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# A characterization of the Riesz distribution

(Running title: **A characterization of the Riesz distribution**)

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## Abstract

Bobecka and Wesolowski (2002) have shown that, in the Olkin and Rubin characterization of the Wishart distribution (See Casalis and Letac (1996)), when we use the division algorithm defined by the quadratic representation and replace the property of invariance by the existence of twice differentiable densities, we still have a characterization of the Wishart distribution. In the present work, we show that, when we use the division algorithm defined by the Cholesky decomposition, we get a characterization of the Riesz distribution.

*Keywords:* Symmetric cone, division algorithm, Wishart distribution, Riesz distribution, Beta-Riesz distribution, functional equation

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## 1 Introduction

A remarkable characterization of the gamma distribution, due to Luckacs (1955), says that if  $U$  and  $V$  are two independent non Dirac and non negative random variables such that  $U+V$  is a.s. positive, then  $\frac{U}{U+V}$  and  $U+V$  are independent if and only if  $U$  and  $V$  have the gamma distribution with the same scale parameter. The classical multivariate version of this characterization concerns the Wishart distribution on the cone of symmetric positive matrices. In this case, there is not a single way to define the quotient of two matrices. For instance if  $Y$  is a positive definite matrix, one can for example, use the quadratic representation, that is write  $Y = Y^{\frac{1}{2}}Y^{\frac{1}{2}}$  and define the ratio  $X$  by  $Y$  as  $Y^{-\frac{1}{2}}XY^{-\frac{1}{2}}$ , or use the Cholesky decomposition  $Y = TT^*$ , where  $T$  is a lower triangular matrix and define the ratio as  $(T^{-1})X(T^{-1})^*$ . A general definition of a division algorithm will

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be given in Section 2, however these two examples are the most usual and most important. In 1962, Olkin and Rubin have shown that, if  $U$  and  $V$  are two independent random variables valued in the cone of symmetric non negative matrices such that  $U + V$  is a.s. positive definite, then, independently of the choice of the division algorithm, the quotient of  $U$  by  $U + V$  is independent of  $U + V$  and its distribution is invariant by the orthogonal group if and only if  $U$  and  $V$  have the Wishart distribution. This result has been extended to the Wishart distribution on any symmetric cone by Casalis and Letac (1996). Recently, Bobecka and Wesolowski (2002) have given another characterization of the Wishart distribution without any invariance assumption for the quotient. More precisely, they have shown that if we use the division algorithm defined by the quadratic representation and we replace in the Olkin and Rubin theorem the condition of invariance for the distribution of the quotient by the existence of twice differentiable densities, then we still have a characterization of the Wishart distribution. The present paper gives a parallel result which starts from the observation that, when the condition of invariance of the distribution of the ratio by the orthogonal group is dropped, the characterization in the Bobecka and Wesolowski way is not independent of the choice of the division algorithm. We show that, when we use the division algorithm defined by the Cholesky decomposition, we get a characterization of the Riesz distribution introduced by Hassairi and Lajmi (2001). Our method of proof is based on some functional equations depending on the triangular group. These equations are more involved than the ones used in the characterization of the Wishart distribution, their solutions are expressed in terms of the generalized power. Our results will be presented in the framework of the Riesz distribution on the symmetric cone of a simple Euclidean Jordan algebra. This will enable us to use some technical results established in the book of Faraut-Korányi (1994) and in Hassairi et al. (2001, 2005). However, to make the paper accessible to a reader who is not familiar with the theory of Jordan algebras, we will give a particular emphasis to the cone of positive definite symmetric matrices.

## 2 Riesz distributions

We first review some facts concerning Jordan algebras and their symmetric cones. Our notations are the ones used in the book of Faraut-Korányi (1994). Let us recall that a Euclidean Jordan algebra is a Euclidean space  $E$  with scalar product  $\langle x, y \rangle$  and a bilinear map

$$E \times E \longrightarrow E, (x, y) \longmapsto xy$$

called Jordan product such that, for all  $x, y, z$  in  $E$ ,

- i)  $xy = yx$ ,
- ii)  $\langle x, yz \rangle = \langle xy, z \rangle$ ,
- iii) there exists  $e$  in  $E$  such that  $ex = x$ ,
- iv)  $x(x^2y) = x^2(xy)$ , where we used the abbreviation  $x^2 = xx$ .

An Euclidean Jordan algebra is said to be simple if it does not contain a nontrivial ideal. Actually to each Euclidean simple Jordan algebra, one attaches the set of Jordan squares

$$\overline{\Omega} = \{x^2; x \in E\}.$$

Its interior  $\Omega$  is a symmetric cone, i.e., a cone which is

i) self dual, i.e.,  $\Omega = \{x \in E; \langle x, y \rangle > 0 \forall y \in \overline{\Omega} \setminus \{0\}\}$ .

ii) homogeneous, i.e., the subgroup  $G(\Omega)$  of the linear group  $GL(E)$  of linear automorphisms which preserves  $\Omega$  acts transitively on  $\Omega$ .

iii) salient, i.e.,  $\Omega$  does not contain a line. Furthermore, it is irreducible in the sense that it is not the product of two cones.

Let now  $x$  be in  $E$ . If  $L(x)$  is the endomorphism of  $E$ ;  $y \mapsto xy$  and  $P(x) = 2L(x)^2 - L(x^2)$ , then  $L(x)$  and  $P(x)$  are symmetric for the Euclidean structure of  $E$  and the map  $x \mapsto P(x)$  is called the quadratic representation of  $E$ .

An element  $c$  of  $E$  is said to be idempotent if  $c^2 = c$ , it is a primitive idempotent if furthermore  $c \neq 0$  and is not the sum  $t + u$  of two non null idempotents  $t$  and  $u$  such that  $t.u = 0$ .

A Jordan frame is a set  $\{c_1, \dots, c_r\}$  of primitive idempotents such that  $\sum_{i=1}^r c_i = e$  and  $c_i c_j = \delta_{ij} c_i$ , for  $1 \leq i, j \leq r$ . It is an important result that the size  $r$  of such a frame is a constant called the rank of  $E$ .

If  $c$  is a primitive idempotent of  $E$ , the only possible eigenvalues of  $L(c)$  are  $0, \frac{1}{2}$  and  $1$ . The corresponding eigenspaces are respectively denoted by  $E(c, 0)$ ,  $E(c, \frac{1}{2})$  and  $E(c, 1)$  and the decomposition

$$E = E(c, 0) \oplus E(c, \frac{1}{2}) \oplus E(c, 1)$$

is called the Peirce decomposition of  $E$  with respect to  $c$ .

Suppose now that  $(c_i)_{1 \leq i \leq r}$  is a Jordan frame in  $E$  and let, for  $1 \leq i, j \leq r$ ,

$$E_{ij} = \begin{cases} E(c_i, 1) = \mathbb{R}c_i & \text{if } i = j \\ E(c_i, \frac{1}{2}) \cap E(c_j, \frac{1}{2}) & \text{if } i \neq j. \end{cases}$$

Then (See Faraut-Korányi (1994), Theorem IV.2.1) we have  $E = \bigoplus_{i \leq j} E_{ij}$  and the dimension of  $E_{ij}$  is, for  $i \neq j$ , a constant  $d$  called the Jordan constant. It is related to the dimension  $n$  and the rank  $r$  of  $E$  by the relation  $n = r + r(r-1)\frac{d}{2}$ .

For  $1 \leq k \leq r$ , let  $P_k$  denote the orthogonal projection on the Jordan subalgebra

$$E^{(k)} = E(c_1 + \dots + c_k, 1),$$

$\det^{(k)}$  the determinant in the subalgebra  $E^{(k)}$  and, for  $x$  in  $E$ ,  $\Delta_k(x) = \det^{(k)}(P_k(x))$ . Then  $\Delta_k$  is called the principal minor of order  $k$  with respect to the Jordan frame  $(c_i)_{1 \leq i \leq r}$ . For  $s = (s_1, \dots, s_r) \in \mathbb{R}^r$ , and  $x$  in  $\Omega$ , we write

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \dots \Delta_r(x)^{s_r}.$$

This is the generalized power function. Note that, if  $x = \sum_{i=1}^r \lambda_i c_i$ , then  $\Delta_s(x) = \lambda_1^{s_1} \lambda_2^{s_2} \dots \lambda_r^{s_r}$  and that  $\Delta_s(x) = (\det x)^p$  if  $s = (p, \dots, p)$  with  $p \in \mathbb{R}$ . It is also easy to see that  $\Delta_{s+s'}(x) = \Delta_s(x)\Delta_{s'}(x)$ . In particular, if  $m \in \mathbb{R}$  and  $s + m = (s_1 + m, \dots, s_r + m)$ , we have  $\Delta_{s+m}(x) = \Delta_s(x)(\det x)^m$ .

As we have mentioned above, one may suppose that  $E$  is the algebra of real symmetric matrices with rank  $r$ . In this case, the Jordan product  $xy$  of two symmetric matrices  $x$  and  $y$  is defined by  $\frac{1}{2}(x.y + y.x)$  where  $x.y$  is the ordinary product of the matrices  $x$  and  $y$ , the cone  $\Omega$  is the cone of positive definite matrices,  $\bar{\Omega}$  is the cone of symmetric non negative matrices and  $d = 1$ . If  $x = (x_{ij})_{1 \leq i, j \leq r}$  is an  $(r, r)$ -symmetric positive definite matrix, and if, for  $1 \leq k \leq r$ , we denote  $P_k(x) = (x_{ij})_{1 \leq i, j \leq k}$  and  $\Delta_k(x) = \det(x_{ij})_{1 \leq i, j \leq k}$ , the generalized power is the function on  $\Omega$  defined by

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2} \Delta_2(x)^{s_2 - s_3} \dots \Delta_r(x)^{s_r}.$$

The definition of the Riesz distribution on a symmetric cone  $\Omega$  relies on the notion of generalized power. In fact, for  $\sigma$  is in  $\Omega$  and  $s$  such that, for all  $i$ ,  $s_i > (i - 1)\frac{d}{2}$ , the measure on  $\Omega$

$$R(s, \sigma)(dx) = \frac{1}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})} e^{-\langle \sigma, x \rangle} \Delta_{s - \frac{d}{2}}(x) \mathbf{1}_\Omega(x) dx$$

where  $\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^r \Gamma(s_j - (j-1)\frac{d}{2})$ , is a probability distribution. It is called the Riesz distribution with parameters  $s$  and  $\sigma$ .

We come now to the general definition of a division algorithm in a symmetric cone  $\Omega$ . Let  $G$  be the connected component of the identity in  $G(\Omega)$ . A division algorithm is defined as a measurable map  $g$  from  $\Omega$  into  $G$  such that, for all  $y$  in  $\Omega$ ,  $g(y)(y) = e$ .

As in the case of symmetric matrices, we will introduce two important division algorithms, the first is based on the quadratic representation  $x \mapsto P(x^{-\frac{1}{2}})$  and the second algorithm takes its values in the triangular group  $T$ . For the definition of  $T$ , we need to introduce some other facts concerning a Jordan algebra. For  $x$  and  $y$  in  $E$ , let  $x \square y$  denote the endomorphism of  $E$  defined by

$$x \square y = L(xy) + [L(x), L(y)] = L(xy) + L(x)L(y) - L(y)L(x). \quad (1)$$

If  $c$  is an idempotent and if  $z$  is an element of  $E(c, \frac{1}{2})$ ,

$$\tau_c(z) = \exp(2z \square c)$$

is called a Frobenius transformation, it is an element of the group  $G$ .

Given a Jordan frame  $(c_i)_{1 \leq i \leq r}$ , the subgroup of  $G$

$$T = \left\{ \tau_{c_1}(z^{(1)}) \dots \tau_{c_{r-1}}(z^{(r-1)}) P\left(\sum_{i=1}^r a_i c_i\right), a_i > 0, z^{(j)} \in \bigoplus_{k=j+1}^r E_{jk} \right\}$$

is called the triangular group corresponding to the Jordan frame  $(c_i)_{1 \leq i \leq r}$ . It is an important result (Faraut-Korányi, p.113, Prop VI.3.8) that the symmetric cone  $\Omega$  of the algebra  $E$  is parameterized by the set

$$E_+ = \left\{ u = \sum_{i=1}^r u_i c_i + \sum_{i < j} u_{ij}, u_i > 0 \right\} \quad (2)$$

More precisely, if

$$t_u = \tau_{c_1}(z^{(1)}) \dots \tau_{c_{r-1}}(z^{(r-1)}) P\left(\sum_{i=1}^r u_i c_i\right) \quad (3)$$

where  $z_{ij} = \frac{u_{ij}}{u_i}$ ,  $i < j$  and  $z^{(j)} = \sum_{k=j+1}^r z_{jk}$ , then the map  $u \mapsto t_u(e)$  is a bijection from  $E_+$  into  $\Omega$  with a Jacobian equal to  $2^r \prod_{i=1}^r u_i^{1+d(r-i)}$ . Also, for all  $x$  in  $E$ , we have

$$\Delta_k(t_u(x)) = u_1^2 \dots u_k^2 \Delta_k(x) = \Delta_k(t_u(e)) \Delta_k(x).$$

It is shown that, for each  $b$  in  $\Omega$ , there exists a unique  $t$  in the triangular group  $T$  such that  $b = t(e)$ . Hence the map

$$g : \Omega \longrightarrow T; b \longmapsto t^{-1} \quad (4)$$

realizes a division algorithm.

### 3 Characterization of the Riesz distribution

In this section we state and prove our main result which may be seen as an extension of the result by Bobecka and Wesolowski (2002) concerning the ordinary Wishart distribution on symmetric matrices. More precisely these authors use the division algorithm defined by the quadratic representation to characterize the Wishart distribution, we use the division algorithm defined by the triangular group to characterize the Riesz distribution.

**Theorem 3.1.** *Let  $X$  and  $Y$  be two independent Riesz random variables  $X \sim R(s, \sigma)$  and  $Y \sim R(s', \sigma)$ . If we set  $V = X + Y$  and  $U = g(X + Y)(X)$ , then*

- i)  $V$  is a Riesz random variable  $V \sim R(s + s', \sigma)$  and is independent of  $U$*
- ii) The density of  $U$  with respect to the Lebesgue measure is*

$$\frac{1}{B_\Omega(s, s')} \Delta_{s-\frac{n}{r}}(x) \Delta_{s'-\frac{n}{r}}(e-x) \mathbf{1}_{\Omega \cap (e-\Omega)}(x),$$

where  $B_\Omega(s, s')$  is the beta function defined on the symmetric cone  $\Omega$  (See Faraut-Korányi, 1994, p.130) by

$$B_\Omega(s, s') = \frac{\Gamma_\Omega(s) \Gamma_\Omega(s')}{\Gamma_\Omega(s + s')}.$$

**Proof.** Consider the transformation:  $\Omega \times \Omega \longrightarrow (\Omega \cap (e - \Omega)) \times \Omega$ ;  $(x, y) \longmapsto (u, v)$ , where  $v = x + y = te$ ,  $t \in T$  and  $u = t^{-1}(x)$ . Its Jacobian is  $\det t^{-1} = (\det t(e))^{-\frac{n}{r}} = (\det v)^{-\frac{n}{r}}$ .

The density of probability of  $(U, V)$  with respect to the Lebesgue measure is then given by

$$\frac{1}{\Gamma_{\Omega}(s)\Gamma_{\Omega}(s')\Delta_{s+s'}(\sigma^{-1})} \Delta_{s-\frac{n}{r}}(t(u))\Delta_{s'-\frac{n}{r}}(t(e-u)) (\det v)^{\frac{n}{r}} e^{-\langle \sigma, v \rangle} \mathbf{1}_K(u, v)$$

where  $K$  is defined by

$$K = \{(u, v)/u \in \Omega \cap (e - \Omega) \text{ and } v \in \Omega\}.$$

Using equality (3.5) in Hassairi and Lajmi (2001), this density may be written as

$$\frac{\Gamma_{\Omega}(s+s')}{\Gamma_{\Omega}(s)\Gamma_{\Omega}(s')} \Delta_{s-\frac{n}{r}}(u) \Delta_{s'-\frac{n}{r}}(e-u) \frac{1}{\Gamma_{\Omega}(s+s')\Delta_{s+s'}(\sigma^{-1})} e^{-\langle \sigma, v \rangle} \Delta_{s+s'-\frac{n}{r}}(v) \mathbf{1}_K(u, v).$$

From this we deduce that  $U$  and  $V$  are independent and that  $V \sim R(s+s', \sigma)$ . Furthermore the distribution of  $U$  is concentrated on  $\Omega \cap (e - \Omega)$  with a density equal to

$$\frac{\Gamma_{\Omega}(s+s')}{\Gamma_{\Omega}(s)\Gamma_{\Omega}(s')} \Delta_{s-\frac{n}{r}}(u) \Delta_{s'-\frac{n}{r}}(e-u) \mathbf{1}_{\Omega \cap (e-\Omega)}(u).$$

□

Note that the distribution of the random variable  $U = g(X+Y)(X)$  is called beta-Riesz distribution with parameters  $s$  and  $s'$  (See Hassairi et al (2005)). In the case of symmetric matrices, the random variable  $U$  is nothing but

$$U = (T^{-1})X(T^{-1})^*,$$

where  $T$  is a lower triangular matrix with positive diagonal such that

$$X + Y = TT^*.$$

**Theorem 3.2.** *Let  $b \longmapsto g(b)$  be the division algorithm defined by (4). Let  $X$  and  $Y$  be independent random variables valued in  $\Omega$  with strictly positive twice differentiable densities. Set  $V = X+Y$  and  $U = g(V)(X)$ . If  $U$  and  $V$  are independent then there exist  $s, s' \in \mathbb{R}^r$ ;  $s_i > (i-1)\frac{d}{2}$ ,  $s'_i > (i-1)\frac{d}{2}$  for all  $i$ , and  $\sigma \in \Omega$  such that  $X \sim R(s, \sigma)$  and  $Y \sim R(s', \sigma)$ .*

The proof of this theorem relies on the resolution of two functional equations given in the following theorems which are interesting in their own rights. The proofs of these theorems are given in Section 4.

**Theorem 3.3.** *Let  $a : \Omega \cap (e - \Omega) \longrightarrow \mathbb{R}$  and  $g : \Omega \longrightarrow \mathbb{R}$  be functions such that, for any  $x \in \Omega \cap (e - \Omega)$  and  $t \in T$ ,*

$$a(x) = g(tx) - g(t(e-x)). \quad (5)$$

Assume that  $g$  is differentiable, then there exist  $p \in \mathbb{R}^r$  and  $c \in \mathbb{R}$  such that, for any  $x \in \Omega \cap (e - \Omega)$  and  $y \in \Omega$ ,

$$a(x) = \log \Delta_p(x) - \log \Delta_p(e - x), \quad g(y) = \log \Delta_p(y) + c.$$

**Theorem 3.4.** Let  $a_1 : \Omega \cap (e - \Omega) \rightarrow \mathbb{R}$  and  $a_2, g : \Omega \rightarrow \mathbb{R}$  be functions satisfying

$$a_1(x) + a_2(te) = g(tx) + g(t(e - x)), \quad (6)$$

for any  $x \in \Omega \cap (e - \Omega)$  and  $t \in T$ . Assume that  $g$  is twice differentiable then there exist  $p' \in \mathbb{R}^r$ ,  $\delta \in E$  and  $c_1, c_2, c_3 \in \mathbb{R}$  such that for any  $x \in \Omega \cap (e - \Omega)$  and  $y \in \Omega$ ,

$$\begin{aligned} g(y) &= \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_1 \\ a_1(x) &= \log \Delta_{p'}(x) + \log \Delta_{p'}(e - x) + c_2, \\ a_2(y) &= 2 \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_3, \end{aligned}$$

where  $2c_1 = c_2 + c_3$ .

**Proof of Theorem 3.2.** We again use the transformation:  $\Omega \times \Omega \rightarrow (\Omega \cap (e - \Omega)) \times \Omega$ ;  $(x, y) \mapsto (u, v)$ , where  $v = x + y = te$ ,  $t \in T$  and  $u = t^{-1}(x)$ . Let  $f_X, f_Y, f_U$  and  $f_V$  be the densities of  $X, Y, U$  and  $V$ , respectively. Then, since  $(X, Y)$  and  $(U, V)$  have independent components, we have that, for all  $u \in \Omega \cap (e - \Omega)$  and  $v \in \Omega$ ,

$$f_U(u)f_V(v) = (\det v)^{\frac{n}{r}} f_X(tu)f_Y(t(e - u)) \quad (7)$$

Taking logarithms in (7) we get

$$g_1(u) + g_2(te) = g_3(tu) + g_4(t(e - u)), \quad (8)$$

where

$$g_1(u) = \log f_U(u), \quad (9)$$

$$g_2(v) = \log f_V(v) - \frac{n}{r} \log \det v, \quad (10)$$

$$g_3(x) = \log f_X(x), \quad (11)$$

and

$$g_4(y) = \log f_Y(y). \quad (12)$$

Inserting  $e - u$  for  $u$  in (8) gives

$$g_1(e - u) + g_2(te) = g_3(t(e - u)) + g_4(tu). \quad (13)$$



Subtracting (13) from (8), we obtain

$$g_1(u) - g_1(e - u) = [g_3(tu) - g_4(tu)] - [g_3(t(e - u)) - g_4(t(e - u))]. \quad (14)$$

Define

$$a(u) = g_1(u) - g_1(e - u), \quad g = g_3 - g_4,$$

then,

$$a(u) = g(t(u)) - g(t(e - u)).$$

Now according to Theorem 3.3, we obtain

$$a(u) = \log \Delta_p(u) - \log \Delta_p(e - u), \quad g(v) = \log \Delta_p(v) + c,$$

for  $p = (p_1, \dots, p_r) \in \mathbb{R}^r$  and  $c \in \mathbb{R}$ .

Hence

$$g_3(v) = g_4(v) + g(v) = g_4(v) + \log \Delta_p(v) + c. \quad (15)$$

Inserting (15) back into (8) gives

$$g_1(u) + g_2(te) = \log \Delta_p(u) + \log \Delta_p(te) + c + g_4(t(u)) + g_4(t(e - u)),$$

which can be rewritten in the form

$$a_1(u) + a_2(te) = g_4(t(u)) + g_4(t(e - u)), \quad (16)$$

where

$$a_1(u) = g_1(u) - \log \Delta_p(u), \quad (17)$$

$$a_2(te) = g_2(te) - \log \Delta_p(te) - c. \quad (18)$$

Hence, by Theorem 3.4, it follows that

$$g_4(v) = \log \Delta_{p'}(v) + \langle \delta, v \rangle + c_1,$$

$$a_1(u) = \log \Delta_{p'}(u) + \log \Delta_{p'}(e - u) + c_2,$$

$$a_2(v) = 2 \log \Delta_{p'}(v) + \langle \delta, v \rangle + c_3,$$

for some  $\delta \in E$ ,  $p' \in \mathbb{R}^r$  and  $c_1, c_2, c_3 \in \mathbb{R}$  such that  $c_3 = 2c_1 - c_2$ .

This with (12) imply

$$\log f_Y(y) = \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_1,$$

that is

$$f_Y(y) = e^{c_1} e^{\langle \delta, y \rangle} \Delta_{p'}(y).$$

Since  $f_Y$  is a probability density, it follows that  $\sigma = -\delta \in \Omega$  and  $s' = p' + \frac{n}{r}$  is such that  $s'_i > (i-1)\frac{d}{2}$ ,  $\forall 1 \leq i \leq r$  so that  $Y \sim R(s', \sigma)$ .

Now by (15), we get

$$\begin{aligned} g_3(v) &= \log \Delta_p(v) + c + \log \Delta_{p'}(v) + \langle \delta, v \rangle + c_1 \\ &= \log \Delta_{p+p'}(v) + \langle \delta, v \rangle + c + c_1. \end{aligned}$$

From (11) it follows that

$$\begin{aligned} f_X(x) &= \Delta_{p+p'}(x) e^{\langle \delta, x \rangle} e^{c+c_1} \\ &= \Delta_{p+s'-\frac{n}{r}}(x) e^{-\langle \sigma, x \rangle} e^{c+c_1} \end{aligned}$$

which implies that  $p_i + s'_i > (i-1)\frac{d}{2}$ ,  $\forall 1 \leq i \leq r$  and consequently  $X \sim R(s, \sigma)$  where  $s = p + s'$ .

## 4 Proofs of Theorems 3.3 and 3.4

### 4.1 Proof of Theorem 3.3

For the proof of Theorem 3.3, we need to establish some preliminary results.

**Proposition 4.1.** *i) The map  $\varphi : \Omega \rightarrow \mathbb{R}$ ;  $x \mapsto \log \Delta_k(x)$  is differentiable on  $\Omega$  and*

$$\nabla \log \Delta_k(x) = (P_k(x))^{-1}.$$

*ii) The differential of the map  $x \mapsto (P_k(x))^{-1}$  is  $-P((P_k(x))^{-1})$ .*

**Proof.** We first observe that if  $c$  is an idempotent of  $E$  and  $x$  is in  $E(c, 1)$ , then

$$P(x)P(c) = P(x)$$

In fact, let  $h = h_1 + h_{12} + h_0$  be the Peirce decomposition of an element  $h$  in  $E$  with respect to  $c$ . As  $x \in E(c, 1)$ , we have that

$$P(x)(h_{12}) = P(x)(h_0) = 0,$$

and it follows that

$$P(x)(h) = P(x)(h_1) = P(x)P(c)(h)$$

i) Let  $\Omega_k = P_k(\Omega)$  and consider the map  $\psi : \Omega_k \rightarrow \mathbb{R}$ ;  $x \mapsto \log \det^{(k)}(x)$ . Then  $\varphi(x) = \psi \circ P_k(x)$ . Since  $\nabla \log \det x = x^{-1}$ , we have

$$\begin{aligned}
(\log \Delta_k(x))'(h) &= \psi'(P_k(x))(P_k(x))'(h), \forall h \in E \\
&= \langle (P_k(x))^{-1}, P_k(h) \rangle \\
&= \langle (P_k(x))^{-1}, h \rangle
\end{aligned}$$

ii) We have that the differential in  $x$  of the map  $\beta : x \mapsto x^{-1}$  is  $-P(x^{-1})$ . As  $(P_k(x))^{-1} = \beta \circ P_k(x)$ ,  $\forall x \in \Omega$ , then the differential in  $x \in \Omega$  of the map  $x \mapsto (P_k(x))^{-1}$  is equal to  $-P((P_k(x))^{-1}) \circ P_k$  which is also equal to  $-P((P_k(x))^{-1})$

**Proposition 4.2.** *Let  $u$  be in  $E_+$  and let  $x = t_u(e)$ . Then, for  $1 \leq k \leq r$ ,*

$$(P_k(x))^{-1} = t_u^{*-1} \left( \sum_{i=1}^k c_i \right).$$

**Proof.** We know, from Faraut-Korányi (Proposition VI.3.10), that for  $x$  in  $E$  and  $t_u$  in  $T$ ,

$$P_k(t_u(x)) = t_{P_k(u)}(P_k(x)).$$

Using (3) we can write

$$\begin{aligned}
P_k(x) &= \tau(\tilde{z}^{(1)}) \dots \tau(\tilde{z}^{(k-1)}) P \left( \sum_{i=1}^k u_i c_i \right) (c_1 + \dots + c_k) \\
&= \tau(\tilde{z}^{(1)}) \dots \tau(\tilde{z}^{(k-1)}) \left( \sum_{i=1}^k u_i^2 c_i \right)
\end{aligned}$$

where  $\tilde{z}^{(j)} = \sum_{l=j+1}^k z_{jl}$  and  $z_{jl} = \frac{u_{jl}}{u_j}$ ,  $j < l$ .

Hence

$$(P_k(x))^{-1} = \tau(-\tilde{z}^{(1)})^* \dots \tau(-\tilde{z}^{(k-1)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right) \quad (19)$$

On the other hand

$$t_u^{*-1} \left( \sum_{i=1}^k c_i \right) = \tau(-z^{(1)})^* \dots \tau(-z^{(r-1)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right).$$

Let us recall that if  $c$  is an idempotent,  $z$  is in  $E(c, \frac{1}{2})$  and  $x_1, x_{12}, x_0$  are the Peirce components of  $x$  with respect to  $c$ . Then the Peirce components of  $y = \tau_c(z)^*(x)$  are

$$\begin{cases} y_1 = 2c[z(zx_0) + zx_{12}] + x_1, \\ y_{12} = 2zx_0 + x_{12}, \\ y_0 = x_0. \end{cases} \quad (20)$$

This, after some elementary calculations, implies that, for  $k + 1 \leq j \leq r - 1$ ,  $z^{(j)} \in E(c_j, \frac{1}{2})$  and

$$\sum_{i=1}^k \frac{1}{u_i^2} c_i \in E(c_j, 0)$$

$$\tau(-z^{(j)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right) = \sum_{i=1}^k \frac{1}{u_i^2} c_i, \quad \forall k + 1 \leq j \leq r - 1,$$

and it follows that,

$$t_u^{*-1} \left( \sum_{i=1}^k c_i \right) = \tau(-z^{(1)})^* \dots \tau(-z^{(k)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right).$$

Using again (20), we have that

$$t_u^{*-1} \left( \sum_{i=1}^k c_i \right) = \tau(-z^{(1)})^* \dots \tau(-z^{(k-1)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right).$$

From this, we deduce that, for  $1 \leq j \leq k - 1$  and  $a \in E(c_1 + \dots + c_k, 1)$ ,

$$\tau(-z^{(j)})^* a = \tau(-\tilde{z}^{(j)})^* a.$$

Finally, we conclude that

$$\begin{aligned} t_u^{*-1} \left( \sum_{i=1}^k c_i \right) &= \tau(-z^{(1)})^* \dots \tau(-z^{(k-1)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right) \\ &= \tau(-\tilde{z}^{(1)})^* \dots \tau(-\tilde{z}^{(k-1)})^* \left( \sum_{i=1}^k \frac{1}{u_i^2} c_i \right) \\ &= (P_k(x))^{-1}. \end{aligned}$$

For the proof of Theorem 3.3, we also need the following result for which a proof is given in Hassairi and Lajmi (2001). For  $h = \sum_{i=1}^r h_i c_i + \sum_{i < j} h_{ij}$  in  $E$ , we denote  $\bar{h} = \sum_{i=1}^r h_i c_i$  and  $h^{(j)} =$

$$\sum_{k=j+1}^r h_{jk}, \quad 1 \leq j \leq r - 1.$$

**Lemma 4.3.** *i) For all  $1 \leq j \leq r - 1$ , the map from  $E_+$  into  $L(E)$ ;  $u \mapsto \tau(u^{(j)})$  is differentiable on  $E_+$ . Its differential in a point  $u$  is the map*

$$h \mapsto K_j^h(u) = 2h^{(j)} \square c_j + 4(h^{(j)} \square c_j)(u^{(j)} \square c_j)$$

*ii) The map  $H : E_+ \rightarrow T$ ;  $u \mapsto t_u$  is differentiable on  $E_+$  and*

$$\begin{aligned}
H'(u)(h) &= \sum_{j=1}^{r-1} \tau(u^{(0)})\tau(u^{(1)})\dots\tau(u^{(j-1)})K_j^h(u)\tau(u^{(j+1)})\dots\tau(u^{(r-1)})P(\bar{u}) \\
&+ 2\tau(u^{(1)})\dots\tau(u^{(r-1)})[2L(\bar{u})L(\bar{h}) - L(\bar{u}\bar{h})]
\end{aligned}$$

where  $\tau(u^{(0)}) = Id$ .

Note that for  $u = e$ , we get

$$K_j^h(e) = 2h^{(j)}\square c_j \text{ and } H'(e)(h) = \sum_{j=1}^{r-1} K_j^h(e) + 2L(\bar{h}) = 2\left[\sum_{j=1}^{r-1} h^{(j)}\square c_j + L(\bar{h})\right] \quad (21)$$

We are now in position to prove Theorem 3.3.

**Proof of Theorem 3.3.** Differentiating (5) with respect to  $x$  gives

$$a'(x) = t^*g'(tx) + t^*g'(t(e-x)). \quad (22)$$

Setting  $x = \frac{e}{2}$  and replacing  $t$  by  $2t$  in (22) give

$$g'(te) = t^{*-1}b, \text{ where } b = \frac{1}{4}a'\left(\frac{e}{2}\right) = g'(e). \quad (23)$$

Taking  $t = Id$  in (22) gives

$$a'(x) = g'(x) + g'(e-x).$$

Inserting this identity back into (22) gives

$$t^{*-1}g'(x) - g'(tx) = -[t^{*-1}g'(e-x) - g'(t(e-x))].$$

For any  $s \in (0, 1)$  and  $x = te$ , we have

$$g'(sx) = g'(ste) = \frac{1}{s}t^{*-1}b = \frac{1}{s}g'(x).$$

Hence for any  $s \in (0, 1)$ ,

$$t^{*-1}g'(x) - g'(tx) = -s[t^{*-1}g'(e-sx) - g'(t(e-sx))].$$

Consequently, on letting  $s \rightarrow 0$ , we obtain for any  $x \in \Omega \cap (e - \Omega)$  and  $t \in T$ , that

$$g'(tx) = t^{*-1}g'(x). \quad (24)$$

Now consider  $u \in E_+$ . Then (5) can be rewritten as

$$a(x) = g(t_u(x)) - g(t_u(e-x)). \quad (25)$$

To differentiate (25) with respect to  $u$ , let us consider the functions

$$H : E_+ \longrightarrow L(E); u \longmapsto t_u \text{ and } \alpha : L(E) \longrightarrow E; f \longmapsto f(x),$$

so that for  $u \in E_+$ , we can write  $g(t_u(x)) = (g \circ \alpha \circ H)(u)$ .

From Lemma 4.3, we have that

$$\begin{aligned} (g \circ \alpha \circ H)'(u)(h) &= g'(t_u(x))\alpha'(H(u))H'(u)(h), \forall h \in E \\ &= g'(t_u(x))\alpha(H'(u)(h)) \\ &= \langle g'(t_u(x)), H'(u)(h)(x) \rangle. \end{aligned}$$

Then, for all  $h \in E$ ,

$$\langle g'(t_u(x)), H'(u)(h)(x) \rangle = \langle g'(t_u(e-x)), H'(u)(h)(e-x) \rangle.$$

Hence by (24) we get

$$\langle t_u^{*-1} g'(x), H'(u)(h)(x) \rangle = \langle t_u^{*-1} g'(e-x), H'(u)(h)(e-x) \rangle,$$

and for any  $s \in (0, 1)$ ,

$$\begin{aligned} \langle t_u^{*-1} g'(sx), H'(u)(h)(sx) \rangle &= \langle t_u^{*-1} g'(x), H'(u)(h)(x) \rangle \\ &= \langle t_u^{*-1} g'(e-sx), H'(u)(h)(e-sx) \rangle. \end{aligned}$$

Letting  $s \longrightarrow 0$ , we get

$$\langle t_u^{*-1} g'(x), H'(u)(h)(x) \rangle = \langle t_u^{*-1} g'(e), H'(u)(h)(e) \rangle.$$

Note that if  $u = e$ , we have for all  $x \in \Omega \cap (e - \Omega)$ ,

$$\langle g'(x), H'(e)(h)(x) \rangle = \langle b, H'(e)(h)(e) \rangle, \forall h \in E. \quad (26)$$

Let  $b = \sum_{i=1}^r b_i c_i + \sum_{i < j} b_{ij}$  be the Peirce decomposition of  $b$  defined in (23) with respect to

the Jordan frame  $(c_i)_{1 \leq i \leq r}$ . We will show that  $b = \sum_{i=1}^r b_i c_i$ . In order to do so, we consider  $x = \frac{e}{2} + \varepsilon x_{ij}$  with  $\varepsilon \in ]0, \frac{1}{2}[$  and  $x_{ij} \in E_{ij}$  such that  $\|x_{ij}\| = 1$ . It is easy to see that  $x = t_v(e)$ , where  $v = \sum_{k=1, k \neq j}^r \frac{c_k}{\sqrt{2}} + \sqrt{\frac{1}{2} - \varepsilon^2} c_j + \sqrt{2}\varepsilon x_{ij} \in E_+$ . This implies that  $x \in \Omega$  and  $e - x = \frac{e}{2} - \varepsilon x_{ij}$  is also in  $\Omega$ .

Inserting  $h = h_{ij}$  in (21), we obtain

$$H'(e)(h) = 2h_{ij} \square c_j.$$

With the notations used above and from (1), we have

$$H'(e)(h)(e) = h_{ij},$$

and

$$H'(e)(h)(x) = \frac{1}{2}h_{ij} + \varepsilon \langle h_{ij}, x_{ij} \rangle c_j.$$

It follows by (23) that

$$\begin{aligned} \langle g'(x), H'(e)(h)(x) \rangle &= \langle t_v^{*-1} b, H'(e)(h)(x) \rangle \\ &= \langle b, t_v^{-1} H'(e)(h)(x) \rangle. \end{aligned}$$

After some standard calculation using (3), we get

$$t_v^{-1} H'(e)(h)(x) = \frac{1}{\sqrt{1-2\varepsilon^2}} h_{ij}.$$

Hence according to (26), we have for all  $h_{ij} \in E_{ij}$

$$\langle b_{ij}, h_{ij} \rangle = \frac{1}{\sqrt{1-2\varepsilon^2}} \langle b_{ij}, h_{ij} \rangle.$$

From this we readily deduce that  $b_{ij} = 0$  for all  $1 \leq i < j \leq r$ .

Now let  $x = t(e)$ , with  $t \in T$  be in  $\Omega$ . Then from (23), we can write

$$\begin{aligned} g'(x) &= t^{*-1} b \\ &= t^{*-1} \left( \sum_{i=1}^r b_i c_i \right) \\ &= t^{*-1} \left( \sum_{k=1}^r (b_k - b_{k+1}) (c_1 + \dots + c_k) \right) \end{aligned}$$

where  $b_{r+1} = 0$ .

And using Proposition 4.2, we get

$$g'(x) = \sum_{k=1}^r (b_k - b_{k+1}) (P_k(x))^{-1}.$$

By the statement (i) of Proposition 4.1, we have

$$g(x) = \sum_{k=1}^r (b_k - b_{k+1}) \log \Delta_k(x) + c, \quad c \in \mathbb{R}$$

$$= \log \Delta_p(x) + c \quad (27)$$

where  $p = (b_1, \dots, b_r)$ .

Finally, inserting (27) into (5) we get

$$a(x) = \log \Delta_p(x) - \log \Delta_p(e - x).$$

## 4.2 Proof of Theorem 3.4

For the resolution of the functional equation (6), we will use second derivatives and the following result.

**Proposition 4.4.** *Let  $x = \sum_{i=1}^r x_i c_i + \sum_{i<j} x_{ij}$  be the Peirce decomposition with respect to  $(c_i)$  of an element  $x$  of  $E$ . Then, for  $q = (q_1, \dots, q_r)$  in  $\mathbb{R}^r$ ,*

$$\sum_{i=1}^r q_i x_i c_i + \sum_{i<j} q_j x_{ij} = [q_r P(c_1 + \dots + c_r) + \sum_{k=1}^{r-1} (q_k - q_{k+1}) P(c_1 + \dots + c_k)](x).$$

**Proof.** We use induction on the rank  $r$  of the algebra. The result is obvious for  $r = 2$ , in fact we have

$$\begin{aligned} q_1 x_1 c_1 + q_2 x_2 c_2 + q_2 x_{12} &= q_2 (x_1 c_1 + x_2 c_2 + x_{12}) + (q_1 - q_2) x_1 c_1 \\ &= [q_2 P(c_1 + c_2) + (q_1 - q_2) P(c_1)](x). \end{aligned}$$

Suppose the result true for a Jordan algebra with a rank equal to  $r - 1$ . We easily verify that

$$\sum_{i=1}^r q_i x_i c_i + \sum_{i<j} q_j x_{ij} = q_r \left( \sum_{i=1}^r x_i c_i + \sum_{i<j} x_{ij} \right) + \sum_{i=1}^{r-1} (q_i - q_r) x_i c_i + \sum_{1 \leq i < j \leq r-1} (q_j - q_r) x_{ij}.$$

As, for  $a = \sum_{i=1}^r a_i c_i$ , we have  $P(a)x = \sum_{i \leq j} a_i a_j x_{ij}$ , then

$$\begin{aligned} \sum_{i=1}^r q_i x_i c_i + \sum_{i<j} q_j x_{ij} &= q_r P(c_1 + \dots + c_r)(x) + \\ &[(q_{r-1} - q_r) P(c_1 + \dots + c_{r-1}) + \sum_{k=1}^{r-2} (q_k - q_{k+1}) P(c_1 + \dots + c_k)] \left( \sum_{i=1}^{r-1} x_i c_i + \sum_{1 \leq i < j \leq r-1} x_{ij} \right) \end{aligned}$$

This with some standard calculation give the result.



**Proof of Theorem 3.4.** Differentiating (6) with respect to  $x$  gives

$$a'_1(x) = t^*g'(tx) - t^*g'(t(e-x)). \quad (28)$$

Differentiating (28) once again with respect to  $x$  gives

$$a''_1(x) = t^*g''(tx)t + t^*g''(t(e-x))t. \quad (29)$$

Substitute  $x = \frac{1}{2}e$  in the above equation then replace  $t$  by  $2t$  to get

$$g''(te) = t^{*-1}Bt^{-1} \quad (30)$$

where  $B = \frac{1}{8}a''(\frac{e}{2}) = g''(e)$ .

Observe that taking  $t = Id$  in (29) gives

$$a''_1(x) = g''(x) + g''(e-x). \quad (31)$$

Inserting this identity back into (29), we get

$$g''(x) - t^*g''(tx)t = -[g''(e-x) - t^*g''(t(e-x))t].$$

For any  $s \in (0, 1)$ , change  $x$  to  $sx$  in the above equation, and use the fact that by (30), we have

$$g''(sx) = s^{-2}g''(x),$$

then on multiplication by  $s^2$ , we deduce that for any  $s \in (0, 1)$

$$g''(x) - t^*g''(tx)t = -s^2[g''(e-sx) - t^*g''(t(e-sx))t].$$

Letting  $s \rightarrow 0$ , we obtain, for any  $x \in \Omega \cap (e - \Omega)$  and  $t \in T$

$$g''(tx) = t^{*-1}g''(x)t^{-1}. \quad (32)$$

Let us now consider  $u \in E_+$ , then (28) can be rewritten in the form

$$a'_1(x) = t_u^*g'(t_u(x)) - t_u^*g'(t_u(e-x)). \quad (33)$$

In order to differentiate (33) with respect to  $u$ , we introduce the functions

$$H : E_+ \rightarrow L(E); u \mapsto t_u, \quad \pi : L(E) \rightarrow L(E); f \mapsto f^* \text{ and } \phi : L(E) \rightarrow E; f \mapsto f(x).$$

We have

$$(\pi \circ H)'(u)(h) = H'(u)(h)^*, \quad \forall h \in E.$$

As, for any  $u \in E_+$ , we have  $g'(t_u(x)) = (g' \circ \phi \circ H)(u)$ , then

$$(g' \circ \phi \circ H)'(u)(h) = g''(t_u(x))H'(u)(h)(x).$$

We also consider the functions

$$\Psi : E_+ \longrightarrow L(E) \times E; u \longmapsto (t_u^*, g'(t_u(x))), \text{ where } x \text{ is fixed, and } \zeta : L(E) \times E \longrightarrow E; (f, z) \longmapsto f(z).$$

Then one can easily see that

$$\Psi'(u)(h) = (H'(u)(h)^*, g''(t_u(x))H'(u)(h)(x)),$$

and it follows that

$$\begin{aligned} (\zeta \circ \Psi)'(u)(h) &= \zeta'(\Psi(u))\Psi'(u)(h) \\ &= \zeta'(t_u^*, g'(t_u(x)))(H'(u)(h)^*, g''(t_u(x))H'(u)(h)(x)) \\ &= \zeta(t_u^*, g''(t_u(x))H'(u)(h)(x)) + \zeta(H'(u)(h)^*, g'(t_u(x))) \\ &= t_u^*g''(t_u(x))H'(u)(h)(x) + H'(u)(h)^*(g'(t_u(x))). \end{aligned}$$

Now differentiating (33) with respect to  $u$  gives

$$t_u^*g''(t_u(x))H'(u)(h)(x) + H'(u)(h)^*(g'(t_u(x))) = t_u^*g''(t_u(e-x))H'(u)(h)(e-x) + H'(u)(h)^*(g'(t_u(e-x))).$$

If we make  $u = e$ , we get

$$g''(x)H'(e)(h)(x) + H'(e)(h)^*g'(x) = g''(e-x)H'(e)(h)(e-x) + H'(e)(h)^*g'(e-x). \quad (34)$$

That is

$$(g''(x) + g''(e-x))H'(e)(h)(x) + H'(e)(h)^*(g'(x) - g'(e-x)) = g''(e-x)H'(e)(h)(e).$$

Using (28) and (31), this becomes

$$a_1''(x)H'(e)(h)(x) + H'(e)(h)^*a_1'(x) = g''(e-x)H'(e)(h)(e).$$

On the other hand, we know that

$$\begin{cases} a_1''(e-x) = a_1''(x) \\ a_1'(e-x) = -a_1'(x). \end{cases}$$

Hence

$$a_1''(x)H'(e)(h)(e-x) - H'(e)(h)^*a_1'(x) = g''(x)H'(e)(h)(e).$$

Given (31), we obtain

$$H'(e)(h)^*a_1'(x) = g''(e-x)H'(e)(h)(e-x) - g''(x)H'(e)(h)(x), \quad \forall h \in E.$$

Setting  $h = e$  in this equality, we obtain by (21),

$$a'_1(x) = g''(e-x)(e-x) - g''(x)(x). \quad (35)$$

Therefore, using (35) and the fact that, for all  $x$  in  $\Omega \cap (e - \Omega)$ ,  $a'_1(x) = g'(x) - g'(e-x)$ , the equality (34) becomes

$$g''(x)H'(e)(h)(x) - H'(e)(h)^*g''(x)(x) = g''(e-x)H'(e)(h)(e-x) - H'(e)(h)^*g''(e-x)(e-x). \quad (36)$$

For  $s \in (0, 1)$  change  $x$  to  $sx$ , to obtain

$$g''(x)H'(e)(h)(x) - H'(e)(h)^*g''(x)(x) = s[g''(e-sx)H'(e)(h)(e-sx) - H'(e)(h)^*g''(e-sx)(e-sx)].$$

Letting  $s \rightarrow 0$ , implies that for any  $x \in \Omega \cap (e - \Omega)$  and  $h \in E$ ,

$$H'(e)(h)^*g''(x)(x) = g''(x)H'(e)(h)(x). \quad (37)$$

If  $x = \frac{e}{2}$ , then, using the fact that  $g''(\frac{e}{2}) = 4B$ , we can write

$$H'(e)(h)^*B(e) = BH'(e)(h)(e), \quad \forall h \in E. \quad (38)$$

Now let

$$B(e) = \sum_{i=1}^r \lambda_i c_i + \sum_{i < j} \lambda_{ij}$$

be the Peirce decomposition of  $B(e)$  with respect to the Jordan frame  $(c_i)_{1 \leq i \leq r}$ . We will show that

$$B(e) = \sum_{i=1}^r \lambda_i c_i.$$

In fact, substituting  $h = \sum_{i=1}^r h_i c_i$  in (38) gives

$$\left( \sum_{i=1}^r h_i c_i \right) B(e) = \sum_{i=1}^r h_i B(c_i)$$

In particular, if  $h = c_i$ , we get

$$B(c_i) = c_i B(e), \quad \forall 1 \leq i \leq r.$$

Consequently

$$B(c_i) = \lambda_i c_i + \frac{1}{2} \left[ \sum_{k=1}^{i-1} \lambda_{ki} + \sum_{k=i+1}^r \lambda_{ik} \right] \quad (39)$$

Using (38) again for  $h = h_{ij}$ , we get

$$\begin{aligned}
B(h_{ij}) &= H'(e)(h)^* B(e) \\
&= 2(h_{ij} \square c_i)^* B(e) \\
&= 2(c_i \square h_{ij}) B(e) \\
&= \langle \lambda_{ij}, h_{ij} \rangle c_i + \lambda_j h_{ij} + 2h_{ij} \left[ \sum_{k=1}^{j-1} \lambda_{kj} + \sum_{k=j+1}^r \lambda_{jk} \right], \forall i < j.
\end{aligned}$$

By symmetry of  $B$ , it follows that

$$\langle B(h_{ij}), c_j \rangle = \frac{1}{2} \langle h_{ij}, \lambda_{ij} \rangle = 0, \forall i < j.$$

This implies that  $\lambda_{ij} = 0, \forall i < j$ .

Hence, we have

$$\begin{cases} B(c_i) = \lambda_i c_i, \forall 1 \leq i \leq r. \\ B(h_{ij}) = \lambda_j h_{ij}, \forall 1 \leq i < j \leq r. \\ B(e) = \sum_{i=1}^r \lambda_i c_i. \end{cases}$$

Let  $x = \sum_{i=1}^r x_i c_i + \sum_{i < j} x_{ij}$  be the Peirce decomposition of  $x$ . By Proposition 4.4, we get

$$\begin{aligned}
B(x) &= \sum_{i=1}^r \lambda_i x_i c_i + \sum_{i < j} \lambda_j x_{ij} \\
&= [\lambda_r P(c_1 + \dots + c_r) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) P(c_1 + \dots + c_k)](x),
\end{aligned}$$

that is

$$B = \lambda_r P(c_1 + \dots + c_r) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) P(c_1 + \dots + c_k).$$

As for each  $x$  in  $\Omega$ , there exists a unique  $t$  in the triangular group  $T$  such that  $x = t(e)$ , then using (30) and the fact that, for all  $x$  in  $E$  and for all  $g \in G$ ,  $P(gx) = gP(x)g^*$ , we can write

$$\begin{aligned}
g''(x) &= t^{*-1} B t^{-1} \\
&= \lambda_r t^{*-1} P(c_1 + \dots + c_r) t^{-1} + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) t^{*-1} P(c_1 + \dots + c_k) t^{-1} \\
&= \lambda_r P(t^{*-1}(c_1 + \dots + c_r)) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) P(t^{*-1}(c_1 + \dots + c_k))
\end{aligned}$$

Using Proposition 4.2, we can write

$$g''(x) = \lambda_r P(x^{-1}) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) P((P_k(x))^{-1}).$$

This, invoking Proposition 4.1, implies that

$$g'(x) = -\lambda_r x^{-1} - \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) (P_k(x))^{-1} + \delta$$

where  $\delta \in E$ .

And by Proposition 4.1, we have that

$$g(x) = -\lambda_r \log \det x - \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) \log \Delta_k(x) + \langle \delta, x \rangle + c_1$$

where  $c_1 \in \mathbb{R}$ .

Hence

$$g(x) = \log \Delta_{p'}(x) + \langle \delta, x \rangle + c_1$$

where  $p' = (p'_1, \dots, p'_r) = (-\lambda_1, \dots, -\lambda_r)$ .

As  $a'_1(x) = g'(x) - g'(e - x)$  (See (28)), we obtain

$$a_1(x) = \log \Delta_{p'}(x) + \log \Delta_{p'}(e - x) + c_2$$

where  $c_2 \in \mathbb{R}$ .

Finally, from (6) we get

$$a_2(te) = 2 \log \Delta_{p'}(te) + 2c_1 - c_2 + \langle \delta, te \rangle.$$

For any  $y \in \Omega$  there exists a unique  $t \in T$  such that  $y = te$ . Then

$$a_2(y) = 2 \log \Delta_{p'}(y) + \langle \delta, y \rangle + c_3,$$

where  $c_3 = 2c_1 - c_2$ .

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