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Some additive applications of the isoperimetric approach

Yahya O. Hamidoune*

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Abstract

Let G be a group and let X be a finite subset. The isoperimetric method investigates the objective function $|(XB) \setminus X|$, defined on the subsets X with $|X| \geq k$ and $|G \setminus (XB)| \geq k$.

In this paper we present all the basic facts about the isoperimetric method. We improve some of our previous results and obtain generalizations and short proofs for several known results. We also give some new applications.

Some of the results obtained here will be used in a coming paper [23] to improve Kempermann structure Theory.

1 Introduction

The starting point of set product estimation is the inequality $|AB| \geq \min(|G|, |A| + |B| - 1)$, where A, B are subsets of a group with a prime order, proved by Cauchy [4] and rediscovered by Davenport [6]. Some of the generalizations of this result are due to Chowla [5], Shepherson [45], Mann [37] and Kemperman [34].

Kneser's generalization of the Cauchy-Davenport Theorem is a basic tool in Additive Number Theory:

Theorem 1 (Kneser [36, 35, 38, 47]) *Let G be an abelian group and let $A, B \subset G$ be finite nonempty subsets such that $|AB| \leq |A| + |B| - 2$. Then $|AB| = |AH| + |BH| - |H|$, where $H = \{x : ABx = AB\}$.*

Among the numerous applications of Kneser's Theorem, we mention one to the Frobenius problem due to Dixmier [9].

Generalizing Kneser's Theorem to non-abelian groups still a challenging problem. Several attempts were made for this purpose. The first one is due to Diderrich:

Theorem 2 (Diderrich [8]) *Let G be a group and let $A, B \subset G$ be finite nonempty subsets such that $|AB| \leq |A| + |B| - 2$. Assume moreover that the elements of B commute. Then $|AB| = |AH| + |BH| - |H|$, where $H = \{x : ABx = AB\}$.*

*Université Pierre et Marie Curie, Paris hamidoune@math.jussieu.fr

It was observed in [16] that this generalization is equivalent to Kneser's Theorem, c.f. Corollary 16. More investigation and some examples, showing that the natural extension fails, can be found in Olson [41].

The critical pair Theory is the description of the subsets A, B with $|AB| = |A| + |B| - 1$. Vosper's Theorem [48, 49, 36, 38] states that in a group with a prime order $|AB| = |A| + |B| - 1 \leq |G| - 2$ holds if and only if A and B are progressions with the same ratio, where $\min(|A|, |B|) \geq 2$. More recently the authors of [27] obtained a description of sets A, B with $|AB| = |A| + |B| \leq |G| - 4$, if $|G|$ is a prime. For general abelian group the classical tools are based on Kemperman's critical pair Theory [33].

The results described above were proved using the transformations introduced by Cauchy, Davenport and Dyson [4, 6, 38, 47]. These transformations were generalized by Kemperman to non-abelian groups [34]. In the non-abelian case only few results, proved using Kemperman transformations, were known until recent years. These results are basically due to Kemperman [34], Olson [39, 40, 41] and Brailowski-Freiman [3].

More recently Károlyi [32] used group extensions and the Feit-Thompson Theorem to obtain a generalization of Vosper's Theorem to the non-abelian case.

The exponential sums method in Additive Number Theory gives some sharp estimates for $|AB|$ in the abelian case if $|A|, |B|$ are relatively small. The reader may find applications of this method in the text books [38, 47] and the papers of Deshouillers-Freiman [7] and Green-Ruzsa [11].

Another method in Additive Number Theory based on Nonstandard Analysis was introduced by Jin. An example of the application of this method may be found in [31].

In this paper we are concerned with the isoperimetric method introduced by the author in [13, 17, 20, 21]. Let us present briefly some special cases of this method:

Let $\Gamma = (V, E)$ be a finite reflexive relation and consider the objective function $X \rightarrow |\Gamma(X) \setminus X|$, defined on the subsets X with $|X| \geq k$ and $|V \setminus \Gamma(X)| \geq k$. The minimal value of this objective function is the k th-connectivity and a k -atom is a set with minimal cardinality where the objective function achieves its minimal value. The main result proved in [20] implies that distinct k -atoms of Γ intersect in at most $k - 1$ elements or that distinct k -atoms of Γ^{-1} intersect in at most $k - 1$ elements. This result, which generalizes some previous results of the author [13, 17, 21], has several applications in Additive number Theory as we shall see in the present paper.

The strong connectivity, usually defined in Graph theory as the minimum cardinality of a cutset, coincides with our first connectivity. The k th-connectivity was introduced in [20] in connection with some additive problems.

Let B be a subset of a finite abelian group G with $1 \in B$ and $B \neq G$. As showed in [17], the main result proved in [13] implies that the objective function $X \rightarrow |(XB) \setminus X|$, defined on the nonempty subsets X with $XB \neq G$, attains its minimal value on a subgroup. If $|G|$ is a prime this value is necessarily $|B| - 1$. The Cauchy-Davenport Theorem follows obviously from this fact.

In this paper we shall present basic facts about the isoperimetric method. We shall improve some of our previous results and obtain generalizations and short proofs for several known

results. We also give some new applications.

The paper's organization is the following:

In section 1, we present the terminology. In section 2, we introduce the concepts of k th-connectivity, k -fragment and k -atom and prove some elementary properties of these objects. In section 3, we give some basic properties of the intersection of fragments. The main result of this section is Theorem 7 which gives conditions implying that the intersection of two k -fragments is a k -fragment. This theorem generalizes results contained in [13, 14, 17, 21, 20]. In section 4, we obtain the structure of 1-atoms and give few applications. Most of the results of this section were proved in [17, 21]. We prove them since they are needed in several parts of this paper in order to make the present work self-contained. In section 5, we investigate the inequality $|AB| \geq |A| + |B|/2$ and its critical pairs. In section 6, Proposition 19 gives the value of κ_2 for sets with a small cardinality. As an application we generalize the result of Károlyi [32] mentioned above. Theorem 22 determines the structure of the 2-atoms in the abelian case. This result extends to the infinite case a previous result of the author [22]. The proof given here is much easier than our first proof. In section 7, we prove more results on the intersection of fragments. We apply these results in section 8 to give an upper bound for the size of a 2-atom. This result could help in proving that a 2-atom containing 1 is a normal subgroup. As an application we generalize to the infinite case a result proved in the finite case by Arad and Muzychuk [1]. In section 8, we investigate minimal cutsets and the corresponding saturated sets.

Some of the results obtained here will be used in a coming paper [23] to improve Kempermann structure Theory.

The reader may find some applications of the isoperimetric method in Serra's survey [43].

2 Terminology and preliminaries

2.1 Groups

Let G be a group and let S be a subset of G . The subgroup generated by S will be denoted by $\langle S \rangle$. Let A, B be subsets of G . The *Minkowski product* is defined as

$$AB = \{xy : x \in A \text{ and } y \in B\}.$$

Let H be a subgroup. A partition $A = \bigcup_{i \in I} A_i$, where A_i is the nonempty intersection of some left- H -coset with A will be called a *H -left-decomposition* of A . A *right-decomposition* is defined similarly.

Let X be a subset of a group G . We write

$$\Pi^r(X) = \{x : Xx = X\} \text{ and } \Pi^l(X) = \{x : xX = X\}.$$

Let G be a group and let S be a subset of G . The set S is said to be *normal* if for every $x \in G$, $xS = Sx$. We shall say that S is *semi-normal* if there is $a \in G$ such that for every $x \in G$, $xS = Sxa$. Obviously S is *semi-normal* if and only if there is $a \in G$ such that Sa is normal.

We use the following well known fact:

Lemma 3 ([36], Theorem 1) *Let G be a finite group and let A, B be subsets such that $|A| + |B| > |G|$. Then $AB = G$.*

2.2 Graphs

The diagonal of V is by definition $\Delta_V = \{(x, x) : x \in V\}$. Let V be a set and let $E \subset V \times V$. The relation $\Gamma = (V, E)$ will be called a *graph*. The elements of V will be called *points* or *vertices*. The graph Γ is said to be *reflexive* if $\Delta_V \subset E$. The *reverse* of Γ is by definition $\Gamma^{-1} = (V, E^{-1})$, where $E^{-1} = \{(x, y) : (y, x) \in E\}$.

Let $a \in V$ and let $A \subset V$. The image of a is by definition

$$\Gamma(a) = \{x : (a, x) \in E\}.$$

The image of A is by definition

$$\Gamma(A) = \bigcup_{x \in A} \Gamma(x).$$

The *valency* of x is by definition $d_\Gamma(x) = |\Gamma(x)|$. We shall say that Γ is *locally finite* if $d_\Gamma(x)$ is finite for all x . We put $\delta(\Gamma) = \min\{d_\Gamma(x); x \in V\}$. The graph Γ will be called *regular* with valency r if the elements of V have the same valency r .

Let $\Gamma = (V, E)$ be a graph. For $X \subset V$, the *boundary* of X is by definition

$$\partial_\Gamma(X) = \Gamma(X) \setminus X.$$

When the context is clear the reference to Γ will be omitted. In this case we write

- $\partial_-(X) = \Gamma^{-1}(X) \setminus X$,
- $X^\wedge = V \setminus (X \cup \Gamma(X))$,
- $X^\vee = V \setminus (X \cup \Gamma^{-1}(X))$.

Most of the time we shall work with reflexive graphs. In this case we have $\Gamma(X) = X \cup \Gamma(X)$.

Notice that there is no arc connecting X to X^\wedge , since any arc starting in X must end in $X \cup \partial(X)$. The reader should always have in mind this obvious fact.

A set T of the form $\partial(F)$, where $F \neq \emptyset$ and $F \cup \partial(F) \neq V$ is called a *cutset*. Notice the deletion of T destroys all the arcs connecting F to $V \setminus (F \cup \partial(F))$.

Let $\Gamma = (V, E)$ be a reflexive graph. We shall say that a subset X induces a *k -separation* if $|X| \geq k$ and $|X^\wedge| \geq k$. We shall say that Γ is *k -separable* if some X induces a *k -separation*.

Observe that for every k , Γ is *k -separable* if V is infinite.

Notice that X induces a *k -separation* of Γ if and only if X^\wedge induces a *k -separation* of Γ^{-1} . In particular Γ is *k -separable* if and only if Γ^{-1} is *k -separable*.

Intuitively speaking Γ is *k -separable* if there is a cutset($\partial(X)$) whose deletion disconnects a set X of size $\geq k$ from another set (X^\wedge) of size $\geq k$.

2.3 Cayley graphs

Let $\Gamma = (V, E)$, $\Phi = (W, F)$ be two graphs. A map $f : V \rightarrow W$ will be called a *homomorphism* if $(f(x), f(y)) \in F$ for all $x, y \in V$ such that $(x, y) \in E$.

The graph Γ will be called *point-transitive* if for all $x, y \in V$, there is an automorphism f such that $y = f(x)$. Clearly a point-transitive graph is regular.

Let G be a group and let $a \in G$. The permutation $\gamma_a : x \rightarrow ax$ of G will be called *left-translation*. Let S be a subset of G . The graph (G, E) , where $E = \{(x, y) : x^{-1}y \in S\}$ is called a *Cayley graph*. It will be denoted by $\text{Cay}(G, S)$.

Let $\Gamma = \text{Cay}(G, S)$ and let $F \subset G$. Clearly $\Gamma(F) = FS$.

The following facts are easily seen:

- $(\text{Cay}(G, S))^{-1} = \text{Cay}(G, S^{-1})$;
- For every $a \in G$, γ_a is an automorphism of $\text{Cay}(G, S)$, and hence $\text{Cay}(G, S)$ is point-transitive.

3 The isoperimetric method revisited

In this section, we introduce the concepts of *kth-connectivity*, *k-fragment* and *k-atom*. We also prove some elementary properties of these objects.

Let $\Gamma = (V, E)$ be a locally finite *k-separable* reflexive graph. The *kth-connectivity* of Γ (called *kth-isoperimetric number* in [20]) is defined as

$$\kappa_k(\Gamma) = \min\{|\partial(X)| : \infty > |X| \geq k \text{ and } |V \setminus \Gamma(X)| \geq k\}. \quad (1)$$

A finite subset X of V such that $|X| \geq k$, $|V \setminus \Gamma(X)| \geq k$ and $|\partial(X)| = \kappa_k(\Gamma)$ is called a *k-fragment* of Γ . A *k-fragment* with minimum cardinality is called a *k-atom*. The cardinality of a *k-atom* of Γ will be denoted by $\alpha_k(\Gamma)$.

These notions, which generalize some concepts in [13, 14, 17, 21], were introduced in [20].

We shall now extend the above notions to non-*k-separable* graphs.

Let $\Gamma = (V, E)$ be a non-*k-separable* graph with $|V| \geq 2k - 1$. Then Γ is necessarily finite. We set in this case $\kappa_k(\Gamma) = |V| - 2k + 1$. In this case, a *k-fragment* (resp. *k-atom*) is a set with cardinality k .

A *k-fragment* of Γ^{-1} will be called a *negative k-fragment*. We use the following notations, where the reference to Γ could be implicit:

- $\alpha_{-k}(\Gamma) = \alpha_k(\Gamma^{-1})$,
- $\kappa_{-k}(\Gamma) = \kappa_k(\Gamma^{-1})$.

Lemma 4 *Let $\Gamma = (V, E)$ be a locally finite reflexive graph such that $|V| \geq 2k - 1$. Then $\kappa_k(\Gamma)$ is the maximal integer j such that for every finite subset $X \subset V$ with $|X| \geq k$,*

$$|\Gamma(X)| \geq \min(|V| - k + 1, |X| + j). \quad (2)$$

Formulae (2) is an immediate consequence of the definitions. We shall call (2) the *isoperimetric inequality*. The reader may use the conclusion of this lemma as a definition of $\kappa_k(\Gamma)$.

Remark. For any locally finite reflexive graph $\Gamma = (V, E)$, we have $\kappa_1(\Gamma) \leq \delta_\Gamma - 1$.

Lemma 5 *Let $\Gamma = (V, E)$ be reflexive finite graph with $|V| \geq 2k - 1$. Then*

$$\kappa_k = \kappa_{-k}. \quad (3)$$

Proof.

Observe that (3) holds by convention if Γ is non- k -separable. Suppose now that Γ is k -separable, and let X be a k -fragment of Γ . We have clearly $\partial_-(X^\wedge) \subset \partial(X)$. Therefore

$$\kappa_k(\Gamma) \geq |\partial(X)| \geq |\partial_-(X^\wedge)| \geq \kappa_{-k}.$$

The reverse inequality follows by applying this one to Γ^{-1} . ■

Lemma 6 *Let $\Gamma = (V, E)$ be a locally finite k -separable reflexive graph. Let X be a k -fragment. Then*

$$\partial_-(X^\wedge) = \partial(X), \quad (4)$$

$$(X^\wedge)^\vee = X. \quad (5)$$

In particular X^\wedge is a negative k -fragment, if V is finite.

Proof.

$$\boxed{X \mid \partial(X) \mid X^\wedge}$$

We have clearly $\partial_-(X^\wedge) \subset \partial(X)$.

We must have $\partial_-(X^\wedge) = \partial(X)$, since otherwise there is $y \in \partial(X) \setminus \partial_-(X^\wedge)$. It follows that $|\partial(X \cup \{y\})| \leq |\partial(X)| - 1$, contradicting the definition of κ_k . This proves (4).

We have $\Gamma^{-1}(X^\wedge) = X^\wedge \cup \partial_-(X^\wedge) = X^\wedge \cup \partial(X) = V \setminus X$. This implies obviously (5).

Assume now that V is finite. We have by Lemma 5, $|\partial_-(X^\wedge)| = |\partial(X)| = \kappa_k = \kappa_{-k}$.

This proves X^\wedge is a negative k -fragment.

■

Remark. Observe that (4) could be stated equivalently by saying that $V \setminus (\Gamma^{-1}(X^\wedge)) = X$. Such a choice was made in [20, 22]. Our present choice is similar to the presentation in [21].

We conclude this section by introducing an important notion:

Let $\Gamma = (V, E)$ be a reflexive graph. We shall say that Γ is a *Vosper graph* if Γ is non-2-separable or $\kappa_2 \geq \delta$.

Clearly Γ is a Vosper graph if and only if for every $X \subset V$ with $|X| \geq 2$,

$$|\Gamma(X)| \geq \min(|V| - 1, |X| + \delta).$$

4 The intersection of fragments

We shall now formulate and prove some basic properties of the intersection of fragments. The main result of this section is Theorem 7 which gives conditions implying that the intersection of two k -fragments is a k -fragment. Theorem 7 implies that two distinct k -atoms intersect in at most $k - 1$ elements if $\alpha_k \leq \alpha_{-k}$ or if the weaker condition $\alpha_k \leq \frac{|V| - \kappa_k}{2}$ is satisfied.

The following result is proved in [21] in the special case $\kappa_2 = \kappa_1$. Indeed the paper [21] was concerned only with Vosper graphs. The concept of κ_k was introduced two years later in [20].

Theorem 7 [21] *Let $\Gamma = (V, E)$ be a reflexive locally finite graph such that $|V| \geq 2k - 1$.*

Let X, Y be k -fragments such that $|X \cap Y| \geq k$ and $|Y^\wedge| \geq |X|$.

Then $X \cap Y$ and $X \cup Y$ are k -fragments.

Proof.

The result is obvious if Γ is non- k -separable since k -fragments are k -subsets. So we may assume that Γ is k -separable.

Suppose that $|X \cap Y| \geq k$.

\times	Y	$\partial(Y)$	Y^\wedge
X	R_{11}	R_{12}	R_{13}
$\partial(X)$	R_{21}	R_{22}	R_{23}
X^\wedge	R_{31}	R_{32}	R_{33}

By the definition of a k -fragment we have

$$\kappa_k = |\partial(X)| = |R_{21}| + |R_{22}| + |R_{23}|.$$

The following inclusion follows by an easy verification:

$$\partial(X \cap Y) \subset R_{12} \cup R_{22} \cup R_{21}.$$

We have clearly $|V \setminus \Gamma(X \cap Y)| \geq |V \setminus \Gamma(X)| \geq k$. By the definition we have $|\partial(X \cap Y)| \geq \kappa_k$. It follows that

$$\begin{aligned}
|R_{21}| + |R_{22}| + |R_{23}| &= \kappa_k \\
&\leq |\partial(X \cap Y)| \\
&\leq |R_{12}| + |R_{22}| + |R_{21}|.
\end{aligned} \tag{6}$$

Therefore

$$|R_{23}| \leq |R_{12}|. \tag{7}$$

Now

$$|V \setminus (X \cup Y)| = |R_{33}| \tag{8}$$

$$\begin{aligned}
&= |Y^\wedge| - |R_{23}| - |R_{13}| \\
&\geq |X| - |R_{12}| - |R_{13}| \\
&= |R_{11}| \geq k.
\end{aligned} \tag{9}$$

By the definition of κ_k , we have

$$|\partial(X \cup Y)| \geq \kappa_k.$$

It follows that

$$\begin{aligned}
|R_{12}| + |R_{22}| + |R_{32}| &= \kappa_k \\
&\leq |\partial(X \cup Y)| \\
&\leq |R_{32}| + |R_{22}| + |R_{23}|.
\end{aligned}$$

Therefore $|R_{12}| \leq |R_{23}|$. By (7) we have $|R_{12}| = |R_{23}|$.

It follows that the inequality $\kappa_k \leq |\partial(X \cap Y)|$, used in the proof of (6), is an equality and hence $X \cap Y$ is a k -fragment of Γ .

It follows also that the inequality $\kappa_k \leq |\partial(X \cup Y)|$, used in the proof of (10), is an equality and hence $X \cup Y$ is a k -fragment of Γ .

■

The next consequence of Theorem 7 will be a main tool in this paper.

Theorem 8 [20] *Let $\Gamma = (V, E)$ be a reflexive locally finite graph such that $|V| \geq 2k - 1$. Let A be a k -atom and let F be a k -fragment such that $|A \cap F| \geq k$ and $\alpha_k \leq \frac{|V| - \kappa_k}{2}$.*

Then $A \subset F$

In particular distinct k -atoms intersect in at most $k - 1$ elements.

Proof. Let us show that $|F^\wedge| \geq |A|$. This holds clearly if V is infinite. Suppose that $|V|$ is finite. Now we have $|V| = |A| + \kappa_k + |A^\wedge| \geq |A| + \kappa_k + |F^\wedge|$. It follows that $|F^\wedge| \geq |V| - \kappa_k - \alpha_k \geq \alpha_k$. The result follows now by Theorem 7. ■

The next lemma shows that $\alpha_k \leq \frac{|V| - \kappa_k}{2}$ or $\alpha_{-k} \leq \frac{|V| - \kappa_{-k}}{2}$.

Lemma 9 *Let $\Gamma = (V, E)$ be a reflexive locally finite graph such that $|V| \geq 2k - 1$. If V is infinite or $\alpha_k \leq \alpha_{-k}$, then $\alpha_k \leq \frac{|V| - \kappa_k}{2}$.*

Proof. The result is obvious if V is infinite. Suppose that V is finite. By Lemma 6 A^\wedge is a negative fragment. We have $|V| = |A| + \kappa_k + |A^\wedge| \geq |A| + \kappa_k + \alpha_{-k} \geq \kappa_k + 2\alpha_k$. ■

5 Estimation of the size of a set product

Most of the results in this section are proved in [17, 21]. We prove them here since they are needed in several parts of this paper in order to make the present work self-contained.

Let G be a group and let S be a subset of G with $1 \in S$. We put

- $\alpha_k(S) = \alpha_k(\text{Cay}(\langle S \rangle, S))$;
- $\kappa_k(S) = \kappa_k(\text{Cay}(\langle S \rangle, S))$.

We shall say that a subset S is *k-separable* (a *Vosper subset*) if $\text{Cay}(\langle S \rangle, S)$ is *k-separable* (a Vosper graph). By a fragment of S we shall mean a fragment of $\text{Cay}(\langle S \rangle, S)$.

We shall consider only generating subset containing 1 in order to avoid degenerate situations where $\kappa_k = 0$. Notice that $\kappa_k(\text{Cay}(G, S)) = 0$, if S generates a finite proper subgroup. However $\kappa_k(S) > 0$ if $|S| \geq 2$ and $|\langle S \rangle| \geq 2k - 1$. This easy fact was observed in [20].

Let us prove a lemma:

Lemma 10 *Let S be a generating subset of a group G with $1 \in S$. Let H be a 2-atom of S with $1 \in H$. Assume that $\alpha_2 \leq \frac{|G| - \kappa_2}{2}$ or $\alpha_2 \leq \alpha_{-2}$. If H is right-periodic then H is a subgroup.*

Proof. By Lemma 9, we have always $\alpha_2 \leq \frac{|G| - \kappa_2}{2}$. Put $Q = \Pi^r(H)$. Take $a \in H$. We have $aH \cap H \supset aQ$. By Theorem 8, $aH = H$. Then $H^2 = H$ and hence H is a subgroup. ■

The intersection property implies easily the following description of 1-atoms, obtained in [17] in the finite case under the stronger assumption $\alpha_1 \leq \alpha_{-1}$ which is enough for several applications. The general case was given later in [21].

Proposition 11 [17, 21] *Let S be a generating subset of a group G with $1 \in S$. Let H be a 1-atom of S with $1 \in H$ and $|H| \leq \frac{|G| - \kappa_1}{2}$. Then H is a subgroup generated by $S \cap H$.*

Proof. Let $a \in H$. The set aH is a 1-atom, since any left-translation is an automorphism of the Cayley graph. Since $|(aH) \cap H| \geq 1$, we have by Theorem 8, $aH = H$. Therefore $HH = H$. Hence H is a subgroup.

Let $H_0 = \langle H \cap S \rangle$. We have clearly $H_0S \cap H \subset H_0$. Therefore $\partial(H_0) \subset H_0S \setminus H \subset HS \setminus H$. It follows that H_0 is a 1-fragment and hence $H_0 = H$. ■

Corollary 12 [17, 21]

Let S be a generating subset of a group G with $1 \in S$. Let H be a 1-atom and let K be a negative 1-atom such that $1 \in H \cap K$.

- If $|H| \leq |K|$ or if $|G|$ is infinite, then H is a subgroup.
- If $|K| \leq |H|$ and G is finite, then K is a subgroup.

In particular there is a finite subgroup $H \neq G$ such that $\kappa_1 = \min(|HS| - |H|, |SH| - |H|)$.

Proof.

Assume first $|H| \leq |K|$ or that G is infinite. By Lemma 9 and Proposition 11 H is a subgroup. By the definition of a 1-atom we have $\kappa_1 = |HS| - |H|$.

Assume now $|H| > |K|$ and that $|G|$ is finite. By Lemma 9 and Proposition 11 K is a subgroup. By Lemma 5 and the definition of a negative 1-atom, we have $\kappa_1 = \kappa_{-1} = |KS^{-1}| - |K| = |SK| - |K|$.

■

The next lemma could be useful when S generates a proper subgroup:

Lemma 13 Let G be group and let A, S be finite nonempty subsets of G with $1 \in S$. Put $K = \langle S \rangle$ and let $A = \bigcup_{i \in I} A_i$ be a left K -decomposition of A . Put $W = \{i : |A_i S| < |K|\}$. Then

$$|W| \leq \frac{|AS| - |A|}{\kappa_1(S)}. \quad (10)$$

Proof. For each i take $a_i \in A_i^{-1}$.

By the isoperimetric inequality, we have $|a_i A_i S| \geq |A_i| + \kappa_1$ for all $i \in W$. Then

$$\begin{aligned} |AS| &= \sum_{i \in I} |A_i S| \\ &\geq \sum_{i \in I \setminus W} |A_i| + \sum_{i \in W} (|A_i| + \kappa_1) = |A| + |W| \kappa_1. \end{aligned}$$

■

6 A universal for κ_1

The main result of this section is Proposition 14. The bound $\kappa_1(S) \geq \frac{|S|}{2}$ holds for all point-transitive graphs (where $|S|$ is replaced by δ). This was proved [14] in the finite case. The general case is proved in [21].

We give here a characterization of the sets S with $\kappa_1(S) = \frac{|S|}{2}$.

Proposition 14 [14, 21] *Let S be a finite generating subset of a group G with $1 \in S$. Let H be a 1-atom and let K be a negative 1-atom. Then*

$$\kappa_1(S) \geq \frac{|S|}{2}. \quad (11)$$

Moreover $\kappa_1(S) = \frac{|S|}{2}$ holds if and only if one of the following holds:

- $|H| \leq |K|$ or G is infinite, and there is a u such that $S = H \cup Hu$;
- G is finite and $|H| \geq |K|$ and there is a u such that $S = K \cup uK$.

Proof.

Assume first that $|H| \leq |K|$ or that G is infinite. By Corollary 12 H is a subgroup. We have $\kappa_1(S) = |HS| - |H| \geq \frac{|HS|}{2} \geq \frac{|S|}{2}$, observing that $|HS| \geq 2|H|$ since S is a generating subset with $1 \in S$. Suppose now that $\kappa_1(S) = \frac{|S|}{2}$. We see that $|HS| = 2|H|$, and hence there is a u such that $S = H \cup Hu$.

Assume now that $|H| \geq |K|$ and that G is finite. By Corollary 12 K is a subgroup. By Lemma 5, we have

$\kappa_1 = \kappa_{-1} = |KS^{-1}| - |K| \geq \frac{|SK|}{2} \geq \frac{|S|}{2}$, observing that $|SK| \geq 2|K|$ since S is a generating subset with $1 \in S$. Suppose now that $\kappa_1(S) = \frac{|S|}{2}$. We see that $|SK| = 2$, and hence there is a u such that $S = K \cup uK$.

■

Zémor constructed in [50] a Cayley graph with $\alpha_1 > \alpha_{-1}$.

The above result suggests more constructions of this type:

Example Consider a finite group G of odd order and consider a non-normal subgroup H and an element u such that $uH \neq Hu$. Put $S = H \cup Hu$. By Proposition 14 H is a 1-atom of S . Let Q be a negative 1-atom. Q can not be a subgroup since otherwise $|Q| = |H|$ and $|QS^{-1}| = 2|Q|$, which is impossible. It follows that $|Q| > \frac{|G| - \kappa_1}{2}$.

Corollary 15 (Olson) [39, 40] *Let A, B be finite nonempty subsets of a group G and put $K = \langle BB^{-1} \rangle$. Then*

$$|B^j| \geq \min(|K|, \frac{(j+1)|S|}{2}), \text{ and} \quad (12)$$

$$|AB| \geq \min(|AK|, |A| + \frac{|S|}{2}). \quad (13)$$

Proof. It would be enough to prove Formulae (13).

Take $b \in B^{-1}$ and put $S = Bb$. Since $BB^{-1} = SS^{-1} \subset \langle S \rangle$ and $S = Bb \subset BB^{-1}$ we have $K = \langle S \rangle$.

Take a left- K -decomposition $A = \bigcup_{i \in I} A_i$. Suppose now that $AB \neq AK$. Then there is an s such that $|A_s B| < |K|$. Take an $u \in A_s^{-1}$. By the isoperimetric inequality and by (11), $|A_s B| = |u A_s B| \geq |A_s| + \frac{|B|}{2}$. Then

$$|AB| = \sum_{i \in I} |A_i B| \geq \sum_{i \in I \setminus \{s\}} |A_i| + |A_s B| \geq |A| + \frac{|B|}{2}.$$

■

Formulae (12) is proved by Olson in [39] as main tool in his proof that a subset of a finite group G with cardinality $\geq 3\sqrt{|G|}$ contains some nonempty subset $\{a_1, \dots, a_k\}$ with $a_1 \cdots a_k = 1$, a result conjectured by Erdős and Heilbronn [10]. Olson's result improves results by Erdős-Heilbronn [10] and Szemerédi [46]. A nice application of (12) to the Frobenius problem is given by Rødseth [42].

Formulae (13) is proved by Olson in [40]. Applications of this formulae to σ -finite groups are given by the authors of [26] and by Hegyváry [30].

The proof given here looks much easier than the proof of Olson [40].

Lemma 13 has the following consequence.

Corollary 16 [16] *Diderrich Theorem 2 is equivalent to Kneser's Theorem 1.*

Proof. Diderrich Theorem implies clearly Kneser's Theorem. Suppose now that any two elements of B commute and let K be the subgroup generated by B . Observe that K is abelian.

Without loss of generality we may assume that $1 \in B$. Take a left- K -decomposition $A = \bigcup_{i \in I} A_i$. By Proposition 14 and Lemma 13 there is a $j \in I$ such that $A_i B = A_i K$ for all $i \neq j$. Without loss of generality we may assume $A_j \subset K$. Since $\Pi^r((A \setminus B_j)B) = K$, we have $H = \Pi^r(AB) = \Pi^r(A_j B)$. Since K is abelian we have by Kneser's Theorem 1, $|A_j B| = |A_j H| + |BH| - |H|$. Therefore $|AB| = |AH| + |BH| - |H|$. ■

7 Some structural properties of the 2-atoms

In this section we start with some results dealing with non necessarily abelian groups. Proposition 19 gives the value of κ_2 for sets with a small cardinality. Theorem 22 determines the structure of the 2-atoms in the abelian case. This result extends to the infinite case a previous result [22]. The proof given here is much easier than our first proof.

The following lemma, which allows proofs by induction, is implicit in [20]:

Lemma 17 [20] *Let S be a finite generating subset of a group G with $1 \in S$ and $|S| \geq 3$. Let H be a 2-atom with $1 \in H$ and $|H| \leq \frac{|G| - \kappa_2}{2}$. Then one of the following holds:*

- (i) $|H| \leq |S| - 1$.

(ii) H is left-periodic.

Proof.

(i) holds clearly if $|H| = 2$. So we may assume $|H| \geq 3$. For each $x \in H$, there is an $a_x \in H$ such that $a_x^{-1}x \in S \setminus \{1\}$, since otherwise $\partial(H \setminus \{x\}) \subset \partial(H)$, and $H \setminus \{x\}$ would be a 2-fragment, a contradiction. Assume contrary to (i) that $|H| \geq |S|$. By the pigeonhole principle there are $x, y \in H$ with $x \neq y$ and $a_x^{-1}x = a_y^{-1}y$. It follows that $a_y, y \in a_y a_x^{-1}H$. Therefore by Theorem 8, $H = a_y a_x^{-1}H$ and hence H is left-periodic.

■

Lemma 18 *Let S be a finite generating subset of a group G with $1 \in S$, $|S| \geq 3$ and $|S| - 1 \leq \kappa_2(S) \leq |S|$. Let H be a 2-atom of S with $1 \in H$, $\kappa_1(H^{-1}) = |H| - 1$ and $|H| \leq \frac{|G| - \kappa_2}{2}$. Then one of the following holds:*

(i) H is left-periodic,

(ii) $\kappa_2(H^{-1}) \leq |H| + |S| - \kappa_2(S)$.

Proof.

Put $t = |S| - \kappa_2(S)$. Notice that

$$0 \geq t.$$

Take $a \in H \setminus \{1\}$.

Put $L = \langle H \rangle$ and take a right- L -decomposition $S = S_1 \cup \dots \cup S_j$. Without loss of generality we may assume $|HS_1| \leq \dots \leq |HS_j|$. Put $W = \{i : |HS_i| < |L|\}$. By Lemma 13 $|W| \leq \frac{|H|}{|H|-1} < 2$. Then we must have $|HS_i| = |L|$, for $i \geq 2$.

We must have

$$|HS_1| \geq |S_1| + \kappa_1(H^{-1}) = |S_1| + |H| - 1, \quad (14)$$

since otherwise by the isoperimetric inequality $|HS_1| = |L|$. It would follow that $HS = LS$. Then $|HS| - |H| = \kappa_2(S) \leq |LS| - |L| = |HS| - |L|$. Therefore $|H| = |L|$, hence H is a subgroup, a contradiction.

We have

$$\begin{aligned} |S| + |H| + t &= |HS| \\ &\geq |S_1| + |H| - 1 + (j-1)|L|. \end{aligned}$$

Therefore

$$|S| - |S_1| \geq (j-1)|L| - t - 1. \quad (15)$$

By (15) and Lemma 3, $|\{1, a\}S_i| = |L|$, for all $i \geq 2$. It follows that $|S_1| \geq 2$, since otherwise $|\{1, a\}S| = (j-1)|L| + 2 = 2 + |S| + t$. This would imply that $\{1, a\}$ is a 2-atom of S a contradiction.

In order to prove that $\kappa_2(H^{-1}) \leq |H|$, we need to show that $|HS_j| \leq |L| - 2$. Suppose that $|HS_j| \geq |L| - 1$. Then $LS \neq G$, since otherwise $|HS| \geq |G| - 1$.

Now we have $|S| + t = \kappa_2(S) \leq |LS| - |L| = (j - 1)|L| \leq |S| - |S_1| + t + 1 \leq |S| + t - 1$, a contradiction.

■

Let $p(G)$ denotes

- the smallest cardinality of a finite nontrivial subgroup of G if G is a torsion group,
- ∞ if G is a torsion free group.

Proposition 19 *Let S be a generating subset of a group G with $1 \in S$ and $|S| \leq p(G)$. Then*

- (i) $\kappa_1(S) = |S| - 1$,
- (ii) $\kappa_2(S) = |S| - 1$ if and only if S is a progression.

Proof. Put $L = \langle H \rangle$. By Corollary 12, there is a subgroup K with $\kappa_1(S) = \min(|SK|, |KS|) - |K| \geq |K|$, since S generates G and since $1 \in S$. We must have $|K| = 1$, since otherwise $\kappa_1(S) \geq |K| \geq p(G) \geq |S|$. Therefore $\kappa_1(S) = \min(|S|, |S|) - 1 = |S| - 1$.

We shall now prove (ii) by induction on $|S|$. Take a 2-atom H of S and a 2-atom K of S^{-1} . Suppose that $\kappa_2(S) = |S| - 1$.

Case 1. $|H| \leq |K|$ or G is infinite (or both).

By Lemma 9, $|H| \leq \frac{|G| - \kappa_2}{2}$.

Let us first prove that H is non-left-periodic. Suppose the contrary. Then $QH = H$, where $|Q| \geq 2$. It follows that $|S| - 1 = \kappa_2 = |QHS| - |QH| \geq |Q| \geq p(G) \geq |S|$, a contradiction.

By Lemma 17, $|H| \leq |S| - 1 \leq p(G) - 2$. It follows by (i) that $\kappa_1(H^{-1}) = |H| - 1$. By Lemma 18 $\kappa_2(H^{-1}) \leq |H| - 1$.

By the induction hypothesis there are u, k such that $H^{-1} = \{a, au, \dots, au^k\}$. We have $|H \cap uH| = k$. Since H is non-left-periodic, we have by Lemma 9 and Theorem 8, $k = 1$.

Put $H = \{1, r\}$. It follows since $|S| \leq p(G) \leq |\langle r \rangle|$ that S is an r -progression.

Case 2. $|H| > |K|$ and G is finite. By Lemma 5, we have $\kappa_2(S) = \kappa_{-2}(S)$. The proof follows as in Case 1.

■

Corollary 20 *Let G be a group and let A, B be subsets of G such that $|A|, |B| \geq 2$, $1 \in A \cap B$ and $|B| \leq p(G)$.*

Assume that $|AB| = |A| + |B| - 1 \leq |K| - 1$, where $K = \langle B \rangle$.

- If $|A| + |B| = |K|$ then there is a such that $A^{-1}a = K \setminus B$;

- If $|A| + |B| \leq |K| - 1$ then there is r such that B and A are r -progressions.

Proof. By Proposition 19, $\kappa_1(B) = |B| - 1$. Take a left K -decomposition $A = A_1 \cup \dots \cup A_j$. By Lemma 13, $|\{i : |A_i B| \neq |K|\}| \leq 1$. We have $|K| - 1 \geq |AB| > (j - 1)|K|$, and hence $j = 1$.

Assume first $|A| + |B| = |K|$. By the isoperimetric inequality $|A| + |B| - 1 \geq |AB| \geq |A| + |B| - 1 = |K| - 1$. Take $\{a\} = K \setminus AB$. We have $A^{-1}a \subset K \setminus B$. Since these two sets have the same cardinality we must have $A^{-1}a = K \setminus B$.

Assume now $|A| + |B| \leq |K| - 1$.

Then B is 2-separable and $\kappa_2(B) \leq |B| - 1 < p(G)$. By Proposition 19, B is a r -progression, for some r . It follows easily that A is an r -progression. ■

Corollary 20 implies a result due to Brailowski-Freiman [3] since $p(G) = \infty$ for a torsion free group. Corollary 20 implies the validity of Károlyi's Theorem 4 [32] for infinite groups. In the finite case Károlyi's condition $|A| + |B| \leq p(G)$ is relaxed in Corollary 20 to the weaker one $|B| \leq p(G)$ and $|A| + |B| \leq |\langle B \rangle| - 1$. Notice that Corollary 20 implies Vosper's Theorem.

Lemma 21 *Let S be finite subset of an abelian group G with $1 \in S$. Then $\kappa_k(S) = \kappa_{-k}(S)$ and $\alpha_k(S) = \alpha_{-k}(S)$.*

Proof. This follows since the map $x \rightarrow x^{-1}$ is an isomorphism from $\text{Cay}(G, S)$ onto its reverse $\text{Cay}(G, S^{-1})$, in the abelian case. ■

The next result is proved for $\kappa_2 = |S| - 1$ in [19], and for finite groups in [22].

Theorem 22 [19, 22]. *Let S be a finite generating 2-separable subset of an abelian group G with $1 \in S$ and $\kappa_2(S) \leq |S|$. Put $t = \kappa_2(S) - |S|$. Also assume that $|S| \neq |G| - 6$ if $t = 0$. Let $1 \in H$ be a 2-atom. If H is not a subgroup then $|H| = 2$.*

Proof.

Put $L = \langle H \rangle$. The result holds clearly if $\kappa_2(S) \leq |S| - 2$, by Lemma 21 and Proposition 11, since in this case a 2-atom is a 1-atom. So we may assume $|S| \geq \kappa_2(S) \geq |S| - 1$. Therefore

$$-1 \leq t \leq 0.$$

The proof is by induction on $|S|$, the result being obvious if $|S| = 2$. Suppose that H is not a subgroup. By Lemma 10, H is aperiodic.

By Lemma 21 and Theorem 8, for every $a \in H \setminus \{1\}$,

$$|aH \cap H| = 1.$$

Claim For every subgroup M of L with $3 \leq |M| < |L|$ we have $|MH| > |M| + |H|$.

Suppose the contrary and take a subset $\{1, b, c\} \subset M$ of cardinality 3. Since $1 \in H$ and H generates L , we have $2|M| \leq |MH| \leq |M| + |H|$ and hence $|M| \leq |H|$. Then

$$\begin{aligned} |H| + |M| &= |HM| \\ &\geq |H + \{1, a, b\}| \\ &\geq |H| + |Ha \setminus H| + |Hb \setminus (H \cup Ha)| \\ &\geq |H| + |H| - 1 + |H| - 2. \end{aligned}$$

Therefore $|M| = |H| = 3$ and hence $M = H$. It follows that H is a subgroup, a contradiction.

We have $\kappa_1(H) = |H| - 1$ since otherwise by Lemma 21 and Proposition 11 there is a subgroup M with $1 < |M| < |L|$ such $|MH| \leq |M| + |H| - 2$, contradicting the claim.

By Lemma 17, $|H| \leq |S| - 1$ and $\kappa_2(H) \leq |H|$. Let K be a 2-atom of H with $1 \in K$. By the claim K can not be a subgroup if $|K| > 2$.

By the induction hypothesis we have $|K| = 2$. Put $K = \{1, k\}$. We must have $|kH \cap H| \geq 2|H| - |KH| = |H| - 2 - t$. Then $|H| \leq 3 + t$. It follows that $|H| = 2$ if $t = -1$.

Assume $t = 0$. Then $|H| \leq 3$.

Suppose $|H| = 3$ and put $H = \{1, c, d\}$. Since $|H^\wedge| \geq |H|$, we have by Lemma 21 and Theorem 7 that $H \cup cH$ is a 2-fragment.

Observe that $|G| = |H| + \kappa_2 + |H^\wedge| \geq 2|H| + |S| \geq 6 + |S|$.

Now we have $|(H \cup cH)^\wedge| \geq |G| - (2|H| - 1) - |S| = |G| - |S| - t - 5 \geq 2$, since otherwise $|S| = |G| - 6$. We must have $|(H \cup cH)^\wedge| \geq 3$, since otherwise $(H \cup cH)^\wedge$ would be a negative 2-fragment by Lemma 6. It follows that $2 = \alpha_{-2} = \alpha_2$, a contradiction. By Lemma 21 and Theorem 7, $|(H \cup cH) \cap dH| \leq 1$. Therefore $|H^2| = |H \cup cH| + |dH \setminus (H \cup cH)| \geq 2|H| - 1 + |H| - 1 = 7$, a contradiction since $|X^2| \leq 6$ for every subset X of an abelian group with $|X| = 3$.

Therefore $|H| = 2$. ■

In the case where $|G|$ is a prime, a proof of Theorem 22 using the Davenport's transform was obtained by the authors of [27]. In [44] Serra and Zémor proved that a 2-atom of S has size $= 2$ if $|G|$ is a prime and $|S| < \binom{4+\kappa_2-|S|}{2}$. A short proof of the last result was given by the authors of [28]. A generalization of this result to arbitrary finite abelian groups was obtained by the authors of [29] when $\kappa_2 - |S| \leq 4$.

An example given by Serra and Zémor [44] shows that in the prime case the 2-atom may have size $= 3$ if $|S| = |G| - 6$.

8 More intersection properties of fragments

In this section we prove more results on the intersection of fragments that we shall need in the next section.

We first prove a result concerning the intersection of an atom and the dual of negative atom. In the finite case this result follows by Theorem 7.

Theorem 23 *Let $\Gamma = (V, E)$ be a reflexive locally finite graph such that $|V| \geq 2k - 1$. Let X be a k -fragment and let Y be a negative k -fragment such that $|Y| \geq |X|$ and $|X \cap Y^\vee| \geq k$. Then $X \cap Y^\vee$ is a k -fragment. In particular $X \subset Y^\vee$ if X is a k -atom.*

Proof.

The result is obvious if Γ is non- k -separable since k -fragments are k -subsets. So we may assume that Γ is k -separable.

Suppose that $|X \cap Y^\vee| \geq k$.

\times	Y^\vee	$\partial^-(Y)$	Y
X	R_{11}	R_{12}	R_{13}
$\partial(X)$	R_{21}	R_{22}	R_{23}
X^\wedge	R_{31}	R_{32}	R_{33}

By the definition of a k -fragment we have

$$\kappa_k = |\partial(X)| = |R_{21}| + |R_{22}| + |R_{23}|.$$

The following inclusion follows by an easy verification:

$$\partial(X \cap Y^\vee) \subset R_{12} \cup R_{22} \cup R_{21}.$$

We have clearly $|V \setminus \Gamma(X \cap Y^\vee)| \geq |V \setminus \Gamma(X)| \geq k$. By the definition we have $|\partial(X \cap Y^\vee)| \geq \kappa_k$. It follows that

$$\begin{aligned} |R_{21}| + |R_{22}| + |R_{23}| &= \kappa_k \\ &\leq |\partial(X \cap Y^\vee)| \\ &\leq |R_{12}| + |R_{22}| + |R_{21}|. \end{aligned}$$

Therefore

$$|R_{12}| \geq |R_{23}|. \tag{16}$$

Now

$$\begin{aligned} |X^\wedge \cap Y| = |R_{33}| &= |Y| - |R_{23}| - |R_{13}| \\ &\geq |X| - |R_{12}| - |R_{13}| \\ &= |R_{11}| \geq k. \end{aligned}$$

By the definition of κ_{-k} we have

$$|\partial(X^\wedge \cap Y)| \geq \kappa_{-k}.$$

It follows that

$$\begin{aligned}
|R_{12}| + |R_{22}| + |R_{32}| &= \kappa_{-k} \\
&\leq |\partial(X^\wedge \cap Y)| \\
&\leq |R_{22}| + |R_{23}| + |R_{32}|.
\end{aligned}$$

Therefore $|R_{12}| \leq |R_{23}|$. By (16) we have $|R_{12}| = |R_{23}|$.

It follows that the inequality $\kappa_k \leq |\partial(X \cap Y^\vee)|$, used in the proof of (16), is an equality and hence $X \cap Y^\vee$ is a k -fragment of Γ .

■

Notice that the last proof is similar to the proof of Theorem 7. A common proof of these two results is given [21] in the special case $\kappa_k = \kappa_1$. We gave a direct proof in order to avoid the introduction of the unnecessary notion of a cofinite fragment.

Lemma 24 *Let $\Gamma = (V, E)$ be a locally finite k -separable reflexive graph. Let X and Y be k -fragments. Then $X \subset Y$ if and only if $Y^\wedge \subset X^\wedge$.*

Proof. Assume first that $X \subset Y$. Then

$$X^\vee = V \setminus \Gamma(X) \supset V \setminus \Gamma(Y) = Y^\vee$$

Assume now that $Y^\vee \subset X^\vee$. Then

$$X = V \setminus \Gamma^{-1}(X^\vee) \subset V \setminus \Gamma^{-1}(Y^\vee) = Y.$$

■

Lemma 25 *Let $\Gamma = (V, E)$ be a reflexive locally finite k -separable graph with $k \geq 2$. Let A, F be k -fragments such that $|A| \leq |F^\wedge|$ and $|A \cap F| \geq k - 1$. Then*

$$|A \cap \partial(F)| \leq |\partial(A) \cap F^\wedge|, \quad (17)$$

$$|\Gamma(A) \cap \Gamma(F)| \leq |A \cap F| + \kappa_k \text{ and} \quad (18)$$

$$|F^\wedge \setminus A^\wedge| \leq |A \setminus F| + \kappa_k - \kappa_{k-1}. \quad (19)$$

Proof.

\times	F	$\partial(F)$	F^\wedge
A	R_{11}	R_{12}	R_{13}
$\partial(A)$	R_{21}	R_{22}	R_{23}
A^\wedge	R_{31}	R_{32}	R_{33}

Suppose that (17) is false. Then $|R_{12}| + |R_{22}| + |R_{21}| > \kappa_k = |R_{23}| + |R_{22}| + |R_{21}|$, and hence

$$|R_{12}| > |R_{23}|.$$

It follows that

$$\begin{aligned} |F^\wedge \cap A^\wedge| &= |F^\wedge| - |R_{23}| - |R_{13}| \\ &> |A| - |R_{12}| - |R_{13}| \\ &= |R_{11}| \geq k - 1, \end{aligned}$$

Now we have

$$\begin{aligned} |R_{32}| + |R_{22}| + |R_{23}| &\geq |\partial(A \cup F)| \geq \kappa_k \\ &= |R_{23}| + |R_{22}| + |R_{21}|, \end{aligned}$$

and hence $|R_{23}| \geq |R_{21}|$ a contradiction proving (17). Now we have

$$\begin{aligned} |\Gamma(A) \cap \Gamma(F)| &= |A \cap F| + |R_{12}| + |R_{21}| + |R_{22}| \\ &\leq |A \cap F| + |R_{23}| + |R_{21}| + |R_{22}| \\ &= |A \cap F| + \kappa_k. \end{aligned}$$

This proves (18).

Since $|A \cap F| \geq k - 1$, we have

$$\begin{aligned} |R_{12}| + |R_{22}| + |R_{21}| &\geq \kappa_{k-1} \\ &= \kappa_k - (\kappa_k - \kappa_{k-1}) \\ &= |R_{21}| + |R_{22}| + |R_{23}| - (\kappa_k - \kappa_{k-1}). \end{aligned}$$

It follows that

$$|R_{23}| \leq |R_{12}| + \kappa_k - \kappa_{k-1}.$$

Hence

$$\begin{aligned} |F^\wedge \setminus A^\wedge| &= |R_{13}| + |R_{23}| \\ &\leq |R_{13}| + |R_{12}| + \kappa_k - \kappa_{k-1} \\ &= |A \setminus F| + \kappa_k - \kappa_{k-1}. \end{aligned}$$

This proves (19).

■

9 More structural properties of the 2-atoms

In this section we investigate the number of 2-atoms containing a given element and obtain an upper bound for the size of a 2-atom.

We obtain some applications including a generalization to the infinite case a result proved in the finite case by Arad and Muzychuk [1].

The smallest number j (possibly null) such that every element $x \in V$ belongs to j pairwise distinct k -atoms will be denoted by $\omega_k(\Gamma)$. We shall write $\omega_k(\Gamma^{-1}) = \omega_{-k}(\Gamma)$. We also write $\omega_k(S) = \omega_k(\text{Cay}(\langle S \rangle, S))$. The next result is a basic tool in the investigation of the 2-atoms structure. The case $\kappa_2 = \kappa_1$ of this result is proved in [21].

Theorem 26 *Let $\Gamma = (V, E)$ be a reflexive locally finite graph. Let H be a 2-atom and let K be a negative 2-atom with $|K| \geq |H| \geq 3$. Also assume that $|K| \leq \frac{|G| - \kappa_2}{2}$. Then one of the following holds:*

- (i) $\omega_2 \leq 2$ or $\omega_{-2} \leq 2$,
- (ii) $|H| \leq 3 + \max(\kappa_2 - \delta, \kappa_{-2} - \delta_-)$.

Proof.

Suppose contrary to (i) that $\omega_2 \geq 3$ and $\omega_{-2} \geq 3$. We have $\alpha_1 = \alpha_{-1} = 1$, since otherwise a 2-atom containing x is a 1-atom containing x and is unique by Theorem 8, contradicting $\omega_2, \omega_{-2} \geq 3$.

Take $v \in V$ and choose two distinct 2-atoms M_1, M_2 such that $v \in M_1 \cap M_2$. By Theorem 8, we have $M_1 \cap M_2 = \{v\}$.

\times	M_2	$\partial_-(M_2)$	M_2^\wedge
M_1	v	R_{12}	R_{13}
$\partial_-(M_1)$	R_{21}	R_{22}	R_{23}
M_1^\wedge	R_{31}	w	

We have $M_1^\wedge \not\subset M_2^\wedge$, by Lemma 24.

Take $w \in M_1^\wedge \setminus M_2^\wedge$, and take three pairwise distinct negative 2-atoms L_1, L_2, L_3 such that $w \in L_1 \cap L_2 \cap L_3$.

Assume first that for some $i \neq j$ we have $L_i \cup L_j \subset M_1^\wedge$.

By Theorem 23, $|L_i \cap M_2^\wedge| \leq 1$, for every i .

Then we have using (19)

$$\begin{aligned}
|H| + \kappa_2 - \delta &= |M_1 \setminus M_2| + \kappa_2 - \kappa_1 \\
&\geq |(M_1^\wedge \setminus M_2^\wedge) \cap (L_i \cup L_j)| \\
&= |L_i \cup L_j| - |(L_i \cup L_j) \cap M_2^\wedge| \\
&\geq |L_i \cup L_j| - 2 \\
&= 2|K| - 3 \geq 2|H| - 3,
\end{aligned}$$

and hence (ii) holds.

We can now assume without loss of generality that

$$L_1, L_2 \not\subset M_1^\wedge.$$

By Lemma 24, we have $M_1 \not\subset L_i^\vee$ for $1 \leq i \leq 2$. By Theorem 23, $|M_1 \cap (L_1^\vee \cup L_2^\vee)| \leq 2$. Then $|M_1 \cap \Gamma^{-1}(L_1) \cap \Gamma^{-1}(L_2)| \geq |M_1| - 2$.

Now $\Gamma^{-1}(L_1) \cap \Gamma^{-1}(L_2) \supset (\Gamma^{-1}(L_1) \cap \Gamma^{-1}(L_2) \cap M_1) \cup \Gamma^{-1}(w)$. Observe that $\Gamma^{-1}(w) \cap M_1 = \emptyset$.

Then we have by (18)

$$\begin{aligned}
\delta_- + |H| - 2 &\leq |\Gamma^{-1}(w)| + |M_1 \cap \Gamma^{-1}(L_1) \cap \Gamma^{-1}(L_2)| \\
&\leq |\Gamma^{-1}(L_1) \cap \Gamma^{-1}(L_2)| \\
&\leq 1 + \kappa_{-2}.
\end{aligned}$$

Therefore $|H| \leq 3 + \kappa_{-2} - \delta_-$.

■

Let us apply this result in the symmetric case.

Corollary 27 *Let S be a finite generating subset of a group G with $1 \in S$ and $S = S^{-1}$. Let H be a 2-atom of S such that $1 \in H$. If $|H| \geq \kappa_2 - |S| + 4$, then H is left-periodic.*

Proof. Since $S = S^{-1}$, H is also a negative 2-atom. Also $\kappa_2 = \kappa_{-2}$. Take a 3-subset $\{a_1, a_2, a_3\}$ contained in H .

By Theorem 26 $\omega_2 \leq 2$. Hence two of the 2-atoms $a_1^{-1}H, a_2^{-1}H, a_3^{-1}H$ must be equal and hence H is left-periodic. ■

Let us now formulate on lemma:

Lemma 28 *Let S be a finite generating semi-normal subset of a group G with $1 \in S$. Then F^{-1} is a negative k -fragment for any k -fragment F . Moreover*

- (1) $\kappa_k = \kappa_{-k}$,
- (2) $\alpha_k = \alpha_{-k}$,

(3) $\omega_k = \omega - k$.

Proof. We have $|F^{-1}S^{-1}| = |SF| = |FS| = |F| + \kappa_k$. It follows that $\kappa_k \geq \kappa_{-k}$. The inequality $\kappa_k \leq \kappa_{-k}$ follows by duality and hence (1) holds. It follows easily that $\alpha_k = \alpha_{-k}$.

■

The next result extends to the infinite case Arad-Muzychuk Theorem 3.1 of [1]. Our terminologies differ slightly:

Corollary 29 *Let S be a finite generating semi-normal subset of a group G with $1 \in S$. Let H be a 2-atom of S such that $1 \in H$. If $|H| \geq \kappa_2 - |S| + 4$, then H is subgroup of G and $[G : N_G(H)] \leq 2$.*

Proof. By Lemma 28, $\alpha_2 = \alpha_{-2}$, $\kappa_2 = \kappa_{-2}$ and $\omega_2 = \omega_{-2}$.

By Theorem 26, $\omega_2 \leq 2$.

Take a 3-subset $\{a_1, a_2, a_3\}$ contained in H . Then two of the 2-atoms $a_1^{-1}H, a_2^{-1}H, a_3^{-1}H$ must be equal and hence H is left-periodic. By Lemma 28 H^{-1} is a negative 2-atom which is right-periodic. By Lemma 10, H is a subgroup.

Clearly xHx^{-1} is a 2-atom for every x . If $xHx^{-1} = H$, for every x then $N_G(H) = G$. Suppose that there is a such that $aHa^{-1} \neq H$,

We have $[G : N_G(H)] \leq 2$, since for every $x \in G$, we have $xHx^{-1} = H$ or $xHx^{-1} = aH$, otherwise $\omega_2 \geq 3$, a contradiction.

■

10 Saturated subsets

In this section we present a more general formalism.

Let $\Gamma = (V, E)$ be a locally finite graph and let X be a subset of V . A Γ -saturation of a subset X is a maximal subset $X' \supset X$ such that $\Gamma(X') = \Gamma(X)$. Such a subset exists and is clearly unique if X is finite or cofinite. We shall denote it by X^*_Γ . The subset X will be called Γ -saturated if $X = X^*_\Gamma$.

The reference to Γ will be omitted when it is clear from the context. Let G be a group and let S be a subset of G . A subset X is said to be saturated with respect S if it is saturated in $\text{Cay}(G, S)$.

It is an easy exercise to show that a k -fragment is a saturated subset.

Our notion of "saturated subset" coincides with the notions of "a cell" in Balandraud [2] and "non-extendible" in Gryniewicz [12].

Lemma 30 *Let Γ be a locally finite graph on a set V and let X be a finite subset. Then*

$$X^*_\Gamma = X^{\wedge \vee}. \quad (20)$$

Moreover the intersection of two saturated subsets is a saturated subset.

The proof is left to the reader.

Let $\Gamma = \text{Cay}(G, S)$.

Then (20) may be formulated equivalently: $X^\star = G \setminus ((G \setminus XB)B^{-1})$. The last formulae is proved in [2].

Lemma 31 *Let Γ be a locally finite graph on a set V and let X be a finite saturated subset. Then*

$$\partial_-(X^\wedge) = \partial(X), \quad (21)$$

$$(X^\wedge)^\vee = X. \quad (22)$$

Moreover X^\wedge is a Γ^{-1} -saturated subset.

The proof is similar to the proof of Lemma 6. Let $\Gamma = \text{Cay}(G, S)$. Then (22) may be formulated equivalently: $G \setminus ((G \setminus XB)B^{-1}) = X$.

This formulae is proved in [2, 12]. Our formalism used here is very close to a proof given in [13].

It was observed in [18] that Kneser's Theorem implies that in an abelian Cayley graph $\text{Cay}(G, S)$ any minimal cutset with cardinality $\leq |S| - 2$ is periodic.

A much precise result was proved recently by Balandraud [2]. Actually Balandraud's result implies a strong form of Kneser's Theorem.

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