\mathbf{u}}}$ is generated by the image of ρ .*

Proof. For any integer $1 \leq i \leq n$, we denote by C_i the canonical image of $K[\mathfrak{S}_{u_i}]$ in $\text{End}_K(V_i^{\otimes u_i})$. The algebra $\text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})^{\mathfrak{S}_{\mathbf{u}}}$ is the commutant of the canonical image of $K[\mathfrak{S}_{\mathbf{u}}]$ in $\text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n}) \cong \text{End}_K(V_1^{\otimes u_1}) \otimes \cdots \otimes \text{End}_K(V_n^{\otimes u_n})$, which is identified with $C_1 \otimes \cdots \otimes C_n$. As the commutant of tensor product equals to the tensor product of commutants, we have

$$\text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})^{\mathfrak{S}_{\mathbf{u}}} \cong C'_1 \otimes \cdots \otimes C'_n,$$

where C'_i is the commutant of C_i . After Lemma 2.1, the sub- K -algebra C'_i of $\text{End}_K(V_i^{\otimes u_i})$ is generated by the image of $\text{GL}_K(V_i^{\otimes u_i})$. Therefore $C'_1 \otimes \cdots \otimes C'_n$ is generated by the image of G . \square

Theorem 2.4 *With the notations above, if $\mathbf{v} = (v_i)_{1 \leq i \leq n}$ is a family of strictly positive integers and if T is a non-zero element in $U := V_1^{\otimes u_1} \otimes V_1^{\vee \otimes v_1} \otimes \cdots \otimes V_n^{\otimes u_n} \otimes V_n^{\vee \otimes v_n}$ which is invariant by the action of G , then $\mathbf{u} = \mathbf{v}$, and T is a linear combination of elements in $\mathfrak{S}_{\mathbf{u}}$.*

Proof. Let $a \in K^\times$ be an element which is not root of unity. For any integer $1 \leq i \leq n$, the action of

$$\text{Id}_{V_1} \times \cdots \times \text{Id}_{V_{i-1}} \times a \text{Id}_{V_i} \times \text{Id}_{V_{i+1}} \times \cdots \times \text{Id}_{V_n}$$

on U is the homothetic transformation by $a^{u_i - v_i}$. As U has a non-zero vector invariant by G , we know that $a^{u_i - v_i} = 1$ and hence $u_i = v_i$. Therefore we can identify U with the underlying vector space of $A := \text{End}_K(V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n})$. Let B be the canonical image of $K[\mathfrak{S}_{\mathbf{u}}]$ by the representation of permutations, which is a sub- K -algebra of A . The algebra A is simple, central and of finite rank over K . The subalgebra B of A is semi-simple since it is a quotient of the algebra of a finite group. After the bicommutant theorem, we have $B'' = B$. Hence the endomorphisms in $V_1^{\otimes u_1} \otimes \cdots \otimes V_n^{\otimes u_n}$ which are invariant by the action of G lie in B , so are linear combinations of elements in $\mathfrak{S}_{\mathbf{u}}$. \square

Let G be an algebraic group over $\text{Spec } K$ and X be a projective variety over $\text{Spec } K$. Suppose that G acts on X and that L is an ample G -linear line bundle on X . We say that a rational point x of X is *semistable* for the action of G relatively to L if there exists an integer $D \geq 1$ and a section $s \in H^0(X, L^{\otimes D})$ invariant under the action of G such that x lies in the open subset of X defined by the non-vanishing of s . Clearly x is semistable for the action of G relatively to L if and only if it is semistable for the action of G relatively to any strictly positive tensor power of L .

Let $(V_i)_{1 \leq i \leq n}$ be a family of finite dimensional non-zero vector space over K , Ξ be a finite family of non-identically zero mappings from $\{1, \dots, n\}$ to \mathbb{N} and $(b_i)_{1 \leq i \leq n}$ be a family of integers. We denote by W the vector space $\bigoplus_{\alpha \in \Xi} \bigotimes_{i=1}^n V_i^{\otimes \alpha(i)}$ and by G the algebraic group $\text{GL}_K(V_1) \times_K \cdots \times_K \text{GL}_K(V_n)$. The group G acts naturally on W . Let L be the $G(K)$ -module $(\det V_1)^{\otimes b_1} \otimes \cdots \otimes (\det V_n)^{\otimes b_n}$. For any integer $D \geq 1$ and any element $\alpha = (\alpha_j)_{1 \leq j \leq D} \in \Xi^D$, let

$$\text{pr}_\alpha : W^{\otimes D} \longrightarrow V_1^{\otimes (\alpha_1(1) + \cdots + \alpha_D(1))} \otimes \cdots \otimes V_n^{\otimes (\alpha_1(n) + \cdots + \alpha_D(n))}$$

be the canonical projection. For any integer $1 \leq i \leq n$, let r_i be the rank of V_i over K . Finally let $\pi : \mathbb{P}(W^\vee) \rightarrow \text{Spec } K$ be the canonical morphism.

Theorem 2.5 *With the notations above, if m is a strictly positive integer and if R is a vector space of rank 1 of W (considered as a rational point of $\mathbb{P}(W^\vee)$) which is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^*L$, then there exists an integer $D \geq 1$ and a family $\alpha = (\alpha_j)_{1 \leq j \leq mD}$ of elements in Ξ such that, by noting $A = \alpha_1 + \dots + \alpha_{mD}$, one has $A(i) = Db_i r_i$ for any $1 \leq i \leq n$ (this implies in particular that $b_i \geq 0$ for any $1 \leq i \leq n$). Furthermore, there exists an element $\sigma \in \mathfrak{S}_{\alpha(1)} \times \dots \times \mathfrak{S}_{\alpha(n)}$ such that the composition of homomorphisms*

$$\begin{array}{ccc}
R^{\otimes mD} \otimes L^{\vee \otimes D} & \longrightarrow & W^{\otimes mD} \otimes L^{\vee \otimes D} \xrightarrow{\text{pr}_\alpha \otimes \text{Id}} V_1^{\otimes A(1)} \otimes \dots \otimes V_n^{\otimes A(n)} \otimes L^{\vee \otimes D} \\
& & \downarrow \sigma \otimes \text{Id} \\
& & V_1^{\otimes A(1)} \otimes \dots \otimes V_n^{\otimes A(n)} \otimes L^{\vee \otimes D} \\
& & \downarrow \det_{V_1}^{\otimes Db_1} \otimes \dots \otimes \det_{V_n}^{\otimes Db_n} \otimes \text{Id} \\
& & L^{\otimes D} \otimes L^{\vee \otimes D} \cong K
\end{array}$$

does not vanish, where the first arrow is induced by the canonical inclusion of $R^{\otimes mD}$ in $W^{\otimes mD}$.

Proof. Since R is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^*L$, there exists an integer $D \geq 1$ and an element $s \in S^D(W^\vee) \otimes L^{\otimes D}$ which is invariant by the action of $G(K)$ such that the composition of homomorphisms

$$R^{\otimes mD} \otimes L^{\vee \otimes D} \longrightarrow \Gamma^{mD}(W) \otimes L^{\vee \otimes D} \xrightarrow{s} K$$

does not vanish, the first arrow being the canonical inclusion. Since $K[G(K)]$ is semisimple, $S^{mD}(W^\vee) \otimes L^{\otimes D}$ is a direct factor (as $G(K)$ -module) of $(W^\vee)^{\otimes mD} \otimes L^{\otimes D}$. Let $s' \in W^{\vee \otimes mD} \otimes L^{\otimes D}$ be an element invariant by the action of $G(K)$, whose class in $S^{mD}(W^\vee) \otimes L^{\otimes D}$ coincides with s . There then exists $\alpha = (\alpha_j)_{1 \leq j \leq mD} \in \Xi^D$ such that the composition

$$R^{\otimes mD} \otimes L^{\vee \otimes D} \longrightarrow W^{\otimes mD} \otimes L^{\vee \otimes D} \xrightarrow{\text{pr}_\alpha \otimes \text{Id}} V_1^{\otimes A(1)} \otimes \dots \otimes V_n^{\otimes A(n)} \otimes L^{\vee \otimes D} \xrightarrow{s'_\alpha} K$$

is non-zero, where $A = \alpha_1 + \dots + \alpha_{mD}$ and s'_α is the component of index α of s' . Choose a preimage s''_α of s'_α in $V_1^{\vee \otimes A(1)} \otimes \dots \otimes V_n^{\vee \otimes A(n)} \otimes V_1^{\otimes Db_1 r_1} \otimes \dots \otimes V_n^{\otimes Db_n r_n}$ which is invariant by $G(K)$. After Theorem 2.4, we obtain that $A(i) = Db_i r_i$ for any $1 \leq i \leq n$, and that s''_α is a linear combination of permutations. Therefore the theorem is proved. \square

3 Upper bound for the degree of a Hermitian line subbundle with hypothesis of semi-stability

Let K be a number field and \mathcal{O}_K be its integer ring. Consider a family $(\overline{E}_i)_{1 \leq i \leq n}$ of non-zero Hermitian vector bundles on $\text{Spec } \mathcal{O}_K$. Let Ξ be a non-empty and finite family of non-identically zero mappings from $\{1, \dots, n\}$ to \mathbb{N} . We define a new Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ as follows

$$\overline{E} := \bigoplus_{\alpha \in \Xi} \overline{E}_1^{\otimes \alpha(1)} \otimes \dots \otimes \overline{E}_n^{\otimes \alpha(n)}.$$

In this section, we shall use the ideas in [Bos94] to obtain an upper bound for the Arakelov degree of a Hermitian line subbundle \overline{M} of \overline{E} under the hypothesis of semi-stability (in geometric invariant theory sense) for M_K . This upper bound is crucial because, as we shall see

later, the general case can be reduced to this special one through an argument of Ramanan and Ramanathan.

For any integer $1 \leq i \leq n$, let r_i be the rank of E_i and let V_i be the vector space $E_{i,K}$. Let $W = E_K$ and $\pi : \mathbb{P}(W^\vee) \rightarrow \text{Spec } K$ be the canonical morphism. We denote by G the algebraic group $\mathbb{G}\mathbb{L}_K(V_1) \times \cdots \times \mathbb{G}\mathbb{L}_K(V_n)$ which acts naturally on $\mathbb{P}(W^\vee)$. Clearly we have $W = \bigoplus_{\alpha \in \Xi} \bigotimes_{i=1}^n V_i^{\alpha(i)}$. Let $(b_i)_{1 \leq i \leq n}$ be a family of strictly positive integers such that r_i divides b_i . Finally let

$$\bar{L} = (\Lambda^{r_1} \bar{E}_1)^{\otimes b_1/r_1} \otimes \cdots \otimes (\Lambda^{r_n} \bar{E}_n)^{\otimes b_n/r_n}.$$

Lemma 3.1 *Let H be a Hermitian space of dimension $d > 0$. The norm of the homomorphism $\det : H^{\otimes d} \rightarrow \Lambda^d H$ equals to $\sqrt{d!}$.*

Proof. In fact, the homomorphism \det can be written as $d!$ orthogonal vectors of norm 1 in $H^{\vee \otimes d}$. So its norm is $\sqrt{d!}$. \square

Theorem 3.2 *With the notations above, if $m \geq 1$ is an integer and if \bar{M} is a Hermitian line subbundle of \bar{E} such that M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^* L_K$, then*

$$\widehat{\deg}(\bar{M}) \leq \frac{1}{m} \widehat{\deg}(\bar{L}) + \frac{1}{2m} \sum_{i=1}^n b_i \log(\text{rk } E_i) = \sum_{i=1}^n \frac{b_i}{m} \left(\widehat{\mu}(\bar{E}_i) + \frac{1}{2} \log(\text{rk } E_i) \right).$$

Proof. Let $R = M_K$. With the notations of Theorem 2.5, using the slope inequality (cf. [Bos96] Appendix) and Lemma 3.1, we get

$$\begin{aligned} mD\widehat{\deg}(\bar{M}) - D\widehat{\deg}(\bar{L}) &= mD\widehat{\deg}(\bar{M}) - \sum_{i=1}^n Db_i \widehat{\mu}(\bar{E}_i) \\ &\leq \sum_{i=1}^n \frac{A(i) \log(r_i!)}{2r_i} = \sum_{i=1}^n \frac{Db_i \log(r_i!)}{2r_i} \leq \frac{1}{2} D \sum_{i=1}^n b_i \log r_i, \end{aligned}$$

where we have used the evident estimation $r! \leq r^r$ in the last inequality. If we divide the inequality by mD , we obtain

$$\widehat{\deg}(\bar{M}) \leq \frac{1}{m} \widehat{\deg}(\bar{L}) + \frac{1}{2m} \sum_{i=1}^n b_i \log r_i = \sum_{i=1}^n \frac{b_i}{m} \left(\widehat{\mu}(\bar{E}_i) + \frac{\log r_i}{2} \right).$$

\square

Let m be a strictly positive integer which is divisible by all r_i . If we apply Theorem 3.2 to the special case where Ξ contains a single mapping α such that $\alpha(i) = 1$ for any $i \in \{1, \dots, n\}$, in other words, $\bar{E} = \bar{E}_1 \otimes \cdots \otimes \bar{E}_n$, and where $b_i = m$ for any $1 \leq i \leq n$, we get the following upper bound:

Corollary 3.3 *If \bar{M} is a Hermitian line subbundle of $\bar{E}_1 \otimes \cdots \otimes \bar{E}_n$ such that M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^* L_K$, then we have*

$$\widehat{\deg}(\bar{M}) \leq \sum_{i=1}^n \left(\widehat{\mu}(\bar{E}_i) + \frac{1}{2} \log(\text{rk } E_i) \right). \quad (3)$$

4 A criterion of semi-stability (for Hermitian vector bundles)

We shall give a semi-stability criterion for Hermitian vector bundles, which is the arithmetic analogue of a result due to Bogomolov in geometric framework (see [Ray81]).

Let \bar{E} be a non-zero Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ and let $V = E_K$. We denote by r its rank. If $\mathcal{D} : V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_d = 0$ is a flag of V , it induces a strictly decreasing sequence of saturated sub- \mathcal{O}_K -modules $E = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_d = 0$ of E . For any integer $0 \leq j < d$, let r_j be the rank of E_j/E_{j+1} . If $\mathbf{a} = (a_j)_{0 \leq j < d}$ is an element in $r\mathbb{Z}^d$, we denote by $\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}$ the Hermitian line bundle (on $\text{Spec } \mathcal{O}_K$)

$$\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}} := \bigotimes_{j=0}^{d-1} \left((\Lambda^{r_j}(\bar{E}_j/\bar{E}_{j+1}))^{\vee \otimes a_j} \otimes (\Lambda^r \bar{E})^{\otimes \frac{r_j a_j}{r}} \right).$$

Proposition 4.1 *The Hermitian vector bundle \bar{E} is semistable (resp. stable) if and only if for any integer $d \geq 1$, any flag \mathcal{D} of length d of V , and any strictly increasing sequence $\mathbf{a} = (a_j)_{0 \leq j < d}$ of integers in $r\mathbb{Z}^d$, one has $\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) \geq 0$ (resp. $\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) > 0$).*

Proof. “ \implies ”: By definition, one has

$$\begin{aligned} \widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) &= - \sum_{j=0}^{d-1} a_j \left[- \frac{\text{rk}(E_j) - \text{rk}(E_{j+1})}{r} \widehat{\deg}(\bar{E}) + \widehat{\deg}(\bar{E}_j) - \widehat{\deg}(\bar{E}_{j+1}) \right] \\ &= - \sum_{j=0}^{d-1} a_j \left[\text{rk}(E_j) (\widehat{\mu}(\bar{E}_j) - \widehat{\mu}(\bar{E})) - \text{rk}(E_{j+1}) (\widehat{\mu}(\bar{E}_{j+1}) - \widehat{\mu}(\bar{E})) \right] \\ &= - \sum_{j=1}^{d-1} (a_j - a_{j-1}) \text{rk}(E_j) (\widehat{\mu}(\bar{E}_j) - \widehat{\mu}(\bar{E})). \end{aligned}$$

If \bar{E} is semistable (resp. stable), then for any integer $1 \leq j < d$, one has $\widehat{\mu}(\bar{E}_j) \leq \widehat{\mu}(\bar{E})$ (resp. $\widehat{\mu}(\bar{E}_j) < \widehat{\mu}(\bar{E})$). Hence $\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) \geq 0$ (resp. $\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) > 0$).

“ \impliedby ”: Let E_1 be a saturated sub- \mathcal{O}_K -module of E . Consider the flag $\mathcal{D} : V \supseteq E_{1,K} \supseteq 0$ and the integer sequence $\mathbf{a} = (0, r)$. One has

$$\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) = r \text{rk}(E_1) (\widehat{\mu}(\bar{E}) - \widehat{\mu}(\bar{E}_1)) \geq 0 \quad (\text{resp. } \widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) = r \text{rk}(E_1) (\widehat{\mu}(\bar{E}) - \widehat{\mu}(\bar{E}_1)) > 0).$$

Therefore $\widehat{\mu}(\bar{E}_1) \leq \widehat{\mu}(\bar{E})$ (resp. $\widehat{\mu}(\bar{E}_1) < \widehat{\mu}(\bar{E})$). As E_1 is arbitrary, the Hermitian vector bundle \bar{E} is semistable (resp. stable). \square

Remark 4.2 If $\mathbf{a} = (a_j)_{0 \leq j < d}$ is a strictly increasing sequence of integers (not necessarily in $r\mathbb{Z}^d$) satisfying $\sum_{j=0}^{d-1} r_j a_j = 0$, then $\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}$ is well defined and equals to $\bigotimes_{j=0}^{d-1} (\Lambda^{r_j}(\bar{E}_j/\bar{E}_{j+1}))^{\vee \otimes a_j}$. The semi-stability of \bar{E} implies also $\widehat{\deg}(\overline{\mathcal{L}}_{\mathcal{D}}^{\mathbf{a}}) \geq 0$.

5 Filtrations of vector spaces

In this section, we introduce some basic notations and results on \mathbb{R} -filtrations of vector spaces, which we shall use in the sequel (see [Che06] or [Che07] for filtrations in general categories).

Let K be a (commutative) field and V be a vector space of finite rank over K . We call (decreasing) \mathbb{R} -filtration of V any family $\mathcal{F} = (\mathcal{F}_\lambda V)_{\lambda \in \mathbb{R}}$ of subspaces of V such that $\mathcal{F}_\lambda V \supset \mathcal{F}_{\lambda'} V$ for $\lambda \leq \lambda'$. In this article, we suppose that all filtrations are *separated* (i.e., $\mathcal{F}_\lambda V = 0$ for λ sufficiently positive), *exhaustive* (i.e., $\mathcal{F}_\lambda V = V$ for λ sufficiently negative) and *left continuous* (i.e., the function $x \mapsto \text{rk}_K(\mathcal{F}_x V)$ on \mathbb{R} is left continuous). A filtration \mathcal{F} of V is equivalent to the data of a flag

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_d = 0$$

of V together with a strictly increasing sequence of real numbers $(\lambda_i)_{0 \leq i < d}$. In fact, we have the relation $\mathcal{F}_\lambda V = \bigcup_{\lambda_i \geq \lambda} V_i$. We denote by \mathbf{Fil}_V the set of all filtrations of V . If \mathcal{F} is a filtration of V , we define a function $\lambda_{\mathcal{F}} : V \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\lambda_{\mathcal{F}}(x) = \sup\{a \in \mathbb{R} \mid x \in \mathcal{F}_a V\}$. The function $\lambda_{\mathcal{F}}$ takes finite value unless at $x = 0$.

Suppose that V is non-zero. If \mathcal{F} is a filtration of V , we denote by $\nu_{\mathcal{F}}$ and we call the *probability measure associated to \mathcal{F}* the unique Borel measure on \mathbb{R} such that

$$\nu_{\mathcal{F}}([x, y]) = \frac{\text{rk}_K(\mathcal{F}_y V) - \text{rk}_K(\mathcal{F}_x V)}{\text{rk}_K V}.$$

Let Z be a subset of \mathbb{R} . We say that the filtration \mathcal{F} is *supported by Z* if $\text{supp}(\nu_{\mathcal{F}}) \subset Z$. We denote by \mathbf{Fil}_V^Z the set of all filtrations of V supported by Z . We define the *expectation* of \mathcal{F} to be $\mathbb{E}[\mathcal{F}] := \int_{\mathbb{R}} x d\nu_{\mathcal{F}}(x)$. If \mathcal{F} corresponds to the flag

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_d = 0$$

together with the sequence $(\lambda_i)_{0 \leq i < d}$, then

$$\nu_{\mathcal{F}} = \sum_{i=0}^{d-1} \frac{\text{rk}_K(V_i/V_{i+1})}{\text{rk}_K V} \delta_{\lambda_i},$$

where δ_x is the Dirac measure at x . If $\mathbf{e} = (e_1, \dots, e_r)^T$ is a base of V , we define $\nu_{\mathcal{F}, \mathbf{e}}$ to be the Borel measure

$$\nu_{\mathcal{F}, \mathbf{e}} := \frac{1}{r} \sum_{i=1}^r \delta_{\lambda(e_i)}.$$

For any bounded increasing function f on \mathbb{R} , we have always

$$\int_{\mathbb{R}} f d\nu_{\mathcal{F}, \mathbf{e}} \leq \int_{\mathbb{R}} f d\nu_{\mathcal{F}}.$$

Moreover, $\nu_{\mathcal{F}, \mathbf{e}} = \nu_{\mathcal{F}}$ if and only if for any $\lambda \in \mathbb{R}$, $\mathbf{e} \cap \mathcal{F}_\lambda V$ generates $\mathcal{F}_\lambda V$. In this case, we say that the base \mathbf{e} is *compatible* with the filtration \mathcal{F} . We denote by $\mathbf{Fil}_{\mathbf{e}}$ the set of all filtrations of V with which \mathbf{e} is compatible. Clearly there exists a one-to-one correspondence $\Phi_{\mathbf{e}}$ from $\mathbf{Fil}_{\mathbf{e}}$ to \mathbb{R}^r , which sends \mathcal{F} to $(\lambda_{\mathcal{F}}(e_1), \dots, \lambda_{\mathcal{F}}(e_r))$. If in addition Z is a subset of \mathbb{R} , we denote by $\mathbf{Fil}_{\mathbf{e}}^Z$ the subset of $\mathbf{Fil}_{\mathbf{e}}$ consisting of the filtrations supported by Z . The restriction of $\Phi_{\mathbf{e}}$ on $\mathbf{Fil}_{\mathbf{e}}^Z$ gives a one-to-one correspondence from $\mathbf{Fil}_{\mathbf{e}}^Z$ to Z^r .

Let F be a subfield of \mathbb{R} . If we fix a non-zero vector l in V , then the function $\mathcal{F} \mapsto \lambda_{\mathcal{F}}(l)$ from $\mathbf{Fil}_{\mathbf{e}}^F$ to \mathbb{R} can be written as the minimal value of a finite number of F -linear forms. In fact, let $l = \sum_{i=1}^r a_i e_i$ be the decomposition of l in the base \mathbf{e} , then for any filtration \mathcal{F} with which \mathbf{e} is compatible, we have

$$\lambda_{\mathcal{F}}(l) = \min_{\substack{1 \leq i \leq r \\ a_i \neq 0}} \lambda_{\mathcal{F}}(e_i).$$

If $\varepsilon > 0$ is a real number and if \mathcal{F} is a filtration of V , then $\psi_\varepsilon \mathcal{F} := (\mathcal{F}_{\varepsilon\lambda} V)_{\lambda \in \mathbb{R}}$ is also a filtration of V , called the *dilation* of \mathcal{F} by ε . Clearly we have

$$\mathbb{E}[\psi_\varepsilon \mathcal{F}] = \varepsilon \mathbb{E}[\mathcal{F}] \quad \text{and} \quad \lambda_{\psi_\varepsilon \mathcal{F}} = \varepsilon \lambda_{\mathcal{F}}. \quad (4)$$

Let $(V^{(i)})_{1 \leq i \leq n}$ be a family of non-zero vector spaces of finite rank and let $V = \bigoplus_{i=1}^n V^{(i)}$ be their direct sum. If for each integer $1 \leq i \leq n$, we choose a filtration $\mathcal{F}^{(i)}$ of $V^{(i)}$, then we can construct a filtration \mathcal{F} of V such that

$$\mathcal{F}_\lambda V = \bigoplus_{i=1}^n \mathcal{F}_\lambda^{(i)} V^{(i)}.$$

Clearly we have the relation

$$\nu_{\mathcal{F}} = \sum_{i=1}^n \frac{\text{rk}_K V^{(i)}}{\text{rk}_K V} \nu_{\mathcal{F}^{(i)}}.$$

The filtration \mathcal{F} is called the *direct sum* of $\mathcal{F}^{(i)}$ and will be denoted by $\mathcal{F}^{(1)} \oplus \dots \oplus \mathcal{F}^{(n)}$. If for each $1 \leq i \leq n$, $\mathbf{e}^{(i)}$ is a base of $V^{(i)}$ which is compatible with $\mathcal{F}^{(i)}$, then the disjoint union $\mathbf{e}^{(1)} \amalg \dots \amalg \mathbf{e}^{(n)}$, which is a base of $V^{(1)} \oplus \dots \oplus V^{(n)}$, is compatible to $\mathcal{F}^{(1)} \oplus \dots \oplus \mathcal{F}^{(n)}$. Similarly, if $W = \bigotimes_{i=1}^n V^{(i)}$ is the tensor product of $V^{(i)}$, we can also construct a filtration \mathcal{G} of W such that

$$\mathcal{G}_\lambda W = \sum_{\lambda_1 + \dots + \lambda_n \geq \lambda} \bigotimes_{i=1}^n \mathcal{F}_{\lambda_i}^{(i)} V^{(i)}$$

The probability $\nu_{\mathcal{G}}$ associated to the filtration \mathcal{G} is equal to the convolution

$$\nu_{\mathcal{F}^{(1)}} * \dots * \nu_{\mathcal{F}^{(n)}}.$$

The filtration \mathcal{G} is called the *tensor product* of $\mathcal{F}^{(i)}$ and will be denoted by $\mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(n)}$. Furthermore, if $\mathbf{e}^{(i)}$ is a base of $V^{(i)}$ which is compatible with the filtration $\mathcal{F}^{(i)}$, then the base

$$\mathbf{e}^{(1)} \otimes \dots \otimes \mathbf{e}^{(n)} := \{e_1 \otimes \dots \otimes e_n \mid \forall 1 \leq i \leq n, e_i \in \mathbf{e}^{(i)}\}$$

of $V^{(1)} \otimes \dots \otimes V^{(n)}$ is compatible to $\mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(n)}$. Finally, if $\varepsilon > 0$ is a positive number, then

$$\psi_\varepsilon(\mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(n)}) = \psi_\varepsilon \mathcal{F}^{(1)} \otimes \dots \otimes \psi_\varepsilon \mathcal{F}^{(n)}. \quad (5)$$

Let V be a non-zero vector space of finite rank over K . If \mathcal{F} and \mathcal{G} are two filtrations of V , we define the *scalar product* of \mathcal{F} and \mathcal{G} to be the number

$$\langle \mathcal{F}, \mathcal{G} \rangle := \int_{\mathbb{R}^2} xy d\nu_{\mathcal{F}, \mathcal{G}}, \quad (6)$$

where $\nu_{\mathcal{F}, \mathcal{G}}$ is the unique Borel measure on \mathbb{R}^2 such that the measure of the square $[x_1, x_2] \times [y_1, y_2]$ is

$$\text{rk}_K(\mathcal{F}_{x_1} V \cap \mathcal{G}_{y_1} V) - \text{rk}_K(\mathcal{F}_{x_1} V \cap \mathcal{G}_{y_2} V) - \text{rk}_K(\mathcal{F}_{x_2} V \cap \mathcal{G}_{y_1} V) + \text{rk}_K(\mathcal{F}_{x_2} V \cap \mathcal{G}_{y_2} V).$$

The number $\|\mathcal{F}\| := \langle \mathcal{F}, \mathcal{F} \rangle^{\frac{1}{2}}$ is called the *norm* of the filtration \mathcal{F} . Notice that $\|\mathcal{F}\| = 0$ if and only if $\nu_{\mathcal{F}} = \delta_0$. In this case, we say that the filtration \mathcal{F} is *trivial*. If $\mathbf{e} = (e_1, \dots, e_r)^T$ is a base in V which is compatible simultaneously with \mathcal{F} and with \mathcal{G} , we have

$$\nu_{\mathcal{F}, \mathcal{G}} = \sum_{i=1}^r \frac{1}{r} \delta_{(\lambda_{\mathcal{F}}(e_i), \lambda_{\mathcal{G}}(e_i))}$$

Hence

$$\langle \mathcal{F}, \mathcal{G} \rangle = \frac{1}{r} \sum_{i=1}^r \lambda_{\mathcal{F}}(e_i) \lambda_{\mathcal{G}}(e_i),$$

which means that, by identifying $\mathbf{Fil}_{\mathbf{e}}$ through $\Phi_{\mathbf{e}}$ with a vector space of dimension r over \mathbb{R} , $\langle \cdot, \cdot \rangle$ define a scalar product on the corresponding vector space.

Let V be a non-zero vector space of finite rank over K and \mathcal{F} be a filtration of V corresponding to the flag

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_d = 0$$

together with the sequence $(\lambda_j)_{0 \leq j < d}$. For any integer $0 \leq j < d$, we pick a base \mathbf{e}^j of the subquotient V_j/V_{j+1} . After choosing a preimage of \mathbf{e}^j in V_j and taking the disjoint union of the preimages, we get a base $\mathbf{e} = (e_1, \dots, e_r)$ of V which is clearly compatible with the filtration \mathcal{F} . The base \mathbf{e} defines a natural isomorphism Ψ from V to $\bigoplus_{j=0}^{d-1} (V_j/V_{j+1})$ which sends e_i to its class in $V_{\tau(i)}/V_{\tau(i)+1}$, where $\tau(i) = \max\{j \mid e_i \in V_j\}$. If for any integer $0 \leq j \leq d-1$, we choose a filtration \mathcal{G}^j of V_j/V_{j+1} which is compatible with the base \mathbf{e}^j , we can then construct a filtration \mathcal{G} on V which is just the direct sum via Ψ of \mathcal{G}^j ($1 \leq j \leq d$). Notice that the base \mathbf{e} is compatible with the new filtration \mathcal{G} . If e_i is an element in \mathbf{e} , then we have $\lambda_{\mathcal{G}}(e_i) = \lambda_{\mathcal{G}^{\tau(i)}}(\Psi(e_i))$. Therefore we have

$$\mathbb{E}[\mathcal{G}] = \frac{1}{r} \sum_{j=0}^{d-1} \mathbb{E}[\mathcal{G}^j] \operatorname{rk}_K(V_j/V_{j+1}), \quad \langle \mathcal{F}, \mathcal{G} \rangle = \frac{1}{r} \sum_{j=0}^{d-1} \lambda_j \mathbb{E}[\mathcal{G}^j] \operatorname{rk}_K(V_j/V_{j+1}). \quad (7)$$

6 More facts in geometric invariant theory

We shall establish in this section the explicit version of a result of Ramanan and Ramanathan [RR84] (Proposition 1.12) for our particular purpose, along the path indicated by Totaro [Tot96] in his proof of Fontaine's conjecture.

Let K be a perfect field. If G is a reductive group over $\operatorname{Spec} K$, we call *one parameter subgroup* of G any morphism of K -group schemes from $\mathbb{G}_{m,K}$ to G . Let X be a K -scheme on which G acts. If x is a rational point of X and if h is a one parameter subgroup of G , then we get a K -morphism from $\mathbb{G}_{m,K}$ to X given by the composition

$$\mathbb{G}_{m,K} \xrightarrow{h} G \xrightarrow{\sim} G \times_K \operatorname{Spec} K \xrightarrow{\operatorname{Id} \times x} G \times_K X \xrightarrow{\sigma} X,$$

where σ is the action of the group. If in addition X is proper over $\operatorname{Spec} K$, this morphism extends in the unique way to a K -morphism $f_{h,x}$ from \mathbb{A}_K^1 to X . We denote by 0 the unique element in $\mathbb{A}^1(K) \setminus \mathbb{G}_m(K)$. The morphism $f_{h,x}$ sends the point 0 to a rational point of X which is invariant under the action of $\mathbb{G}_{m,K}$. If L is a G -linear line bundle on X , then the action of $\mathbb{G}_{m,K}$ on $L|_{f_{h,x}(0)}$ defines a character of $\mathbb{G}_{m,K}$ of the form $t \mapsto t^{\mu(x,h,L)}$ where $\mu(x,h,L) \in \mathbb{Z}$. Furthermore, if we denote by $\operatorname{Pic}^G(X)$ the group of isomorphism classes of all G -linear line bundles, then $\mu(x,h, \cdot)$ is a homomorphism of groups from $\operatorname{Pic}^G(X)$ to \mathbb{Z} .

Recall a well known result which gives a semistability criterion for rational points in a projective variety equipped with an action of a reductive group.

Theorem 6.1 (Hilbert-Mumford-Kempf-Rousseau) *Let G be a reductive group which acts on a projective variety X over $\operatorname{Spec} K$, L be a G -linear bundle on X and $x \in X(K)$ be a rational point. The point x is semistable for the action of G relatively to L if and only if $\mu(x,h,L) \geq 0$ for any one parameter subgroup h of G .*

This theorem has been originally proved by Mumford (see [MFK94]) for the case where K is algebraically closed. Then it has been independently proved in all generality by Kempf [Kem78] and Rousseau [Rou78], where Kempf's approach has been revisited by Ramanan and Ramanathan [RR84] to prove that the tensor product of two semistable vector bundle on a smooth curve (over a perfect field) is also semistable. The idea of Kempf is to choose a special one parameter subgroup h_0 of G destabilizing x , which minimizes a certain function. The uniqueness of his construction enabled us to descend to a smaller field. Later Totaro [Tot96] introduced a new approach of Kempf's construction and thus found an elegant proof of Fontaine's conjecture.

In the following of this section, we recall Totaro's approach of Hilbert-Mumford criterion in our setting. We begin with calculating explicitly the number $\mu(x, h, L)$ by using filtrations introduced in the previous section.

Let V be a vector space of finite rank over K and $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G on V . If $h : \mathbb{G}_{m,K} \rightarrow G$ is a one parameter subgroup, then the multiplicative group $\mathbb{G}_{m,K}$ acts on V via h and ρ . Hence we can decompose V into direct sum of eigenspaces. More precisely, we have the decomposition $V = \bigoplus_{i \in \mathbb{Z}} V(i)$, where the action of $\mathbb{G}_{m,K}$ on $V(i)$ is given by the composition

$$\mathbb{G}_{m,K} \times_K V(i) \xrightarrow{(t \mapsto t^i) \times \mathrm{Id}} \mathbb{G}_{m,K} \times_K V(i) \longrightarrow V(i),$$

the second arrow being the scalar multiplication structure on $V(i)$. The existence of such decomposition is easy when K is algebraically closed. The case where K is merely perfect follows by using Galois descent. We then define a filtration $\mathcal{F}^{\rho,h}$ (supported by \mathbb{Z}) of V such that

$$\mathcal{F}_\lambda^{\rho,h} V = \sum_{i \geq \lambda} V(i) \quad (\lambda \in \mathbb{R}),$$

called the *filtration associated to h* relatively to the representation ρ . If there is no ambiguity on the representation, we also write \mathcal{F}^h instead of $\mathcal{F}^{\rho,h}$ to simplify the notations. We emphasize that if $G = \mathrm{GL}(V)$ and if ρ is the canonical representation, then for any filtration \mathcal{F} of V supported by \mathbb{Z} , there exists a one-parameter subgroup h of G such that the filtration associated to h equals to \mathcal{F} .

From the scheme-theoretical point of view, the algebraic group G acts via the representation ρ on the projective space $\mathbb{P}(V^\vee)$ (with Grothendieck's convention). Any rational point x of $\mathbb{P}(V^\vee)$ corresponds to a 1-dimensional subspace of V in which we pick a non-zero element l_x .

Proposition 6.2 *One has*

$$\mu(x, h, \mathcal{O}_{V^\vee}(1)) = -\lambda_{\mathcal{F}^{\rho,h}}(l_x)$$

Proof. Let $l_x = \sum_{i \in \mathbb{Z}} l_x(i)$ be the canonical decomposition of l_x . Let $i_0 = \lambda_{\mathcal{F}^{\rho,h}}(l_x)$. It is the maximal index i such that $l_x(i)$ is non-zero. Furthermore, $f_{h,x}(0)$ is just the rational point corresponding to the subspace of V generated by $l_x(i_0)$. By definition, the restriction of $\mathcal{O}_{V^\vee}(1)$ on x identifies with the quotient $(Kl_x)^\vee$ of V^\vee . So $\mu(x, h, \mathcal{O}_{V^\vee}(1)) = -i_0 = -\lambda_{\mathcal{F}^{\rho,h}}(l_x)$. \square

Let $(V_i)_{1 \leq i \leq n}$ be a finite family of non-zero vector spaces of finite rank over K . For each integer $1 \leq i \leq n$, let r_i be the rank of V_i . Let G be the algebraic group $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. We suppose that the algebraic group G acts on a vector space V . Let $\pi : \mathbb{P}(V^\vee) \rightarrow \mathrm{Spec} K$ be the canonical morphism. For each integer $1 \leq i \leq n$, we choose an integer m_i which is divisible by r_i . Let M be the G -linear line bundle on $\mathbb{P}(V^\vee)$ as follows

$$M := \bigotimes_{i=1}^n \pi^*(\Lambda^{r_i} V_i)^{\otimes m_i/r_i}.$$

It is a trivial line bundle on $\mathbb{P}(V^\vee)$ with possibly non-trivial G -action. Notice that any one parameter subgroup of G is of the form $h = (h_1, \dots, h_n)$, where h_i is a one parameter subgroup of $\mathbb{GL}(V_i)$. Let \mathcal{F}^{h_i} be the filtration of V_i associated to h_i relatively to the canonical representation of $\mathbb{GL}(V_i)$ on V_i . The action of $\mathbb{G}_{m,K}$ via h_i on $\Lambda^{r_i} V_i$ is nothing other than the multiplication by $t^{r_i \mathbb{E}[\mathcal{F}^{h_i}]}$. Therefore, we get the following result:

Proposition 6.3 *With the notations above, for any rational point x of $\mathbb{P}(V^\vee)$, we have*

$$\mu(x, h, M) = \sum_{i=1}^n m_i \mathbb{E}[\mathcal{F}^{h_i}].$$

We now introduce the Kempf's destabilizing flag for the action of a finite product of general linear groups. Consider a family $(V^{(i)})_{1 \leq i \leq n}$ of finite dimensional non-zero vector space over K . Let W be the tensor product $V^{(1)} \otimes_K \dots \otimes_K V^{(n)}$ and G be the algebraic group $\mathbb{GL}(V^{(1)}) \times \dots \times \mathbb{GL}(V^{(n)})$. For any integer $1 \leq i \leq n$, let $r^{(i)}$ be the rank of $V^{(i)}$. The group G acts naturally on W and hence on $\mathbb{P}(W^\vee)$. We denote by $\pi : \mathbb{P}(W^\vee) \rightarrow \text{Spec } K$ the canonical morphism. Let m be a strictly positive integer which is divisible by all $r^{(i)}$ and L be a G -linear line bundle on $\mathbb{P}(W^\vee)$ as follows:

$$L := \mathcal{O}_{W^\vee}(m) \otimes \bigotimes_{i=1}^n \pi^*(\det V^{(i)})^{\otimes (m/r^{(i)})}.$$

Suppose that $h = (h_1, \dots, h_n)$ is a one-parameter subgroup of G . The filtration \mathcal{F}^h of W associated to h coincides with the tensor product filtration $\mathcal{F}^{h_1} \otimes \dots \otimes \mathcal{F}^{h_n}$, where \mathcal{F}^{h_i} is the filtration of $V^{(i)}$ associated to h_i . After Proposition 6.2 and 6.3, for any rational point x of $\mathbb{P}(W^\vee)$, one has

$$\mu(x, h, L) = m \left(\sum_{i=1}^n \mathbb{E}[\mathcal{F}^{h_i}] - \lambda_{\mathcal{F}^h}(l_x) \right). \quad (8)$$

We define a function $\Lambda_x : \mathbf{Fil}_{V^{(1)}}^{\mathbb{Q}} \times \dots \times \mathbf{Fil}_{V^{(n)}}^{\mathbb{Q}} \rightarrow \mathbb{R}$ such that

$$\Lambda_x(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)}) = \frac{\mathbb{E}[\mathcal{F}^{(1)}] + \dots + \mathbb{E}[\mathcal{F}^{(n)}] - \lambda_{\mathcal{F}^{(1)} \otimes \dots \otimes \mathcal{F}^{(n)}}(l_x)}{(\|\mathcal{F}^{(1)}\|^2 + \dots + \|\mathcal{F}^{(n)}\|^2)^{\frac{1}{2}}} \quad (9)$$

if at least one filtration among the $\mathcal{F}^{(i)}$'s is non-trivial, and $\Lambda_x(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)}) = 0$ otherwise. Notice that the function Λ_x is invariant under dilation. In other words, if $\varepsilon > 0$ is a positive number, then

$$\Lambda_x(\psi_\varepsilon \mathcal{F}^{(1)}, \dots, \psi_\varepsilon \mathcal{F}^{(n)}) = \Lambda_x(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)}).$$

The Hilbert-Mumford criterion, joint with the equalities (8), (5) and (4), implies the following assertion:

Proposition 6.4 *The point x is not semistable for the action of G relatively to L if and only if the function Λ_x defined above takes at least one strictly negative value.*

The following result generalizes Proposition 2 of [Tot96], and the proof is the same.

Proposition 6.5 *With the notations of Proposition 6.4, if x is not semistable for the action of G relatively to L , then the function Λ_x attains its minimal value. Furthermore, the element in $\mathbf{Fil}_{V^{(1)}}^{\mathbb{Q}} \times \dots \times \mathbf{Fil}_{V^{(n)}}^{\mathbb{Q}}$ minimizing Λ_x is unique up to dilations. Finally, if $(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)})$*

is an element in $\mathbf{Fil}_{V^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{V^{(n)}}^{\mathbb{Q}}$ minimizing Λ_x and if c is the minimal value of Λ_x , then for any element $(\mathcal{G}^{(1)}, \dots, \mathcal{G}^{(n)})$ in $\mathbf{Fil}_{V^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{V^{(n)}}^{\mathbb{Q}}$, we have the inequality

$$\sum_{i=1}^n \mathbb{E}[\mathcal{G}^{(i)}] - \lambda_{\mathcal{G}^{(1)} \otimes \cdots \otimes \mathcal{G}^{(n)}}(l_x) \geq c \frac{\left(\langle \mathcal{F}^{(1)}, \mathcal{G}^{(1)} \rangle + \cdots + \langle \mathcal{F}^{(n)}, \mathcal{G}^{(n)} \rangle \right)}{\left(\|\mathcal{F}^{(1)}\|^2 + \cdots + \|\mathcal{F}^{(n)}\|^2 \right)^{\frac{1}{2}}} \quad (10)$$

Proof. For each integer $1 \leq i \leq n$, let $\mathbf{e}^{(i)}$ be a base of $V^{(i)}$. Consider the restriction of Λ_x on $\mathbf{Fil}_{\mathbf{e}^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}^{\mathbb{Q}}$. By embedding canonically $\mathbf{Fil}_{\mathbf{e}^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}^{\mathbb{Q}}$ in the Euclidean space $\mathbf{Fil}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}$, we extend the restriction of Λ_x to a function on $\mathbf{Fil}_{\mathbf{e}^{(1)}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}$, whose numerator part is the maximal value of a finite number of linear forms with rational coefficients (as we have shown in Section 5) and whose denominator part is just the norm of vector in the Euclidean space. Then after Lemma 3 of [Tot96], the restriction of Λ_x on $\mathbf{Fil}_{\mathbf{e}^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}^{\mathbb{Q}}$ attains its minimal value, and its minimizing element $(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)})$ in $\mathbf{Fil}_{\mathbf{e}^{(1)}}^{\mathbb{Q}} \times \cdots \times \mathbf{Fil}_{\mathbf{e}^{(n)}}^{\mathbb{Q}}$ is unique up to dilations. As pointed out by Totaro, there are only a finite number of functions on Euclidean space of dimension $r^{(1)} + \cdots + r^{(n)}$ arising from the restrictions as above. Thus we deduce that the function Λ_x attains globally its minimal value. Finally the uniqueness of minimizing filtrations (up to dilations) comes from the fact that for any two filtrations of $V^{(i)}$ we can choose a base of $V^{(i)}$ which is simultaneously compatible to them. This is just a consequence of the Bruhat's decomposition for general linear group. The same fact combining with Lemma 3 of [Tot96] deduces the inequality (10). \square

Although the minimizing filtrations $(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)})$ in Proposition 6.5 are *a priori* supported by \mathbb{Q} , it is always possible to choose them to be supported by \mathbb{Z} after a dilation. Now let $(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)})$ be an element in $\mathbf{Fil}_{V^{(1)}}^{\mathbb{Z}} \times \cdots \times \mathbf{Fil}_{V^{(n)}}^{\mathbb{Z}}$ minimizing Λ_x . We suppose that $\mathcal{F}^{(i)}$ corresponds to the flag

$$\mathcal{G}^{(i)} : V^{(i)} = V_0^{(i)} \supseteq V_1^{(i)} \supseteq \cdots \supseteq V_{d^{(i)}}^{(i)} = 0$$

and the strictly increasing sequence of integers $\lambda^{(i)} = (\lambda_j^{(i)})_{0 \leq j < d^{(i)}}$. Let \tilde{G} be the algebraic group

$$\tilde{G} := \prod_{i=1}^n \prod_{j=0}^{d^{(i)}-1} \mathbb{G}\mathbb{L}(V_j^{(i)}/V_{j+1}^{(i)}).$$

Let $\mathcal{F} = \mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(n)}$ and $\beta = \lambda_{\mathcal{F}}(l_x)$, which is the largest integer i such that $l_x \in \mathcal{F}_i W$. Let $\tilde{W} := \mathcal{F}_i W / \mathcal{F}_{i+1} W$ and let \tilde{l}_x be the canonical image of l_x in \tilde{W} . Notice that

$$\tilde{W} = \sum_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} \geq \beta} \bigotimes_{i=1}^n V_{j_i}^{(i)} \Big/ \sum_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} > \beta} \bigotimes_{i=1}^n V_{j_i}^{(i)} \cong \bigoplus_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} = \beta} \bigotimes_{i=1}^n \left(V_{j_i}^{(i)} / V_{j_i+1}^{(i)} \right).$$

So the algebraic group \tilde{G} acts naturally on \tilde{W} . Let \tilde{x} be the rational point of $\mathbb{P}(\tilde{W})$ corresponding to the subspace generated by \tilde{l}_x . We shall discuss the semistability of \tilde{x} for the action of \tilde{G} .

Suppose that for any integers $1 \leq i \leq n$ and $0 \leq j < d^{(i)}$, we choose an arbitrary filtration $\mathcal{G}^{(i),j}$ of $V_j^{(i)}/V_{j+1}^{(i)}$ supported by \mathbb{Z} . We have explained in Section 5 how to construct a new filtration $\mathcal{G}^{(i)}$ of $V^{(i)}$ from $\mathcal{G}^{(i),j}$ ($0 \leq j < d^{(i)}$). Let

$$\mathcal{G} = \bigotimes_{i=1}^n \mathcal{G}^{(i)}, \quad \tilde{\mathcal{G}} = \bigoplus_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} = \beta} \bigotimes_{i=1}^n \mathcal{G}^{(i),j_i}.$$

From the construction we know that $\lambda_{\mathcal{G}}(l_x) = \lambda_{\tilde{\mathcal{G}}}(\tilde{l}_x)$. For all integers $1 \leq i \leq n$ and $0 \leq j < d^{(i)}$, let $r_j^{(i)}$ be the rank of $V_j^{(i)}/V_{j+1}^{(i)}$ over K . Using (7), the inequality (10) implies:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{r_j^{(i)}}{r^{(i)}} \mathbb{E}[\mathcal{G}^{(i),j}] - \lambda_{\tilde{\mathcal{G}}}(\tilde{l}_x) &\geq \tilde{c} \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{\lambda_j^{(i)} r_j^{(i)}}{r^{(i)}} \mathbb{E}[\mathcal{G}^{(i),j}], \\ \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{r_j^{(i)}}{r^{(i)}} \mathbb{E}[\mathcal{G}^{(i),j}] - \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{\tilde{c} \lambda_j^{(i)} r_j^{(i)}}{r^{(i)}} \mathbb{E}[\mathcal{G}^{(i),j}] - \lambda_{\tilde{\mathcal{G}}}(\tilde{l}_x) &\geq 0, \end{aligned} \quad (11)$$

where $\tilde{c} := \frac{c}{(\|\mathcal{F}^{(1)}\|^2 + \dots + \|\mathcal{F}^{(n)}\|^2)^{\frac{1}{2}}}$ is a strictly negative rational number. Choose a strictly positive integer N divisible by all $r^{(i)}$ such that for any integers $1 \leq i \leq n$ and $0 \leq j < d^{(i)}$, $a_j^{(i)} := -\frac{N\tilde{c}\lambda_j^{(i)}}{r^{(i)}}$ is an integer. As the sequence $(\lambda_j^{(i)})_{0 \leq j < d^{(i)}}$ is strictly increasing, also is $\mathbf{a}^{(i)} := (a_j^{(i)})_{0 \leq j < d^{(i)}}$. With this notation the inequality (11) becomes

$$\sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{Nr_j^{(i)}}{r^{(i)}} \mathbb{E}[\mathcal{G}^{(i),j}] + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} \mathbb{E}[\mathcal{G}^{(i),j}] - N\lambda_{\tilde{\mathcal{G}}}(\tilde{l}_x) \geq 0. \quad (12)$$

We are now able to establish an explicit version of Proposition 1.12 in [RR84] for product of general linear groups. We will see that the explicite form of \tilde{L} is crucial in Section 7.

Proposition 6.6 *Let $\tilde{\pi} : \mathbb{P}(\tilde{W}^\vee) \rightarrow \text{Spec } K$ be the canonical morphism and let*

$$\tilde{L} := \mathcal{O}_{\tilde{W}^\vee}(N) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=0}^{d^{(i)}-1} \tilde{\pi}^* (\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)}))^{\otimes N/r^{(i)}} \right) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=0}^{d^{(i)}-1} \tilde{\pi}^* (\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)}))^{\otimes a_j^{(i)}} \right).$$

The rational point \tilde{x} of $\mathbb{P}(\tilde{W}^\vee)$ is semistable for the action of \tilde{G} relatively to the G -linear line bundle \tilde{L} .

Proof. Let h be an arbitrary one parameter subgroup of \tilde{G} corresponding to filtrations $\mathcal{G}^{(i),j}$. By Proposition 6.2, Proposition 6.3, and the fact that $\mu(\tilde{x}, h, \cdot)$ is a homomorphism of groups, we obtain

$$\begin{aligned} \mu(\tilde{x}, h, \tilde{L}) &= \mu(\tilde{x}, h, \mathcal{O}_{\tilde{W}^\vee}(N)) + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \left(\frac{Nr_j^{(i)}}{r^{(i)}} + a_j^{(i)} r_j^{(i)} \right) \mathbb{E}[\mathcal{G}^{(i),j}] \\ &= -N\lambda_{\tilde{\mathcal{G}}}(\tilde{l}_x) + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \left(\frac{Nr_j^{(i)}}{r^{(i)}} + a_j^{(i)} r_j^{(i)} \right) \mathbb{E}[\mathcal{G}^{(i),j}] \geq 0. \end{aligned}$$

By Hilbert-Mumford's criterion, the point \tilde{x} is semistable by the action of \tilde{G} relatively to \tilde{L} . \square

Finally we point out the following consequence of the inequality (11), which shall be useful.

Proposition 6.7 *The minimizing filtrations $(\mathcal{F}^{(1)}, \dots, \mathcal{F}^{(n)})$ satisfy*

$$\mathbb{E}[\mathcal{F}^{(1)}] = \dots = \mathbb{E}[\mathcal{F}^{(n)}] = 0.$$

In other words, the equality $\sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} = 0$ holds, or equivalently, the equality $\sum_{j=0}^{d^{(i)}-1} \lambda_j^{(i)} r_j^{(i)} = 0$ holds for any integer $1 \leq i \leq n$.

Proof. Let $(u_i)_{1 \leq i \leq n}$ be an arbitrary sequence of integers. For all integers $1 \leq i \leq n$ and $0 \leq j < d^{(i)}$, let $\mathcal{G}^{(i),j}$ be the filtration of $V_j^{(i)}/V_j^{(i+1)}$ the measure associated to which is δ_{u_i} . Notice that in this case $\nu_{\mathcal{G}} = \delta_{u_1 + \dots + u_n}$. The inequality (12) gives

$$\sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{N r_i^{(j)} u_i}{r^{(i)}} + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} u_i - N \sum_{i=1}^n u_i = \sum_{i=1}^n u_i \sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} \geq 0.$$

Since $(u_i)_{1 \leq i \leq n}$ is arbitrary, we know that $\sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} = 0$, and therefore $\sum_{j=0}^{d^{(i)}-1} \lambda_j^{(i)} r_j^{(i)} = 0$. \square

7 Upper bound for the degree of a Hermitian line subbundle

In this section, we shall give an upper bound for the Arakelov degree of a Hermitian line subbundle of a finite tensor product of Hermitian vector bundles. As explained in the introduction, we shall use the results established in the previous section to reduce our problem to the case with semistability condition (in geometric invariant theory sense), which has already been discussed in Section 3. We point out that, in order to obtain the same estimation as (3) in full generality, we should assume that all Hermitian vector bundles \overline{E}_i are semistable, as a price paid for removing the semistability condition (in geometric invariant theory sense) for M_K .

We denote by K a number field and by \mathcal{O}_K its integer ring. Let $(\overline{E}^{(i)})_{1 \leq i \leq n}$ be a family of **semistable** Hermitian vector bundles on $\text{Spec } \mathcal{O}_K$. For each integer $1 \leq i \leq n$, let $r^{(i)}$ be the rank of $E^{(i)}$ and $V^{(i)} = E_K^{(i)}$. Let $\overline{E} = \overline{E}^{(1)} \otimes \dots \otimes \overline{E}^{(n)}$ and $W = E_K$. We denote by $\pi : \mathbb{P}(W^\vee) \rightarrow \text{Spec } K$ the natural morphism. The algebraic group $G := \mathbb{G}_{\mathbb{L}_K}(V^{(1)}) \times_K \dots \times_K \mathbb{G}_{\mathbb{L}_K}(V^{(n)})$ acts naturally on $\mathbb{P}(W^\vee)$. Let \overline{M} be a Hermitian line subbundle of \overline{E} and m be a strictly positive integer which is divisible by all $r^{(i)}$'s.

Proposition 7.1 *For any Hermitian line subbundle \overline{M} of $\overline{E}^{(1)} \otimes \dots \otimes \overline{E}^{(n)}$, one has*

$$\widehat{\text{deg}}(\overline{M}) \leq \sum_{i=1}^n \left(\widehat{\mu}(\overline{E}^{(i)}) + \frac{1}{2} \log(\text{rk } E^{(i)}) \right).$$

Proof. We have proved that if M_K is semistable for the action of G relatively to $\mathcal{O}_{W^\vee}(m) \otimes \pi^* \left(\bigotimes_{i=1}^n (\Lambda^{r^{(i)}} V^{(i)})^{\otimes m/r^{(i)}} \right)$, where m is a strictly positive integer which is divisible by all $r^{(i)}$, then the following inequality holds:

$$\widehat{\text{deg}}(\overline{M}) \leq \sum_{i=1}^n \left(\widehat{\mu}(\overline{E}_i) + \frac{1}{2} \log r^{(i)} \right).$$

If this hypothesis of semistability is not fulfilled, after Proposition 6.6, there exists two strictly positive integers N and β , and for each integer $1 \leq i \leq n$,

1) a flag

$$\mathcal{D}^{(i)} : V^{(i)} = V_0^{(i)} \supseteq V_1^{(i)} \supseteq \cdots \supseteq V_{d^{(i)}}^{(i)} = 0$$

of $V^{(i)}$ corresponding to the sequence

$$E^{(i)} = E_0^{(i)} \supseteq E_1^{(i)} \supseteq \cdots \supseteq E_{d^{(i)}}^{(i)} = 0$$

of saturated sub- \mathcal{O}_K -modules of E ,

2) two strictly increasing sequence $\lambda^{(i)} = (\lambda_j^{(i)})_{0 \leq j < d^{(i)}}$ and $\mathbf{a}^{(i)} = (a_j^{(i)})_{0 \leq j < d^{(i)}}$ of integers, such that

i) N is divisible by all $r^{(i)}$'s,

ii) for any integer $1 \leq i \leq n$, $\sum_{j=0}^{d^{(i)}-1} a_j^{(i)} r_j^{(i)} = 0$, where $r_j^{(i)} = \text{rk}(V_j^{(i)}/V_{j+1}^{(i)})$,

iii) the inclusion of M in E factorizes through $\sum_{\lambda_{i_1}^{(1)} + \cdots + \lambda_{i_n}^{(n)} \geq \beta} E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_n}^{(n)}$,

iv) the canonical image of M_K in

$$\widetilde{W} := \sum_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} \geq \beta} \bigotimes_{i=1}^n V_{j_i}^{(i)} \Big/ \sum_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} > \beta} \bigotimes_{i=1}^n V_{j_i}^{(i)} \cong \bigoplus_{\lambda_{j_1}^{(1)} + \cdots + \lambda_{j_n}^{(n)} = \beta} \bigotimes_{i=1}^n (V_{j_i}^{(i)}/V_{j_i+1}^{(i)}).$$

is non-zero, and is semistable for the action of the group

$$\widetilde{G} := \prod_{i=1}^n \prod_{j=0}^{d^{(i)}-1} \text{GL}(V_j^{(i)}/V_{j+1}^{(i)})$$

relatively to

$$\mathcal{O}_{\widetilde{W}^\vee}(N) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=0}^{d^{(i)}-1} \widetilde{\pi}^* (\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)}))^{\otimes N/r^{(i)}} \right) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=0}^{d^{(i)}-1} \widetilde{\pi}^* (\Lambda^{r_j^{(i)}}(V_j^{(i)}/V_{j+1}^{(i)}))^{\otimes a_j^{(i)}} \right),$$

where $\widetilde{\pi} : \mathbb{P}(\widetilde{W}^\vee) \rightarrow \text{Spec } K$ is the canonical morphism.

Notice that $\bigotimes_{j=0}^{d^{(i)}-1} (\Lambda^{r_j^{(i)}}(\overline{E}_j^{(i)}/\overline{E}_{j+1}^{(i)}))^{\otimes a_j^{(i)}}$ is nothing other than $\overline{\mathcal{L}}_{\mathcal{D}^{(i)}}^{\mathbf{a}^{(i)\vee}}$ defined in Section 4 (see Remark 4.2 and Proposition 6.7 *infra*).

Applying Theorem 3.2, one gets

$$\begin{aligned} \widehat{\text{deg}}(\overline{M}) &\leq \frac{1}{N} \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{N}{r^{(i)}} (\widehat{\text{deg}}(\overline{E}_j^{(i)}) - \widehat{\text{deg}}(\overline{E}_{j+1}^{(i)})) - \frac{1}{N} \sum_{i=1}^n \widehat{\text{deg}} \overline{\mathcal{L}}_{\mathcal{D}^{(i)}}^{\mathbf{a}^{(i)}} + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{Nr_j^{(i)}/r^{(i)} + r_j^{(i)} a_j^{(i)}}{2N} \log r_j^{(i)} \\ &= \sum_{i=1}^n \widehat{\mu}(\overline{E}^{(i)}) - \frac{1}{N} \sum_{i=1}^n \widehat{\text{deg}} \overline{\mathcal{L}}_{\mathcal{D}^{(i)}}^{\mathbf{a}^{(i)}} + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{Nr_j^{(i)}/r^{(i)} + r_j^{(i)} a_j^{(i)}}{2N} \log r_j^{(i)} \\ &\leq \sum_{i=1}^n \widehat{\mu}(\overline{E}^{(i)}) + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{Nr_j^{(i)}/r^{(i)} + r_j^{(i)} a_j^{(i)}}{2N} \log r_j^{(i)}, \end{aligned}$$

the last inequality is because $\overline{E}^{(i)}$'s are semistable (see Proposition 4.1 *infra*). The semistability of the canonical image of M_K implies that $Nr_j^{(i)}/r^{(i)} + r_j^{(i)}a_j^{(i)} \geq 0$ for any integer $1 \leq i \leq n$. Therefore

$$\widehat{\deg}(\overline{M}) \leq \sum_{i=1}^n \widehat{\mu}(\overline{E}^{(i)}) + \sum_{i=1}^n \sum_{j=0}^{d^{(i)}-1} \frac{Nr_j^{(i)}/r^{(i)} + r_j^{(i)}a_j^{(i)}}{2N} \log r^{(i)}.$$

As for any integer $1 \leq i \leq n$, $\sum_{j=0}^{d^{(i)}-1} r_j^{(i)}a_j^{(i)} = 0$, we have proved the Proposition. \square

For any non-zero Hermitian vector bundle \overline{F} on $\text{Spec } \mathcal{O}_K$ we denote by $\widehat{\text{udeg}}(\overline{F})$ the maximal degree of line subbundles of \overline{F} . We recall a result of J.-B. Bost and K. Künnemann [BK07] comparing the maximal degree and the maximal slope of \overline{F} which states as follows:

$$\widehat{\text{udeg}}(\overline{F}) \leq \widehat{\mu}_{\max}(\overline{F}) \leq \widehat{\text{udeg}}(\overline{F}) + \frac{1}{2} \log(\text{rk } F) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]}. \quad (13)$$

This is actually a variant of the first theorem of Minkowski. Using this inequality, Proposition 7.1 implies:

Corollary 7.2 *The following inequality is verified:*

$$\widehat{\mu}_{\max}(\overline{E}^{(1)} \otimes \cdots \otimes \overline{E}^{(n)}) \leq \sum_{i=1}^n \left(\widehat{\mu}(\overline{E}^{(i)}) + \log(\text{rk } E^{(i)}) \right) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]}.$$

8 Proof of Theorem 1.1

We finally give the proof of Theorem 1.1.

Lemma 8.1 *Let K be a number field and \mathcal{O}_K be its integer ring. Suppose given a finite family $(\overline{E}_i)_{1 \leq i \leq n}$ of non-zero Hermitian vector bundles (non-necessarily semistable). Let $\overline{E} = \overline{E}_1 \otimes \cdots \otimes \overline{E}_n$. Then the following inequality holds:*

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(E_i) + \log(\text{rk } E_i) \right) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]}.$$

Proof. Let F be a sub- \mathcal{O}_K -module of E . By taking Harder-Narasimhan flags of E_i 's (cf. [Bos96]) we obtain that there exists, for any $1 \leq i \leq n$, a semistable subquotient $\overline{F}_i/\overline{G}_i$ of E_i such that

- 1) $\widehat{\mu}(\overline{F}_i/\overline{G}_i) \leq \widehat{\mu}_{\max}(\overline{E}_i)$,
- 2) the homomorphism of inclusion from F to E factorise through $F_1 \otimes \cdots \otimes F_n$,
- 3) the canonical image of F in $(F_1/G_1) \otimes \cdots \otimes (F_n/G_n)$ does not vanish.

Combining with the slope inequality, Corollary 7.2 implies

$$\begin{aligned} \widehat{\mu}_{\min}(\overline{F}) &\leq \sum_{i=1}^n \left(\widehat{\mu}(\overline{F}_i/\overline{G}_i) + \log(\text{rk}(F_i/G_i)) \right) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} \\ &\leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(\overline{E}_i) + \log(\text{rk } E_i) \right) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]} \end{aligned}$$

Since F is arbitrary, the proposition is proved. \square

Proof of Theorem 1.1 Let $N \geq 1$ be an arbitrary integer, after Lemma 8.1, one has, in considering $\overline{E}^{\otimes N}$ as $\underbrace{\overline{E}_1 \otimes \cdots \otimes \overline{E}_1}_{N \text{ copies}} \cdots \otimes \underbrace{\overline{E}_n \otimes \cdots \otimes \overline{E}_n}_{N \text{ copies}}$

$$\widehat{\mu}_{\max}(\overline{E}^{\otimes N}) \leq \sum_{i=1}^n N \left(\widehat{\mu}_{\max}(\overline{E}_i) + \log(\text{rk } E_i) \right) + \frac{\log |\Delta_K|}{2[K : \mathbb{Q}]}$$

On the other hand, it is clear that $\widehat{\mu}_{\max}(\overline{E}^{\otimes N}) \geq N \widehat{\mu}_{\max}(\overline{E})$. Therefore, one has

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(\overline{E}_i) + \log(\text{rk } E_i) \right) + \frac{\log |\Delta_K|}{2N[K : \mathbb{Q}]}.$$

As N is arbitrary, we obtain by taking $N \rightarrow +\infty$,

$$\widehat{\mu}_{\max}(\overline{E}) \leq \sum_{i=1}^n \left(\widehat{\mu}_{\max}(\overline{E}_i) + \log(\text{rk } E_i) \right),$$

the theorem is thus proved.

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