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Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -modules of higher level

Nicolas Jacon* and Cédric Lecouvey†

Abstract

We study the crystal graphs of irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -modules of higher level ℓ . Generalizing results of the first author, we obtain a simple description of the bijections between the classes of multipartitions which naturally label these graphs: the Uglov multipartitions. By works of Ariki, Grojnowski and Lascoux-Leclerc-Thibon, it is then known that these bijections permit also to link the distinct parametrizations of the simple modules in modular representation theory of Ariki-Koike algebras. Our main tool is to make explicit an embedding of the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -crystals of level ℓ into $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -crystals associated to highest weight modules.

1 Introduction

Let $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ be the affine quantum group of type $A_{e-1}^{(1)}$ and $\{\Lambda_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ its set of fundamental weights. Consider $\ell \in \mathbb{N}$ and $\underline{s} = (s_0, \dots, s_{\ell-1}) \in (\mathbb{Z}/e\mathbb{Z})^\ell$. Denote by $V_e(\Lambda_{\underline{s}})$ the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -module of highest weight $\Lambda_{\underline{s}} = \sum_{i=0}^{\ell-1} \Lambda_{s_i}$. The general theory of Kashiwara provides a crystal basis and a global basis for $V_e(\Lambda_{\underline{s}})$. The crystal basis of $V_e(\Lambda_{\underline{s}})$ comes equipped with the crystal graph $B_e(\Lambda_{\underline{s}})$ which encodes much information on the module structure.

There are different possible realizations of $V_e(\Lambda_{\underline{s}})$ depending on the choice of a representative $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$ of the class $\underline{s} \in (\mathbb{Z}/e\mathbb{Z})^\ell$. Indeed, to \underline{s} is associated a Fock space $\mathfrak{F}_e^{\underline{s}}$ which provides an explicit construction of $V_e(\Lambda_{\underline{s}})$. We will denote it $V_e^{\underline{s}}(\Lambda_{\underline{s}})$ and write $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ for the corresponding crystal graph. The crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ with $\underline{s} \in \underline{s}$ are thus all isomorphic to the abstract crystal $B_e(\Lambda_{\underline{s}})$.

The purpose of the paper is to make explicit the isomorphisms between the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ when \underline{s} runs over \underline{s} . By the works of Ariki [2], Grojnowski [9] and Lascoux-Leclerc-Thibon [19], an important application of these isomorphisms is to provide bijections between the distinct parametrizations of the simple modules in modular representation theory of Ariki-Koike algebras. To be more precise, let η be a primitive e^{th} -root of unity. Write $\mathcal{H} = \mathcal{H}(\eta; \eta^{s_0}, \dots, \eta^{s_{\ell-1}})$ for the Ariki-Koike algebra defined over an algebraically closed field F of characteristic 0. This algebra is

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generated by T_0, \dots, T_{n-1} subject to the relations $(T_0 - \eta^{s_0}) \dots (T_0 - \eta^{s_{\ell-1}}) = 0$, $(T_i - \eta)(T_i + 1) = 0$, for $1 \leq i \leq n$ and the type B braid relations

$$\begin{aligned} (T_0 T_1)^2 &= (T_1 T_0)^2, & T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (1 \leq i < n), \\ T_i T_j &= T_j T_i \quad (j \geq i + 2). \end{aligned}$$

The algebra \mathcal{H} is not semisimple in general, and, by a deep Theorem of Ariki, its representation theory is intimately connected to the global bases of the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_\ell)$ -modules. In particular, the simple modules of \mathcal{H} are labelled by the vertices of any crystal $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ such that $\underline{s} \in \underline{\mathfrak{s}}$. In fact the Fock space $\mathfrak{F}_e^{\underline{s}}$ admits a crystal basis indexed by multipartitions of length ℓ . This implies that the vertices of the crystal $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ can be identified with certain multipartitions of length ℓ which are called the Uglov multipartitions. Note that when $\ell = 2$, remarkable results of Geck [8] show that these multipartitions also naturally appear in the context of Kazhdan-Lusztig theory and cellular structure of Hecke algebras. Other connections between Uglov multipartitions and modular representation theory of Ariki-Koike algebras are also known to hold for $\ell > 2$ [12]. Although nice properties have been given in particular cases (see [3] and [4]), the combinatorics of the Uglov multipartitions and their associated crystal graphs are not really well understood. For example, we do not even have a non recursive characterization of the Uglov multipartitions for any $\underline{s} \in \underline{\mathfrak{s}}$.

This paper extends the results obtained in [13] for $\ell = 2$ by giving a combinatorial description of the isomorphisms between the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$. Nevertheless, the ideas we use are quite different. Indeed, we show that the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ can be embedded in crystals corresponding to irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules. This result allows us to prove that most of the crystal isomorphisms in type $A_{e-1}^{(1)}$ can be derived from crystal isomorphisms in type A_∞ . Now the combinatorial description of the isomorphisms of A_∞ -crystals is very close to that of the isomorphisms of A_r -crystals in finite rank r . This permits us to use some elegant results of Nakayashiki and Yamada [23] on combinatorial R -matrices in type A_r . One of the advantages of this new method is to avoid cumbersome case by case verifications unavoidable in [12].

We would like to mention also that there is another way to realize the abstract crystals $B_e(\Lambda_{\underline{s}})$ by using Fock spaces of level ℓ which are tensor products of Fock spaces of level 1. The crystal $B_e^a(\Lambda_{\underline{s}})$ so obtained notably appears in the works by Ariki (see [2]). The vertices of $B_e^a(\Lambda_{\underline{s}})$ are parametrized by multipartitions called the ‘‘Kleshchev multipartitions’’ [17]. The crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ and $B_e^a(\Lambda_{\underline{s}})$ do not coincide in general. In particular, our method does not permit one to embed $B_e^a(\Lambda_{\underline{s}})$ in a crystal of type A_∞ (but see the remark after Theorem 4.2.2).

The present paper is organized as follows. The second section is devoted to the combinatorial description of certain isomorphisms of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -crystals. In section 3, we recall basic results on $\mathcal{U}_v(\widehat{\mathfrak{sl}}_\ell)$ -crystals. By using two natural parametrizations of the Dynkin diagram in type $A_{e-1}^{(1)}$, we link in particular the two usual presentations of the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ which appear in the literature. We then show in Section 4 that the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ can be embedded in crystals corresponding to irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules. This embedding allows us to give in Section 5 a description of the isomorphisms between the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}})$ and to obtain another characterization for the sets of Uglov multipartitions. This characterization does not necessitate an induction on the sum of the parts of the Uglov multipartitions contrary to the original one [24].

2 Crystals in type A_∞

In this section, we study crystal isomorphisms in type A_∞ .

2.1 Background on $\mathcal{U}_v(\mathfrak{sl}_\infty)$

Let \mathfrak{sl}_∞ be the Lie algebra associated to the doubly infinite Dynkin diagram in type A_∞ (see [15] and [16]).

$$A_\infty : \cdots - \overset{-m}{\circ} - \overset{1-m}{\circ} - \cdots - \overset{-1}{\circ} - \overset{0}{\circ} - \overset{1}{\circ} - \cdots - \overset{m-1}{\circ} - \overset{m}{\circ} - \cdots. \quad (1)$$

We denote by $\mathcal{U}_v(\mathfrak{sl}_\infty)$ the corresponding quantum group and we write $\omega_i, i \in \mathbb{Z}$ for the fundamental weights of the corresponding root system. We associate to the sequence $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$ the dominant weight $\omega_{\underline{s}} = \sum_{i=0}^{\ell-1} \omega_{s_i}$. Then the irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules are parametrized by the sequences \underline{s} of arbitrary length ℓ . We denote by $V_\infty(\omega_{\underline{s}})$ the irreducible $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module of highest weight $\omega_{\underline{s}}$. The module $V_\infty(\omega_{\underline{s}})$ admits a crystal basis. We refer the reader to [11] for a complete review on crystal bases. We write $B(\omega_{\underline{s}})$ for the crystal graph corresponding to $V_\infty(\omega_{\underline{s}})$. When $\underline{s} = (s_0)$ we write for short $V_\infty(\omega_{s_0})$ and $B_\infty(\omega_{s_0})$ instead of $V(\omega_{\underline{s}})$ and $B(\omega_{\underline{s}})$.

In addition to the irreducible highest weight modules $V_\infty(\omega_{\underline{s}})$, it will be convenient to consider also irreducible modules $V_\infty(k)$ indexed by nonnegative integers which are not of highest weight. By a column of height k , we mean a column shaped Young diagram

$$C = \begin{array}{|c|} \hline x_1 \\ \hline \cdot \\ \hline \cdot \\ \hline x_k \\ \hline \end{array} \quad (2)$$

of height k filled by integers $x_i \in \mathbb{Z}$ such that $x_1 > \cdots > x_k$. When $a \in C$ and $b \notin C$, we write $C - \{a\} + \{b\}$ for the column obtained by replacing in C the letter a by the letter b . The module $V_\infty(k)$ is defined as the vector space with basis $\mathcal{B}_k = \{v_C \mid C \text{ is a column of height } k\}$. The actions of the Chevalley generators $e_i, f_i, k_i, i \in \mathbb{Z}$ of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ are given by

$$\begin{aligned} f_i(v_C) &= \begin{cases} 0 & \text{if } \delta_i(C) = 0 \text{ or } \delta_{i+1}(C) = 1 \\ v_{C'} & \text{with } C' = C \setminus \{i\} \cup \{i+1\} \text{ otherwise} \end{cases} , \\ e_i(v_C) &= \begin{cases} 0 & \text{if } \delta_i(C) = 1 \text{ or } \delta_{i+1}(C) = 0 \\ v_{C'} & \text{with } C' = C \setminus \{i+1\} \cup \{i\} \text{ otherwise} \end{cases} , \\ k_i(v_C) &= v^{\delta_i(C) - \delta_{i+1}(C)} v_C. \end{aligned}$$

where for any $i \in \mathbb{Z}$, $\delta_i(C) = 1$ if $i \in C$ and $\delta_i(C) = 0$ otherwise.

Remark: Consider a, b two integers such that $a < b$. Denote by $[a, b]$ the set of integers i such that $a \leq i \leq b - 1$. Write $\mathcal{U}_v(\mathfrak{sl}_{a,b})$ for the subalgebra of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ generated by the Chevalley generators e_i, f_i, k_i with $i \in [a, b]$. Then $\mathcal{U}_v(\mathfrak{sl}_{a,b})$ can be identified with the quantum group associated to the Dynkin diagram obtained by deleting the nodes $i \notin [a, b]$ in (1). In particular

$\mathcal{U}_v(\mathfrak{sl}_{a,b})$ is isomorphic to the quantum group $\mathcal{U}_v(\mathfrak{sl}_d)$ with $d = b - a + 1$. The subspace $V_{a,b}(k)$ of $V_\infty(k)$ generated by the basis vectors v_C such that C contains only letters x_i with $a \leq x_i \leq b$ has the structure of a $\mathcal{U}_v(\mathfrak{sl}_{a,b})$ -module. Moreover $V_{a,b}(k)$ is isomorphic to the k^{th} fundamental module of $\mathcal{U}_v(\mathfrak{sl}_{a,b})$.

We follow the convention of [18] and consider $\mathcal{U}_v(\mathfrak{sl}_\infty)$ as a Hopf algebra with coproduct given by

$$\begin{aligned}\Delta(k_i) &= k_i \otimes k_i, \\ \Delta(e_i) &= e_i \otimes k_i + 1 \otimes e_i \text{ and } \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i.\end{aligned}$$

Given M_1 and M_2 two $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules with crystal graphs B_1 and B_2 , the crystal graph structure on $B_1 \otimes B_2$ is then given by

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(v) \leq \varepsilon_i(u) \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(v) > \varepsilon_i(u) \end{cases}, \quad (3)$$

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varphi_i(v) \geq \varepsilon_i(u) \\ \tilde{e}_i(u) \otimes v & \text{if } \varphi_i(v) < \varepsilon_i(u) \end{cases}. \quad (4)$$

Note that this convention is the reverse of that used in many references on crystals bases (see for example [10]) but it is the natural one for working with multipartitions.

2.2 The crystals graphs of irreducible $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules

For any $s \in \mathbb{Z}$, the $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module $V_\infty(\omega_s)$ can be obtained as an irreducible component of the Fock space $F_\infty(\omega_s)$ defined by considering semi-infinite wedge products of $V_\infty(1)$. The crystal $B_\infty(\omega_s)$ is identified with the graph whose vertices are the infinite columns

$$\mathcal{C} = \begin{array}{|c|} \hline x_1 \\ \hline \cdot \\ \hline \cdot \\ \hline x_k \\ \hline s - k + 1 \\ \hline s - k \\ \hline \dots \\ \hline \end{array}$$

that is, the infinite columns shaped Young diagrams filled by decreasing integers x_i from top to bottom and such that $x_k = s - k + 1$ for k sufficiently large. Given \mathcal{C}_1 and \mathcal{C}_2 in $B_\infty(\omega_s)$, we have an arrow $\mathcal{C}_1 \xrightarrow{i} \mathcal{C}_2$ if and only if $i \in \mathcal{C}_1$, $i + 1 \notin \mathcal{C}_2$ and $\mathcal{C}_2 = \mathcal{C}_1 - \{i\} + \{i + 1\}$. The highest weight vertex of $B_\infty(\omega_s)$ is the column $\mathcal{C}^{(s)}$ such that $x_k = s - k + 1$ for all $k \geq 1$.

We associate to the $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module $V_\infty(k)$ a graph $B_\infty(k)$. Its vertices are the columns of height k (see (2)) and we draw an oriented arrow $\mathcal{C}_1 \xrightarrow{i} \mathcal{C}_2$ if and only if $i \in \mathcal{C}_1$, $i + 1 \notin \mathcal{C}_2$ and $\mathcal{C}_2 = \mathcal{C}_1 - \{i\} + \{i + 1\}$. The graph $B_\infty(k)$ can be regarded as the crystal graph of $V_\infty(k)$. In fact, one can define a notion of crystal basis for the module $V_\infty(k)$ despite the fact it is not

of highest weight. This can be done essentially by considering the direct limit of the directed system formed by the crystal bases of the $\mathcal{U}_v(\mathfrak{sl}_{a,b})$ -modules $V_{a,b}(k)$ [21]. The corresponding crystal graph then coincide with $B_\infty(k)$. In the sequel we will only need the crystal $B_\infty(k)$ and not the whole crystal basis of $B_\infty(k)$.

Consider an infinite column $\mathcal{C} \in B_\infty(\omega_s)$ with letters x_i , $i \geq 1$. We write $\pi_a(\mathcal{C})$ for the finite column obtained by deleting the infinite sequence of letters x_i such that $x_i < a$ in \mathcal{C} . Note that the height of $\pi_a(\mathcal{C})$ depends then of the integer a chosen.

Let $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$. The abstract crystal $B_\infty(\omega_{\underline{s}})$ is then isomorphic to the connected component $B_\infty^{\underline{s}}(\omega_{\underline{s}})$ of $B_\infty(\underline{s}) = B_\infty(\omega_{s_0}) \otimes \dots \otimes B_\infty(\omega_{s_{\ell-1}})$ with highest weight vertex $b(\underline{s}) = \mathcal{C}^{(s_0)} \otimes \dots \otimes \mathcal{C}^{(s_{\ell-1})}$. Consider $b = \mathcal{C}_0 \otimes \dots \otimes \mathcal{C}_{\ell-1} \in B_\infty^{\underline{s}}(\omega_{\underline{s}})$ with $\mathcal{C}_k \in B_\infty(\omega_{s_k})$, $k = 0, \dots, \ell - 1$. For any integer a and any $k = 0, \dots, \ell - 1$, let h_k be the height of the finite column $\pi_a(\mathcal{C}_k)$. We set

$$\pi_a(b) = \pi_a(\mathcal{C}_0) \otimes \dots \otimes \pi_a(\mathcal{C}_{\ell-1}) \in B_\infty(h_0) \otimes \dots \otimes B_\infty(h_{\ell-1}).$$

Consider b_1 and b_2 two vertices of $B_\infty^{\underline{s}}(\omega_{\underline{s}})$. Let \tilde{K} be any path between b_1 and b_2 in $B_\infty^{\underline{s}}(\omega_{\underline{s}})$, that is any sequence of crystal operators such that $b_1 = \tilde{K}(b_2)$.

Lemma 2.2.1 *With the previous notation, for any integer $a \geq 1$ sufficiently large, we have $\pi_a(b_1) \in B_\infty(h_0) \otimes \dots \otimes B_\infty(h_{\ell-1})$ and $\pi_a(b_2) \in B_\infty(h_0) \otimes \dots \otimes B_\infty(h_{\ell-1})$. In this case $\pi_a(b_1) = \tilde{K}(\pi_a(b_2))$ in $B_\infty(h_0) \otimes \dots \otimes B_\infty(h_{\ell-1})$.*

Proof. We can choose a sufficiently large so that the infinite columns appearing in every vertex b of the path joining b_1 to b_2 contain all the letters $x < a$. Then, for each $k = 0, \dots, \ell - 1$, the height of the column obtained by deleting the letters $x < a$ in the k -th column of b does not depend on b . Set $h_k = s_k + 1 - a$. The deleted letters $x < a$ do not interfere during the computation of the crystal operators defining the path between b_1 and b_2 . This implies that the path joining $\pi_a(b_1)$ to $\pi_a(b_2)$ obtained by applying the same sequence \tilde{K} of crystal operators also exists in $B_\infty(h_0) \otimes \dots \otimes B_\infty(h_{\ell-1})$. Thus $\pi_a(b_1) = \tilde{K}(\pi_a(b_2))$. ■

The vertices of $B_\infty(\omega_s)$ are also naturally labelled by partitions. Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_p)$ of length p is a weakly decreasing sequence $\lambda_1 \geq \dots \geq \lambda_p$ of nonnegative integers called the parts of λ . In the sequel we identify the partitions having the same nonzero parts and denote by \mathcal{P} the set of all partitions. It is convenient to represent λ by its Young diagram. In the sequel we use the French convention for the Young diagram (see example below). To any pair (a, b) of integers, we associate the node $\gamma = (a, b)$ which is the box obtained after a north moves and b east moves starting from the south-west box of λ . Observe that the node γ does not necessarily belong to λ . We say that γ is removable when $\gamma = (a, b) \in \lambda$ and $\lambda \setminus \{\gamma\}$ is a partition. Similarly γ is said addable when $\gamma = (a, b) \notin \lambda$ and $\lambda \cup \{\gamma\}$ is a partition.

We associate to the infinite column $\mathcal{C} \in B_\infty(\omega_s)$ with letters x_k the partition λ such that $\lambda_k = x_k - s + k - 1$. Since $\lambda_k = 0$ for k sufficiently large, this permits us to label the vertices of $B_\infty(\omega_s)$ by \mathcal{P} . Let $\gamma = (a, b)$ be a node for $\lambda \in B_\infty(\omega_s)$. The content of γ is

$$c(\gamma) = b - a + s. \tag{5}$$

In $B_\infty(\omega_s)$, we have an arrow $\lambda \xrightarrow{i} \mu$ if and only if $\mu/\lambda = \gamma$ where γ is addable in λ with $c(\gamma) = i$. For any partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ we set $|\lambda| = \lambda_1 + \dots + \lambda_p$. Then $|\lambda|$ is equal to the length of the directed path joining \emptyset to λ in the crystal $B_\infty(\omega_s)$.

Example 2.2.2

$$\begin{array}{c}
 \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 4 \\ \hline 1 \\ \hline \bar{1} \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array} \\
 \text{To } \mathcal{C} = \in B_\infty(\omega_3) \text{ corresponds } \lambda = \begin{array}{|c|c|c|c|} \hline R & A & & \\ \hline & & R & A \\ \hline & & & R \\ \hline & & & \\ \hline & & & A \\ \hline \end{array} . \\
 \\
 \begin{array}{c}
 \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 1 \\ \hline \bar{1} \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array} \\
 \text{We have } \tilde{f}_4(\mathcal{C}) = \text{ or equivalently } \tilde{f}_4(\lambda) = \begin{array}{|c|c|c|c|} \hline R & A & & \\ \hline & & & R \\ \hline & & & \\ \hline & & & \\ \hline & & & A \\ \hline \end{array}
 \end{array}
 \end{array}$$

for the node $\gamma = (3, 4)$ is addable in λ with content $c(\gamma) = 4 - 3 + 3 = 4$. Here we have written \bar{k} instead of $-k$ for any positive integer k .

Since $B_\infty^s(\omega_s)$ is labelled by partitions, the crystal $B_\infty(\underline{s})$ is labelled by multipartitions $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$. More precisely, to each vertex $b = \mathcal{C}_0 \otimes \dots \otimes \mathcal{C}_{\ell-1} \in B_\infty(\underline{s})$ such that $\mathcal{C}_k \in B(\omega_{s_k})$ for $k = 0, \dots, \ell - 1$, corresponds the multipartition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ where $\lambda^{(k)}$ is obtained from \mathcal{C}_k as described above. By a part of the multipartition $\underline{\lambda}$, we mean a part of one of the partitions $\lambda^{(k)}$. The action of the operators \tilde{e}_i and \tilde{f}_i on b are deduced from (3) and (4). One verifies easily that they can be also obtained by applying the following algorithm. Let w_i be the word obtained by considering the letters i and $i + 1$ successively in the columns $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{\ell-1}$ of b when these columns are read from top to bottom and left to right. The word w_i is called the i -signature of b . Encode in w_i each integer i by a symbol $+$ and each integer $i + 1$ by a symbol $-$. Choose any factor of consecutive $-+$ and delete it. Repeat this procedure until no factor $-+$ can be deleted. The final sequence \tilde{w}_i is uniquely determined and has the form $\tilde{w}_i = +^p -^q$. It is called the reduced i signature of b . Then $\tilde{f}_i(b)$ is obtained from b by replacing the integer i corresponding to the rightmost symbol $+$ in \tilde{w}_i by $i + 1$. Similarly, $\tilde{e}_i(b)$ is obtained from λ by replacing the integer $i + 1$ corresponding to the leftmost symbol $-$ in \tilde{w}_i by i .

One can describe similarly the action of \tilde{e}_i and \tilde{f}_i on b considered as the multipartition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$. This time, one considers the word W_i obtained by reading the addable and removable nodes of $\underline{\lambda}$ with content i successively in the partitions $\lambda^{(0)}, \dots, \lambda^{(\ell-1)}$. This reading is well defined since a partition cannot both content addable and removable nodes with content i . Encode in W_i each removable node by R and each addable node by A . Let \widetilde{W}_i be the word obtained by deleting the factors RA successively in the encoding. One can write

$$\widetilde{W}_i = A^p R^q. \tag{6}$$

Then $\tilde{f}_i(b)$ is obtained from $\underline{\lambda}$ by adding the node γ corresponding to the rightmost symbol A in \widetilde{W}_i .

2.3 The crystal isomorphism between $B_\infty(\omega_k) \otimes B_\infty(\omega_l)$ and $B_\infty(\omega_l) \otimes B_\infty(\omega_k)$

Consider k and l two integers. We denote by $\psi_{k,l}$ the crystal graph isomorphism

$$\psi_{k,l} : B_\infty(\omega_k) \otimes B_\infty(\omega_l) \xrightarrow{\cong} B_\infty(\omega_l) \otimes B_\infty(\omega_k). \quad (7)$$

To give the explicit combinatorial description of $\psi_{k,l}$, we start by considering the crystal graph isomorphism $\theta_{k,l}$ between $B_\infty(k) \otimes B_\infty(l)$ and $B_\infty(l) \otimes B_\infty(k)$. Consider $C_1 \otimes C_2$ in $B_\infty(k) \otimes B_\infty(l)$. We are going to associate to $C_1 \otimes C_2$ a vertex $C'_2 \otimes C'_1 \in B_\infty(l) \otimes B_\infty(k)$.

Suppose first $k \geq l$. Consider $x_1 = \min\{t \in C_2\}$. We associate to x_1 the integer $y_1 \in C_1$ such that

$$y_1 = \begin{cases} \max\{z \in C_1 \mid z \leq x_1\} & \text{if } \min\{z \in C_1\} \leq x \\ \max\{z \in C_1\} & \text{otherwise} \end{cases}. \quad (8)$$

We repeat the same procedure to the columns $C_1 \setminus \{y_1\}$ and $C_2 \setminus \{x_1\}$. By induction this yields a sequence $\{y_1, \dots, y_l\} \subset C_1$. Then we define C'_2 as the column obtained by reordering the integers of $\{y_1, \dots, y_l\}$ and C'_1 as the column obtained by reordering the integers of $C_1 \setminus \{y_1, \dots, y_l\} + C_2$. Now, suppose $k < l$. Consider $x_1 = \min\{t \in C_1\}$. We associate to x_1 the integer $y_1 \in C_2$ such that

$$y_1 = \begin{cases} \min\{z \in C_2 \mid x_1 \leq z\} & \text{if } \max\{z \in C_2\} \geq x \\ \min\{z \in C_2\} & \text{otherwise} \end{cases}. \quad (9)$$

We repeat the same procedure to the columns $C_1 \setminus \{x_1\}$ and $C_2 \setminus \{y_1\}$ and obtain a sequence $\{y_1, \dots, y_k\} \subset C_2$. Then we define C'_1 as the column obtained by reordering the integers of $\{y_1, \dots, y_k\}$ and C'_2 as the column obtained by reordering the integers of $C_2 \setminus \{y_1, \dots, y_k\} + C_1$. We denote by $\theta_{k,l}$ the map defined from $B_\infty(k) \otimes B_\infty(l)$ to $B_\infty(l) \otimes B_\infty(k)$ by setting $\theta_{k,l}(C_1 \otimes C_2) = C'_2 \otimes C'_1$.

Example 2.3.1 Consider $C_1 = \begin{array}{|c|} \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline \end{array}$ and $C_2 = \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}$. We obtain $\{y_1, y_2, y_3, y_4, y_5\} = \{9, 2, 4, 5, 7\}$. This

gives $C'_2 = \begin{array}{|c|} \hline 9 \\ \hline 7 \\ \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline \end{array}$ and $C'_1 = \begin{array}{|c|} \hline 8 \\ \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 3 \\ \hline 1 \\ \hline \end{array}$. Thus $\theta_{6,5}(C_1 \otimes C_2) = C'_2 \otimes C'_1$. One can also easily verifies that $\theta_{5,6}(C'_2 \otimes C'_1) = C_1 \otimes C_2$.

Proposition 2.3.2 The map $\theta_{k,l}$ is an isomorphism of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -crystals that is, for any integer i and any vertex $C_1 \otimes C_2 \in B_\infty(k) \otimes B_\infty(l)$, we have

$$\theta_{k,l} \circ \tilde{e}_i(C_1 \otimes C_2) = \tilde{e}_i(C'_2 \otimes C'_1) \quad \text{and} \quad \theta_{k,l} \circ \tilde{f}_i(C_1 \otimes C_2) = \tilde{f}_i(C'_2 \otimes C'_1). \quad (10)$$

Proof. Choose $a < b$ two integers such that the letters of C_1 and C_2 belong to $[a, b]$. It follows from Proposition 3.21 of [23], that the isomorphism between the finite $\mathcal{U}_v(\mathfrak{sl}_{a-1, b+2})$ -crystals $B_{a-1, b+2}(k) \otimes B_{a-1, b+2}(l)$ and $B_{a-1, b+2}(l) \otimes B_{a-1, b+2}(k)$ is given by the map $\theta_{k, l}$. Since the actions of the crystal operators \tilde{e}_i and \tilde{f}_i with $i \in [a-1, b+2[$ on the crystals $B_{a-1, b+2}(l)$ and $B_{a-1, b+2}(k)$ can be obtained from the crystal structures of $B_\infty(k)$ and $B_\infty(l)$, this implies the commutation relations (10) for any $i \in [a-1, b+2[$. Now if $i \notin [a-1, b+2[$ we have $\tilde{e}_i(C_1 \otimes C_2) = \tilde{f}_i(C_1 \otimes C_2) = 0$ because the letters of C_1 and C_2 belong to $[a, b]$. Similarly $\tilde{e}_i(C'_2 \otimes C'_1) = C'_2 \otimes C'_1 = 0$ because the letters of $C_1 \cup C_2$ are the same as those of $C'_1 \cup C'_2$. Hence (10) holds for any integer i . ■

Now consider $C_1 \otimes C_2 \in B_\infty(\omega_k) \otimes B_\infty(\omega_l)$. Let a be any integer such that C_1 and C_2 both contain all the integers $x < a$. Set $C_1 = \pi_a(C_1)$, $C_2 = \pi_a(C_2)$ and $\theta_{k, l}(C_1 \otimes C_2) = C'_2 \otimes C'_1$. Since C_1 and C_2 both contain only letters $x \geq a$, it is also the case for the columns C'_1 and C'_2 by the previous combinatorial description of the isomorphism $\theta_{k, l}$. Write C'_1 and C'_2 for the infinite columns obtained respectively from C'_1 and C'_2 by adding boxes containing all the letters $x < a$. Then $C'_1 \in B_\infty(\omega_k)$ and $C'_2 \in B_\infty(\omega_l)$. Indeed $C_1 \in B_\infty(\omega_k)$, $C_2 \in B_\infty(\omega_l)$ and C_1, C'_1 (resp. C_2, C'_2) have the same height.

Corollary 2.3.3 (of Proposition 2.3.2) *For any $C_1 \otimes C_2 \in B_\infty(\omega_k) \otimes B_\infty(\omega_l)$ we have with the above notation $\psi_{k, l}(C_1 \otimes C_2) = C'_2 \otimes C'_1$.*

Proof. Recall that $\mathcal{C}^{(k)} \otimes \mathcal{C}^{(l)}$ is the highest weight vertex of $B_\infty(\omega_k) \otimes B_\infty(\omega_l)$. Write $C_1 \otimes C_2 = \tilde{F}(\mathcal{C}^{(k)} \otimes \mathcal{C}^{(l)})$ where \tilde{F} is a sequence of crystal operators \tilde{f}_i , $i \in \mathbb{Z}$. The crystal graph isomorphism $\psi_{k, l}$ must send the highest vertex of $B_\infty(\omega_k) \otimes B_\infty(\omega_l)$ on the highest weight vertex of $B_\infty(\omega_l) \otimes B_\infty(\omega_k)$. Thus, we have $\psi_{k, l}(\mathcal{C}^{(k)} \otimes \mathcal{C}^{(l)}) = \mathcal{C}^{(l)} \otimes \mathcal{C}^{(k)}$. In particular, the Corollary is true for $C_1 \otimes C_2 = \mathcal{C}^{(k)} \otimes \mathcal{C}^{(l)}$.

Now consider $C_1 \otimes C_2 \in B_\infty(\omega_k) \otimes B_\infty(\omega_l)$ and choose a an integer such that C_1 and C_2 both contain all the integers $x < a$. Set $C^{(k)} = \pi_a(C^{(k)})$ and $C^{(l)} = \pi_a(C^{(l)})$. Similarly write $C_1 = \pi_a(C_1)$, $C_2 = \pi_a(C_2)$. Since $C_1 \otimes C_2 = \tilde{F}(\mathcal{C}^{(k)} \otimes \mathcal{C}^{(l)})$ we obtain from Lemma 2.2.1 the equality $C_1 \otimes C_2 = \tilde{F}(C^{(k)} \otimes C^{(l)})$. By definition of the crystal isomorphism $\theta_{k, l}$, we thus have $C'_2 \otimes C'_1 = \tilde{F}(C^{(l)} \otimes C^{(k)})$. The columns $C^{(k)}, C^{(l)}, C'_1$ and C'_2 contains only letters $x \geq a$. Hence we can write $\tilde{F} = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}$ with $i_m \geq a$ for any $m \in \{1, \dots, r\}$. Now the infinite columns $\mathcal{C}^{(l)}, \mathcal{C}^{(k)}, \mathcal{C}'_1$ and \mathcal{C}'_2 are respectively obtained from $C^{(k)}, C^{(l)}, C'_1$ and C'_2 by adding all the letters $x < a$. We can thus deduce from the equality $C'_2 \otimes C'_1 = \tilde{F}(C^{(l)} \otimes C^{(k)})$, the equality $C'_2 \otimes C'_1 = \tilde{F}(\mathcal{C}^{(l)} \otimes \mathcal{C}^{(k)})$. This implies that $\psi_{k, l}(C_1 \otimes C_2) = C'_2 \otimes C'_1$. ■

Example 2.3.4 *Consider $\lambda^{(1)} = (5, 5, 5, 4, 4, 3) \in B_\infty(\omega_4)$ and $\lambda^{(2)} = (4, 4, 4, 3, 2) \in B_\infty(\omega_3)$. The*

columns C_1, C_2 such that $C_1 \otimes C_2 = (\lambda^{(1)}, \lambda^{(2)}) \in B_\infty(\omega_4) \otimes B_\infty(\omega_3)$ are

$$C_1 = \begin{array}{|c|} \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array} \text{ and } C_2 = \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 3 \\ \hline 1 \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array}. \text{ Hence } C_1 = \begin{array}{|c|} \hline 9 \\ \hline 8 \\ \hline 7 \\ \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline \end{array} \text{ and } C_2 = \begin{array}{|c|} \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \text{ for } a = \bar{2}.$$

We deduce from Example 2.3.1 that

$$C'_2 = \begin{array}{|c|} \hline 9 \\ \hline 7 \\ \hline 5 \\ \hline 4 \\ \hline 2 \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array} \text{ and } C'_1 = \begin{array}{|c|} \hline 8 \\ \hline 7 \\ \hline 6 \\ \hline 5 \\ \hline 3 \\ \hline 1 \\ \hline \bar{2} \\ \hline \dots \\ \hline \end{array}.$$

Thus we can write $\psi_{4,3}(\lambda^{(1)}, \lambda^{(2)}) = (\mu^{(2)}, \mu^{(1)})$ with $\mu^{(1)} = (4, 4, 4, 4, 3, 2)$ and $\mu^{(2)} = (6, 5, 4, 4, 3)$.

Remark: The columns C_1 and C_2 obtained in the previous algorithm depend on the integer a considered. Nevertheless, this is not the case for the resulting infinite columns C'_1 and C'_2 as long as $a \ll 0$.

3 Crystals in type $A_{e-1}^{(1)}$

We now turn to the problem of studying the crystals in type $A_{e-1}^{(1)}$. In this section, we recall and show basic facts on their combinatorial descriptions.

3.1 Background on $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$

In order to link the different labellings of the crystal graphs in type $A_{e-1}^{(1)}$ appearing in the literature, we shall need the two following Dynkin diagrams

$$A_{e-1}^{(1),+} : \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \quad \dots \quad \circ \\ | \quad \quad \quad | \\ 1 \quad \quad \quad e-1 \end{array} \text{ and } A_{e-1}^{(1),-} : \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \quad \dots \quad \circ \\ | \quad \quad \quad | \\ e-1 \quad \quad \quad 1 \end{array}. \quad (11)$$

Let $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ be the affine quantum groups defined respectively from the root systems in type $A_{e-1}^{(1),+}$ and $A_{e-1}^{(1),-}$ (see for example [22], Chapter 6). Observe that this notation,

which is not related to the triangular decomposition of $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$, only means we are considering two copies of $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ with Dynkin diagrams $A_{e-1}^{(1),+}$ and $A_{e-1}^{(1),-}$. Write $\{\Lambda_0^+, \Lambda_1^+, \dots, \Lambda_{e-1}^+\}$ and $\{\Lambda_0^-, \Lambda_1^-, \dots, \Lambda_{e-1}^-\}$ for the dominant weights of the root systems $A_{e-1}^{(1),+}$ and $A_{e-1}^{(1),-}$. We have then $\Lambda_k^+ = \Lambda_{e-k}^-$. Write $\{E_i^+, F_i^+, K_i^+ \mid i = 0, \dots, e-1\}$ and $\{E_i^-, F_i^-, K_i^- \mid i = 0, \dots, e-1\}$ for the sets of Chevalley generators respectively in $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$. Let $v^{d^+} \in \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ and $v^{d^-} \in \mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ be the quantum derivation operators. Then the map

$$\iota : \begin{cases} \mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e) \rightarrow \mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e) \\ v^{d^+} \mapsto v^{d^-} \\ E_i^+ \mapsto E_{e-i}^-, F_i^+ \mapsto F_{e-i}^- \text{ and } K_i^+ \mapsto K_{e-i}^- \end{cases} \quad (12)$$

is an isomorphism of algebras.

We associate to $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$ the dominant weights $\Lambda_{\underline{s}}^+ = \sum_{k=0}^{\ell-1} \Lambda_{s_k(\bmod e)}^+$ and $\Lambda_{\underline{s}}^- = \sum_{k=0}^{\ell-1} \Lambda_{s_k(\bmod e)}^-$. Let $V_e(\Lambda_{\underline{s}}^+)$ (resp. $V_e(\Lambda_{\underline{s}}^-)$) be the irreducible $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ -module (resp. $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ -module) of highest weight $\Lambda_{\underline{s}}^+$ (resp. $\Lambda_{\underline{s}}^-$). These modules can be constructed by using the Fock space representation $\mathfrak{F}_e^{\underline{s}}$ of level ℓ . Let $\Pi_{\ell, n}$ be the set of multipartitions $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ of length ℓ with rank n , i.e. such that $|\lambda^{(0)}| + \dots + |\lambda^{(\ell-1)}| = n$. The Fock space $\mathfrak{F}_e^{\underline{s}}$ is defined as the $\mathbb{C}(v)$ -vector space generated by the symbols $|\underline{\lambda}, \underline{s}\rangle$ with $\underline{\lambda} \in \Pi_{\ell, n}$

$$\mathfrak{F}_e^{\underline{s}} = \bigoplus_{n \geq 0} \bigoplus_{\underline{\lambda} \in \Pi_{\ell, n}} \mathbb{C}(v) |\underline{\lambda}, \underline{s}\rangle.$$

The Fock space $\mathfrak{F}_e^{\underline{s}}$ can be endowed with the structure of a $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ -module or equivalently with the structure of a $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ -module. These actions are defined by introducing total orders $\prec_{\underline{s}}^+$ and $\prec_{\underline{s}}^-$ on the i -nodes of the multipartitions. Consider $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ a multipartition and suppose \underline{s} (called the multicharge) is fixed. Then the nodes of $\underline{\lambda}$ can be identified with the triplet $\gamma = (a, b, c)$ where $c \in \{0, \dots, \ell-1\}$ and a, b are respectively the row and column indices of the node γ in $\lambda^{(c)}$. The content of γ is the integer $c(\gamma) = b - a + s_c$ and the residue $\text{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$\text{res}(\gamma) \equiv c(\gamma) \pmod{e}. \quad (13)$$

We will say that γ is an i -node of $\underline{\lambda}$ when $\text{res}(\gamma) \equiv i \pmod{e}$. Let $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ be two i -nodes in $\underline{\lambda}$. We define the total order $\prec_{\underline{s}}^+$ and $\prec_{\underline{s}}^-$ on the addable and removable i -nodes of $\underline{\lambda}$ by setting

$$\gamma_1 \prec_{\underline{s}}^+ \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 < c_2 \end{cases}, \quad (14)$$

$$\gamma_1 \prec_{\underline{s}}^- \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2 \end{cases}. \quad (15)$$

Using these orders, it is possible to define an action of $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ -module and an action of $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ -module on $\mathfrak{F}_e^{\underline{s}}$. These modules will be denoted by $\mathfrak{F}_e^{\underline{s},+}$ and $\mathfrak{F}_e^{\underline{s},-}$. For these actions the empty multipartition $\underline{\emptyset} = (\emptyset, \dots, \emptyset)$ is a highest weight vector respectively of highest weight $\Lambda_{\underline{s}}^+$ and $\Lambda_{\underline{s}}^-$.

We denote by $V_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $V_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$ the irreducible components with highest weight vector \emptyset in $\mathfrak{F}_e^{\underline{s},+}$ and $\mathfrak{F}_e^{\underline{s},-}$. We refer the reader to [14] for a detailed exposition. Note that the modules $V_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ (resp. $V_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$) such that $\underline{s} \in \underline{\mathfrak{s}}$ are all isomorphic to the abstract module $V_e(\Lambda_{\underline{s}}^+)$ (resp. $V_e(\Lambda_{\underline{s}}^-)$). However, the corresponding actions of the Chevalley operators do not coincide in general. This will yield different parametrizations of the associated crystals.

3.2 Crystal graph of the Fock space $\mathfrak{F}_e^{\underline{s},+}$ and $\mathfrak{F}_e^{\underline{s},-}$

The modules $\mathfrak{F}_e^{\underline{s},+}$ and $\mathfrak{F}_e^{\underline{s},-}$ are integrable modules, thus by the general theory of Kashiwara, they admit crystal bases. Write $B_e^{\underline{s},+}$ and $B_e^{\underline{s},-}$ for the crystal graphs corresponding to the action of $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v^-(\widehat{\mathfrak{sl}}_e)$ on $\mathfrak{F}_e^{\underline{s}}$. These crystals are labelled by multipartitions and their crystal graph structure can be explicitly described by using the total orders $\prec_{\underline{s}}^+$ and $\prec_{\underline{s}}^-$. Consider two multipartitions $\underline{\lambda}, \underline{\mu}$ and an integer $i \in \{0, \dots, e-1\}$. The crystal graph $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ (resp. $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$) of $V_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ (resp. $V_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$) is the connected component of $B_e^{\underline{s},+}$ (resp. $B_e^{\underline{s},-}$) whose highest weight vertex is the empty multipartition.

3.2.1 Crystal structure on $B_e^{\underline{s},+}$

Consider the set of addable and removable i -nodes of $\underline{\lambda}$ (see Section 2.2 for the definition of removable and addable nodes). Let w_i be the word obtained by writing the addable and removable i -nodes of $\underline{\lambda}$ in *decreasing* order with respect to $\prec_{\underline{s}}^+$, next by encoding each addable i -node by the letter A and each removable i -node by the letter R . Write $\tilde{w}_i = A^p R^q$ for the word obtained from w_i by deleting as many of the factors RA as possible. If $p > 0$, let γ be the rightmost addable i -node in \tilde{w}_i . The node γ is called the good i -node. Then we have an arrow $\underline{\lambda} \xrightarrow{i} \underline{\mu}$ in $B_e^{\underline{s},+}$ if and only if $\underline{\mu}$ is obtained from $\underline{\lambda}$ by adding the good i -node γ .

3.2.2 Crystal structure on $B_e^{\underline{s},-}$

Consider the set of addable and removable i -nodes of $\underline{\lambda}$. Let w_i be the word obtained by writing these i -nodes in *increasing* order with respect to $\prec_{\underline{s}}^-$, next by encoding each addable i -node by the letter A and each removable i -node by the letter R . Write $\tilde{w}_i = A^p R^q$ for the word obtained from w_i by deleting as many of the factors RA as possible. If $p > 0$, let γ be the rightmost addable i -node in \tilde{w}_i . Then we have an arrow $\underline{\lambda} \xrightarrow{i} \underline{\mu}$ in $B_e^{\underline{s},-}$ if and only if $\underline{\mu}$ is obtained from $\underline{\lambda}$ by adding the good i -node γ .

3.2.3 Link between the crystals $B_e^{\underline{s},+}$ and $B_e^{\underline{s},-}$

For any multicharge $\underline{s} = (s_0, \dots, s_{\ell-1})$, we denote by \underline{s}^* the multicharge $\underline{s}^* = (e - s_0, \dots, e - s_{\ell-1})$. According to the isomorphism (12), there should exist some bijections $\nu : B_e^{\underline{s},+} \rightarrow B_e^{\underline{s}^*,-}$ such that, given any two multipartitions $\underline{\lambda}, \underline{\mu}$,

$$\underline{\lambda} \xrightarrow{i} \underline{\mu} \text{ in } B_e^{\underline{s},+} \iff \nu(\underline{\lambda}) \xrightarrow{e-i} \nu(\underline{\mu}) \text{ in } B_e^{\underline{s}^*,-}. \quad (16)$$

Note that ν cannot be unique since each of the crystals $B_e^{\underline{s},+}$ and $B_e^{\underline{s}^*,-}$ contains isomorphic connected components. To each multipartition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ in $B_e^{\underline{s},+}$, we associate its

conjugate multipartition $\underline{\lambda}' = (\lambda'^{(0)}, \dots, \lambda'^{(\ell-1)})$ in $B_e^{\underline{s}^*, +}$ where for any $k = 0, \dots, \ell - 1$, $\lambda'^{(k)}$ is the conjugate partition of $\lambda^{(k)}$.

Proposition 3.2.1 *The map $\xi : \underline{\lambda} \mapsto \underline{\lambda}'$ from $\mathfrak{F}_e^{\underline{s}^*, +}$ to $\mathfrak{F}_e^{\underline{s}^*, -}$ is a bijection and satisfies (16).*

Proof. Consider $\underline{\lambda}$ and $\underline{\lambda}'$ as vertices respectively of the crystals $\mathfrak{F}_e^{\underline{s}^*, +}$ and $\mathfrak{F}_e^{\underline{s}^*, -}$. Let $\gamma = (a, b, c)$ be a node appearing at the right end of the a -th row of $\lambda^{(c)}$ in $\underline{\lambda}$. One associates to γ the node γ' appearing on the top of the a -th column of $\lambda'^{(k)}$ in $\underline{\lambda}'$. Then by definition of the node γ' , we have $\gamma' = (b, a, c)$. This gives for the content of the nodes γ and γ'

$$c(\gamma) = b - a + s_c \text{ and } c(\gamma') = a - b + e - s_c.$$

Now observe that γ is a removable (resp. addable) node if and only if γ' is a removable (resp. addable) node. We have thus

$$\gamma \text{ is an } A \text{ (resp. } R) \text{ } i\text{-node} \iff \gamma' \text{ is an } A \text{ (resp. } R) \text{ } (e - i)\text{-node.} \quad (17)$$

Let w_i be the word obtained by writing the addable or removable i -nodes of $\underline{\lambda}$ in *decreasing* order with respect to $\prec_{\underline{s}}^+$ as in Section 3.2.1. Similarly, let w'_{e-i} be the word obtained by writing the addable or removable $(e - i)$ -nodes of $\underline{\lambda}'$ in *increasing* order with respect to $\prec_{\underline{s}}^+$ as in Section 3.2.2. Write $w_i = \gamma_1 \cdots \gamma_r$ where for any $m = 1, \dots, r$, γ_m is an i -node that is addable or removable. Then by definition of the order $\prec_{\underline{s}}^+$ and $\prec_{\underline{s}}^-$ (see (14) and (15)), we have

$$w'_{e-i} = \gamma'_1 \cdots \gamma'_r.$$

Write $\tilde{w}_i = A^p R^q$ for the word obtained from w_i by the cancellation process of the pairs RA . We deduce from (17) that the word $\tilde{w}'_{e-i} = (A')^p (R')^q$ coincides with the word obtained from w'_{e-i} by the cancellation process of the letters RA . In particular, γ is the good i -node for $\underline{\lambda}$ if and only if γ' is the good $(e - i)$ -node for $\underline{\lambda}'$. Hence the bijection ξ satisfies (16). ■

Since $V_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $V_e^{\underline{s}^*}(\Lambda_{\underline{s}^*}^-)$ are the irreducible components with highest weight vector \emptyset in $\mathfrak{F}_e^{\underline{s}^*, +}$ and $\mathfrak{F}_e^{\underline{s}^*, -}$, their crystals graphs $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\underline{s}^*}(\Lambda_{\underline{s}^*}^-)$ can be realized as the connected components of highest weight vertex \emptyset in $B_e^{\underline{s}^*, +}$ and $B_e^{\underline{s}^*, -}$. Since $\xi(\emptyset) = \emptyset$, we derive from the previous proposition the equivalence

$$\underline{\lambda} \xrightarrow{i} \underline{\mu} \text{ in } B_e^{\underline{s}}(\Lambda_{\underline{s}}^+) \iff \xi(\underline{\lambda}) \xrightarrow{e-i} \xi(\underline{\mu}) \text{ in } B_e^{\underline{s}^*}(\Lambda_{\underline{s}^*}^-). \quad (18)$$

Definition 3.2.2 *The vertices of $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\underline{s}^*}(\Lambda_{\underline{s}^*}^-)$ are called the Uglov multipartitions.*

Example 3.2.3 *Suppose $\ell = 1$ and $\underline{s} = (s)$.*

- *The vertices of $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ are the e -restricted partitions, that is the partitions $\lambda = (\lambda_1, \dots, \lambda_p)$ such that $\lambda_i - \lambda_{i+1} \leq e - 1$ for any $i = 1, \dots, p - 1$.*
- *The vertices of $B_e^{\underline{s}^*}(\Lambda_{\underline{s}^*}^-)$ are the e -regular partitions, that is the partitions λ with at most $e - 1$ parts equal.*

Remarks:

- (i) : The crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ are essentially those used in [2] whereas the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$ appear in [7] and [13].
- (ii) : Consider $\underline{s} = (s_0, \dots, s_{\ell-1})$ and $\underline{s}' = (s'_0, \dots, s'_{\ell-1})$ two multicharges such that $s_k \equiv s'_k$ for any $k = 0, \dots, \ell - 1$. Then $\Lambda_{\underline{s}}^+ = \Lambda_{\underline{s}'}^+$, thus the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\underline{s}'}(\Lambda_{\underline{s}}^+)$ are isomorphic. The combinatorial description of this isomorphism is complicated in general (but see Section 5). Nevertheless, in the case when there exists an integer d such that $s_k = s'_k + de$ for any $k = 0, \dots, \ell - 1$, one derives easily from the description of the crystal structure on $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ that the relevant isomorphism coincide with the identity map. The situation is the same for the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$ and $B_e^{\underline{s}'}(\Lambda_{\underline{s}}^-)$.

When the multicharge $\underline{s} = (s_0, \dots, s_{\ell-1})$ satisfies $0 \leq s_0 \leq \dots \leq s_{\ell-1} \leq e - 1$, there exists a combinatorial description of the Uglov multipartitions labelling $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$ due to Foda, Leclerc, Okado, Thibon and Welsh.

Proposition 3.2.4 [6] [14] *When $0 \leq s_0 \leq \dots \leq s_{\ell-1} \leq e - 1$, the multipartition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ belongs to $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$ if and only if:*

1. $\underline{\lambda}$ is cylindric, that is, for every $k = 0, \dots, \ell - 2$ we have $\lambda_i^{(k)} \geq \lambda_{i+s_{k+1}-s_k}^{(k+1)}$ for all $i > 1$ (the partitions are taken with an infinite numbers of empty parts) and $\lambda_i^{(\ell-1)} \geq \lambda_{i+e+s_0-s_{\ell-1}}^{(0)}$ for all $i > 1$
2. for all $r > 0$, among the residues appearing at the right ends of the length r rows of $\underline{\lambda}$, at least one element of $\{0, 1, \dots, e - 1\}$ does not appear.

Remarks:

- (i) : By using (18), it is easy to deduce a combinatorial characterization of the Uglov multipartitions appearing in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ when $0 \leq s_{\ell-1} \leq \dots \leq s_0 \leq e - 1$. These multipartitions are called the FLOTW multipartitions.
- (ii) : In the sequel we will essentially consider the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ because they can be embedded in a natural way in $\mathcal{U}_v(\mathfrak{sl}_{\infty})$ -crystals. Thanks to (18), it will be easy to translate our statements to make them compatible with the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^-)$.
- (iii) : There exists also another realization $B_e^a(\Lambda_{\underline{s}}^+)$ of the abstract crystal $B_e(\Lambda_{\underline{s}}^+)$ which is in particular used by Ariki [2]. This realization is obtained by defining the Fock space of level ℓ as a tensor product of Fock spaces of level 1. It is suited to the Specht modules theory for Ariki-Koike algebras introduced by Dipper, James and Mathas [5]. Note that in the level 2 case, Geck [8] has generalized this theory by defining Specht modules adapted to the definition of the Fock spaces used in this paper. It is expected that similar results hold in higher level.
- (iv) : The multipartitions which label the vertices of the crystals $B_e^a(\Lambda_{\underline{s}}^+)$ are called the Kleshchev multipartitions. For any nonnegative integer n , the subgraph of the crystal $B_e^a(\Lambda_{\underline{s}}^+)$ containing all the Kleshchev multipartitions of rank $m \leq n$ coincide with the corresponding subgraph of $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ when the multicharge \underline{s} satisfies $s_i - s_{i+1} > n - 1$ for any $i = 0, \dots, \ell - 2$. As a consequence Kleshchev multipartitions are particular cases of Uglov multipartitions.

4 Embedding of $B_e^s(\Lambda_{\underline{s}}^+)$ in $B_{\infty}^{s^\diamond}(\omega_{\underline{s}})$

The aim of this section is to show that we have an embedding of crystals from $B_e^s(\Lambda_{\underline{s}}^+)$ to $B_{\infty}^{s^\diamond}(\omega_{\underline{s}})$ with $\underline{s}^\diamond = (s_{\ell-1}, \dots, s_0)$.

4.1 Embedding of $B_e^s(\Lambda_{\underline{s}}^+)_{\leq n}$ in $B_f^s(\Lambda_{\underline{s}}^+)_{\leq n}$

In the sequel we denote by \tilde{E}_i and \tilde{F}_i , $i = 0, \dots, e-1$ the crystal operators corresponding to the $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ -crystals. Consider $\underline{\lambda}$ and $\underline{\mu}$ two multipartitions in $\mathfrak{F}_e^{s,+}$ and $i \in \{0, \dots, e-1\}$ such that $\tilde{F}_i(\underline{\lambda}) = \underline{\mu}$. Then $\underline{\mu}$ is obtained by adding an i -node γ in $\underline{\lambda}$. Set $j = c(\gamma)$. In the sequel we slightly abuse the notation and write $\tilde{F}_j(\underline{\lambda}) = \underline{\mu}$ although j does not belong to $\{0, \dots, e-1\}$ in general. Note that j is uniquely determined from the equality $\tilde{F}_i(\underline{\lambda}) = \underline{\mu}$. Observe that with this convention, an arrow in $\mathfrak{F}_e^{s,+}$ can be labelled by any integer. To recover the original labelling of a $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ -crystal, it suffices to read these labels modulo e . With this convention, when there exists an arrow from the multipartition $\underline{\lambda}$ to the multipartition $\underline{\mu}$ in both $F_e^{s,+}$ and $F_f^{s,+}$ (with $e \neq f$), this arrows can be pictured $\underline{\lambda} \xrightarrow{j} \underline{\mu}$ where j is the content of the node added to $\underline{\lambda}$ to obtain $\underline{\mu}$. In particular the label so obtained is independent of e and f and it makes sense to write $\tilde{F}_j(\underline{\lambda}) = \underline{\mu}$ in $B_e^s(\Lambda_{\underline{s}}^+)$ and $B_f^s(\Lambda_{\underline{s}}^+)$.

For any multicharge $\underline{s} = (s_0, \dots, s_{\ell-1})$ in \mathbb{Z}^ℓ , we set $\|\underline{s}\| = \max\{|s_k| \mid k = 0, \dots, \ell-1\}$ where $|s_k| = s_k$ if s_k is nonnegative and $|s_k| = -s_k$ otherwise. The following proposition is a generalization in level ℓ of Proposition 4.1 in [13].

Proposition 4.1.1 *Let n be a nonnegative integer. Consider $\underline{\lambda}$ a multipartition in $B_e^s(\Lambda_{\underline{s}}^+)$ such that $\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_n}(\emptyset)$. Then $|\underline{\lambda}| = n$. Moreover, for any integer $f \geq n + \|\underline{s}\|$, we have $\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_n}(\emptyset)$ in $B_f^s(\Lambda_{\underline{s}}^+)$.*

Proof. Since each crystal operator \tilde{F}_{j_k} adds a node on the multipartition $\tilde{F}_{j_{k+1}} \cdots \tilde{F}_{j_n}(\emptyset)$, we have $|\underline{\lambda}| = n$.

To prove the second assertion of our proposition, we proceed by induction on n . When $n = 0$, the proposition is immediate. Suppose our statement true for any multipartition $\underline{\lambda}$ with $|\underline{\lambda}| = n-1$. Consider $\underline{\mu} \in B_e^s(\Lambda_{\underline{s}}^+)$ with $|\underline{\mu}| = n$.

With the above convention, there exist an integer j and a multipartition $\underline{\lambda} \in B_e^s(\Lambda_{\underline{s}}^+)$ such that $\tilde{F}_j(\underline{\lambda}) = \underline{\mu}$ and $\underline{\mu}$ is obtained by adding a node with content j to $\underline{\lambda}$. Let $i \in \{0, \dots, e-1\}$ be such that $i \equiv j \pmod{e}$. Write w_i^e for the word obtained by writing the addable or removable i -nodes of $\underline{\lambda}$ in *decreasing* order with respect to $\prec_{\underline{s}}^+$ as in Section 3.2.1.

Choose any integer $f \geq n + \|\underline{s}\|$. Observe that each j -node $\gamma = (a, b, c)$ in $\underline{\lambda}$ verifies $c(\gamma) = \text{res}(\gamma)$ when its residue is computed modulo f . Indeed, we have $c(\gamma) = b - a + s_c$ and thus

$$-f < 1 - n + s_c \leq c(\gamma) \leq n - 1 + s_c < f. \quad (19)$$

Write w_j^f for the word obtained by writing the addable or removable j -nodes of $\underline{\lambda}$ in *decreasing* order with respect to $\prec_{\underline{s}}^+$. Then the nodes contributing to w_j^f are the addable and removable

nodes of $\underline{\lambda}$ with content j . This implies that w_j^f is a subword of w_i^e , that is each node appearing in w_j^f also appears in w_i^e . If γ is a node in w_i^e which does not belong to w_j^f , we must thus have $c(\gamma) \neq j$. Since the nodes of w_i^e are ordered according to $\prec_{\underline{s}}^+$, we deduce that w_j^f is a factor of w_i^e (i.e., there is no node γ such that $c(\gamma) \neq j$ between two nodes of w_j^f).

Write \widehat{w}_i^e and \widehat{w}_j^f respectively for the words obtained from w_i^e and w_j^f by the cancellation process of the factors RA . Since these words do not depend on the order of the consecutive factors RA which are deleted, \widehat{w}_j^f is a factor of \widehat{w}_i^e . Let γ_0 be the good i -node in \widehat{w}_i^e . Suppose that γ_0 is not a node contributing to \widehat{w}_j^f . Then, there exists a node δ_0 in w_j^f which can be paired with γ_0 to produce a factor RA . In this case, γ_0 and δ_0 are nodes of w_i^e because w_j^f is a factor of w_i^e . This gives a contradiction. Indeed, the word w_i^e does not depend on the cancellation process for the factors RA . So, if we can pair γ_0 and δ_0 together in order to obtain a factor RA in w_j^f , we can do the same pairing in w_i^e (for w_j^f is a factor of w_i^e) and γ_0 cannot be the good i -node in \widehat{w}_i^e . Hence we have shown that γ_0 is a j -node of \widehat{w}_j^f . Since \widehat{w}_j^f is a factor of \widehat{w}_i^e , γ_0 is the rightmost addable j -node in \widehat{w}_j^f , thus is the good j -node for w_j^f . By the induction hypothesis, one has $\underline{\lambda} = \widetilde{F}_{j_1} \cdots \widetilde{F}_{j_n}(\emptyset)$ in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$ because $f \geq n + \|\underline{s}\| > n - 1 + \|\underline{s}\|$. Moreover, we have $\widetilde{F}_j(\underline{\lambda}) = \underline{\mu}$ in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$ by the previous arguments. Thus $\underline{\mu} = \widetilde{F}_j \widetilde{F}_{j_1} \cdots \widetilde{F}_{j_n}(\emptyset)$ in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$. ■

For any fixed nonnegative integer n , write $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)_{\leq n} = \{\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+) \mid |\underline{\lambda}| \leq n\}$. We deduce from the above proposition, that the identity map $\iota_{e,f} : \underline{\lambda} \mapsto \underline{\lambda}$ from $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)_{\leq n}$ to $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)_{\leq n}$ is an embedding of crystals.

4.2 Embedding of $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ in $B_{\infty}^{\underline{s}^{\diamond}}(\omega_{\underline{s}})$

For any multipartition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ we write $\underline{\lambda}^{\diamond} = (\lambda^{(\ell-1)}, \dots, \lambda^{(0)})$.

Proposition 4.2.1 *Consider a multicharge \underline{s} and f, n two nonnegative integers such that $f \geq n + \|\underline{s}\|$. Let $\underline{\lambda}$ be a multipartition in $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$ with $|\underline{\lambda}| = n$. Suppose $\underline{\lambda} = \widetilde{F}_{j_1} \cdots \widetilde{F}_{j_n}(\emptyset)$ in $B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$. Then we have $\underline{\lambda}^{\diamond} = \widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n}(\emptyset)$ in the $\mathcal{U}_v(\mathfrak{sl}_{\infty})$ -crystal $B_{\infty}^{\underline{s}^{\diamond}}(\omega_{\underline{s}})$.*

Proof. We proceed by induction on n . When $n = 0$, the proposition is immediate. Suppose our statement true for any multipartition $\underline{\lambda}$ with $|\underline{\lambda}| = n$. Consider $\underline{\mu} \in B_f^{\underline{s}}(\Lambda_{\underline{s}}^+)$ with $|\underline{\mu}| = n$. There exists an integer j and a multipartition $\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ such that $\widetilde{F}_j(\underline{\lambda}) = \underline{\mu}$ and $\underline{\mu}$ is obtained by adding a node with content j to $\underline{\lambda}$. Since $f \geq n + \|\underline{s}\|$, we have $\text{res}(\gamma) = c(\gamma)$ for each node γ in $\underline{\lambda}$ as in (19). Hence the word w_j^f obtained by writing the addable or removable j -nodes of $\underline{\lambda}$ in *decreasing* order with respect to $\prec_{\underline{s}}^+$ contains exactly the addable and removable nodes of $\underline{\lambda}$ with content equal to j . Set $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$. By definition of the order $\prec_{\underline{s}}^+$ (see 14), this means that w_j^f is the word obtained by reading the addable and removable nodes with content equal to j successively in the partitions $\lambda^{(\ell-1)}, \lambda^{(\ell-2)}, \dots, \lambda^{(0)}$ of $\underline{\lambda}$. Moreover each partition $\lambda^{(c)}$, $c = 0, \dots, \ell - 1$ contains at most one addable or removable node with content j because the nodes

with the same content in $\lambda^{(c)}$ must belong to the same diagonal. By the induction hypothesis, we know that $\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_n}(\emptyset)$ in $B_f^s(\Lambda_{\underline{s}}^+)$ and $\underline{\lambda}^\diamond = \tilde{f}_{j_1} \cdots \tilde{f}_{j_n}(\emptyset)$ in $B_\infty^{s^\diamond}(\omega_{\underline{s}})$. The previous arguments show that the word w_j^f coincide with the word W_j obtained by reading the addable and removable nodes of content j in $\underline{\lambda}^\diamond$ as described in Section 2.2 just before (6). Thus we obtain $\underline{\mu} = \tilde{F}_j \tilde{F}_{j_1} \cdots \tilde{F}_{j_n}(\emptyset)$ in $B_f^s(\Lambda_{\underline{s}}^+)$ and $\underline{\mu}^\diamond = \tilde{f}_j \tilde{f}_{j_1} \cdots \tilde{f}_{j_n}(\emptyset)$ in $B_\infty^{s^\diamond}(\omega_{\underline{s}})$. ■

Set $B_\infty^{s^\diamond}(\omega_{\underline{s}})_{\leq n} = \{\underline{\lambda} \in B_\infty^{s^\diamond}(\omega_{\underline{s}}) \mid |\underline{\lambda}| \leq n\}$. We deduce from the previous proposition that the map

$$\varphi_{f,\infty}^{(n)} : \begin{cases} B_e^s(\Lambda_{\underline{s}}^+)_{\leq n} \rightarrow B_\infty^{s^\diamond}(\omega_{\underline{s}})_{\leq n} \\ \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)}) \mapsto \underline{\lambda}^\diamond = \lambda^{(\ell-1)} \otimes \dots \otimes \lambda^{(0)} \end{cases}$$

is an embedding of crystals for any $f \geq n + \|s\|$.

Theorem 4.2.2 *Given any positive integer e and any multicharge \underline{s} the map*

$$\varphi_{e,\infty} : \begin{cases} B_e^s(\Lambda_{\underline{s}}^+) \rightarrow B_\infty^{s^\diamond}(\omega_{\underline{s}}) \\ \underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)}) \mapsto \underline{\lambda}^\diamond = \lambda^{(\ell-1)} \otimes \dots \otimes \lambda^{(0)} \end{cases}$$

is an embedding of crystals : for any $\underline{\lambda} \in B_e^s(\Lambda_{\underline{s}}^+)$ we have

$$\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_r}(\emptyset) \text{ in } B_e^s(\Lambda_{\underline{s}}^+) \implies \underline{\lambda}^\diamond = \tilde{f}_{j_1} \cdots \tilde{f}_{j_r}(\emptyset) \text{ in } B_\infty^{s^\diamond}(\omega_{\underline{s}}).$$

Proof. Consider $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)}) \in B_e^s(\Lambda_{\underline{s}}^+)$ such that $\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_r}(\emptyset)$ and set $|\underline{\lambda}| = n$. By Proposition 4.1.1 we have $\underline{\lambda} = \tilde{F}_{j_1} \cdots \tilde{F}_{j_r}(\emptyset)$ in any crystal $B_f^s(\Lambda_{\underline{s}}^+)$ with $f \geq n + \|s\|$. Fix such an integer f . We derive from Proposition 4.2.1 that

$$\underline{\lambda}^\diamond = \tilde{f}_{j_1} \cdots \tilde{f}_{j_r}(\emptyset) \text{ in } B_\infty^{s^\diamond}(\omega_{\underline{s}}).$$

Clearly the map $\varphi_{e,\infty}$ is injective. Thus it is an embedding from the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -crystal $B_e^s(\Lambda_{\underline{s}}^+)$ to the $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -crystal $B_\infty^{s^\diamond}(\omega_{\underline{s}})$. ■

Remark: According to the previous theorem, the crystal $B_e^s(\Lambda_{\underline{s}}^+)$ can be embedded in $B_\infty^{s^\diamond}(\omega_{\underline{s}})$. The situation is more complicated for the crystal $B_e^a(\Lambda_{\underline{s}}^+)$ labelled by Kleshchev multipartitions (see Remark (iv) after Proposition 3.2.4). Indeed, the subcrystal $B_e^a(\Lambda_{\underline{s}}^+)_{\leq n}$ can be embedded in a crystal $B_\infty(\omega_{\underline{s}(n)})$ where the multicharge $\underline{s}(n) = (s_0(n), \dots, s_{l-1}(n))$ verifies $s_k(n) - s_{k+1}(n) > n - 1$ for any $k = 0, \dots, l - 2$. Since $s(n)$ depends on n , this procedure cannot provide an embedding of the whole crystal $B_e^a(\Lambda_{\underline{s}}^+)$ in a crystal $B_\infty(\omega_{\underline{t}})$ where \underline{t} is a fixed multicharge.

5 Isomorphism class of a multipartition

We can now use the above embedding to obtain a simple characterization of the set of Uglov multipartitions.

5.1 The extended affine symmetric group \widehat{S}_ℓ

We write \widehat{S}_ℓ for the extended affine symmetric group in type $A_{\ell-1}$. The group \widehat{S}_ℓ can be regarded as the group generated by the elements $\sigma_1, \dots, \sigma_{\ell-1}$ and $y_0, \dots, y_{\ell-1}$ together with the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1, \quad \sigma_i^2 = 1$$

with all indices in $\{1, \dots, \ell-1\}$ and

$$y_i y_j = y_j y_i, \quad \sigma_i y_j = y_j \sigma_i \text{ for } j \neq i, i+1, \quad \sigma_i y_i \sigma_i = y_{i+1}.$$

We identify the subgroup of \widehat{S}_ℓ generated by the transpositions σ_i , $i = 1, \dots, \ell-1$ with the symmetric group S_ℓ of rank ℓ . For any $i \in \{0, \dots, \ell-1\}$, we set $z_i = y_1 \cdots y_i$. Write also $\xi = \sigma_{\ell-1} \cdots \sigma_1$ and $\tau = y_\ell \xi$. Since $y_i = z_{i-1}^{-1} z_i$, \widehat{S}_ℓ is generated by the transpositions σ_i with $i \in \{1, \dots, \ell-1\}$ and the elements z_i with $i \in \{1, \dots, \ell\}$. Observe that for any $i \in \{1, \dots, \ell-1\}$, we have

$$z_i = \xi^{\ell-i} \tau^i. \quad (20)$$

This implies that \widehat{S}_ℓ is generated by the transpositions σ_i with $i \in \{1, \dots, \ell-1\}$ and τ . Consider e a fixed positive integer. We obtain a faithful action of \widehat{S}_ℓ on \mathbb{Z}^ℓ by setting for any $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$

$$\sigma_i(\underline{s}) = (s_0, \dots, s_i, s_{i-1}, \dots, s_{\ell-1}) \text{ and } y_i(\underline{s}) = (s_0, \dots, s_{i-1}, s_i + e, \dots, s_{\ell-1}).$$

Then $\tau(\underline{s}) = (s_1, s_2, \dots, s_{\ell-1}, s_0 + e)$. We denote by $\mathcal{C}(\underline{s})$ the orbit of the multicharge \underline{s} under the action of \widehat{S}_ℓ on \mathbb{Z}^ℓ . Clearly each class $\mathcal{C}(\underline{s})$ contains a unique multicharge $\tilde{\underline{s}} = (\tilde{s}_0, \dots, \tilde{s}_{\ell-1})$ such that

$$0 \leq \tilde{s}_{\ell-1} \leq \cdots \leq \tilde{s}_0 \leq e-1. \quad (21)$$

Hence the orbits $\mathcal{C}(\underline{s})$ are parametrized by the multicharges verifying (21). Given any multicharge $\underline{s} = (s_0, \dots, s_{\ell-1}) \in \mathbb{Z}^\ell$, it is easy to determinate $w \in \widehat{S}_\ell$ such that $\tilde{\underline{s}} = w(\underline{s})$. To do this, we compute a sequence of multicharges as follows. Choose $k \in \mathbb{N}$ minimal to have $s_i + ke \geq 0$ for any $i = 0, \dots, \ell-1$. Then $z_{\ell-1}^k(\underline{s}) \in \mathbb{N}^\ell$. Consider $\sigma \in S_\ell$ such that the coordinates of $\underline{s}^{(\ell-1)} = \sigma z_{\ell-1}^k(\underline{s})$ weakly decrease. Write $r_{\ell-2}$ for the quotient of the division of $\underline{s}_{\ell-2}^{(\ell-2)}$ by e and set $\underline{s}^{(\ell-3)} = z_{\ell-2}^{-r_{\ell-2}}(\underline{s}^{(\ell-2)})$. By induction one can compute a sequence $\underline{s}^{(\ell-2)}, \dots, \underline{s}^{(0)}$ such that, for any $i = 1, \dots, \ell-2$, $\underline{s}^{(i-1)} = z_i^{-r_i}(\underline{s}^{(i)})$ where r_i is the quotient of the division of $s_i^{(i)} - s_{i+1}^{(i)}$ by e . We have then $\tilde{\underline{s}} = \underline{s}^{(0)}$ and

$$\tilde{\underline{s}} = w(\underline{s}) = z_0^{-r_0} \cdots z_{\ell-2}^{-r_{\ell-2}} \sigma z_{\ell-1}^k(\underline{s}). \quad (22)$$

5.2 Action of the transformations s_i and τ on a multipartition

Consider a multicharge $\underline{s} = (s_0, \dots, s_{\ell-1})$ and w an element of the extended affine symmetric group. Set $\underline{s}' = w(\underline{s})$. Since the indices of the fundamental weights of $\mathcal{U}_v^+(\widehat{\mathfrak{sl}}_e)$ belong to $\mathbb{Z}/e\mathbb{Z}$, we have $\Lambda_{\underline{s}'}^+ = \Lambda_{\underline{s}}^+$. This implies that the crystals graphs $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\underline{s}'}(\Lambda_{\underline{s}'}^+)$ are isomorphic. Write $\Gamma_{\underline{s}, \underline{s}'}$ for the isomorphism between $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\underline{s}'}(\Lambda_{\underline{s}'}^+)$. Given a multipartition $\underline{\lambda}$ in

$B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$, we are going to see how it is possible to determinate $\mu = \Gamma_{\underline{s}, \underline{s}'}(\underline{\lambda})$ in a non-inductive way, that is, without computing a path joining $\underline{\lambda}$ to $\underline{\emptyset}$ in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$. According to Section 5.1, w decomposes as a product of the elements σ_i $i = 1, \dots, \ell - 1$ and τ . We write for short $\Xi_{\underline{s}}$ and $\Sigma_{\underline{s}, i}$ respectively for the crystal graph isomorphisms $\Gamma_{\underline{s}, \tau(\underline{s})}$ and $\Gamma_{\underline{s}, \sigma_i(\underline{s})}$. The following proposition is a generalization of [13, Prop. 3.1].

Proposition 5.2.1 *Consider $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ a multipartition and \underline{s} a multicharge. Then*

$$\Xi_{\underline{s}}(\underline{\lambda}) = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \lambda^{(0)}).$$

Proof. Set $\underline{s} = (s_0, \dots, s_{\ell-1})$. Then $\tau(\underline{s}) = (s_1, \dots, s_{\ell-1}, s_0 + e)$. Let $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ a multipartition and set $\underline{\lambda}^{\#} = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \lambda^{(0)})$. Consider $i \in \{0, 1, \dots, e - 1\}$ and $\gamma_1 = (a_1, b_1, c_1)$, $\gamma_2 = (a_2, b_2, c_2)$ two i -nodes of $\underline{\lambda}$. Then $\gamma_1^{\#} = (a_1, b_1, c_1 - 1 \pmod{e})$ and $\gamma_2^{\#} = (a_2, b_2, c_2 - 1 \pmod{e})$ are two i -nodes of $\underline{\lambda}^{\#}$. We then easily check that $\gamma_2 \prec_{\underline{s}}^+ \gamma_1$ if and only if $\gamma_2^{\#} \prec_{\tau(\underline{s})}^+ \gamma_1^{\#}$. This implies that $\Xi_{\underline{s}}(\underline{\lambda}) = \underline{\lambda}^{\#}$. ■

Consider $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ a multipartition and \underline{s} a multicharge such that $\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$. Then we know that $\underline{\lambda}^{\diamond} = (\lambda^{(\ell-1)}, \dots, \lambda^{(0)}) \in B_{\infty}^{\underline{s}^{\diamond}}(\omega_{\underline{s}})$. Recall that, by definition of $B_{\infty}^{\underline{s}^{\diamond}}(\omega_{\underline{s}})$, we can write $\underline{\lambda}^{\diamond} = \lambda^{(\ell-1)} \otimes \dots \otimes \lambda^{(0)}$. For any integer $i \in \{0, \dots, n - 1\}$. Set

$$\psi_{s_{i+1}, s_i}(\lambda^{(i+1)} \otimes \lambda^{(i)}) = \tilde{\lambda}^{(i)} \otimes \tilde{\lambda}^{(i+1)} \quad (23)$$

where ψ_{s_{i+1}, s_i} is the crystal graph isomorphism defined in (7).

Proposition 5.2.2 *With the above notation, we have*

$$\Sigma_{\underline{s}, i}(\underline{\lambda}) = (\lambda^{(0)}, \dots, \tilde{\lambda}^{(i+1)}, \tilde{\lambda}^{(i)}, \dots, \lambda^{(\ell-1)})$$

that is $\Sigma_{\underline{s}, i}(\underline{\lambda})$ is obtained by replacing in $\underline{\lambda}$, $\lambda^{(i)}$ by $\tilde{\lambda}^{(i+1)}$ and $\lambda^{(i+1)}$ by $\tilde{\lambda}^{(i)}$.

Proof. We have to prove that the diagram

$$\begin{array}{ccc} B_e^{\underline{s}}(\Lambda_{\underline{s}}^+) & \xrightarrow{\varphi_{e, \infty}} & B_{\infty}(\omega_{s_{\ell-1}}) \otimes \dots \otimes B_{\infty}(\omega_{s_{i+1}}) \otimes B_{\infty}(\omega_{s_i}) \otimes \dots \otimes B_{\infty}(\omega_{s_0}) \\ \Sigma_{\underline{s}, i} \downarrow & & \downarrow \psi_{s_{i+1}, s_i} \\ B_e^{\sigma_i(\underline{s})}(\Lambda_{\underline{s}}^+) & \xrightarrow{\varphi_{e, \infty}} & B_{\infty}(\omega_{s_{\ell-1}}) \otimes \dots \otimes B_{\infty}(\omega_{s_i}) \otimes B_{\infty}(\omega_{s_{i+1}}) \otimes \dots \otimes B_{\infty}(\omega_{s_0}) \end{array} \quad (24)$$

commutes. Consider a multipartition $\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$. Set $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$. Let $\tilde{F}_{i_1}, \dots, \tilde{F}_{i_r}$ be a sequence of crystal operators such that $\underline{\lambda} = \tilde{F}_{i_1} \dots \tilde{F}_{i_r}(\underline{\emptyset})$. Then we can consider the multipartition $\underline{\mu} = \tilde{F}_{i_1}, \dots, \tilde{F}_{i_r}(\underline{\emptyset})$ in the crystal $B_e^{\sigma_i(\underline{s})}(\Lambda_{\underline{s}}^+)$. Observe first that

$$\psi_{s_{i+1}, s_i} \circ \varphi_{e, \infty}(\underline{\emptyset}) = \varphi_{e, \infty} \circ \Sigma_{\underline{s}, i}(\underline{\emptyset}). \quad (25)$$

Moreover, the maps $\varphi_{e, \infty}$, $\Sigma_{\underline{s}, i}$ and ψ_{s_{i+1}, s_i} commute with the crystal operators. This permits us to write

$$\psi_{s_{i+1}, s_i} \circ \varphi_{e, \infty}(\underline{\lambda}) = \psi_{s_{i+1}, s_i} \circ \varphi_{e, \infty}(\tilde{F}_{i_1} \dots \tilde{F}_{i_r}(\underline{\emptyset})) = \tilde{f}_{i_1} \dots \tilde{f}_{i_r}(\psi_{s_{i+1}, s_i} \circ \varphi_{e, \infty}(\underline{\emptyset})).$$

One the other hand we have

$$\varphi_{e,\infty} \circ \Sigma_{\underline{s},i}(\underline{\lambda}) = \varphi_{e,\infty} \circ \Sigma_{\underline{s},i}(\tilde{F}_{i_1} \cdots \tilde{F}_{i_r}(\emptyset)) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(\varphi_{e,\infty} \circ \Sigma_{\underline{s},i}(\emptyset)).$$

Hence we derive the equality $\psi_{s_{i+1},s_i} \circ \varphi_{e,\infty}(\underline{\lambda}) = \varphi_{e,\infty} \circ \Sigma_{\underline{s},i}(\underline{\lambda})$ from (25). This shows that the diagram (24) commutes and establish our proposition. ■

Example 5.2.3 Take $\ell = 3$. Suppose $\underline{s} = (4, 0, 1)$ and $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$ with $\lambda^{(0)} = (4, 3, 3, 2)$, $\lambda^{(1)} = (3, 3, 1)$ and $\lambda^{(2)} = (5, 3, 2)$. Let us compute $\Sigma_{\underline{s},2}(\underline{\lambda})$. The infinite columns associated to $\lambda^{(1)}$ and $\lambda^{(2)}$ are respectively

$$C_1 = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \bar{1} \\ \hline \bar{3} \\ \hline \bar{4} \\ \hline \dots \\ \hline \end{array} \text{ and } C_2 = \begin{array}{|c|} \hline 6 \\ \hline 3 \\ \hline 1 \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \bar{4} \\ \hline \dots \\ \hline \end{array}.$$

For $a = \bar{3}$ the corresponding finite columns are

$$C_1 = \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \bar{1} \\ \hline \bar{3} \\ \hline \end{array} \text{ and } C_2 = \begin{array}{|c|} \hline 6 \\ \hline 3 \\ \hline 1 \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \end{array}.$$

We have to determinate the image of $C_2 \otimes C_1$ under the isomorphism $\theta_{5,4}$ of Proposition 2.3.2. Note that the image of $C_1 \otimes C_2$ under $\theta_{4,5}$ is not relevant here because we must take into account the swap \diamond . We obtain $\{y_1, y_2, y_3, y_4\} = \{3, 1, \bar{2}, \bar{3}\}$. This gives $\theta_{5,4}(C_2 \otimes C_1) = C'_1 \otimes C'_2$ with

$$C'_1 = \begin{array}{|c|} \hline 3 \\ \hline \bar{1} \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \end{array} \text{ and } C'_2 = \begin{array}{|c|} \hline 6 \\ \hline 3 \\ \hline 2 \\ \hline \bar{1} \\ \hline \bar{3} \\ \hline \end{array}. \text{ Hence } \psi_{1,0}(C_2 \otimes C_1) = C'_1 \otimes C'_2 \text{ where}$$

$$C'_1 = \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \bar{2} \\ \hline \bar{3} \\ \hline \bar{4} \\ \hline \dots \\ \hline \end{array} \text{ and } C'_2 = \begin{array}{|c|} \hline 6 \\ \hline 3 \\ \hline 2 \\ \hline \bar{1} \\ \hline \bar{3} \\ \hline \bar{4} \\ \hline \dots \\ \hline \end{array}.$$

Finally we derive $\Sigma_{\underline{s},2}(\underline{\lambda}) = (\lambda^{(0)}, \tilde{\lambda}^{(2)}, \tilde{\lambda}^{(1)})$ with $\tilde{\lambda}^{(1)} = (3, 2)$ and $\tilde{\lambda}^{(2)} = (5, 3, 3, 1)$.

Remark: Assume that $\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ is a bipartition such that $\underline{\lambda}$ belongs to $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ where the multicharge $\underline{s} = (s_0, s_1)$ verifies $s_0 \leq s_1$. Then the combinatorial procedure illustrated by the previous example which permits to compute the crystal isomorphisms $\Sigma_{\underline{s}, i}$, essentially reduces, up to a renormalization due to the change of labelling of the Dynkin diagram in type $A_{e-1}^{(1)}$ (see (11)), to the algorithm depicted in Theorem 4.6 of [13].

5.3 A non recursive characterization of the Uglov multipartitions

Consider a multicharge \underline{s} and define the multicharge $\tilde{\underline{s}}$ as in (21). Then the crystals $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ and $B_e^{\tilde{\underline{s}}}(\Lambda_{\tilde{\underline{s}}}^+)$ are isomorphic. For any multipartition $\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$, write $I(\underline{\lambda}) \in B_e^{\tilde{\underline{s}}}(\Lambda_{\tilde{\underline{s}}}^+)$ for its image under this crystal isomorphism. It is possible to obtain $I(\underline{\lambda})$ from $\underline{\lambda}$ by using results of §5.2. We keep the notations of §5.1. For any $i = 0, \dots, \ell - 1$, we have $z_i = \xi^{(\ell-i)} \tau^{(i)}$. This permits to compute $\Gamma_{\underline{s}, z_i(\underline{s})}(\underline{\lambda})$ by using Propositions 5.2.1 and 5.2.2. We have then

$$I(\underline{\lambda}) = \Gamma_{\underline{s}, w(\underline{s})}(\underline{\lambda})$$

with the notation (22).

Conversely, given any FLOTW multipartition $\underline{\mu}$ and any multicharge \underline{s} , one can compute the multipartition $\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ such that $I(\underline{\lambda}) = \underline{\mu}$. Indeed, we have then $\underline{\lambda} = \Gamma_{\tilde{\underline{s}}, w^{-1}(\tilde{\underline{s}})}(\underline{\mu})$. By remark (i) following Proposition 3.2.4, we thus derive a non recursive combinatorial description of the Uglov multipartitions labelling $B_e^+(\Lambda_{\underline{s}}^+)_n = \{\underline{\lambda} \in B_e^{\underline{s}}(\Lambda_{\underline{s}}^+) \mid |\underline{\lambda}| = n\}$.

Proposition 5.3.1 *For any multicharge \underline{s}*

$$B_e^+(\Lambda_{\underline{s}}^+)_n = \{\Gamma_{\tilde{\underline{s}}, w^{-1}(\tilde{\underline{s}})}(\underline{\mu}) \mid \underline{\mu} \in B_e^{\tilde{\underline{s}}}(\Lambda_{\tilde{\underline{s}}}^+)_n\}$$

where w is obtained from \underline{s} as in (22).

5.4 Isomorphism class of a multipartition

Suppose that e is a fixed positive integer and \underline{s} a multicharge of level ℓ . Consider $\underline{\lambda}$ a multipartition in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$. The isomorphism class of $\underline{\lambda}$ is the set

$$\mathfrak{C}(\underline{\lambda}) = \{\Gamma_{\underline{s}, \underline{s}'}(\underline{\lambda}) \mid \underline{s}' \in \mathcal{C}(\underline{s})\}.$$

Thus $\mathfrak{C}(\underline{\lambda})$ is the set of all multipartitions $\underline{\mu}$ which appear at the same place as $\underline{\lambda}$ in a crystal $B_e^{\underline{s}'}(\Lambda_{\underline{s}'}^+)$ where \underline{s}' is a multicharge of the orbit of \underline{s} under the action of \widehat{S}_ℓ . Then $\mathfrak{C}(\underline{\lambda})$ can be determined from $\underline{\lambda}$ by applying successive elementary transformations using Propositions 5.2.1 and 5.2.2. Observe that for any $\underline{\mu} \in \mathfrak{C}(\underline{\lambda})$ we must have $|\underline{\lambda}| = |\underline{\mu}|$. This implies in particular that $\mathfrak{C}(\underline{\lambda})$ is finite. The cardinality of $\mathfrak{C}(\underline{\lambda})$ is in general rather complicated to evaluate without computing the whole class $\mathfrak{C}(\underline{\lambda})$. Nevertheless, we are going to see in Theorem 5.4.2, that it is possible to obtain an upper bound for $\text{card}(\mathfrak{C}(\underline{\lambda}))$ and to determinate a finite subset $S_{\underline{\lambda}}$ of \widehat{S}_ℓ such that

$$\mathfrak{C}(\underline{\lambda}) = \{\Gamma_{\underline{s}, \underline{s}'}(\underline{\lambda}) \mid \underline{s}' \in S_{\underline{\lambda}} \cdot \underline{s}\}.$$

Lemma 5.4.1 *Let $\underline{\lambda}$ be a multipartition of rank n . Assume that \underline{s} is a multicharge of level ℓ verifying :*

$$s_j - s_{j+1} > n - 1 \quad (26)$$

for $j = 0, \dots, \ell - 2$. Then for any $k \in \{0, \dots, \ell - 1\}$ we have $\Gamma_{\underline{s}, z_k(\underline{s})}(\underline{\lambda}) = \underline{\lambda}$.

Proof. Let $\underline{s} = (s_0, \dots, s_{\ell-1})$ be such that $s_j - s_{j+1} > n - 1$ for $j = 0, \dots, \ell - 2$ and let $\underline{\mu}$ be a multipartition in $B_e^{\underline{s}^+}$ such that $|\underline{\mu}| \leq n$. Let $i \in \{0, 1, \dots, e - 1\}$ and consider $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ two i -nodes in $\underline{\mu}$ such that $c_1 < c_2$. We have $b_1 - a_1 + s_{c_1} - (b_2 - a_2 + s_{c_2}) > b_1 - a_1 - (b_2 - a_2) + n - 1 \geq 0$. Hence, the contains of γ_1 and γ_2 considered as nodes of $\underline{\mu} \in B_e^{\underline{s}^+}$ are such that $c(\gamma_1) > c(\gamma_2)$. Hence we have $\gamma_2 \prec_{\underline{s}}^+ \gamma_1$.

Now, put $(s'_0, \dots, s'_{\ell-1}) := z_k(\underline{s}) = (s_0 + e, \dots, s_k + e, s_{k+1}, \dots, s_{\ell-1})$. As we have $s'_j - s'_{j+1} > n - 1$ for $j = 0, \dots, \ell - 2$, the above discussion shows that the order $\prec_{\underline{s}}^+$ and $\prec_{z_k(\underline{s})}^+$ on the i -nodes of $\underline{\mu}$ coincide. ■

Theorem 5.4.2 *Suppose that e is a fixed positive integer and \underline{s} a multicharge of level ℓ such that*

$$0 \leq s_{\ell-1} \leq \dots \leq s_0 < e$$

Consider $\underline{\lambda}$ a FLOTW multipartition in $B_e^{\underline{s}}(\Lambda_{\underline{s}}^+)$ of order n . For any $j \in \{0, \dots, \ell - 2\}$, let p_j be the minimal nonnegative integer such that

$$s_j + p_j e - s_{j+1} > n - 1.$$

Then we have :

$$\mathfrak{C}(\underline{\lambda}) = \{\Gamma_{\underline{s}, z_0^{a_0} \dots z_{\ell-2}^{a_{\ell-2}} \sigma(\underline{s})}(\underline{\lambda}) \mid \sigma \in S_{\ell} \text{ and } 0 \leq a_j \leq p_j \text{ for any } j \in \{0, \dots, \ell - 2\}\}.$$

In particular, $\mathfrak{C}(\underline{\lambda})$ is finite and

$$\text{card}(\mathfrak{C}(\underline{\lambda})) \leq \ell! \prod_{j=0}^{\ell-2} (p_j + 1).$$

Proof. Consider $\underline{\mu} \in \mathfrak{C}(\underline{\lambda})$. According to (22), one can write

$$\underline{\mu} = \Gamma_{\underline{s}, w^{-1}(\underline{s})}(\underline{\lambda}) \text{ with } w^{-1} = z_{\ell-1}^{-k} \sigma z_{\ell-2}^{r_{\ell-2}} \dots z_0^{r_0}.$$

By Remark (ii) following Example 3.2.3, we have $\underline{\mu} = \Gamma_{\underline{s}, w^{-1}(\underline{s})}(\underline{\lambda}) = \Gamma_{\underline{s}, u(\underline{s})}(\underline{\lambda})$ where $u = \sigma z_{\ell-2}^{r_{\ell-2}} \dots z_0^{r_0}$. By Lemma 5.4.1, we can thus derive

$$\mathfrak{C}(\underline{\lambda}) = \{\Gamma_{\underline{s}, z_0^{a_0} \dots z_{\ell-2}^{a_{\ell-2}} \sigma(\underline{s})}(\underline{\lambda}) \mid \sigma \in S_{\ell} \text{ and } 0 \leq a_j \leq p_j \text{ for any } j \in \{0, \dots, \ell - 2\}\}$$

and our theorem follows. ■

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