Axiomatic Rewriting Theory I: A Diagrammatic Standardization Theorem

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Paul-André Melliès

Equipe Preuves, Programmes et Systèmes
CNRS, Université Paris 7 Denis Diderot

Dedicated to Jan Willem Klop

Abstract. By extending *nondeterministic* transition systems with *concurrency* and *copy mechanisms*, Axiomatic Rewriting Theory provides a uniform framework for a variety of rewriting systems, ranging from higher-order systems to Petri nets and process calculi. Despite its generality, the theory is surprisingly simple, based on a mild extension of transition systems with independence: an axiomatic rewriting system is defined as a 1-dimensional transition graph $\mathcal{G}$ equipped with 2-dimensional transitions describing the *redex permutations* of the system, and their orientation. In this article, we formulate a series of elementary axioms on axiomatic rewriting systems, and establish a diagrammatic standardization theorem.

Foreword by the author

Many concepts of Rewriting Theory started in the λ-calculus — which is by far the most studied rewriting system in history. A remarkable illustration is the ***confluence theorem***. The theorem was formulated by A. Church and J.B. Rosser in the early years of the λ-calculus [7]. The theorem was then generalized and applied extensively to other rewriting systems. It became eventually an object of study in itself, in a line of research pioneered by H.-B. Curry and R. Feys in their book on Combinatory Logic (1958). This culminated in a series of beautiful papers by G. Huet, J. W. Klop, and J.-J. Lévy published at the end of the 1970s and beginning of the 1980s. Today, more than half a century after its appearance in the λ-calculus, the confluence property is universally accepted as the theoretical principle underlying *deterministic* computations.

The article is concerned with another key property of the λ-calculus: the ***standardization theorem***, which was discovered by A. Church and J.B. Rosser quite at the same time as the confluence property. We advocate in this article that, in the same way as confluence underlies deterministic computations, standardization guides *causal* computations. It is worth clarifying here what kind of causality we have in mind, since the concept has been used in so many different ways. First of all, by *computation*, we mean a rewriting path

$$M_1 \stackrel{u_1}{\longrightarrow} M_2 \stackrel{u_2}{\longrightarrow} M_3 \longrightarrow \cdots \longrightarrow M_{n-1} \stackrel{u_n}{\longrightarrow} M_n$$

in which every term M_k describes a particular state of the system, and in which every redex u_k describes a particular transition on states, for $1 \leq k \leq n$. Then, by *causal*

computation, we mean a computation in which every transition u_k is enabled by a chain or cascade of previous transitions. We are particularly interested in situations where the chain of causality leading to u_k is not necessarily the whole rewriting path

$$M_1 \overset{u_1}{\longrightarrow} M_2 \overset{u_2}{\longrightarrow} M_3 \longrightarrow \cdots \longrightarrow M_{k-1} \overset{u_{k-1}}{\longrightarrow} M_k. \tag{1}$$

At this point, we advise the reader to practice the following spiritual exercise: think of today as a particular sequence of transitions (1) starting from your bedroom (state M_1) and leading you to the current position in the day (state M_k). Then, call $v = u_k$ the transition consisting in reading this very article:

$$v = u_k \quad : \quad M_k \longrightarrow M_{k+1}.$$

You must admit that some transitions performed today among the $u_1, \ldots, u_{k-1}$ are not necessary to read this article. And that it seems particularly difficult to disentangle the necessary transitions from the unnecessary ones. This is the point of this article: we investigate how to perform this task in Rewriting Theory by *permuting* transitions — in the spirit of true concurrency and Mazurkiewicz traces. Suppose for instance that your last action $u = u_{k-1}$ today has been to drink coffee:

$$u = u_{k-1} \quad : \quad M_{k-1} \longrightarrow M_k.$$

Do you really need that coffee to read these lines? The simplest way to answer is to check whether the transition v may be permuted before the transition u. If this is the case, then coffee is not necessary. Of course, you may reply that you have already drunk your coffee ten minutes ago, and thus, that it is far too late *now* to permute the order of events! You are certainly right... but this is not what matters here: the very fact that permuting the transition v before the transition u is possible *in principle* is sufficient to establish that performing transition u is not necessary in order to perform transition v.

Suppose on the other hand that your last action u has been to fetch this article from the library. In that case, performing the transition u is absolutely necessary in order to perform the transition v. There is no way indeed (either in reality or in principle) to permute the order of the two transitions... and this is precisely the reason why you went to the library on the first hand!

Of course, separating the necessary transitions from the unnecessary ones may involve more than just one permutation. Suppose for instance that you have drunk coffee just before fetching the article from the library. In that case, it takes two permutations (permute your coffee time after your visit to the library, and then after your exploration of the article) in order to demonstrate that drinking coffee is not necessary.

Everyday life shows that chains of causality may be reconstructed by applying relevant series of permutations on transitions. Now, Rewriting Theory complicates matters by implementing a symbolic universe in which computations may be *erased* or *duplicated* at will. New situations arise, which often defy common sense! We illustrate this with a simple example, involving a coffee machine M producing a cup of coffee C, and a duplicator. The situation proceeds in three transitions:

1. M produces the cup of coffee C,

2. Duplicator replicates M in two exact copies M_1 and M_2, each one containing its own cup of coffee C_1 and C_2,
3. You fetch the cup of coffee C_1 from M_1, and drink it.

The situation is particularly intricate from a conceptual point of view. On the one hand, producing the cup of coffee C (first transition) is necessary to fetch the cup of coffee C_1 (last transition) since the cup C_1 is just a copy of the cup C. On the other hand, the first two transitions produce the cup of coffee C_2 which is not necessary to fetch the cup of coffee C_1 in the last transition. The only way to clarify things here is to permute the duplication of the machine M (second transition) *before* the production of the cup of coffee C (first transition). From this results a series of four transitions:

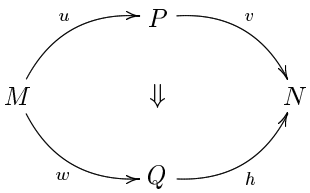
1. Duplicator replicates M in two exact copies M_1 and M_2,
2. M_1 produces the cup of coffee C_1,
3. M_2 produces the cup of coffee C_2,
4. You fetch the cup of coffee C_1 from M_1, and drink it.

There is more work for everybody now (except for Duplicator possibly) since each machine M_1 and M_2 has to produce its own cup of coffee C_1 and C_2. On the other hand, starting by duplicating the machine M enables to disentangle now the necessary part (producing the cup of coffee C_1) from the unnecessary part (producing the cup of coffee C_2). The chain of causality leading to the cup of coffee C_1 is exhibited by permuting the two last steps in the previous sequence of transitions, to obtain:

1. Duplicator replicates M in two exact copies M_1 and M_2,
2. M_1 produces the cup of coffee C_1,
3. You fetch the cup of coffee C_1 from M_1, and drink it.

This long discussion explains why standardization reorganizes computations by giving priority to duplicators and erasers: duplication and erasure are an inherent part of disentanglement. This aspect of causality is fundamental but subtle, and thus often misunderstood, even by specialists.

Technically speaking, the article is built on a seminal observation made by Jan Willem Klop in his PhD thesis, more than twenty-five years ago. The PhD thesis, published in 1980, contains two proofs of the standardization theorem for the leftmost-outermost λ-calculus. In the second proof, Jan Willem Klop reduces standardization to strong normalization and confluence of a 2-*dimensional* rewriting process on the β-rewriting paths, understood here as 1-*dimensional* entities. The process consists in permuting the so-called *anti-standard* pairs of β-redexes u and v in the following way:



The 2-dimensional transition $f \Longrightarrow g$ transforms the β-rewriting path $f = u \cdot v$ into the β-rewriting path $g = w \cdot h$ where:

- the β-redex w is the ancestor of the β-redex v before β-reduction of the β-redex u,
- the β-rewriting path h develops the residuals of the β-redex u after β-reduction of the β-redex w.

By anti-standard pair, one means that the β-redex w lies outside or to the left of the β-redex u. Jan Willem Klop shows that the 2-dimensional procedure $\Longrightarrow$ strongly normalizes and converges on a unique normal form for every β-rewriting path. The resulting normal form is precisely the standard (that is, leftmost-outermost) β-rewriting path associated to the original β-rewriting path.

In this article, we generalize the construction to a wide class of rewriting systems, ranging from higher-order systems to Petri nets or process calculi. This provides evidence that *causality* is a general phenomenon in Rewriting Theory, and that its scope is not limited to deterministic computations. We proceed in a purely diagrammatic way: we start by formulating a series of *3-dimensional principles* which regulate the *2-dimensional permutations* acting on the *1-dimensional rewriting paths.* We then show that every Rewriting System satisfying these elementary principles (called axioms) satisfies our diagrammatic standardization theorem. The theorem states that applying 2-dimensional permutations to a rewriting path f leads eventually to a *unique* rewriting path g — modulo a fundamental notion of reversible permutation introduced in the course of the article. The standard rewriting path is finally *defined* as the unique normal form obtained at the end of the 2-dimensional procedure.

I have had several occasions to appreciate the extraordinary quality and insight of Jan Willem Klop's contribution to Rewriting Theory. It is thus a great pleasure and honour for me to dedicate today this article to Jan Willem Klop, on the occasion of his 60th birthday.

1 Standardization: From Syntax to Diagrams

1.1 Computing leftmost outermost is judicious... in the λ-calculus

The λ-calculus is the pure calculus of functions. It has a unique reduction rule, called the β-rule,

$$(\lambda x.M)P \longrightarrow M[x := P] \tag{2}$$

which substitutes every free variable x in the λ-term M with the λ-term P. Despite its simplicity, the β-rule enables an extraordinary range of behaviours. For instance, depending on the number of times the variable x occurs in M, the β-redex (2) duplicates its argument P, or erases it... Typically, the λ-term $\Delta = (\lambda x.xx)$ defines a duplicator, while the λ-term $K = (\lambda x.\lambda y.x)$ defines an eraser, with the following behaviours:

$$\Delta P \longrightarrow PP, \qquad\qquad KPQ \longrightarrow (\lambda y.P)Q \longrightarrow P.$$

Amusingly, the duplicator Δ applied to itself defines a λ-term $\Delta\Delta$ whose computation loops:

$$\Delta\Delta \longrightarrow \Delta\Delta \longrightarrow \cdots$$

The λ-term $Ka(\Delta\Delta)$ obtained by applying the eraser K to the variable a and to the loop $\Delta\Delta$ is particularly interesting, because its behaviour depends on the strategy chosen to compute it. When computed from left to right, the λ-term $Ka(\Delta\Delta)$ reduces in two steps to its result a:

$$Ka(\Delta\Delta) \longrightarrow (\lambda y.a)(\Delta\Delta) \longrightarrow a \tag{3}$$

On the other hand, when computed from right to left, the same λ-term $Ka(\Delta\Delta)$ loops for ever on the unnecessary computation of its subterm $\Delta\Delta$:

$$Ka(\Delta\Delta) \longrightarrow Ka(\Delta\Delta) \longrightarrow \cdots \tag{4}$$

To summarize: applying the "wrong" strategy on the λ-term $Ka(\Delta\Delta)$ computes it for ever, whereas applying the more judicious strategy (3) transforms it into its result a. This raises a very pragmatic question: does there exist a "judicious" strategy for every λ-term? This strategy would avoid useless computations, and reach the result of the λ-term, whenever this result exists. Remarkably, such a "judicious" strategy exists, and its recipe is surprisingly uniform: reduce at each step the *leftmost outermost* β-redex of the λ-term! Note that this is precisely the strategy applied successfully in (3) to compute the λ-term $Ka(\Delta\Delta)$.

We recall below the definition of the leftmost outermost strategy, formulated originally by A. Church and J. B. Rosser in the λI-calculus (the λ-calculus without erasers) then adapted to the λ-calculus by H.-B. Curry and R. Feys. A *β-redex* is a pattern $(\lambda x.P)Q$ occurring in the syntactical tree of a λ-term. The λ-terms $(\lambda x.P)$ and Q are called respectively the *function* and the *argument* of the β-redex $(\lambda x.P)Q$. A λ-term which does not contain any β-redex is called a *normal form*: it cannot be computed further. Now, consider a λ-term M containing a β-redex at least. Its *leftmost outermost* β-redex is defined by induction on the size of the λ-term M:

1. as $(\lambda x.P)Q$ when $M = \lambda x_1 ... \lambda x_k.((\lambda x.P)QR_1...R_m)$,
2. as the leftmost outermost β-redex of Q when

$$M = \lambda x_1 ... \lambda x_k.(xP_1...P_mQR_1...R_n)$$

and every P_i is a normal form.

Theorem 1 (Curry-Feys) *Suppose that there exists a rewriting path from a λ-term M to a normal form P. The strategy consisting in rewriting at each step M_i the leftmost outermost β-redex in M_i constructs a rewriting path*

$$M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{k-1} \longrightarrow M_k = P$$

from M to P.

Theorem 1 may be stated alternatively by defining $\dashrightarrow$ as the least relation between λ-terms satisfying the inductive steps of Figure 1, then by establishing that $M \twoheadrightarrow P$ is equivalent to $M \dashrightarrow P$, for every λ-term M and normal form P. We leave the reader check as exercise that the definition of $\dashrightarrow$ constructs the rewriting path (3) in the case of $M = Ka(\Delta\Delta)$.

1.2 Computing leftmost outermost is not necessarily judicious... in other rewriting systems

This clarifies how a term should be computed in the λ-calculus: from left to right. It appears however that this orientation is very particular to the λ-calculus. Consider for instance the term rewriting system defined by the rules

$$\begin{array}{c} A \to A \\ B \to C \\ F(x, C) \to D \end{array} \tag{5}$$

Then, the rightmost outermost strategy (6) rewrites the term $F(A, B)$ to a result D:

$$F(A, B) \longrightarrow F(A, C) \longrightarrow D \tag{6}$$

whereas the leftmost outermost strategy loops for ever on the term $F(A, B)$:

$$F(A, B) \longrightarrow F(A, B) \longrightarrow \cdots \tag{7}$$

One must admit here that there exists no universal "syntactic orientation" in Rewriting Theory. This should not be a surprise: after all, the "syntactic orientation" of a rewriting system is extremely sensitive to its notation! Think only of the λ-calculus written through the Looking Glass, in a reverse notation: now, the calculus is oriented right to left, instead of left to right... The general case is even worse. A rewriting system does not enjoy any uniform orientation in general, and finding the "judicious" strategy, even if we know that it exists, is a non decidable problem, see [18].

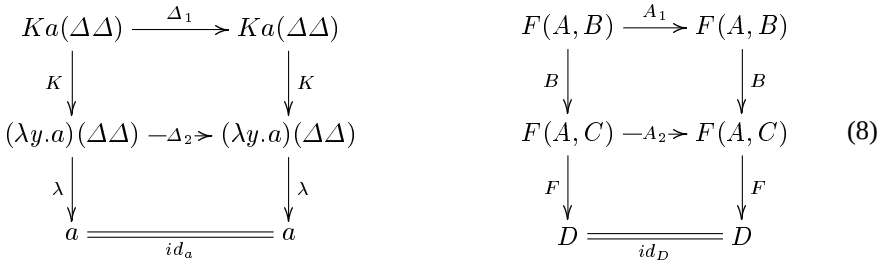
Despite the apparent mess, we will initiate in this article a *generic* theory of orientations and causality in rewriting systems. But on what foundations? Obviously, we need to abstract away from syntax in order to describe uniformly examples (3), (4), (6) and (7). We are thus compelled to reason *diagrammatically* instead of *syntactically*, and to develop a *syntax-free* Rewriting Theory, based on a 2-dimensional refinement of the traditional notion of *Abstract Rewriting System* developed in [32, 17, 21].

$$\begin{array}{ll} (\textsc{Var}) & x \dashrightarrow x \\ (\textsc{Beta}) & \dfrac{M \dashrightarrow \lambda x.P \qquad P[x := N] \dashrightarrow Q}{MN \dashrightarrow Q} \\ (\textsc{App}) & \dfrac{M \dashrightarrow xP_1 \ldots P_k \qquad N \dashrightarrow Q}{MN \dashrightarrow xP_1 \ldots P_k Q} \\ (\textsc{Xi}) & \dfrac{M \dashrightarrow P}{\lambda x.M \dashrightarrow \lambda x.P} \end{array}$$

Fig. 1. An inductive definition of Curry and Feys' leftmost outermost strategy.

1.3 Forget syntax, think diagrammatically!

The diagrammatic approach to Rewriting Theory which we plan to develop starts with a simple but surprising observation: despite their syntactic differences, the two terms $Ka(\Delta\Delta)$ and $F(A,B)$ define *exactly* the same transition system, which we draw below.

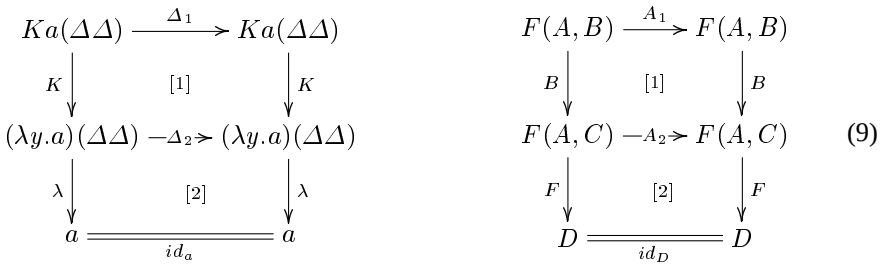


(8)

Apparently, the *dynamical analogy* between the two terms $Ka(\Delta\Delta)$ and $F(A,B)$ goes beyond the equality of their transition systems. Observe indeed that in the lefthand side and the righthand side of the diagram:

- the steps Δ_1 and A_1 are "unnecessary" because they may be "erased" by the paths $K \cdot \lambda$ and $B \cdot F$,
- the paths $K \cdot \lambda$ and $B \cdot F$ are more "judicious" than the paths $\Delta_1 \cdot K \cdot \lambda$ and $A_1 \cdot B \cdot F$ because they avoid computing the "unnecessary" redexes Δ_1 and A_1.

This analogy between the two terms $Ka(\Delta\Delta)$ and $F(A,B)$ is too strong to be reflected by the transition systems of Diagram (8). Nevertheless, it is possible to *refine* the notion of transition system, in order to capture the analogy. The refinement is based on the concept of *redex permutation* introduced by J.-J. Lévy in his work on the λ-calculus and on term rewriting systems, see [24, 18, 3]. Permuting redexes inside rewriting paths enables to express by local transformations that two different rewriting paths compute the same *events*, but in a different order. Typically, the transition system of the terms $Ka(\Delta\Delta)$ and $F(A,B)$ may be equipped with the two permutations [1] and [2] indicated below:



(9)

Consider for instance the transition system of the λ-term $Ka(\Delta\Delta)$ on the lefthand side of Diagram 9:

- the two paths $\Delta_1 \cdot K \cdot \lambda$ and $K \cdot \Delta_2 \cdot \lambda$ are *equivalent* modulo permutation [1] of the β-redexes Δ_1 and K, and

– the two paths $K \cdot \Delta_2 \cdot \lambda$ and $K \cdot \lambda$ are *equivalent* modulo permutation [2] of the β-redexes Δ_2 and λ.

All put together, the two paths $f = \Delta_1 \cdot K \cdot \lambda$ and $g = K \cdot \lambda$ are equivalent modulo the two permutations [1] and [2]. In particular, they compute the same *events*, but in a different order. Note however that the redex Δ_1 has disappeared in the process of reorganizing the rewriting path f into the rewriting path g. Remarkably, the same story may be told of the term $F(A, B)$: the redex A_1 has disappeared during the process of reorganizing the rewriting path $f = A_1 \cdot B \cdot F$ into the rewriting path $g = B \cdot F$ using the two permutations [1] and [2].

The process of reorganizing a path $f : P \longrightarrow\!\!\!\!\!\rightarrow Q$ into the properly oriented path $g : P \longrightarrow\!\!\!\!\!\rightarrow Q$ is known as *the standardization procedure*. The rewriting path g obtained at the end of the procedure is called *the standard path* associated to the path f. J.-J. Lévy introduced the idea of an *equivalence relation* between rewriting paths *modulo redex permutation*. Here, we *orient* the redex permutations and thus refine Lévy equivalence relation into a *preorder* on rewriting paths. We call this preorder the *standardization preorder*. This enables us to describe standardization in a purely diagrammatic way, as an *extremal problem*:

standard paths $=$ *minimal paths wrt. the standardization preorder.*

All this is explained here in Sections 1.4—1.8, and illustrated by the λ-calculus in three different ways in Section 1.9. A concise and subjective history of the standardization theorem is provided in Section 1.10.

1.4 Standardization as 2-dimensional rewriting "modulo"

Standardization is too often explained syntactically, and this complicates matters... In order to understand the reorganization of redexes in a simple and diagrammatic way, we decide to *orient* the permutations [1] and [2], and to define standardization as the 2-dimensional process of transforming the path $\Delta_1 \cdot K \cdot \lambda$ into the path $K \cdot \lambda$. During that transformation, each permutation [1] and [2] plays the role of a 2-dimensional rewriting step $\Longrightarrow$ reducing a rewriting path into another "more standard" rewriting path:

$$\Delta_1 \cdot K \cdot \lambda \implies K \cdot \Delta_2 \cdot \lambda \implies K \cdot \lambda. \tag{10}$$

The normal form of $\Delta_1 \cdot K \cdot \lambda$ is the standard path $K \cdot \lambda$. In this way, we define uniformly — for the first time — standardization for a wide class of existing rewriting system. The 2-dimensional perspective unifies already our two favourite examples: the rewriting path $A_1 \cdot B \cdot F$ is rewritten as the "rightmost outermost" rewriting path $B \cdot F$ by the same 2-dimensional procedure as example (10):

$$A_1 \cdot B \cdot F \implies B \cdot A_2 \cdot F \implies B \cdot F.$$

The interpretation of standardization as 2-dimensional rewriting is the author's rediscovery of an old idea published fifteen years earlier by J. W. Klop in his PhD thesis. At the time of J. W. Klop's PhD thesis (1975-80) standardization was limited to the

λ-calculus and similar "leftmost-outermost" standardization theorems. J. W. Klop observed that standardization could be expressed nicely as a *plain* 2-dimensional rewriting system. Quite at the same time, G. Huet and J.-J. Lévy reshaped the field entirely by establishing a revolutionary standardization theorem for term rewriting systems, in [18]. Unfortunately, the richer standardization mechanisms disclosed by G. Huet and J.-J. Lévy cannot be expressed as a plain 2-dimensional rewriting system anymore — and J. W. Klop's elegant idea would simply not work.

It is only fifteen years later, trying to abstract away from the syntactical details of [18] that the 2-dimensional approach took shape again. This was a completely independent discovery originating from a long and obsessive reflexion on the diagrammatic presentation of [13]. Already in germ there and in the author's PhD thesis [27] the idea emerged finally that the standardization mechanism described by G. Huet and J.-J. Lévy reduces to distinguishing two classes of permutations:

- the *reversible* permutations — for instance, permutation [1] in Diagram (9),
- the *irreversible* permutations — for instance, permutation [2] in Diagram (9).

In this way, the standardization mechanisms disclosed by G. Huet and J.-J. Lévy can be reformulated as a 2-dimensional rewriting system *modulo reversible permutations* — which then specializes to a plain 2-dimensional rewriting system in the case of the "leftmost-outermost" standardization theorems studied by J. W. Klop in his PhD thesis.

At this point, it is worth explaining briefly and informally the difference between a reversible and an irreversible permutation. Permutation [1] is called *reversible* because it permutes two *disjoint* rewriting steps K and Δ_1, or B and A_1 — disjoint in the syntactic sense that no redex contains the other redex in the tree nesting order. The permutation is thus *neutral* from the point of view of standardization.

$$\begin{array}{ccc} Ka(\Delta\Delta) & \xrightarrow{\Delta_1} & Ka(\Delta\Delta) \\ {\scriptstyle K}\downarrow & [1] & \downarrow{\scriptstyle K} \\ (\lambda y.a)(\Delta\Delta) & \xrightarrow[\Delta_2]{} & (\lambda y.a)(\Delta\Delta) \end{array} \qquad \begin{array}{ccc} F(A,B) & \xrightarrow{A_1} & F(A,B) \\ {\scriptstyle B}\downarrow & [1] & \downarrow{\scriptstyle B} \\ F(A,C) & \xrightarrow[A_2]{} & F(A,C) \end{array}$$

Permutation [2] is called *irreversible* because it replaces the "inside-out" computation $\Delta_2 \cdot \lambda$ or $A_2 \cdot F$ by its "outside-in" equivalent λ or F — thus *strictly improving* the computation from the point of view of standardization.

$$\begin{array}{ccc} (\lambda y.a)(\Delta\Delta) & \xrightarrow{\Delta_2} & (\lambda y.a)(\Delta\Delta) \\ {\scriptstyle \lambda}\downarrow & [2] & \downarrow{\scriptstyle \lambda} \\ a & \underset{\mathrm{id}_a}{=\!=} & a \end{array} \qquad \begin{array}{ccc} F(A,C) & \xrightarrow{A_2} & F(A,C) \\ {\scriptstyle F}\downarrow & [2] & \downarrow{\scriptstyle F} \\ D & \underset{\mathrm{id}_D}{=\!=} & D \end{array}$$

1.5 The basic vocabulary of Axiomatic Rewriting Theory

It is time to introduce several key definitions related to our diagrammatic theory of standardization.

Definition 1 (transition system) *A transition system (or oriented graph) $\mathcal{G}$ is a quadruple*

$$(\mathbf{terms}, \mathbf{redexes}, \mathbf{source}, \mathbf{target})$$

consisting of a set $\mathbf{terms}$ *of vertices (=* terms*), a set* $\mathbf{redexes}$ *of edges (= rewriting steps, or* redexes*), and two functions* $\mathbf{source}, \mathbf{target} : \mathbf{redexes} \to \mathbf{terms}$ *(= the source and target functions). We write*

$$u : M \longrightarrow N \quad \textit{when } \mathbf{source}(u) = M \textit{ and } \mathbf{target}(u) = N.$$

Recall that a *path* in a transition system $\mathcal{G}$ is a sequence

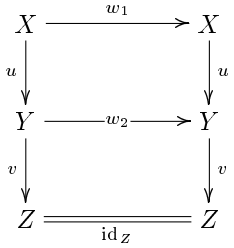
$$f = (M_1, u_1, M_2, ..., M_m, u_m, M_{m+1}) \tag{11}$$

where $u_i : M_i \longrightarrow M_{i+1}$ for every $i \in [1...m]$. We write $f : M_1 \longrightarrow M_{m+1}$. The *length* of f is m and f is said to be empty when $m = 0$. Two paths $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ and $g : P \longrightarrow\!\!\!\!\!\rightarrow Q$ are coinitial (resp. cofinal) when $M = P$ (resp. $N = Q$). The path $f;g : M \longrightarrow\!\!\!\!\!\rightarrow Q$ denotes the concatenation of two paths $f : M \longrightarrow\!\!\!\!\!\rightarrow P$ and $g : P \longrightarrow\!\!\!\!\!\rightarrow Q$.

Definition 2 (2-dimensional transition system) *A* 2-dimensional transition system *is a pair $(\mathcal{G}, \rhd)$ consisting of a transition system $\mathcal{G}$ and a binary relation $\rhd$ on the paths of $\mathcal{G}$. The relation $\rhd$ is required to relate coinitial and cofinal paths:*

$$\forall f : M \longrightarrow\!\!\!\!\!\rightarrow N,\ g : P \longrightarrow\!\!\!\!\!\rightarrow Q, \qquad f \rhd g \ \Rightarrow\ (M, N) = (P, Q)$$

The starting point of Axiomatic Rewriting Theory is to replace a concrete rewriting system by its 2-dimensional transition system. This has the effect of revealing unexpected similarities: typically, the two terms $Ka(\Delta\Delta)$ and $F(A, B)$ behave differently syntactically (left to right vs. right to left) but induce the same 2-dimensional transition system (drawn below) in the λ-calculus and in the term rewriting system (5).



$$\begin{array}{l} w_1 \cdot u \rhd u \cdot w_2 \\ u \cdot w_2 \rhd w_1 \cdot u \\ w_2 \cdot v \rhd \quad v \end{array} \tag{12}$$

It should be obvious at this point of the exposition that the dynamical analogy observed previously between the terms $Ka(\Delta\Delta)$ and $F(A, B)$ (Section 1.3) follows from the identity of their 2-dimensional transition system (12).

Definition 3 (permutation) *A* permutation *(f, g) in a 2-dimensional transition system $(\mathcal{G}, \rhd)$ is a pair of paths such that $f \rhd g$. We often use the more explicit (and overloaded) notation $f \rhd g$ for a permutation (f, g).*

Definition 4 (standardization step, $\overset{1}{\Longrightarrow}$) *A* standardization step *from a path $d : M \longrightarrow\!\!\!\!\!\rightarrow N$ to a coinitial and cofinal path $e : M \longrightarrow\!\!\!\!\!\rightarrow N$ in a 2-dimensional transition system $(\mathcal{G}, \triangleright)$, is a triple $(d_1, f \triangleright g, d_2)$ consisting of a permutation $f \triangleright g$ and two paths d_1, d_2 such that:*

$$d \;=\; M \xrightarrow{d_1}\!\!\!\!\!\rightarrow P \xrightarrow{f}\!\!\!\!\!\rightarrow Q \xrightarrow{d_2}\!\!\!\!\!\rightarrow N \qquad\qquad e \;=\; M \xrightarrow{d_1}\!\!\!\!\!\rightarrow P \xrightarrow{g}\!\!\!\!\!\rightarrow Q \xrightarrow{d_2}\!\!\!\!\!\rightarrow N$$

We write $d \overset{1}{\Longrightarrow} e$ when there exists a standardization step from d to e.

Definition 5 (standardization preorder $\Longrightarrow$, Lévy equivalence $\equiv$) *In every 2-dimensional transition system $(\mathcal{G}, \triangleright)$*

- *the* standardization preorder $\Longrightarrow$ *is the least transitive reflexive relation containing $\overset{1}{\Longrightarrow}$. We say that a path $e : M \longrightarrow\!\!\!\!\!\rightarrow N$ is* more standard *than a path $d : M \longrightarrow\!\!\!\!\!\rightarrow N$ when $d \Longrightarrow e$.*
- *the* Lévy permutation equivalence $\equiv$ *is the least equivalence relation containing $\Longrightarrow$. Alternatively, the equivalence relation $\equiv$ is the least equivalence relation containing $\triangleright$ and closed under composition.*

To illustrate our definitions with Diagram (12), one shows that the path $u \cdot v$ is more standard than the path $w_1 \cdot u \cdot v$ by exhibiting the sequence of standardization steps:

$$w_1 \cdot u \cdot v \overset{1}{\Longrightarrow} u \cdot w_2 \cdot v \overset{1}{\Longrightarrow} u \cdot v.$$

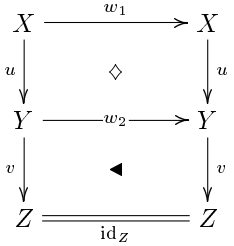
1.6 Reversible and irreversible permutations

Permutations of $(\mathcal{G}, \triangleright)$ are discriminated in two classes, reversible and irreversible, according to the following definition.

Definition 6 (reversible, irreversible permutation) *In every 2-dimensional transition system $(\mathcal{G}, \triangleright)$*

1. *A permutation (f, g) is* reversible *when $g \triangleright f$. A box $\diamondsuit$ signals reversible permutations $f \diamondsuit g$ in text and diagrams.*
2. *A permutation (f, g) is* irreversible *when $\neg(g \triangleright f)$. A triangle $\blacktriangleright$ signals irreversible permutations $f \blacktriangleright g$ in text and diagrams.*

Check that the definition matches the previous qualification in Section 1.4 of permutation [1] as reversible, and permutation [2] as irreversible, in Diagrams (9) and (12). We illustrate our new diagrammatic conventions on the 2-dimensional transition system (12).



$$\begin{array}{ccc} w_1 \cdot u & \diamondsuit & u \cdot w_2 \\ w_2 \cdot v & \blacktriangleright & v \end{array} \qquad (13)$$

In the definition below, the discrimination on permutations generalizes to the obvious discrimination on standardization steps. The key concept of reversible permutation equivalence $\simeq$ is revealed, as a stronger version of usual Lévy permutation equivalence $\equiv$.

Definition 7 ($\stackrel{REV}{\Longrightarrow}, \stackrel{IRR}{\Longrightarrow}$, reversible permutation equivalence $\simeq$) *In every 2-dimensional transition system $(\mathcal{G}, \triangleright)$*

- *A standardization step $(e, f \triangleright g, h)$ is* reversible *(resp.* irreversible*) when the permutation $f \triangleright g$ is reversible (resp. irreversible). We write*

$$d \stackrel{REV}{\Longrightarrow} e \qquad\qquad d \stackrel{IRR}{\Longrightarrow} e$$

  when there exists a Reversible (resp. Irreversible) standardization step from d to e.
- *The* reversible permutation equivalence $\simeq$ *is the least equivalence relation containing the relation $\stackrel{REV}{\Longrightarrow}$.*

1.7 Standard rewriting paths

Definition 8 (standard path) *A rewriting path $d : M \longrightarrow\!\!\!\!\!\rightarrow N$ is* standard *when there does not exist any sequence of standardization steps*

$$d \stackrel{REV}{\Longrightarrow} d_1 \stackrel{REV}{\Longrightarrow} \cdots \stackrel{REV}{\Longrightarrow} d_k \stackrel{IRR}{\Longrightarrow} d_{k+1}$$

consisting of a series of k Reversible steps followed by an Irreversible step.

So, a standard path is just a normal form of the standardization process, modulo reversible steps. Consequently, when a rewriting path d is standard, and when $d \Longrightarrow e$, then $d \simeq e$ and the rewriting path e is standard.

For instance, the path $X \stackrel{w_1}{\longrightarrow} X \stackrel{u}{\longrightarrow} Y \stackrel{v}{\longrightarrow} Z$ in Diagram (12) is transformed in two steps in the standard path $X \stackrel{u}{\longrightarrow} Y \stackrel{v}{\longrightarrow} Z$. The rewriting path $X \stackrel{w_1}{\longrightarrow} X \stackrel{u}{\longrightarrow} Y$ is another example of standard path, because every standardization sequence from it to itself or to $X \stackrel{u}{\longrightarrow} Y \stackrel{w_2}{\longrightarrow} Y$ is reversible.

1.8 The standardization theorem

One main challenge of Axiomatic Rewriting Theory is to capture the diagrammatic properties of redex permutations in *syntactic* rewriting systems, in order to establish the following diagrammatic *standardization theorem*: for every rewriting path $d : M \longrightarrow\!\!\!\!\!\rightarrow P$ in the transition system $\mathcal{G}$,

1. **existence:** there exists a standardization sequence

$$d \Longrightarrow e$$

   transforming the rewriting path d into a standard path e,

2. **uniqueness:** every standardization sequence

$$d \Longrightarrow f$$

may be extended to a standardization sequence leading to the standard path e:

$$d \Longrightarrow f \Longrightarrow e.$$

The uniqueness property has a series of remarkable consequences. Suppose for instance that the rewriting path f is standard. In that case, the standardization sequence

$$f \Longrightarrow e$$

consists of Reversible steps. Thus,

$$f \simeq e.$$

From this follows that there exists a unique standard path e such that

$$d \Longrightarrow e$$

modulo reversible permutation equivalence. In fact, the uniqueness property ensures that there exists a unique standard path, modulo reversible permutation equivalence, in the Lévy equivalence class of the rewriting path d.

In this article, we formulate a series of nine elementary axioms on the 2-dimensional transition system $(\mathcal{G}, \triangleright)$ and deduce from them the diagrammatic standardization theorem stated above. The axioms uncover a series of simple and elegant *principles* of causality in computations. They also illustrate that a purely *diagrammatic* and *syntax-free* theory of computations is possible, and useful, since it enscopes almost every existing rewriting system, from Petri nets to higher-order rewriting systems.

1.9 Illustration: the λ-calculus and its three standardization orders

There are at least three different ways to interpret the λ-calculus as a 2-dimensional transition system, each one associated to a particular *nesting order* on the β-redexes of λ-terms. The underlying transition system $\mathcal{G}_\lambda$ is the same in the three cases. It is defined in [10, 24] as follows:

- its vertices are the λ-terms, modulo α-conversion,
- its edges are the β-redexes $u : M \longrightarrow N$.

Recall that a β-redex $u = (M, o, N)$ is a triple consisting of a λ-term M, the occurrence o of a β-pattern $(\lambda x.P)Q$ in M and the λ-term N obtained after β-reducing

$$(\lambda x.P)Q \longrightarrow P[x := Q]$$

in the λ-term M.

It is worth noting that there are two different edges $I(Ia) \longrightarrow Ia$ in the graph $\mathcal{G}_\lambda$: each edge corresponds to the reduction of a particular identity combinator $I = (\lambda x.x)$ in the λ-term $I(Ia)$.

There are at least three different ways to refine the transition system $\mathcal{G}_\lambda$ as a 2-dimensional transition system, depending on the order chosen on β-redexes:

– the *tree-order*: a β-redex u is smaller than a β-redex v when v occurs in the function or argument part of u; or equivalently, when the occurrence of u is a strict prefix of the occurrence of v. We use the notation: $u \preceq_{\mathrm{tree}} v$.
– the *left-order*: a β-redex u is smaller than a β-redex v when v occurs in the function or argument part of u, or when there exists an occurrence o of an application node PQ in the λ-term M, such that u occurs in P and v occurs in Q. We use the notation: $u \preceq_{\mathrm{left}} v$.
– the *argument-order*: a β-redex u is smaller than a β-redex v when v occurs in the argument of u. We use the notation: $u \preceq_{\mathrm{arg}} v$.

Each order induces in turn its own permutation relation $\rhd_{\mathrm{tree}}$, $\rhd_{\mathrm{left}}$ and $\rhd_{\mathrm{arg}}$ on the transition system $\mathcal{G}_\lambda$. The order considered in the literature is generally the *left-order*, see [10, 24, 20]. However, we prefer to study here the tree-order, because this seems the most natural choice after the work by G. Huet and J.-J. Lévy on term rewriting systems [18]. The two alternative orders $\preceq_{\mathrm{left}}$ and $\preceq_{\mathrm{arg}}$ are discussed briefly in Section 8.

We define the relation $\rhd_{\mathrm{tree}}$ as follows. Two paths f, g are related as $f \rhd_{\mathrm{tree}} g$ precisely when:

1. the paths f and g factor as $f = v \cdot u'$ and $g = u \cdot h$ where u, v, u' are β-redexes and h is a path,
2. the two β-redexes u and v are coinitial, and $\neg(v \preceq_{\mathrm{tree}} u)$,
3. the β-redex u' is the (unique) residual of u after v, and the path h develops the (possibly) several residuals of v after u. [For a definition of residual and complete development, see [10, 24, 18, 3, 21, 22] or Section 6.]

Thus, every permutation $f \rhd_{\mathrm{tree}} g$ is of the form:

$$\begin{array}{ccc} M & \xrightarrow{\ v\ } & Q \\ {\scriptstyle u}\downarrow & \Leftarrow_{\mathrm{tree}} & \downarrow{\scriptstyle u'} \\ P & \xrightarrow[\ h\ ]{} & N \end{array} \qquad \begin{array}{l} f = v \cdot u' \\ g = u \cdot h \end{array} \tag{14}$$

where u and v are different β-redexes, u' is a β-redex and h is a path. The three paradigmatic examples of β-redex permutation $f \rhd_{\mathrm{tree}} g$ are:

$$\begin{array}{ccc} PQ & \xrightarrow{\ v\ } & P'Q \\ {\scriptstyle u}\downarrow & \Leftarrow_{\mathrm{tree}} & \downarrow{\scriptstyle u'} \\ PQ' & \xrightarrow[\ v'\ ]{} & P'Q' \end{array} \qquad \begin{array}{ccc} (\lambda x.a)P & \xrightarrow{\ v\ } & (\lambda x.a)P' \\ {\scriptstyle u}\downarrow & \Leftarrow_{\mathrm{tree}} & \downarrow{\scriptstyle u'} \\ a & = & a \end{array} \qquad \begin{array}{ccc} \Delta P & \xrightarrow{\ v\ } & \Delta P' \\ {\scriptstyle u}\downarrow & \Leftarrow_{\mathrm{tree}} & \downarrow{\scriptstyle u'} \\ PP & \xrightarrow[\ v_1 \cdot v_2\ ]{} & P'P' \end{array}$$

where $P \longrightarrow P'$ and $Q \longrightarrow Q'$ are two β-redexes. The three permutations are respectively reversible, irreversible and irreversible in the 2-dimensional transition system $(\mathcal{G}_\lambda, \rhd_{\mathrm{tree}})$.

Remark: the argument-order $\preceq_{\mathrm{arg}}$ is included in the tree-order $\preceq_{\mathrm{tree}}$ which is included in the left-order $\preceq_{\mathrm{left}}$. From this follows that the permutation relation $\rhd_{\mathrm{arg}}$ contains the

permutation relation $\rhd_{\mathrm{tree}}$ which contains in turn the permutation relation $\rhd_{\mathrm{left}}$. It is not difficult then to establish that every rewriting path standard wrt. the left-order $\preceq_{\mathrm{left}}$ is standard wrt. the tree-order $\preceq_{\mathrm{tree}}$, and that every rewriting path standard wrt. the tree-order $\preceq_{\mathrm{tree}}$ is standard wrt. the argument-order $\preceq_{\mathrm{arg}}$. The converse is obviously false in the two cases.

1.10 A concise history of the standardization theorem

Many authors have written on the standardization theorem. We do not draw below a comprehensive list, but deliver a concise history of the subject, in eight key steps.

[1936] *A. Church* and *J.B. Rosser* introduce the λI-calculus, a λ-calculus without erasure, and prove that the number of β-steps from a λI-term to its normal form is *bounded* by the length of the leftmost outermost computation. This result is the ancestor of all later standardization theorems.

[1958] *H.B. Curry* and *R. Feys* formulate the first standardization theorem for the λ-calculus: the two authors prove that every time a λ-term P β-reduces to a λ-term Q, there exists also a *standard* way to β-reduce P to Q. The theorem extends Church and Rosser result for the λI-calculus, and plays a role in Curry and Feys' defense of their erasing combinator K.

[1978] *J.-J. Lévy* formulates the standardization theorem in its modern algebraic form: using an equivalence relation on rewriting paths — called today Lévy permutation equivalence — Lévy proves that there exists a *unique* standard rewriting path in each equivalence class. The uniqueness result was so striking at the time that the theorem was called the *strong* standardization theorem by subsequent authors. Despite its conceptual novelty, the theorem is still limited to the λ-calculus and to its leftmost-outermost order.

[1979] *G. Huet* and *J.-J. Lévy* formulate and establish a standardization theorem for term rewriting systems without critical pairs. This is probably the most revolutionary step in the history of standardization, the first time at least that another standardization order is considered than the "leftmost outermost" order of the λ-calculus. The theorem is still limited to term rewriting systems — because its proof relies heavily on syntactical notions like tree-occurrence — but the article delivers the message that standardization is a general property of rewriting systems, related to causality and domain-theoretic notions like stability and sequentiality.

[1980] *J. W. Klop* introduces a 2-dimensional rewriting system on paths, consisting in permuting "anti-standard" paths of length 2 into "standard" paths of arbitrary length. In this way, Klop deduces Lévy's *strong* standardization theorem for leftmost-outermost λ-calculus, by establishing confluence and strong normalization of the 2-dimensional rewriting process: the standard path is obtained as the normal form of the procedure. Another important contribution of J. W. Klop is to stress the role of the finite development lemma in the proof of standardization, and to extend to any "left-regular" Combinatory Reduction System the standardization theorem for leftmost-outermost λ-calculus.

[Early 1980s] *G. Boudol* extends G. Huet and J.-J. Lévy standardization theorem to term rewriting systems with critical pairs. This is another decisive step, because it extends the principle of standardization to non deterministic rewriting systems.

[1992] *G. Gonthier* and *J.-J. Lévy* and *P-A. Melliès* deliver an axiomatic standardization theorem, where the syntactical proof of [18] is replaced by diagrammatic arguments on redexes, residuals and the nesting relation. Subsequently reworked by the author in his PhD thesis [27], the theorem extends G. Huet and J.-J. Lévy's original theorem to a great variety of rewriting systems with and without critical pairs — with the remarkable and puzzling exception (as first noted by R. Kennaway) of rewriting systems based on directed acyclic graphs.

[1996] *D. Clark* and *R. Kennaway* adapt the syntactical works of G. Huet, J.-J. Lévy and G. Boudol and establish a standardization theorem for (possibly conflicting) rewriting systems based on directed acyclic graphs (dags).

It took the author nine years to derive the current axiomatics from [13]. One difficulty was to find the simplest possible description of rewriting systems with critical pairs. The trinity of residual, compatibility and nesting relations operating in [13] was certainly too complicated. Slowly, the 2-dimensional presentation emerged, leading the author to the elementary axiomatics of this article. Twenty-five years ago, the work of [18, 6] on term rewriting systems revealed that the "conflict-free left-regular" rewriting systems considered earlier was the emerged part of the much wider and exciting world of *causal computations*. This is that world and its boundaries which we will explore here in our 2-dimensional diagrammatic language.

Structure of the paper

Axiomatic Rewriting Systems (AxRS) are introduced in Section 2, along with their nine standardization axioms. A less innovative but more traditional axiomatics based on residuals, critical pairs and nesting is formulated in Section 6. Standard paths are characterized in Section 3 as the paths which do not contain a particular "anti-standard" pattern, just as in [13, 27]. The standardization theorem is proved in Section 4, and reformulated 2-categorically in Section 5. An alternative axiomatization based on residuals and nesting orders is formulated in Section 6. A few additional hypotheses on axiomatic rewriting systems are discussed in Section 7. Finally, we illustrate our definition of AxRS with a series of examples in Section 8, like asynchronous transition systems, term rewriting systems, call-by-value λ-calculus, λ-calculus with explicit substitutions.

2 The Standardization Axiomatics

An *Axiomatic Rewriting System (AxRS)* is defined as a 2-dimensional transition system $(\mathcal{G}, \triangleright)$ which satisfies moreover the series of nine *standardization axioms* presented in this section. Each axiom of the section is illustrated by the λ-calculus and its 2-dimensional transition system $(\mathcal{G}_\lambda, \triangleright_{\mathrm{tree}})$ defined in Section 1.9.

2.1 Axiom 1: shape

The first axiom generalizes to every AxRS the shape of permutations encountered in the λ-calculus — see Diagram (14)in Section 1.9.

Axiom 1 (Shape) *We ask that in every permutation $f \rhd g$,*

- *the path f is of length 2,*
- *the path g is of length at least 1,*
- *the initial redexes of f and g are different.*

Thus, every permutation $f \rhd g$ in the 2-dimensional transition system $(\mathcal{G}, \rhd)$ has the following shape:

$$\begin{array}{ccc} M & \xrightarrow{\ v\ } & Q \\ {\scriptstyle u}\downarrow & \Leftarrow & \downarrow{\scriptstyle u'} \\ P & \xrightarrow[\ h\ ]{} & N \end{array} \qquad \begin{array}{l} f = v \cdot u' \\ g = u \cdot h \end{array} \tag{15}$$

where u and v are different redexes, u' is a redex and h is a path. In case of a *reversible* permutation $f \diamondsuit g$, this shape specializes to a 2×2 square:

$$\begin{array}{ccc} M & \xrightarrow{\ v\ } & Q \\ {\scriptstyle u}\downarrow & \diamondsuit & \downarrow{\scriptstyle u'} \\ P & \xrightarrow[\ v'\ ]{} & N \end{array} \qquad \begin{array}{l} f = v \cdot u' \\ g = u \cdot v' \end{array}$$

where u, u', v and v' are redexes, u and v different.

2.2 Axioms 2, 3, 4, 5: ancestor, reversibility, irreversibility and cube

The standardization theorem is usually established by a fine-grained analysis of syntactic mechanisms like erasure, duplication, etc... related to Lévy theory of residuals. The fragment of Lévy theory necessary to the theorem, e.g. the finite development property, appears in our axiomatics... but reformulated, because the more geometric idea of "oriented permutation" replaces the traditional concept of "residual of a redex". The residual theory is particularly visible in the four Axioms **ancestor**, **reversibility**, **irreversibility** and **cube** introduced below, as well as in Axiom **termination** of Section 2.6.

Axiom **ancestor** incorporates two properties of the λ-calculus, traditionally called *uniqueness of ancestor* and *finite development*. The existence of a permutation $f \rhd_{\mathrm{tree}} g$ between two β-rewriting paths:

$$f = M \xrightarrow{v} Q \xrightarrow{u'} N \qquad g = M \xrightarrow{u} P \xrightarrow{h} N$$

means that the β-redex u' is the unique residual of the β-redex u after β-reduction of the redex v, and that the path h is a complete development of the residuals of the redex v after β-reduction of the redex u. In that case, we say that the redex u is an *ancestor* of the redex u' before β-reduction of the redex v. The *uniqueness of ancestor* property states that the redex u is the unique such ancestor of the redex u'. Besides, the *finite development* property of the λ-calculus, recalled in Section 6, states that two complete developments of the same set of β-redexes, are Lévy equivalent. From this follows that any rewriting path g' involved in a permutation $f \rhd_{\mathrm{tree}} g'$ factors as $g' = u' \cdot h'$ where $u = u'$ and $h \equiv_{\mathrm{tree}} h'$. This leads us to formulate the

Axiom 2 (Ancestor) *Suppose that u, u' are redexes, that f, h, h' are rewriting paths, forming together permutations $f \rhd u \cdot h$ and $f \rhd u' \cdot h'$. We ask that $u = u'$ and $h \equiv h'$.*

Axiom **reversibility** indicates that every permutation $f \rhd g$ is either reversible, or reduces to a rewriting path g for which there exists no permutation of the form $g \rhd h$. This mirrors the following property of the λ-calculus. Suppose that $f, g, h : M \longrightarrow\!\!\!\!\!\rightarrow N$ are three β-rewriting paths involved in permutations $f \rhd_{\mathrm{tree}} g$ and $g \rhd_{\mathrm{tree}} h$. The paths f and g are of length 2, the path h is of length at least 1, and the paths f, g, h decompose as

$$f \;=\; M \xrightarrow{v} Q \xrightarrow{u'} N, \qquad g \;=\; M \xrightarrow{u} P \xrightarrow{v'} N, \qquad h \;=\; M \xrightarrow{v''} O \xrightarrow{h_u} N$$

where the two redexes v and v'' are ancestor of the same redex v', and thus $v = v''$; and where the β-redex u' is the unique residual of the β-redex u, and the rewriting path h_u is a development of the residuals of u after v, and thus $h_u = u'$. It follows that $f = h$.

Axiom 3 (Reversibility) *We ask that $f = h$ when $f \rhd g$ and $g \rhd h$.*

Axiom **irreversibility** completes the two previous axioms. The axiom mirrors the fact that in the λ-calculus and in many rewriting systems, standardization preserves complete developments — see [24, 18] or Section 6 for a definition of complete developments. Let us explain briefly what we mean here. Consider any β-rewriting path $h : M \longrightarrow\!\!\!\!\!\rightarrow N$ which defines a complete development of a multi-redex (M, U) in the λ-calculus, and suppose that the path h factors as

$$h \quad = \quad M \xrightarrow{h_1} M' \xrightarrow{h_2} N' \xrightarrow{h_3} N$$

where the β-rewriting path h_2 is involved in a standardization permutation

$$h_2 \rhd h_2'.$$

By definition of $\rhd_{\mathrm{tree}}$, the two β-rewriting paths h_2 and h_2' decompose as

$$h_2 = M' \xrightarrow{v} P \xrightarrow{u'} N' \qquad \text{and} \qquad h_2' = M' \xrightarrow{u} Q \xrightarrow{h''} N'.$$

We claim here that the resulting β-rewriting path

$$h' \quad = \quad M \xrightarrow{h_1} M' \xrightarrow{h_2'} N' \xrightarrow{h_3} N$$

defines a complete development of (M, U). How do we prove this? We establish first that the two redexes u and v are residual of a redex in U after the β-rewriting path h_1. The very definition of the path h as a complete development of the multi-redex (M, U) induces already that:

– the redex v is residual of a redex $v_0 \in U$ after the β-rewriting path h_1; and

– the redex u' is residual of a redex $u_0 \in U$ after the β-rewriting path $h_1 \cdot v$.

We know moreover that the β-redex u is the *unique* ancestor of the β-redex u' before reduction of the β-redex v. This uniqueness property ensures that the β-redex u is residual of the redex $u_0 \in U$ after the β-rewriting path h_1. This establishes that the two redexes u and v are residual of a redex in U after the β-rewriting path h_1. Now, we know by definition of $\rhd_{\mathrm{tree}}$ that the two paths h_2 and h_2' define complete developments of the multi-redex $(M', \{u, v\})$. The *finite development* property of the λ-calculus states moreover that the two β-rewriting paths h_2 and h_2' define the same residual relation. It follows quite immediately that, as we claimed, the β-rewriting path $h_1 \cdot h_2' \cdot h_3$ defines a complete development of the multi-redex (M, U). We conclude more generally that every path more standard than the path h is also a complete development of the multi-redex (M, U).

How is this result interpreted in our axiomatic setting? Consider an irreversible permutation $f \blacktriangleright_{\mathrm{tree}} g$ between two β-rewriting paths

$$f = M \xrightarrow{v} Q \xrightarrow{u'} N \qquad g = M \xrightarrow{u} P \xrightarrow{h_v} N$$

and a β-rewriting path h such that

$$g \Longrightarrow h.$$

It follows from our previous argument that, just like the β-rewriting path f and g, the β-rewriting path h is a complete development of the multi-redex $(M, \{u, v\})$. Besides, the first β-redex reduced in the path h is not the β-redex v. Thus, the β-rewriting path h decomposes necessarily as

$$h = M \xrightarrow{u} P \xrightarrow{h'_v} N$$

where

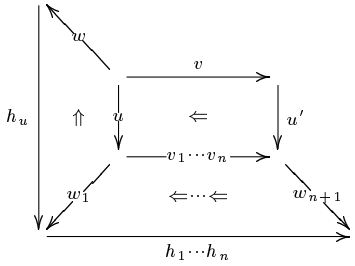
$$h_v \Longrightarrow h'_v.$$

Here, we apply our previous argument another time, and deduce from $h_v \Longrightarrow h'_v$ that, just like the β-rewriting path h_v, the β-rewriting path h'_v is a complete development of the residuals of the β-redex v after reduction of the β-redex u. This shows in particular that $f \blacktriangleright_{\mathrm{tree}} h$. This leads to

Axiom 4 (Irreversibility) *We ask that $f \blacktriangleright h$ when $f \blacktriangleright g$ and $g \Longrightarrow h$.*

Axiom **cube** incorporates the *cube lemma* established in [24, 18] as well as a careful analysis of nesting in the λ-calculus. Suppose that $C[-]$ is a context, see [3] for a definition, and that a β-rewriting path $g : C[M] \longrightarrow\!\!\!\!\!\rightarrow C[N]$ computes only inside M, never inside $C[-]$. Then, just as the β-rewriting path g, every Lévy equivalent β-rewriting path $f : C[M] \longrightarrow\!\!\!\!\!\rightarrow C[N]$ computes only inside M, never inside $C[-]$. So, every β-redex w inside $C[-]$ has the same (unique) residual w'' after the β-rewriting paths f and g. Diagrammatically speaking, the property amounts to the cube property stated in the next axiom, when $f \rhd_{\mathrm{tree}} g$ and $f = v \cdot u'$ and $g = u \cdot v_1 \cdots v_n$ and $w'' = w_{n+1}$. The axiom requires that the property holds in every AxRS.

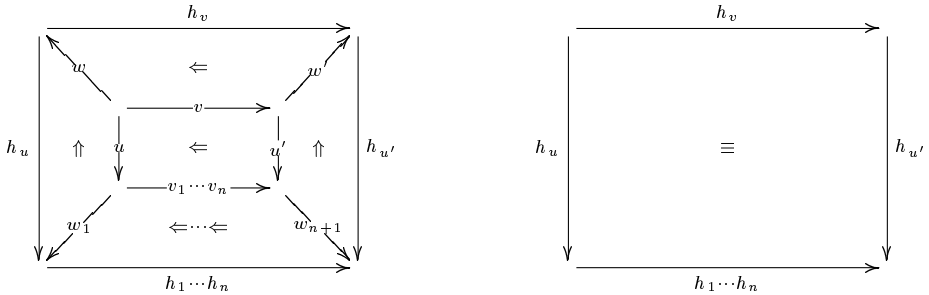
Axiom 5 (Cube) *We ask that every diagram*



with u, u', v and $v_1, ..., v_n$ and $w, w_1, ..., w_n, w_{n+1}$ a series of redexes and $h_1, ..., h_n$ a series of paths forming permutations

$$v \cdot u' \ \rhd \ u \cdot v_1 \cdots v_n \qquad u \cdot w_1 \rhd w \cdot h_u \qquad v_i \cdot w_{i+1} \ \rhd \ w_i \cdot h_i \qquad \textit{for } 1 \leq i \leq n$$

may be completed as a diagram:



where w' is a redex and $h_v, h_{u'}$ are paths which form permutations

$$u' \cdot w_{n+1} \rhd w' \cdot h_{u'} \qquad v \cdot w' \rhd w \cdot h_v$$

and induce the equivalence

$$h_v \cdot h_{u'} \equiv h_u \cdot h_1 \cdots h_n.$$

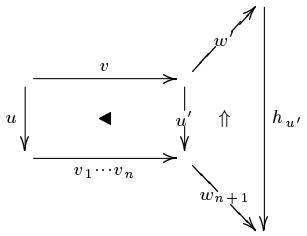
2.3 Axiom 6: enclave

Axiom **enclave** is based on a fundamental property of the λ-calculus, observed for the first time in the preliminary work of [13]. Suppose that a β-redex v is nested under a β-redex u — that is $u \preceq_{\mathrm{tree}} v$ — and that the β-redex v creates a β-redex w'. By *creation*, we mean that the β-redex w' has no ancestor before reduction of the β-redex v. In that case, the β-redex w' is necessarily nested under the (unique) residual u' of the β-redex u after reduction of the β-redex v. The next axiom formulates the property as its contrapose. The existence of the permutation

$$u' \cdot w_{n+1} \rhd_{\mathrm{tree}} w' \cdot h_{u'}$$

means that the β-redex w' is *not* nested under the β-redex u'. And from this follows that the β-redex w' is not created, and thus, has an ancestor w before reduction of the β-redex v. The axiom requires that this *enclave property* holds in every AxRS.

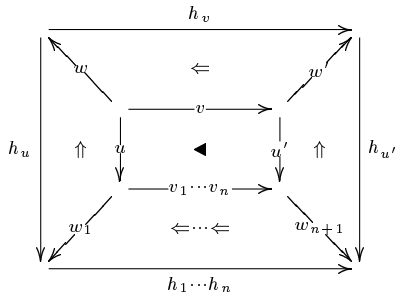
Axiom 6 (Enclave) *We ask that every diagram*



where u, v, u' and $v_1, ..., v_n$ and w', w_{n+1} are redexes, and $h_{u'}$ is a path, forming the permutations (recalling our convention, the symbol $\blacktriangleright$ means that the permutation is irreversible)

$$v \cdot u' \blacktriangleright u \cdot v_1 \cdots v_n \qquad\qquad u' \cdot w_{n+1} \rhd w' \cdot h_{u'}$$

may be completed as a diagram:



with $w, w_1, ..., w_n$ a series of redexes and h_u, h_v and $h_1, ..., h_n$ a series of paths, forming the $n + 2$ permutations

$$v \cdot w' \rhd w \cdot h_v \qquad u \cdot w_1 \rhd w \cdot h_u \qquad v_i \cdot w_{i+1} \rhd w_i \cdot h_i \qquad \text{for } 1 \leq i \leq n$$

2.4 Axioms 7 and 8: stability and reversible stability

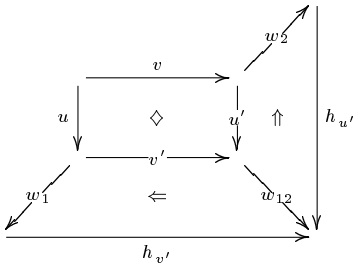
Axiom **stability** incorporates another key property of the λ-calculus, also observed for the first time in the preliminary work of [13]. Consider any reversible permutation

$$M \xrightarrow{u} P \xrightarrow{v'} N \quad \diamondsuit_{\text{tree}} \quad M \xrightarrow{v} Q \xrightarrow{u'} N$$

in which the β-redex u creates a β-redex w_1 and the β-redex v creates a β-redex w_2. It is not difficult to establish that there exists no β-redex w_{12} in the λ-term N which would

be at the same time residual of the β-redex w_1 after reduction of the β-redex v', and residual of the β-redex w_2 after reduction of the β-redex u'. The property is axiomatized below as its contrapose. The axiom states that the *characteristic function* of the *event* of creating the β-redex w_{12} (or equivalently the β-redex w_1, or the β-redex w_2) is *stable* in the sense of G. Berry, see [5]. Axiom **reversible-stability** repeats the axiom in the reversible case.

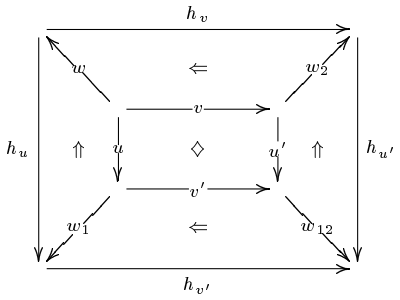
Axiom 7 (Stability) *We ask that every diagram*



where u, v, u', v' and w_1, w_2, w_{12} are redexes and $h_{u'}, h_{v'}$ are paths, forming the permutations (recalling our convention, the symbol $\Diamond$ means that the permutation is reversible)

$$v \cdot u' \Diamond u \cdot v' \qquad u' \cdot w_{12} \rhd w_2 \cdot h_{u'} \qquad v' \cdot w_{12} \rhd w_1 \cdot h_{v'}$$

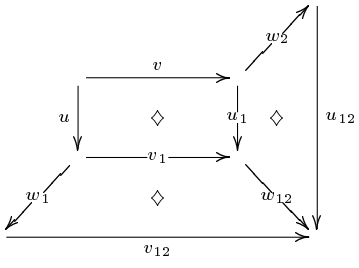
may be completed as a diagram



where w is a redex and h_u, h_v are two paths, forming two permutations

$$v \cdot w_2 \rhd w \cdot h_v \qquad u \cdot w_1 \rhd w \cdot h_u$$

Axiom 8 (Reversible stability) *We ask that every diagram*

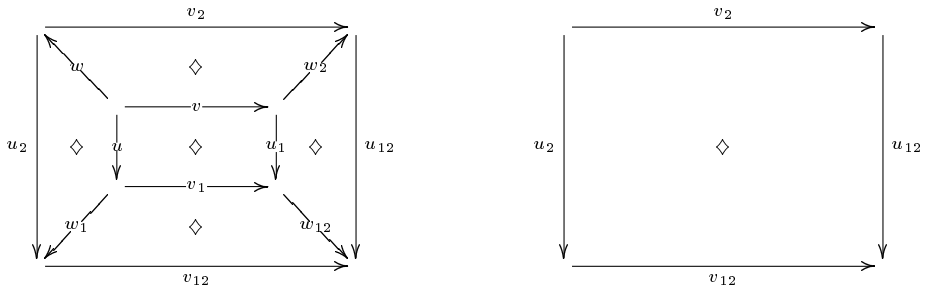


(16)

where u, v, u_1, v_1 *and* $w_1, w_2, w_{12}, u_{12}, v_{12}$ *are redexes forming the reversible permutations*

$$v \cdot u_1 \diamondsuit u \cdot v_1 \qquad u_1 \cdot w_{12} \diamondsuit w_2 \cdot u_{12} \qquad v_1 \cdot w_{12} \diamondsuit w_1 \cdot v_{12}$$

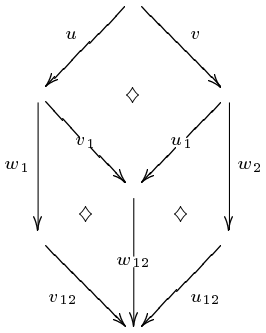
may be completed as a diagram



where w, u_2, v_2 *are three redexes forming the reversible permutations*

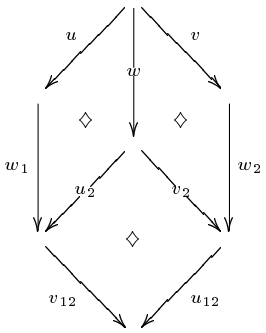
$v \cdot w_2 \diamondsuit w \cdot v_2$ *and* $u \cdot w_1 \diamondsuit w \cdot u_2$ *and* $v_2 \cdot u_{12} \diamondsuit u_2 \cdot v_{12}$

Remark: Axiom **reversible-stability** may be understood as a converse of the reversible variant of Axiom **cube** formulated in Section 7.3. Indeed, Axiom **reversible-stability** states that every diagram



(17)

may be completed into the diagram



(18)

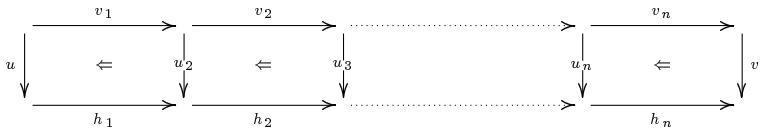


Fig. 2. The path $f = v_1 \cdots v_n$ drags the redex v to the redex u.

and conversely, Axiom **reversible-cube** formulated in Section 7.3 states that Diagram (18) may be completed as Diagram (17). Besides, it is remarkable that the two Axioms **reversible-stability** and **reversible-cube** are *dual* in the sense that each axiom may be obtained from the other one by *reversing* the orientations of all the arrows in diagrams.

2.5 Drag and extraction

We need to introduce a few definitions related to standardization in order to state the last axiom of the theory (Axiom 9).

Definition 9 (drag) *A path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$* drags *a redex v outgoing from N to a redex u outgoing from M, when*

- *$f = \mathrm{id}_M$ and $v = u$,*
- *or $f = v_1 \cdots v_n$ and there exists $n+1$ redexes $u_1, ..., u_{n+1}$ and n paths $h_1, ..., h_n$ such that:*
  - *$u_1 = u$ and $u_{n+1} = v$,*
  - *the rewriting paths $v_i \cdot u_{i+1}$ and $u_i \cdot h_i$ form a permutation $v_i \cdot u_{i+1} \rhd u_i \cdot h_i$ for every index $1 \leq i \leq n$.*

Notation: we write $u \overset{f}{\longleftarrow\!\!\shortmid} v$ when the rewriting path f drags the redex v to the redex u. See Figure 2.

Lemma 10 (preservation of drag) *For every path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$, the relation $\overset{f}{\longleftarrow\!\!\shortmid}$ is a partial function, from the redexes outgoing from N to the redexes outgoing from M. Moreover, the relation is invariant by permutation on f:*

$$\forall g : M \longrightarrow\!\!\!\!\!\rightarrow N, \qquad f \equiv g \;\Rightarrow\; \overset{f}{\longleftarrow\!\!\shortmid} \;=\; \overset{g}{\longleftarrow\!\!\shortmid}\,.$$

Proof. Suppose that $u \overset{f}{\longleftarrow\!\!\shortmid} v$ and $u' \overset{f}{\longleftarrow\!\!\shortmid} v$. Then $u = u'$ by Axiom **ancestor**, and an easy induction on the length of f. Now, by Axiom **cube**, the relation increases by *anti-standardization*: if the rewriting path g drags the redex v to the redex u, and $f \Longrightarrow g$, then the rewriting path f drags the redex v to the redex u. By Axiom **enclave**, the relation increases also by *standardization*: if the rewriting path f drags the redex v to the redex u, and $f \Longrightarrow g$, then the rewriting path g drags the redex v to the redex u as well. We conclude. □

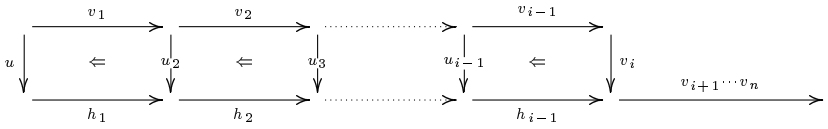


Fig. 3. The redex u is extractible from the path $f = v_1 \cdots v_n$ and the path $g = h_1 \cdots h_{i-1} \cdot v_{i+1} \cdots v_n$ is a projection of the rewriting path f by extraction of the redex u.

Definition 11 (extraction, projection, $\searrow_u$) *A redex $u : M \longrightarrow P$ is* extractible *from a path $f = v_1 \cdots v_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ when there exists an index $1 \leq i \leq n$ such that the path $v_1 \cdots v_{i-1}$ drags the redex v_i to the redex u. In that case, we call* projection *of the rewriting path f by extraction of the redex $u : M \longrightarrow P$ any rewriting path $g : P \longrightarrow\!\!\!\!\!\rightarrow N$ which decomposes as*

$$g = h_1 \cdots h_{i-1} \cdot v_{i+1} \cdots v_n$$

where there exists redexes $u_1, ..., u_i$ with $u_1 = u$ and $u_i = v_i$ and a permutation

$$v_j \cdot u_{j+1} \rhd u_j \cdot h_j$$

for every index $1 \leq j \leq i - 1$.

Notation: We write $f \searrow_u g$ when the redex u is extractible from the path f, and g is a projection of f by extraction of the redex u. See figure 3.

Lemma 12 (preservation of extraction) *Suppose that a redex u is extractible from a path $g : M \longrightarrow\!\!\!\!\!\rightarrow N$ more standard than a path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$. Then the redex u is also extractible from the path f. Moreover, every projection of f by extraction of u and every projection of g by extraction of u are Lévy equivalent.*

Proof. Suppose that the redex u is extractible from the path $f = v_1 \cdots v_n : M \longrightarrow\!\!\!\!\!\rightarrow N$. By definition, there exists an index $1 \leq i \leq n$ such that the path $v_1 \cdots v_{i-1}$ drags the redex v_i to the redex u. We show that the index i is unique. Suppose that there exists another index $1 \leq j \leq n$ such that $v_1 \cdots v_{j-1}$ drags the redex v_j to the redex u. We may suppose without loss of generality that $i < j$. Let the rewriting path h be a projection of the rewriting path $v_1 \cdots v_i$ by extraction of the redex u at position i. By definition of extraction and projection, the two rewriting paths $v_1 \cdots v_i$ and $u \cdot h$ are Lévy equivalent. From this follows that the two paths

$$v_1 \cdots v_{j-1} = v_1 \cdots v_i \cdot v_{i+1} \cdots v_{j-1} \qquad \text{and} \qquad u \cdot h \cdot v_{i+1} \cdots v_{j-1}$$

are Lévy equivalent. Here comes the contradiction. By Lemma 10 (*preservation of drag*), the path $u \cdot h \cdot v_{i+1} \cdots v_{j-1}$ drags the redex v_j to the redex u. This may be decomposed in two steps: first, the path $h \cdot v_{i+1} \cdots v_{j-1}$ drags the redex v_j to a redex v, then the redex u drags the redex v to the redex u. This very last point means that there

exists a permutation of the form $u \cdot v \rhd u \cdot h'$. This contradicts the Axiom **shape**. We thus conclude that the index i is unique for a given u.

We may suppose without loss of generality that there exists a unique standardization step from the rewriting path f to the rewriting path g. The remainder of the lemma follows then from Axioms **reversibility** and **cube** when the standardization step from f to g is reversible, and from Axioms **irreversibility**, **ancestor** and **cube** when the standardization step is irreversible. □

Remark: the uniqueness of the index i in the proof of Lemma 12 is not really necessary to establish the property, but it is a safeguard, since after all, we have not supposed anything like the optional hypothesis **descendant** formulated in Section 7.1.

2.6 Axiom 9: termination

Axiom **termination** mirrors in our theory the *finite development* property of the λ-calculus, which states that every development of a set of β-redexes terminates. Jan Willem Klop uses the property in his PhD thesis to deduce that it is not possible to extract infinitely many times a β-redex from a fixed β-rewriting path, see [20] as well as Section 6.

Axiom 9 (Termination) *There exists no infinite sequence*

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} \cdots \searrow_{u_{k-1}} f_k \searrow_{u_k} \cdots$$

where f_i are paths and u_i are redexes.

3 A Direct Characterization of the Standard Paths

In this section, we establish a key preliminary step in our proof of the standardization theorem, performed in Section 4, by characterizing standard rewriting path in a more direct and explicit way. In Section 3.1, we introduce the notions of *starts* and *stops* of a rewriting path, and analyze their properties. From this, we deduce in Section 3.2 that every path is epi (left cancellable) with relation to the Reversible permutation relation $\simeq$. In Section 3.3, we introduce the notion of *anti-standard* path and establish that a rewriting path is standard if and only if it does not contain any occurrence of such anti-standard path.

3.1 The structure of starts and stops

Definition 13 (starts and stops) *A redex $u : M \longrightarrow P$* starts *a path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ when there exists a path $g : P \longrightarrow\!\!\!\!\!\rightarrow N$ such that $f \simeq u \cdot g$. A redex $v : Q \longrightarrow N$* stops *a path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ with* remainder *$g : M \longrightarrow\!\!\!\!\!\rightarrow Q$ when $f \simeq g \cdot v$. A redex $v : Q \longrightarrow N$* stops *a path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ when the redex v stops the path f with some remainder $g : M \longrightarrow\!\!\!\!\!\rightarrow Q$.*

Definition 14 (reversible permutation of path and redex) *A path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ followed by a redex $v : N \longrightarrow Q$* permutes reversibly *to a redex $u : M \longrightarrow P$ followed by a path $g : P \longrightarrow\!\!\!\!\!\rightarrow Q$, when*

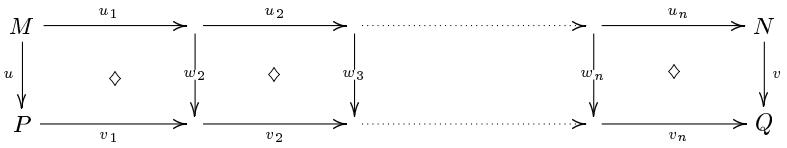


Fig. 4. The path $f = u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ followed by the redex $v : N \longrightarrow Q$ permutes reversibly to the redex $u : M \longrightarrow P$ followed by the path $g = v_1 \cdots v_n : P \longrightarrow\!\!\!\!\!\rightarrow Q$. Alternatively, the redex $u : M \longrightarrow P$ followed by the path $g = v_1 \cdots v_n : P \longrightarrow\!\!\!\!\!\rightarrow Q$ permutes reversibly to the path $f = u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ followed by the redex $v : N \longrightarrow Q$.

- $f = \mathrm{id}_M$ *and* $g = \mathrm{id}_P$ *and* $v = u : M \longrightarrow P$,
- *or* $f = u_1 \cdots u_n$ *and* $g = v_1 \cdots v_n$ *and there exists a series of* $n + 1$ *redexes* $w_1, ..., w_{n+1}$ *such that*
  - $w_1 = u$ *and* $w_{n+1} = v$,
  - *the two paths* $u_i \cdot w_{i+1}$ *and* $w_i \cdot v_i$ *form a reversible permutation* $u_i \cdot w_{i+1} \diamondsuit w_i \cdot v_i$ *for every index* $1 \leq i \leq n$.

In that case, we say also that the redex $u : M \longrightarrow P$ *followed by the path* $g : P \longrightarrow\!\!\!\!\!\rightarrow Q$ permutes reversibly *to the path* $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ *followed by the redex* $v : N \longrightarrow Q$. *See Figure 4.*

Remark: in Definition 14, the redex u and the rewriting path g are uniquely determined by the rewriting path f and the redex v — and conversely, the rewriting path f and the redex v are uniquely determined by the redex u and the rewriting path g. The one-to-one relationship follows from Axiom **reversibility**.

Lemma 15 (structure of stops) *A redex* $v : Q \longrightarrow N$ *stops a path* $f = u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ *with remainder* $g : M \longrightarrow\!\!\!\!\!\rightarrow Q$ *iff there exists an index* $1 \leq i \leq n$ *and a path* $v_{i+1} \cdots v_n$ *such that*

- *the redex* u_i *followed by the path* $u_{i+1} \cdots u_n$ *permutes reversibly to the path* $v_{i+1} \cdots v_n$ *followed by the redex* v,
- *the rewriting path* $(u_1 \cdots u_{i-1}) \cdot (v_{i+1} \cdots v_n)$ *is equivalent to the path* g *modulo* $\simeq$.

Proof. We declare that a redex $v : Q \longrightarrow N$ *super-stops* a path $f = u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ at position $1 \leq i \leq n$ with remainder $g : M \longrightarrow\!\!\!\!\!\rightarrow Q$ when there exists a path $v_{i+1} \cdots v_n$ such that

- the redex u_i followed by the path $u_{i+1} \cdots u_n$ permutes reversibly to the path $v_{i+1} \cdots v_n$ followed by the redex v,
- the rewriting path $(u_1 \cdots u_{i-1}) \cdot (v_{i+1} \cdots v_n)$ is equivalent to the path g modulo $\simeq$.

We declare that a redex v *super-stops* a path f with remainder g when it super-stops the path f with remainder g at some position i.

The lemma states that a redex v stops a path f with remainder a path g iff the redex v super-stops f with remainder g. Right-to-left implication ($\Leftarrow$) is immediate. The other direction ($\Rightarrow$) reduces to showing that whenever the two assertions below holds:

– a redex $v : Q \longrightarrow N$ super-stops a path $f = u_1 \cdots u_n$ with remainder g, and
– the path f' is equivalent to the path f modulo reversible permutations,

then the redex v super-stops the path f' with remainder the same rewriting path g. This elementary but fundamental preservation property is established in the following way. We may suppose without loss of generality that the two rewriting paths $f = u_1 \cdots u_n$ and $f' = u'_1 \cdots u'_n$ are related by a unique reversible permutation

$$f \overset{REV}{\Longrightarrow} f'$$

occurring at a position $1 \leq j \leq n-1$ in the rewriting path f. We thus have:

– $u'_k = u_k$ for every index $1 \leq k \leq n$ different to j and $j+1$, and
– $u_j \cdot u_{j+1} \diamondsuit u'_j \cdot u'_{j+1}$.

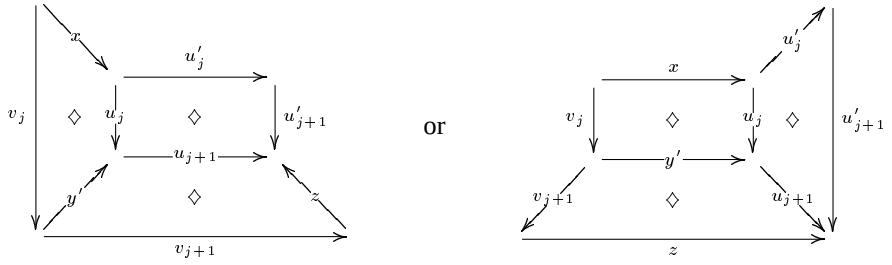
Now, call i any position (there exists in fact only one of these positions, $1 \leq i \leq n$, but nobody cares about that here) such that the redex $v : Q \longrightarrow N$ super-stops the path $f = u_1 \cdots u_n$ at position i with remainder g. We show by case analysis on the indices i and j that there exists an index $1 \leq k \leq n$ such that the redex $v : Q \longrightarrow N$ super-stops the path

$$f' = u'_1 \cdots u'_{k-1} \cdot u'_k \cdot u'_{k+1} \cdots u'_n$$

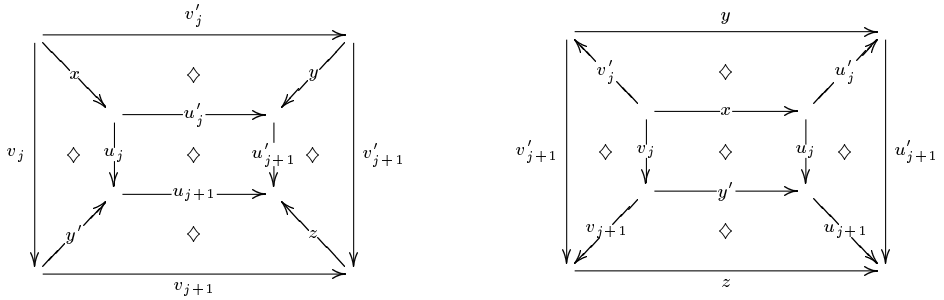
at position k with remainder g. To that purpose, we define a rewriting path $v'_{k+1} \cdots v'_n$ consisting of $n-k$ redexes, such that:

a. the redex u'_k followed by the path $u'_{k+1} \cdots u'_n$ permutes reversibly to the path $v'_{k+1} \cdots v'_n$ followed by the redex v,
b. the rewriting path $(u'_1 \cdots u'_{k-1}) \cdot (v'_{k+1} \cdots v'_n)$ is equivalent to the path g modulo $\simeq$.

∘ The construction is immediate when $j+1 \leq i$: simply take $k = i$ and $v'_i \cdots v'_n = v_i \cdots v_n$.
∘ The construction is also nearly immediate when $j = i$: simply take $k = i+1$ and $v'_{i+2} \cdots v'_n = v_{i+2} \cdots v_n$, then apply Axiom **reversibility** to establish the two properties a. and b.
∘ The difficult case is the remaining case when $j > i$. In that case, let the redex x denote the unique redex such that the redex u_i followed by the path $u_{i+1} \cdots u_{j-1}$ permutes reversibly to the path $v_{i+1} \cdots v_{j-1}$ followed by the redex x. Consider the diagram below, which describes in two perspectives how the redex x followed by the path $u_j \cdot u_{j+1}$ permutes reversibly to the path $v_j \cdot v_{j+1}$ followed by the redex z:



By Axiom **reversible-stability**, the diagram may be completed in the following way



where y and v'_j and v'_{j+1} denote three redexes involved in the three reversible permutations:

$$x \cdot u'_j \diamondsuit v'_j \cdot y, \quad \text{and} \quad v_j \cdot v_{j+1} \diamondsuit v'_j \cdot v'_{j+1} \quad \text{and} \quad y \cdot u'_{j+1} \diamondsuit v'_{j+1} \cdot z.$$

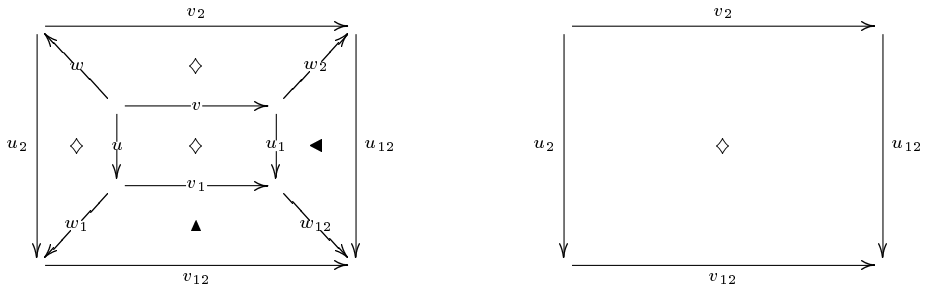
The completed diagram shows (in two perspectives again) that the redex x followed by the path $u'_j \cdot u'_{j+1}$ permutes reversibly to the path $v'_j \cdot v'_{j+1}$ followed by the redex z. So, by taking $k = i$ and by defining $v'_l = v_l$ for every index $i + 1 \leq l \leq n$ different to j and $j + 1$, one obtains that:

a. the redex u_i followed by the path $u'_{i+1} \cdots u'_n$ permutes reversibly to the path $v'_i \cdots v'_n$ followed by the redex v,
b. the rewriting path $(u_1 \cdots u_{i-1}) \cdot (v'_{i+1} \cdots v'_n)$ is equivalent to the path g modulo $\simeq$. This very last point follows from the series of equivalence

$$g \simeq (u_1 \cdots u_{i-1}) \cdot (v_{i+1} \cdots v_n) \quad \text{and} \quad v_{i+1} \cdots v_n \simeq v'_{i+1} \cdots v'_n.$$

□

Unfortunately, the characterization of *starts* is not as simple as the characterization of *stops*. The main reason is that the following 2-dimensional transition system



where

$$\begin{array}{lll} u \cdot v_1 \diamondsuit v \cdot u_1 & v \cdot w_2 \diamondsuit w \cdot v_2 & w_2 \cdot u_{12} \blacktriangleright u_1 \cdot w_{12} \\ u_2 \cdot v_{12} \diamondsuit v_2 \cdot u_{12} & u \cdot w_1 \diamondsuit w \cdot u_2 & w_1 \cdot v_{12} \blacktriangleright v_1 \cdot w_{12} \end{array}$$

satisfies the nine properties required of an axiomatic rewriting system in Section 2. The series of equivalence

$$u \cdot w_1 \cdot v_{12} \simeq w \cdot u_2 \cdot v_{12} \simeq w \cdot v_2 \cdot u_{12} \simeq v \cdot w_2 \cdot u_{12}$$

illustrates then that a redex u may start the path $v \cdot w_2 \cdot u_{12}$ even if the path $v \cdot w_2$ followed by the redex u_{12} does not permute reversibly. However, the situation is not entirely hopeless: observe that the path $v \cdot w_2$ is $\simeq$-equivalent to the path $w \cdot v_2$ which followed by the redex u_{12} permutes reversibly to the redex u followed by the path $w_1 \cdot v_{12}$. Next lemma shows that the property characterizes *starts* in any axiomatic rewriting system.

Lemma 16 (structure of starts) *A redex $u : M \longrightarrow P$ starts a path $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ if and only there exists an index $1 \leq i \leq n$ and two paths $v_1 \cdots v_{i-1}$ and $w_1 \cdots w_{i-1}$ such that*

- *the path $v_1 \cdots v_{i-1}$ is equivalent to the path $u_1 \cdots u_{i-1}$ modulo $\simeq$,*
- *the path $v_1 \cdots v_{i-1}$ followed by the redex u_i permutes reversibly to the redex u followed by the path $w_1 \cdots w_{i-1}$.*

Proof. We declare that a redex $u : M \longrightarrow P$ *super-starts* a path $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ when there exists an index $1 \leq i \leq n$ and two paths $v_1 \cdots v_{i-1}$ and $w_1 \cdots w_{i-1}$ such that

- $u_1 \cdots u_{i-1} \simeq v_1 \cdots v_{i-1}$,
- the path $v_1 \cdots v_{i-1}$ followed by the redex u_i permutes reversibly to the redex u followed by the path $w_1 \cdots w_{i-1}$.

We prove that a redex u starts a path f iff the redex u super-starts f. Right-to-left implication ($\Leftarrow$) is immediate: the redex u super-starts the path f implies the redex u starts the path f. The converse implication ($\Rightarrow$) reduces to the following preservation property: when a redex u super-starts a path f, and when the path g is obtained from the path f by applying a reversible permutation, then the redex u super-starts also the path g.

So, consider a redex $u : M \longrightarrow P$ and a path $f = u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ such that the redex u super-starts the path f. By definition, there exists an index $1 \leq i \leq n$ and two paths $v_1 \cdots v_{i-1}$ and $w_1 \cdots w_{i-1}$ such that

- $u_1 \cdots u_{i-1} \simeq v_1 \cdots v_{i-1}$,
- the redex u followed by the path $w_1 \cdots w_{i-1}$ permutes reversibly to the path $v_1 \cdots v_{i-1}$ followed by the redex u_i.

Consider any reversible standardization step

$$f \stackrel{REV}{\Longrightarrow} g$$

or equivalently, any index $1 \leq j \leq n-1$ and reversible permutation $u_j \cdot u_{j+1} \diamondsuit u'_j \cdot u'_{j+1}$. We claim that the redex u super-starts the path

$$g = (u_1 \cdots u_{j-1}) \cdot (u'_j \cdot u'_{j+1}) \cdot (u_{j+2} \cdots u_n).$$

We proceed by case analysis.
◦ The two first cases, when $j \leq i-2$ or when $j \geq i$, are immediate.
◦ The remaining case, when $j = i-1$, is the only difficult case. The equivalence

$$u_1 \cdots u_{i-1} \simeq v_1 \cdots v_{i-1}$$

shows that the redex u_{i-1} stops the path $v_1 \cdots v_{i-1}$ with remainder $u_1 \cdots u_{i-2}$. By Lemma 15, there exists an index $1 \leq k \leq i-1$ and a path $v'_{k+1} \cdots v'_{i-1}$ such that

– the redex v_k followed by the path $v_{k+1} \cdots v_{i-1}$ permutes reversibly to the path $v'_{k+1} \cdots v'_{i-1}$ followed by the redex u_{i-1},
– the path $(v_1 \cdots v_{k-1}) \cdot (v'_{k+1} \cdots v'_{i-1})$ is equivalent to the path $u_1 \cdots u_{i-2}$ modulo $\simeq$.

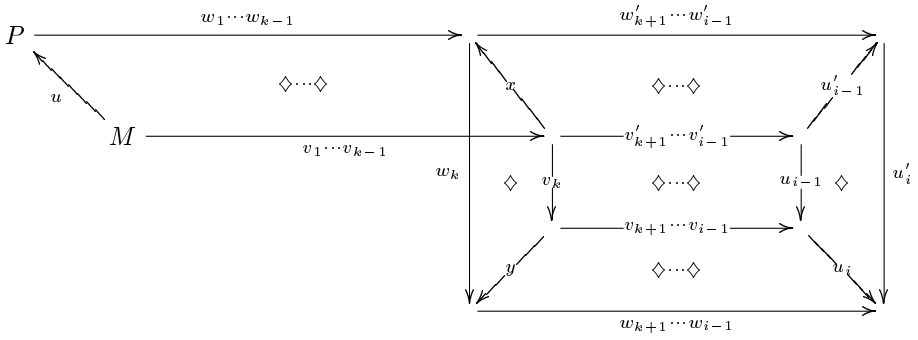
We are also in a situation where

– there exists a reversible permutation $u_{i-1} \cdot u_i \diamondsuit u'_{i-1} \cdot u'_i$
– the path $v_{k+1} \cdots v_{i-1}$ followed by the redex u_i permutes reversibly to a redex y followed by the path $w_{k+1} \cdots w_{i-1}$.

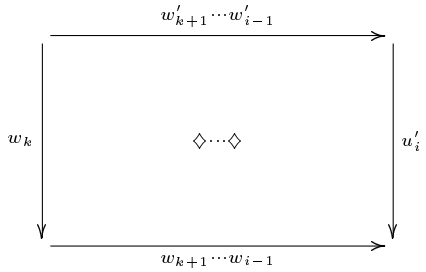
All put together, we deduce by applying Axiom **reversible-stability** $i-k-1$ times, and Axiom **reversibility** once, that there exists a redex x and path $w'_{k+1} \cdots w'_{i-1}$ such that

a. the redex u followed by the path $w_1 \cdots w_{k-1}$ permutes reversibly to the path $v_1 \cdots v_{k-1}$ followed by the redex x,
b. the redex x followed by the redex w_k permutes reversibly to the redex v_k followed by the redex y,
c. the redex y followed by the path $w_{k+1} \cdots w_{i-1}$ permutes reversibly to the path $v_{k+1} \cdots v_{i-1}$ followed by the redex u_i,
d. the redex x followed by the path $w'_{k+1} \cdots w'_{i-1}$ permutes reversibly to the path $v'_{k+1} \cdots v'_{i-1}$ followed by the redex u'_{i-1},
e. the redex w_k followed by the path $w_{k+1} \cdots w_{i-1}$ permutes reversibly to the path $w'_{k+1} \cdots w'_{i-1}$ followed by the redex u'_i.

Points a–d. are summarized in the diagram below.



Point e. completes the diagram above by providing the front face of the cuboid generated by the redexes x and v_k and the path $v'_{k+1} \cdots v'_{i-1}$.



It appears now that the redex u super-starts the path

$$g = (u_1 \cdots u_{i-2}) \cdot (u'_{i-1} \cdot u'_i) \cdot (u_{i+1} \cdots u_n).$$

because

- the path $u_1 \cdots u_{i-2}$ is equivalent to the path $(v_1 \cdots v_{k-1}) \cdot (v'_{k+1} \cdots v'_{i-1})$ modulo $\simeq$,
- the path $(v_1 \cdots v_{k-1}) \cdot (v'_{k+1} \cdots v'_{i-1})$ followed by the redex u'_{i-1} permutes reversibly to the redex u followed by the path $(w_1 \cdots w_{k-1}) \cdot (w'_{k+1} \cdots w'_{i-1})$.

This establishes the equivalence between starting and super-starting a path. Since this is precisely what our lemma asserts, we conclude. □

3.2 Application: every rewriting path is epi wrt. $\simeq$

We illustrate the previous section with an application of Lemma 15.

Lemma 17 (epi wrt. $\simeq$) *If $f \cdot g_1 \simeq f \cdot g_2$ then $g_1 \simeq g_2$.*

Proof. We may suppose without loss of generality that the rewriting path f is a redex u. We prove that $u \cdot g_1 \simeq u \cdot g_2$ implies $g_1 \simeq g_2$ by induction on the length of g_1 (and of g_2). The property is immediate when g_1 (and therefore g_2) is empty. Otherwise, the path g_1 factors as $g_1 = h_1 \cdot v$ for some path h_1 and redex v. By Lemma 15, because the redex v stops the path $u \cdot g_2$ with remainder $u \cdot h_1$, one of the two following cases occurs:

- either there exists a path h_2 such that $g_2 \simeq h_2 \cdot v$ and $u \cdot h_1 \simeq u \cdot h_2$,
- or there exists a path h_2 such that the redex u followed by the path g_2 permutes reversibly to the path h_2 followed by the redex v, and such that $h_2 \simeq u \cdot h_1$.

In the first case, we deduce that $h_1 \simeq h_2$ by induction hypothesis on $u \cdot h_1 \simeq u \cdot h_2$, and conclude that $g_1 \simeq g_2$ by the series of equivalence:

$$g_1 = h_1 \cdot v \simeq h_2 \cdot v \simeq g_2$$

Fig. 5. The definition of an anti-standard path $u \cdot u_1 \cdots u_n \cdot y$: the redex u followed by the path $u_1 \cdots u_n$ permutes reversibly to the path $v_1 \cdots v_n$ followed by the redex v which permutes irreversibly with the redex y, as follows: $v \cdot y \blacktriangleright x \cdot h$.

Now, we prove that the second case does not occur. Obviously, the path h_2 drags the redex v to the redex u. By Lemma 10 (*preservation of drag*) and equivalence $h_2 \simeq u \cdot h_1$, the path $u \cdot h_1$ drags the redex v to the redex u. In particular, there exists a redex w and a path h such that $u \cdot w \rhd u \cdot h$. This contradicts Axiom **shape**, and we conclude. □

Remark: in Section 7.2 an additional hypothesis of **reversible-shape** is required to complete the property to an epi-mono property wrt. $\simeq$.

3.3 Characterization lemma

We introduce below the fundamental notion of *anti-standard* path. These anti-standard paths are called *conflicts* in [13, 27]. We change the terminology here because the word *conflict* is generally understood as *non determinism*, and because the notion of *anti-standard path* specializes to the notion of *anti-standard pair* introduced by J. W. Klop in the particular case of the λ-calculus equipped with the left-order $\preceq_{\text{left}}$ — see [20] and Section 1.9.

Definition 18 *A path is* anti-standard *(see Figure 5) when it factors as*

$$M \xrightarrow{u} P \xrightarrow{f} Q \xrightarrow{y} N$$

where u and y are redexes and f is a rewriting path, and

- *the redex u followed by the path f permutes reversibly to the path g followed by the redex v,*
- *the redex v and the redex y induce an irreversible permutation $v \cdot y \blacktriangleright x \cdot h$, for some redex x and rewriting path h.*

The β-rewriting path taken earlier as illustration

$$Ka(\Delta\Delta) \xrightarrow{\Delta_1} Ka(\Delta\Delta) \xrightarrow{K} (\lambda x.a)(\Delta\Delta) \xrightarrow{\lambda} a$$

is a typical example of anti-standard path in the axiomatic rewriting system $(\mathcal{G}_\lambda, \rhd_{\text{tree}})$. Compare indeed Diagrams (9) and (13) to Figure 5.

This leads us to the main result of the section.

Lemma 19 (characterization) *A path $u_1 \cdots u_n$ is standard if and only if there exists no pair of indices $1 \leq i < j \leq n$ such that $u_i \cdots u_j$ defines an anti-standard path.*

Proof. Left-to-Right implication ($\Rightarrow$) is immediate. Proving the converse direction ($\Leftarrow$) reduces to showing that:

- when two rewriting paths f and g are equivalent modulo reversible permutations $\simeq$, and
- when the path f contains an anti-standard path,

then the path g contains also an anti-standard path.

So, consider two rewriting paths $f = u_1 \cdots u_n$ and $g = u'_1 \cdots u'_n$, and suppose that the path g is obtained after a unique reversible standardization step on the path f:

$$f \overset{REV}{\Longrightarrow} g. \tag{19}$$

Let $1 \leq k \leq n-1$ denote the index where the reversible permutation occurs in the path f. Obviously,

$$u'_1 \cdots u'_{k-1} = u_1 \cdots u_{k-1} \text{ and } u'_k \cdot u'_{k+1} \diamondsuit u_k \cdot u_{k+1} \text{ and } u'_{k+2} \cdots u'_n = u_{k+2} \cdots u_n.$$

Now, suppose that the path f contains an anti-standard path, in the sense that there exist two indices $1 \leq i < j \leq n$ such that the path $u_i \cdots u_j$ is anti-standard. Let y denote the redex u_j. By definition of an anti-standard path, there exists a path $v_{i+1} \cdots v_{j-1}$ and redex w such that:

- the redex u_i followed by the path $u_{i+1} \cdots u_{j-1}$ permutes reversibly to the path $v_{i+1} \cdots v_{j-1}$ followed by the redex w,
- the redexes w and y form an irreversible permutation $w \cdot y \blacktriangleright x \cdot h$ for some redex x and path h.

We establish now that there exist two indices $1 \leq I < J \leq n$ such that the path $u'_I \cdots u'_J$ is anti-standard. This will show in particular that the path g contains an anti-standard path.

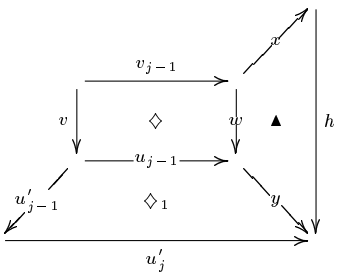
∘ The property is immediate when $k > j$: simply take $(I, J) = (i, j)$.

∘ The property follows from Lemma 15 when $k + 1 < j$:

- take $(I, J) = (i-1, j)$ when $k = i - 1$,
- take $(I, J) = (i+1, j)$ when $k = i$,
- take $(I, J) = (i, j)$ otherwise.

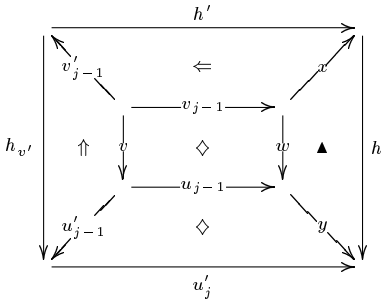
There remain only two difficult cases to treat: when $k = j - 1$ and when $k = j$.

∘ We treat the first case, when $k = j - 1$. The situation is summarized by the diagram:



where the reversible permutation $\diamondsuit_1$ relates the rewriting paths f and g in Equation (19) and where the irreversible permutation $w \cdot y \blacktriangleright x \cdot h$ between the redex w and the redex y witnesses the fact that the path $u_i \cdots u_{j-1} \cdot y$ (or equivalently the path $u_i \cdots u_{j-1} \cdot u_j$) is anti-standard.

The diagram may be completed by Axiom **stability** in the following way:



where v'_{j-1} is a redex, where h' and $h_{v'}$ are two rewriting paths, forming permutations

$$v \cdot u'_{j-1} \rhd v'_{j-1} \cdot h_{v'} \qquad \text{and} \qquad v_{j-1} \cdot x \rhd v'_{j-1} \cdot h'.$$

We proceed by case analysis on the permutation $v \cdot u'_{j-1} \rhd v'_{j-1} \cdot h_{v'}$:

— Either the permutation is irreversible. In that case, the path $u_i \cdots u_{j-2} \cdot u'_{j-1}$ is anti-standard, and we may thus conclude with $(I, J) = (i, j-1)$.

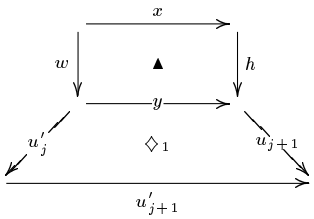
— Or the permutation is reversible. In that case, the path $h_{v'}$ is a redex; we write it v' for clarity's sake. We claim that the path $u_i \cdots u_{j-2} \cdot u'_{j-1} \cdot u'_j$ is anti-standard. Indeed, the redex u_i followed by the path $u_{i+1} \cdots u_{j-2} \cdot u'_{j-1}$ permutes reversibly to the path $v_{i+1} \cdots v_{j-2} \cdot v'_{j-1}$ followed by the redex v', and we establish now that the redexes v' and u'_j are involved in an irreversible permutation $v' \cdot u'_j \blacktriangleright v'_j \cdot h''$ for some redex v'_j and rewriting path h''. First of all, the rewriting path $v \cdot u'_{j-1}$ drags the redex u'_j to the redex v_{j-1}. So, by Lemma 10 (*preservation of drag*), the path $v'_{j-1} \cdot v'$ which is Lévy equivalent to the path $v \cdot u'_{j-1}$, drags the redex u'_j to the redex v_{j-1}. From this follows that there exists a permutation of the form $v' \cdot u'_j \rhd v'_j \cdot h''$ for some redex v'_j and rewriting path h''. There remains to show that this permutation is irreversible in order to establish our claim. We proceed by contradiction and suppose that the permutation $v' \cdot u'_j \rhd v'_j \cdot h''$ is reversible. Then, it follows from Axiom **reversible-stability** applied around the permutation $v \cdot u'_{j-1} \diamondsuit v'_{j-1} \cdot v'$ that:

- there exists a reversible permutation starting from the rewriting path $v \cdot u_{j-1}$; this permutation is necessarily the permutation $v \cdot u_{j-1} \diamondsuit v_{j-1} \cdot w$ by Axiom **reversibility**,
- there exists a reversible permutation starting from the rewriting path $w \cdot y$.

By Axiom **reversibility**, this last assertion contradicts the fact that there exists an irreversible permutation starting from the rewriting path $w \cdot y$. From this, we conclude that the permutation $v' \cdot u'_j \rhd v'_j \cdot h''$ starting from the rewriting path $v' \cdot u'_j$ is irreversible, and thus that the rewriting path $u_i \cdots u_{j-2} \cdot u'_{j-1} \cdot u'_j$ is anti-standard. We may thus take $(I, J) = (i, j)$.

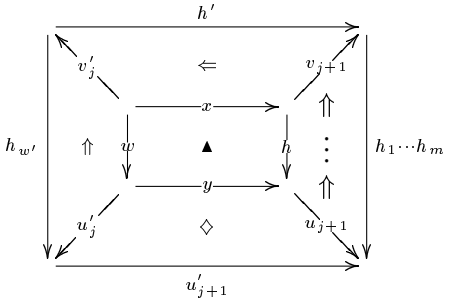
∘ We treat the second case, when $k = j$, and thus, the two redexes u_j and u_{j+1} are permuted reversibly in the path f to obtain the path g. Again, we let y denote the redex u_j. So, the redex u_i followed by the path $u_{i+1} \cdots u_{j-1}$ permutes reversibly to the path $v_{i+1} \cdots v_{j-1}$ followed by the redex w; and the redex w induces the irreversible permutation $w \cdot y \blacktriangleright x \cdot h$ with the redex y, witnessing the fact that the path $u_i \cdots u_{j-1} \cdot y$ (or equivalently the path $u_i \cdots u_{j-1} \cdot u_j$) is anti-standard.

The situation is summarized in the diagram below:



where the reversible permutation $\diamondsuit_1$ relates the rewriting paths f and g in Equation (19).

Here, we apply Axiom **enclave** and complete the diagram in the following way:



with two redex v'_j and two rewriting paths $h_{w'}$ and h' inducing permutations:

$$w \cdot u'_j \rhd v'_j \cdot h_{w'} \qquad \text{and} \qquad x \cdot v_{j+1} \rhd v'_j \cdot h'.$$

Note moreover that the path h grabs the redex u_{j+1} to a redex v_{j+1}, and that the redex x grabs the redex v_{j+1} to the redex v'_j.

We proceed by case analysis on the permutation $w \cdot u'_j \rhd v'_j \cdot h_{w'}$:

— Either the permutation is irreversible. In that case, the rewriting path $u_i \cdots u_{j-1} \cdot u'_j$ is anti-standard, and we may thus conclude with $(I, J) = (i, j)$.

— Or the permutation is reversible. In that case, the path $h_{w'}$ is a redex; we thus write it w' for clarity's sake. We claim that the rewriting path $u_i \cdots u_{j-1} \cdot u'_j \cdot u'_{j+1}$ is anti-standard. Indeed, the redex u_i followed by the path $u_{i+1} \cdots u_{j-1} \cdot u'_j$ permutes reversibly to the path $v_{i+1} \cdots v_{j-1} \cdot v'_j$ followed by the redex w', and we establish now that the redexes w and u'_{j+1} induce together an irreversible permutation starting

from the path $w' \cdot u'_{j-1}$. The path $w \cdot u'_j$ grabs the redex u'_{j+1} to the redex x. By Lemma 10 (*preservation of drag*), the path $v'_j \cdot w'$ which is Lévy equivalent to the path $w \cdot u'_j$, drags the redex u'_{j+1} to the redex x. This ensures that the two redexes w' and u'_{j+1} induce together a permutation starting from the rewriting path $w' \cdot u'_{j+1}$. There remains to show that this permutation is irreversible. We proceed by contradiction and suppose that the permutation $v' \cdot u'_j \rhd v'_j \cdot h''$ is reversible. Then, it follows from Axiom **reversible-stability** applied around the permutation $w \cdot u'_j \diamondsuit v'_j \cdot w'$ that there exists a reversible permutation starting from the rewriting path $w \cdot y$. This together with Axiom **reversibility** contradicts the existence of the irreversible permutation $w \cdot y \blacktriangleright x \cdot h$ which starts also from the rewriting path $w \cdot y$. We conclude that, as claimed, the two redexes w' and u'_{j+1} are involved in an irreversible permutation starting from the rewriting path $w' \cdot u'_{j+1}$. Thus, the rewriting path $u_i \cdots u_{j-1} \cdot u'_j \cdot u'_{j+1}$ is anti-standard. This concludes the proof, with $(I, J) = (i, j+1)$.

Conclusion: we have just established that when a path f contains an anti-standard path, then every path g equivalent to the path f modulo reversible permutations $\simeq$ contains also an anti-standard path. Lemma 19 follows immediately. □

Lemma 20 (interface) *Suppose that two paths $f : M \longrightarrow\!\!\!\!\!\rightarrow P$ and $g : P \longrightarrow\!\!\!\!\!\rightarrow N$ are standard. Then, the composite path $f \cdot g : M \longrightarrow\!\!\!\!\!\rightarrow N$ is standard if and only if the path $u \cdot g$ is standard, for every redex u which stops f.*

Proof. Follows immediately from Lemma 19. □

4 The Standardization Theorem

All along this section, we suppose that the 2-dimensional transition system $(\mathcal{G}, \rhd)$ defines an axiomatic rewriting system — equivalenly, that it satisfies the nine axioms formulated in Section 2. From this assumption, we deduce the diagrammatic standardization theorem (Theorem 2) evocated in the Introduction — in Section 1.8.

4.1 The outermost redex

For every nonempty path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$, we define a redex $outm(f) : M \longrightarrow P$ extractible from the path f, in these sense of Definition 11. This redex is called the *outermost* redex of the rewriting path f. We will see at the later stage of the proof that the redex $outm(f)$ is the first redex of a particular standard path g associated to the path f. The definition of the redex $outm(f)$ is by induction on the length of the path f.

Definition 21 (outermost redex) *For every non-empty path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$, the redex $outm(f)$ is defined as follows:*

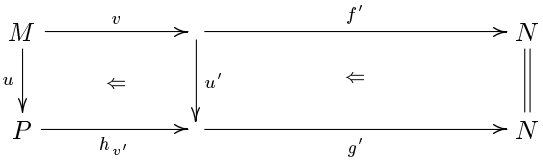
$$outm(v) = v \quad \text{for a redex } v,$$
$$outm(v \cdot f) = \begin{cases} u \text{ when the redex } v \text{ drags the redex } outm(f) \text{ to the redex } u, \\ v \text{ when there is no permutation of the form } v \cdot outm(f) \rhd h. \end{cases}$$

Lemma 22 (preservation of outermost) *Let $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ be a path. Suppose that $u : M \longrightarrow P$ is a redex extractible from f, and that g is a projection of f by extraction of u. Then,*

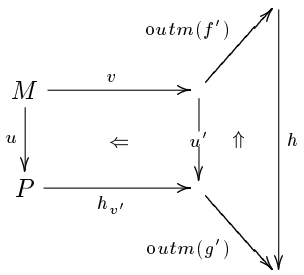
- *either $outm(f) = u$,*
- *or the path g is nonempty, and $outm(g) \overset{u}{\longleftarrow\!\!\!\!\!\shortmid}\, outm(f)$.*

Proof. By induction on the length of the path f. The property is immediate when the path f is a redex. Otherwise, suppose that the path f factors as $f = v \cdot f'$ where v is a redex and where f' is a nonempty path satisfying the property stated in the lemma. Suppose moreover that the redex u is extractible from the path f, and that $f \searrow_u g$ (see Definition 11 for a definition of the notation $\searrow_u$.)

We proceed by case analysis, depending whether the two redexes u and v coincide.
∘ Suppose that $u = v$, and thus, that the redex u is the first redex rewritten in the path f. Then, by definition of the redex $outm(-)$, either $u = outm(f)$ or $outm(f') \overset{u}{\longleftarrow\!\!\!\!\!\shortmid}\, outm(f)$. We conclude because the equality $f' = g$ holds.
∘ Suppose now that $u \neq v$. By definition of $f \searrow_u g$, there exists a redex u' and two paths $h_{v'}$ and g' such that (1) the path g factors as $g = h_{v'} \cdot g'$, and (2) $f' \searrow_{u'} g'$ and (3) $v \cdot u' \rhd u \cdot h_{v'}$. The situation is summarized in the diagram below:



Since the proof is finished when $outm(f) = u$, we suppose from now on that $outm(f) \neq u$. From this follows that $outm(f') \neq u'$ by definition of $outm(-)$ and by Axiom **ancestor**. Here, we apply our induction hypothesis on the path f', and deduce that $outm(f') \overset{u'}{\longleftarrow\!\!\!\!\!\shortmid}\, outm(g')$. The diagram below describes the situation:



From now on, we proceed by case analysis on the permutation $v \cdot u' \rhd u \cdot h_{v'}$.

— Either the permutation $v \cdot u' \rhd u \cdot h_{v'}$ is irreversible. In that case, we apply Axiom **enclave**, and deduce that

1. the redex v drags the redex $outm(f')$ to the redex $outm(g)$, and

2. the path $h_{v'}$ drags the redex $outm(g')$ to the redex $outm(g)$, and
3. the redex u drags the redex $outm(g)$ to the redex $outm(f)$.

The third assertion concludes the proof.

— Or the permutation $v \cdot u' \rhd u \cdot h_{v'}$ is reversible. In that case, the path $h_{v'}$ is a redex. We write it v' for clarity's sake. Again, we proceed by case analysis, depending on whether the redex v coincides with the redex $outm(f)$.

1. Suppose that the redex v does not coincide with $outm(f)$. By definition of $outm(-)$, the redex v drags the redex $outm(f')$ to the redex $outm(f)$. From this follows that the path $v \cdot u'$ drags the redex $outm(g')$ to the redex $outm(f)$. By Lemma 10 (*preservation of drag*), the path $u \cdot v'$ which is Lévy equivalent to the path $v \cdot u'$, the path $u \cdot v'$ drags the redex $outm(g')$ to the redex $outm(f)$. From this follows that the redex v' drags the redex $outm(g')$ to the redex $outm(g)$, and that the redex u drags the redex $outm(g)$ to the redex $outm(f)$. This concludes the proof.
2. Suppose that the redex v is equal to the redex $outm(f)$. In that case, we claim that the redex v' coincides with the redex $outm(g)$. We proceed by contradiction and suppose that $v' \neq outm(g)$. By definition of $outm(-)$, the redex v' drags the redex $outm(g')$ to the redex $outm(g)$. It follows from Axiom **stability** applied around the reversible permutation $v \cdot u' \diamondsuit u \cdot v'$, that the redex v drags the redex $outm(f')$ to a redex w. This contradicts the equality $v = outm(f)$. We conclude that $v' = outm(g)$, and thus, that the redex u drags the redex $v' = outm(g)$ to the redex $v = outm(f)$. We conclude.

All this concludes our proof by induction on the length of the path f. □

Lemma 23 *Let $f : M \twoheadrightarrow N$ be a path. The redex $outm(f)$ is extractible from any path $u_1 \cdots u_n : M \twoheadrightarrow N$ obtained as follows:*

$$f \searrow_{u_1} f_2 \searrow_{u_2} \cdots f_n \searrow_{u_n} \mathrm{id}_N.$$

Proof. Immediate consequence of Lemma 22. □

4.2 Uniqueness

Lemma 24 *Suppose that $(M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1})$ is a standard path. Suppose moreover that, for every index $1 \leq i \leq n$, the path $u_i \cdots u_n$ is more standard than every path in its Lévy equivalence class:*

$$\forall\, 1 \leq i \leq n,\ \ \forall h : M_i \twoheadrightarrow M_{n+1}, \qquad h \equiv u_i \cdots u_n \ \textit{ implies } \ h \Longrightarrow u_i \cdots u_n.$$

Then, for every path $f_1 : M_1 \twoheadrightarrow M_{n+1}$ Lévy equivalent to the path $u_1 \cdots u_n$, there exists a series of rewriting paths $f_i : M_i \twoheadrightarrow M_{n+1}$ indexed by $1 \leq i \leq n$ and a sequence of extractions:

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} \cdots f_n \searrow_{u_n} \mathrm{id}_{M_{n+1}}.$$

Proof. We proceed by induction on the length n of the rewriting path $u_1 \cdots u_n$. Suppose that $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a rewriting path Lévy equivalent to the path $u_1 \cdots u_n$. Note that the redex u_1 is extractible from the path $u_1 \cdots u_n$ with resulting projection the path $u_2 \cdots u_n$. Now, by hypothesis, the path $u_1 \cdots u_n$ is more standard than the path f. From this and Lemma 12 (*preservation of extraction*) follows that the redex u_1 is extractible from the path $f_1 = f$ with projection a path f_2 Lévy equivalent to the path $u_2 \cdots u_n$. We know by induction that there exists a sequence of extractions

$$f_2 \searrow_{u_2} f_3 \searrow_{u_3} \cdots f_n \searrow_{u_n} \mathrm{id}_{M_{n+1}}.$$

We have thus established that there exists a sequence of extractions

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} \cdots f_n \searrow_{u_n} \mathrm{id}_{M_{n+1}}.$$

This concludes our proof by induction. □

Lemma 25 (uniqueness) *A standard path is more standard than every path in its Lévy equivalence class.*

Proof. We proceed by induction on the length of the standard path. Suppose from now on that the property is satisfied for every path of length $n-1$, and suppose that

$$f \;=\; (M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1})$$

is a standard path of length n. We establish that the path f is more standard than every path in its Lévy equivalence class.

Step 1. First of all, we claim that in order to establish that property of the path f, we only need to show that the redex u_1 is extractible from every path Lévy equivalent to the path f. Suppose indeed that this is the case, and consider a path g Lévy equivalent to the standard path f. By definition of Lévy equivalence, there exists a sequence of permutations

$$f = f_1 \overset{1}{\equiv} f_2 \overset{1}{\equiv} \cdots \overset{1}{\equiv} f_m \overset{1}{\equiv} f_{m+1} = g$$

of standardization steps $f_i \overset{1}{\Longrightarrow} f_{i+1}$ or $f_i \overset{1}{\Longleftarrow} f_{i+1}$, for every $1 \leq i \leq m$. For each such index i, the rewriting path f_i is Lévy equivalent to the path f. We have just assumed that the redex u_1 is thus extractible from each path f_i. Now, we may apply Lemma 12 (*preservation of extraction*) as many times as there are permutation steps from the path f to the path g to deduce that the two paths f and g have the same projections (modulo Lévy equivalence) after extraction of the redex u_1. Now, the path $u_2 \cdots u_n$ is the unique projection of the path f by extraction of the redex u_1. We conclude that any projection g' of the rewriting path g obtained by extraction of the redex u_1 is Lévy equivalent to the path $u_2 \cdots u_n$. By applying our induction hypothesis on the path $u_2 \cdots u_n$, we know that the path $u_2 \cdots u_n$ is more standard than the path g'. It follows that the path $f = u_1 \cdots u_n$ is more standard than the path $u_1 \cdot g'$, which is, by construction, more standard than the path g. This establishes that the path f is more standard than every path in its Lévy equivalence class.

Step 2. We have just shown in Step 1. that we only need to prove here that the redex u_1 is extractible from every path Lévy equivalent to the path $f = u_1 \cdots u_n$. We introduce

the necessary notation to that purpose. The proof proceeds by contradiction. We suppose that the redex u_1 is not extractible from a particular path in the Lévy equivalence class of the path f. By definition of Lévy equivalence, there exists a sequence

$$f_1 \overset{1}{\equiv} f_2 \overset{1}{\equiv} \cdots \overset{1}{\equiv} f_m \overset{1}{\equiv} f_{m+1}$$

of standardization steps $f_i \overset{1}{\Longrightarrow} f_{i+1}$ or $f_i \overset{1}{\Longleftarrow} f_{i+1}$, for every $1 \leq i \leq m$, such that:

- $f_1 = f$,
- the redex u_1 is extractible from the path f_j, for every index $1 \leq j \leq m$,
- the redex u_1 is not extractible from the path f_{m+1}.

For each index $1 \leq i \leq m$, we define the path g_i as any projection of the path f_i by extraction of the redex u_1. So,

$$\forall 1 \leq i \leq m, \qquad f_i \searrow_{u_1} g_i.$$

Note that Lemma 12 (*preservation of extraction*) implies that all the paths $g_1 = u_2 \cdots u_n$, and g_2, ... , g_m are Lévy equivalent.

Step 3. Here, we will be slightly more explicit than in Step 2. Let p denote the length of the path f_m. Thus, the path f_m factors as

$$f_m = v_1 \cdots v_p$$

where each v_i denotes a redex, for $1 \leq i \leq p$. We know by construction that $f_m \overset{1}{\equiv} f_{m+1}$. It follows from Lemma 12 (*preservation of extraction*) that in fact

$$f_m \overset{1}{\Longrightarrow} f_{m+1}$$

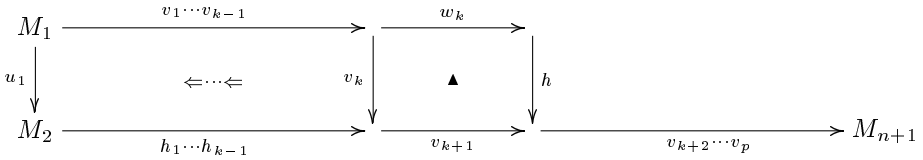
because the redex u_1 is extractible from the path f_m but not from the path f_{m+1}. By definition of $\overset{1}{\Longrightarrow}$, the paths f_m and f_{m+1} factor as:

$$f_m = v_1 \cdots v_{k-1} \cdot (v_k \cdot v_{k+1}) \cdot v_{k+2} \cdots v_p \qquad f_{m+1} = v_1 \cdots v_{k-1} \cdot (w_k \cdot h) \cdot v_{k+2} \cdots v_p$$

for some index $1 \leq k \leq p-1$, where w_k is a redex and h is a path involved in a permutation $v_k \cdot v_{k+1} \rhd w_k \cdot h$. Now, it follows from Lemma 10 (*preservation of drag*) and Axiom **ancestor** that:

- the permutation $v_k \cdot v_{k+1} \blacktriangleright w_k \cdot h$ is irreversible,
- the path $v_1 \cdots v_{k-1}$ drags the redex v_k to the redex u_1.

The situation is summarized in the diagram below:



Step 4. We establish the equality $outm(f_m) = outm(f_{m+1})$. We proceed by case analysis, depending whether the redex v_{k+1} coincides with the redex $outm(v_{k+1} \cdots v_p)$.

– Suppose that the redex v_{k+1} is not equal to the redex $outm(v_{k+1} \cdots v_p)$. By Lemma 22, the path $v_{k+2} \cdots v_p$ is nonempty, and the redex v_{k+1} drags the redex $outm(v_{k+2} \cdots v_p)$ to the redex $outm(v_{k+1} \cdots v_p)$. By Axiom **enclave** applied around the irreversible permutation $v_k \cdot v_{k+1} \blacktriangleright w_k \cdot h$, the two paths $v_k \cdot v_{k+1}$ and $w_k \cdot h$ drag the redex $outm(v_{k+2} \cdots v_p)$ to the same redex $outm(v_k \cdots v_p) = outm(w_k \cdot h \cdot v_{k+2} \cdots v_p)$. The inductive definition of $outm(-)$ ensures then that $outm(f_m) = outm(f_{m+1})$. We conclude.
– Suppose now that the redex v_{k+1} coincides with the redex $outm(v_{k+1} \cdots v_p)$. In that case, $outm(v_k \cdots v_p) = w_k$ because the redex v_k drags the redex $v_{k+1} = outm(v_{k+1} \cdots v_p)$ to the redex w_k. Now, we claim that $outm(w_k \cdot h \cdot v_{k+2} \cdots v_p) = w_k$. First of all, it follows from Axioms **ancestor** and **irreversibility** and from $v_k \cdot v_{k+1} \blacktriangleright w_k \cdot h$ that the redex w_k is the only redex extractible from the path $w_k \cdot h$. So, there only remains to prove that the redex $outm(w_k \cdot h \cdot v_{k+2} \cdots v_p)$ is extractible from the path $w_k \cdot h$. Suppose that it is not. In that case, the path $w_k \cdot h$ drags the redex $outm(v_{k+2} \cdots v_p)$ to the redex $outm(w_k \cdot h \cdot v_{k+2} \cdots v_p)$. By Lemma 10 (*preservation of drag*) the path $v_k \cdot v_{k+1}$ which is Lévy equivalent to the path $w_k \cdot h$, drags the redex $outm(v_{k+2} \cdots v_p)$ to the same redex $outm(v_k \cdots v_p) = outm(w_k \cdot h \cdot v_{k+2} \cdots v_p)$. This contradicts the equality $w_k = outm(v_k \cdots v_p) = v_{k+1}$. We conclude that $outm(v_k \cdots v_p) = w_k = outm(w_k \cdot h \cdot v_{k+2} \cdots v_p)$ and thus that $outm(f_m) = outm(f_{m+1})$.

Step 5. We deduce from Step 4 that the redex u_1 drags the redex $outm(g_m)$ to the redex $outm(f_m)$. We have just proved that $outm(f_m) = outm(f_{m+1})$. From this follows that the redex $outm(f_m)$ is extractible from the path f_{m+1}. Since by construction of the path f_{m+1}, the redex u_1 is not extractible from that path, the two redexes u_1 and $outm(f_m)$ are necessarily different. We may thus apply Lemma 22 on the extraction $f_m \searrow_{u_1} g_m$. This establishes our claim: the redex u_1 drags the redex $outm(g_m)$ to the redex $outm(f_m)$.

Step 6. We prove that the redex $outm(g_m)$ is extractible from the path $g_1 = u_2 \cdots u_n$. By induction hypothesis, each path $u_i \cdots u_n$ is more standard than any of its Lévy equivalent paths, for $2 \leq i \leq n$. We may thus apply Lemma 24 to the paths g_1 and $u_2 \cdots u_n$, and deduce that there exists a series of extractions

$$g_1 \searrow_{u_2} \cdots \searrow_{u_n} \mathrm{id}_{M_{n+1}}.$$

By Lemma 23, the series implies that the redex $outm(g_m)$ is extractible from the path $u_2 \cdots u_n$.

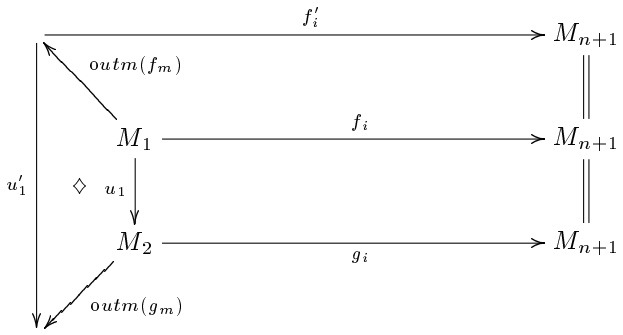
Step 7. We deduce from Step 6 that the redex $outm(g_m)$ is extractible from all the paths $g_1, ..., g_m$. We have already noted at the end of Step 2 that all the paths $g_1 = u_2 \cdots u_n$, g_2, ..., g_m are Lévy equivalent. By induction hypothesis, the standard path $g_1 = u_2 \cdots u_n$ is more standard than every path g_i, for every index $1 \leq i \leq m$. We also know that the redex $outm(g_m)$ is extractible from the path g_1. By Lemma 12 (*preservation of extraction*), the redex $outm(g_m)$ is thus extractible from the path g_i, for every index $1 \leq i \leq m$.

Step 8. We deduce from Steps 4, 5 and 7 that the redex $outm(f_m)$ is extractible from the paths $f_1, ..., f_m, f_{m+1}$. By Step 4, the redex $outm(f_m)$ is extractible from the path f_{m+1}. So, there remains to show that the redex $outm(f_m)$ is extractible from

the paths $f_1, ..., f_m$. By Step 5, the redex u_1 drags the redex $outm(g_m)$ to the redex $outm(f_m)$. By Step 7, the redex $outm(g_m)$ is extractible from all the paths g_1, ..., g_m. From this follows that the redex g_m is extractible from the paths $u_1 \cdot g_1$, ..., $u_1 \cdot g_m$. Now, for every index $1 \leq i \leq m$, the path $u_1 \cdot g_i$ is more standard than the path f_i because $f_i \searrow_{u_1} g_i$. We conclude by Lemma 12 (*preservation of extraction*) that the redex $outm(f_m)$ is extractible from the paths $f_1, ..., f_m$.

Step 9. By Step 8, we may define for every index $1 \leq i \leq m+1$ the path f'_i as an (arbitrary) projection of the path f_i by extraction of $outm(f_m)$. We thus have $f_i \searrow_{outm(f_m)} f'_i$. By Lemma 12 (*preservation of extraction*) applied m times, the rewriting paths f'_1 ,..., f'_{m+1} are Lévy equivalent.

Step 10. In order to reach a contradiction with our hypothesis, we prove that the redex u_1 is extractible from the rewriting path f_{m+1}. We have already noted in Step 9 that the paths $f'_1, ..., f'_{m+1}$ are Lévy equivalent. The path f'_1 is standard of length $n-1$ since it is defined as the projection of the standard path $f_1 = u_1 \cdots u_n$ by extraction of the redex $outm(f_m)$. By induction hypothesis, the path f'_1 is more standard than all the paths $f'_1, ..., f'_{m+1}$. Besides, the rewriting path f'_1 is not empty. We have proved indeed in Step 5 that the redexes u_1 and $outm(f_m)$ are different redexes, and more precisely, that the redex u_1 drags the redex $outm(g_m)$ to the redex $outm(f_m)$. From this follows that the extraction of the redex $outm(f_m)$ from the standard path $f_1 = u_1 \cdots u_n$ induces a reversible permutation $u_1 \cdot outm(g_m) \diamondsuit outm(f_m) \cdot u'_1$. The redex u'_1 is the first redex of the path f'_1, and the path f'_1 is more standard than all the paths $f'_1, ..., f'_{m+1}$. By Lemma 12 (*preservation of extraction*), the redex u'_1 is extractible from all the paths $f'_1, ..., f'_{m+1}$. The diagram below summarizes the situation:



All this has the remarkable consequence that the redex u'_1 is extractible from the rewriting path f'_{m+1}. From this follows that the redex u_1 is extractible from the rewriting path $outm(f_m) \cdot f'_{m+1}$. Now, the path $outm(f_m) \cdot f'_{m+1}$ is more standard than the path f_{m+1} by definition of $f_{m+1} \searrow_{outm(f_m)} f'_{m+1}$. We conclude by Lemma 12 (*preservation of extraction*) that the redex u_1 is extractible from the rewriting path f_{m+1}.

Step 11. This is the concluding step. We deduce from the contradiction reached in Step 10 that the redex u_1 is extractible from every path Lévy equivalent to the rewriting path f. By the preliminary discussion of Step 1, this concludes our proof by induction of Lemma 25. □

4.3 Existence

Lemma 26 (towards existence) *Suppose that $f : M_1 \twoheadrightarrow M_{n+1}$ is a non-empty path whose projection by extraction of the redex $outm(f) : M_1 \longrightarrow M_2$ is Lévy equivalent to a standard path*

$$M_2 \xrightarrow{u_2} M_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1}.$$

Then, the rewriting path

$$M_1 \xrightarrow{outm(f)} M_2 \xrightarrow{u_2} M_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1}$$

is standard.

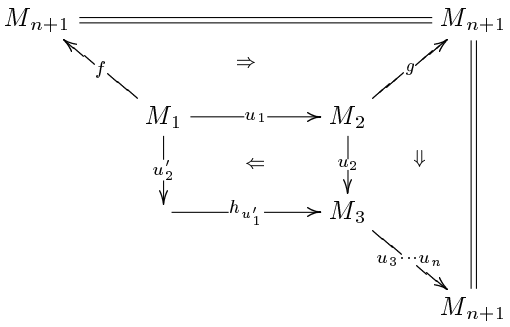
Proof. By induction on n. The lemma is immediate when $n = 1$ because the path $outm(f)$ is standard, like every path of length 1. Suppose that the property is established for every standard path of length $n-2$, and consider a standard path

$$M_2 \xrightarrow{u_2} M_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1}$$

of length $n-1$. Consider moreover a nonempty path $f : M_1 \twoheadrightarrow M_{n+1}$, and suppose that (one of) its projection g by extraction of the redex $outm(f) : M_1 \longrightarrow M_2$ is Lévy equivalent to the standard path $u_2 \cdots u_n$. We write u_1 for the redex $outm(f)$.

We want to prove that the path $u_1 \cdot u_2 \cdots u_n$ is standard. We proceed by contradiction, and suppose that the path $u_1 \cdot u_2 \cdots u_n$ is *not* standard. By Lemma 19 (*characterization lemma*) there exists an anti-standard path inside the rewriting path $u_1 \cdot u_2 \cdots u_n$. Since the path $u_2 \cdots u_n$ is standard, this anti-standard path is necessarily of the form $u_1 \cdots u_{k+1}$ for some index $1 \leq k \leq n-1$.

By definition of an anti-standard path, and whatever the value of the index k, there exists a redex u_2' and a path $h_{u_1'}$ forming a permutation $u_1 \cdot u_2 \rhd u_2' \cdot h_{u_1'}$. The situation is summarized in the the diagram below:



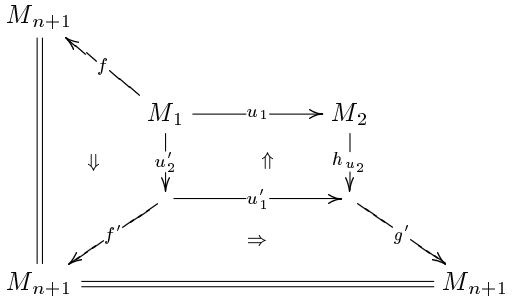
(20)

We show in Steps 2, 3, 4, 5 and 6 that the permutation $u_1 \cdot u_2 \rhd u_2' \cdot h_{u_1'}$ is reversible, or equivalently, that $k \geq 2$.

Step 2. We show that the redex u_2' is extractible from the path f. By Lemma 25 (*uniqueness*), the path $u_2 \cdots u_n$ is more standard than every Lévy equivalent path. In particular,

the path $u_2 \cdots u_n$ is more standard than the path g. It follows from Lemma 12 (*preservation of extraction*) that the redex u_2 which is extractible from the path $u_2 \cdots u_n$ is also extractible from the path g. This and the existence of the permutation $u_1 \cdot u_2 \rhd u'_2 \cdot h_{u'_1}$ implies that the redex u'_2 is extractible from the path $u_1 \cdot g$. The path $u_1 \cdot g$ is more standard than the path f by definition of extraction $f \searrow_{u_1} g$. Thus, by applying Lemma 12 (*preservation of extraction*) again, the redex u'_2 is extractible from the path f.

Step 3. Let the path f' denote an arbitrary projection of the path f by extraction of the redex u'_2. By construction, and Axiom **shape**, the redex u'_2 does not coincide with the redex $outm(f) = u_1$. By Lemma 22, the path f' is non-empty and the redex u'_2 drags the redex $outm(f')$ (denoted u'_1 from now) to the redex $u_1 = outm(f)$. More explicitly, the two redexes u'_1 and u'_2 are involved in a permutation $u'_2 \cdot u'_1 \rhd u_1 \cdot h_{u_2}$ for some path h_{u_2}. Let the path g' denote an arbitrary projection of the path f' by extraction of the redex u'_1. The situation is summarized in the diagram below:



(21)

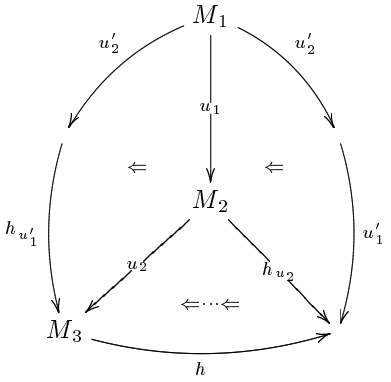
In the next Steps 4–7, we analyze the relationship between the two Diagrams (20) and (21). We establish in Steps 4–6 that the paths $h_{u'_1}$ and h_{u_2} coincide respectively with the redexes u'_1 and u_2, and thus, that the permutation $u_1 \cdot u_2 \rhd u'_2 \cdot h_{u'_1}$ is reversible. We establish in Step 7 that the path g' is Lévy equivalent to the path $u_3 \cdots u_n$. This enables to combine the two Diagrams (20) and (21) in a larger diagram.

Step 4. Here, we deduce from Lemma 25 (*uniqueness*) that the redex u_2 is extractible from the path $h_{u_2} \cdot g'$. By construction, the path $u_1 \cdot h_{u_2} \cdot g'$ is more standard than the path f. The paths $h_{u_2} \cdot g'$ and g are the projections of the paths $u_1 \cdot h_{u_2} \cdot g'$ and f by extraction of the redex u_1, respectively. By Lemma 12 (*preservation of extraction*), the two paths $h_{u_2} \cdot g'$ and g are Lévy equivalent. Now, the path g is also Lévy equivalent to the standard path $u_2 \cdots u_n$. From this and Lemma 25 (*uniqueness*) follows that the path $u_2 \cdots u_n$ is more standard than the path $h_{u_2} \cdot g'$. By Lemma 12 (*preservation of extraction*), we conclude that the redex u_2 is extractible from the path $h_{u_2} \cdot g'$.

Step 5. We deduce from Step 4 that the redex u_2 is extractible from the path h_{u_2}. We proceed by contradiction, and suppose that it is not. The redex u_2 is extractible from the path $h_{u_2} \cdot g'$. By definition of extraction, there exists a redex v extractible from the path g' such that the path h_{u_2} drags the redex v to the redex u_2. From this follows that the path $u_1 \cdot h_{u_2}$ drags the redex v to the redex u'_2. Now, the path $u_1 \cdot h_{u_2}$ is Lévy equivalent to the path $u'_2 \cdot u'_1$. By Lemma 10 (*preservation of drag*), the path $u'_2 \cdot u'_1$ drags the redex v to the redex u'_2. More explicitly, there exists a redex w such that: (a) the redex u'_1 drags the redex v to the redex w; and (b) the redex u'_2 drags the redex w to the redex u'_2. This very last statement (b) contradicts the Axiom **shape** since it implies

that there exists a path h and permutation $u'_2 \cdot w \rhd u'_2 \cdot h$. We conclude that the redex u_2 is extractible from the path h_{u_2}.

Step 6. We deduce from Step 5 that the paths $h_{u'_1}$ and h_{u_2} coincide respectively with the redexes u'_1 and u_2, and that the permutation $u_1 \cdot u_2 \rhd u'_2 \cdot h_{u'_1}$ is reversible. By definition of extraction, there exists a path h such that $h_{u_2} \Longrightarrow u_2 \cdot h$. From this follows that $u'_2 \cdot u'_1 \rhd u_1 \cdot h_{u_2}$ and $u_1 \cdot h_{u_2} \Longrightarrow u'_2 \cdot h_{u'_1} \cdot h$. Diagrammatically,



Suppose that the permutation $u'_2 \cdot u'_1 \rhd u_1 \cdot h_{u_2}$ is irreversible. In that case, it follows from Axiom **irreversibility** that $u'_2 \cdot u'_1 \rhd u'_2 \cdot h_{u'_1} \cdot h$. This last statement contradicts Axiom **shape**, and we thus conclude that the permutation $u'_2 \cdot u'_1 \rhd u_1 \cdot h_{u_2}$ is reversible. From this follows that the path h_{u_2} is a redex. The equality $h_{u_2} = u_2$ follows immediately from the fact that the redex u_2 is extractible from the path h_{u_2}. We conclude that $u'_2 \cdot u'_1 \rhd u_1 \cdot u_2$. At this point, there only remains to apply Axiom **reversibility** on the permutations $u'_2 \cdot u'_1 \rhd u_1 \cdot u_2$. $u_1 \cdot u_2 \rhd u'_2 \cdot h_{u'_1}$, from which we deduce that $h_{u'_1} = u'_1$ and that the permutation $u_1 \cdot u_2 \rhd u'_2 \cdot h_{u'_1}$ is reversible.

Step 7. We have just established that the permutation $u_1 \cdot u_2 \diamondsuit u'_2 \cdot u'_1$ is reversible. In Step 4, we have also proved that $u_2 \cdots u_n$ is more standard than the path $h_{u_2} \cdot g'$. We know now that the path $h_{u_2} \cdot g'$ is equal to the path $u_2 \cdot g'$. The two paths $u_3 \cdots u_n$ and g' are respectively the projections of the paths $u_2 \cdots u_n$ and $u_2 \cdot g'$ by extraction of the redex u_2. By Lemma 12 (*preservation of extraction*), the path g' is Lévy equivalent to the path $u_3 \cdots u_n$.

Step 8. We have just established in Step 7 that the projection g' of the path f' by extraction of the redex $u'_1 = outm(f')$ is Lévy equivalent to the path $u_3 \cdots u_n$. This enables to apply our induction hypothesis on the standard path $u_3 \cdots u_n$. We deduce that the path $u'_1 \cdot u_3 \cdots u_n$ is standard. In particular, the path $u'_1 \cdot u_3 \cdots u_{k+1}$ is *not* anti-standard. From this follows that the path $u_1 \cdot u_2 \cdots u_{k+1}$ is *not* anti-standard. This contradicts our original hypothesis. The path $u_1 \cdot u_2 \cdots u_n$ is thus standard. This concludes the reasoning by induction, and the proof of Lemma 4.3. □

Lemma 27 (existence) *For every path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ there exists a standard path $g : M \longrightarrow\!\!\!\!\!\rightarrow N$ such that $f \Longrightarrow g$.*

Proof. First, we show that every rewriting path $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ is standard when it is obtained as a sequence of extractions from a path $f_1 : M \longrightarrow\!\!\!\!\!\rightarrow N$:

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} f_3 \cdots f_n \searrow_{u_n} \mathrm{id}_N \tag{22}$$

where $u_i = outm(f_i)$ for every index $1 \leq k \leq n$. The proof is nearly immediate, by induction on the length n. Suppose that the property is established for every path of length $n-1$, and consider a path $u_1 \cdots u_n$ obtained as a series of extractions (22). By induction hypothesis, the path

$$f_2 \searrow_{u_2} f_3 \searrow_{u_3} f_4 \cdots f_n \searrow_{u_n} \mathrm{id}_N$$

is standard. By Lemma 26, the path $u_1 \cdot u_2 \cdots u_n = outm(f_1) \cdot u_2 \cdots u_n$ is also standard. We conclude.

Now, suppose that $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ is an arbitrary rewriting path. By Axiom **termination**, every sequence of extractions

$$f = f_1 \searrow_{outm(f_1)} f_2 \searrow_{outm(f_2)} f_3 \cdots f_n \searrow_{outm(f_n)} \cdots$$

is finite. Thus, there exists an index n such that

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} f_3 \cdots f_n \searrow_{u_n} \mathrm{id}_N$$

where $u_i = outm(f_i)$, for all $1 \leq i \leq n$. By construction, the path $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ is more standard than the path f, and it is standard by the previous argument. We conclude. □

4.4 Standardization theorem

Theorem 2 (standardization) *Suppose that $(\mathcal{G}, \rhd)$ is an axiomatic rewriting system and that $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a path in the transition system $\mathcal{G}$. Then:*

- *there exists a standard path $g : M \longrightarrow\!\!\!\!\!\rightarrow N$ more standard than f,*
- *every standard path Lévy equivalent to f is equal to g modulo reversible permutation equivalence $\simeq$.*

The standard path of any path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ may be computed by extracting recursively the outermost redex $outm(f_i)$ in a sequence of rewriting paths

$$f = f_1 \searrow_{outm(f_1)} f_2 \searrow_{outm(f_2)} f_3 \searrow_{outm(f_3)} \quad \cdots \quad f_n \searrow_{outm(f_n)} \mathrm{id}_N.$$

We call this algorithm **STD** as in [13]. Note that the algorithm is non deterministic because it depends at each step f_i on the choice of the next rewriting path f_{i+1}.

Corollary 28 *The relation $\Longrightarrow$ on paths is confluent modulo $\simeq$. The $\Longrightarrow$-normal form of a path is computed by the algorithm* **STD**.

5 Standardization from the 2-Categorical Point of View

In Sections 1—4. we interpret standardization as a 2-dimensional rewriting procedure on 1-dimensional paths, and establish a confluence and normalization property for that procedure. However, we say nothing there about the 2-dimensional reductions $f \Longrightarrow g$ themselves. Intuitively, each such reduction $f \Longrightarrow g$ describes a possible way to tile the 2-dimensional *surface* lying between the two rewriting paths f and g. In this section is to show that all tilings $f \Longrightarrow g$ from a path f to its standard path g, are equivalent in an intuitive sense. We refer the reader to the last chapter of [25] (second edition) for a nice and motivated introduction to 2-categories.

5.1 Tiling graph, tiling paths, and partial injections

To every 2-dimensional transition system $(\mathcal{G}, \triangleright)$ we associate a *tiling graph* in the following way:

Definition 29 (tiling graph, path, step) *The graph* ***tiling-graph***$(\mathcal{G}, \triangleright)$ *has the paths of* $\mathcal{G}$ *as vertices, and the standardization steps* $(e, f \triangleright g, h)$ *as edges* $e \cdot f \cdot h \Longrightarrow e{\cdot}g{\cdot}h$. *The paths in* ***tiling-graph***$(\mathcal{G}, \triangleright)$ *are called* tiling paths *to avoid confusion with the* rewriting paths *of the transition system* $\mathcal{G}$. *According to that spirit, we often call* tiling step *a standardization step. In the graph* ***tiling-graph***$(\mathcal{G}, \triangleright)$, *we write* $\mathrm{id}^f : f \Longrightarrow f$ *for the identity of* f, *and* $\alpha * \beta : f \Longrightarrow h$ *for the composite of two paths* $\alpha : f \Longrightarrow g$ *and* $\beta : g \Longrightarrow h$.

Definition 30 (canonical equivalence on tiling path) *To every tiling path* $\alpha : f \Longrightarrow g$, *we associate a partial injection* $[\alpha] : [g] \rightharpoonup [f]$ *as follows.*

- *to every vertex of* ***tiling-graph***$(\mathcal{G}, \triangleright)$ *we associate the finite set* $[f] = \{1, ..., n\}$ *of cardinal* n *the length of* f *as 1-dimensional path,*
- *to every edge* $\alpha = (e, f \triangleright g, h)$ *of* ***tiling-graph***$(\mathcal{G}, \triangleright)$ *where* e, f, g *and* h *decompose as:*

$$e = u_1 \cdots u_m \qquad f = v \cdot u' \qquad g = v_1 \cdots v_n \qquad h = w_1 \cdots w_p$$

we associate the partial injection $[\alpha] : [e \cdot g \cdot h] \rightharpoonup [e \cdot f \cdot h]$ *defined as*

- *when* $f \diamondsuit g$*:*

$$\begin{cases} k & \mapsto & k & \text{for every } 1 \leq k \leq m \\ m+1 & \mapsto & m+2 & \\ m+2 & \mapsto & m+1 & \\ m+2+k & \mapsto & m+2+k & \text{for every } 1 \leq k \leq p \end{cases}$$

- *when* $f \blacktriangleright g$*:*

$$\begin{cases} k & \mapsto & k & \text{for every } 1 \leq k \leq m \\ m+1 & \mapsto & m+2 & \\ m+n+k & \mapsto & m+2+k & \text{for every } 1 \leq k \leq p \end{cases}$$

The partial injection $[\alpha] : \{1, ..., n\} \rightharpoonup \{1, ..., m\}$ *associated to a tiling path*

$$\alpha : u_1 \cdots u_m \Longrightarrow v_1 \cdots v_n$$

is defined by composing the partial injections $[\alpha_i]$*'s:*

$$[\alpha] = [\alpha_n] \circ \cdots \circ [\alpha_1]$$

Intuitively, the function $[\alpha]$ *traces every redex* v_k *back to its unique "ancestor"* $u_{[\alpha](k)}$ *in the 1-dimensional path* $u_1 \cdots u_m$*, when this redex exists.*

The main result of the section states that

Theorem 3 *Suppose that* g *is a standard rewriting path in an axiomatic rewriting system* $(\mathcal{G}, \triangleright)$*. Then, every two tiling paths* $\alpha, \beta : f \Longrightarrow g$ *from a rewriting path* f *to the rewriting path* g *define the same partial injection* $[\alpha] = [\beta]$*.*

Reformulated 2-categorically, the theorem states that in the 2-category **2-cat**$(\mathcal{G}, \triangleright)$ defined at the beginning of Section 5.3, the standard path $g : M \longrightarrow\!\!\!\!\!\rightarrow N$ is terminal in its connected component in the hom-category **2-cat**$(\mathcal{G}, \triangleright)(M, N)$. The standard path g is in fact strongly terminal, in the sense that in every cell $g \Longrightarrow h$, the path h is also standard, and thus terminal.

We proceed methodologically, and prove the theorem in two steps. In Section 5.2, we give a series of conditions on an equivalence relation $\cong$ on the paths of **tiling-graph**$(\mathcal{G}, \triangleright)$ to ensure that every two tiling paths $\alpha, \beta : f \Longrightarrow g$ from a path f to a standard path g, are equal modulo $\cong$. In Section 5.3, we prove that the equivalence relation $\alpha \cong \beta$ induced by the equality $[\alpha] = [\beta]$ of partial injections, satisfies the formal conditions of Section 5.2.

Remark: Theorem 3 repeats in dimension 2 the observation by J.-J. Lévy in the λ-calculus, or in any conflict-free (term) rewriting system, that there exists a unique path from a term to its normal form, modulo permutation. Here, objects are 1-dimensional, paths are 2-dimensional, permutations are 3-dimensional — and the concept of a conflict-free 2-dimensional system remains to be clarified.

5.2 Standard=strong terminal

Definition 31 (horizontal composition) *The horizontal composite* $\alpha \cdot h$ *of a tiling step (=standardization step)*

$$\alpha = (e, f \triangleright g, h) : e \cdot f \cdot h \Longrightarrow e \cdot g \cdot h : M \longrightarrow\!\!\!\!\!\rightarrow N$$

and of a 1-dimensional path $h' : N \longrightarrow\!\!\!\!\!\rightarrow P$ *is defined as the tiling step:*

$$\alpha \cdot h = (e, f \triangleright g, h \cdot h') : e \cdot f \cdot h \cdot h' \Longrightarrow e \cdot g \cdot h : M \longrightarrow\!\!\!\!\!\rightarrow P$$

The horizontal composite $\alpha \cdot h$ *of a tiling path*

$$\alpha = \alpha_1 * \cdots * \alpha_n : f \Longrightarrow g : M \longrightarrow\!\!\!\!\!\rightarrow N$$

and a 1-dimensional path $h : N \longrightarrow\!\!\!\!\!\rightarrow P$ is defined as the tiling path

$$\alpha \cdot h = (\alpha_1 \cdot h) * \cdots * (\alpha_n \cdot h) : M \longrightarrow\!\!\!\!\!\rightarrow P$$

The horizontal composite $e \cdot \alpha$ of a 1-dimensional path $e : L \longrightarrow\!\!\!\!\!\rightarrow M$ and a tiling path $\alpha : f \cdot g : M \longrightarrow\!\!\!\!\!\rightarrow N$ is defined symmetrically.

From now on, we consider an equivalence relation $\cong$ between the tiling paths of **tiling-graph**$(\mathcal{G}, \triangleright)$, satisfying the four properties below:

1. for all tiling paths $\alpha : f \Longrightarrow f'$ and $\beta : g \Longrightarrow g'$,

$$\alpha \cong \beta \;\Rightarrow\; f = g \text{ and } f' = g'$$

2. for all tiling paths $\alpha, \alpha' : f \Longrightarrow g$ and $\beta, \beta' : g \Longrightarrow h$,

$$\alpha \cong \alpha' \text{ and } \beta \cong \beta' \;\Rightarrow\; \alpha * \beta \cong \alpha' * \beta'$$

3. for all tiling paths $\alpha, \beta : g \Longrightarrow g' : M \longrightarrow\!\!\!\!\!\rightarrow N$ and all 1-dimensional paths $f : L \longrightarrow\!\!\!\!\!\rightarrow M$ and $h : N \longrightarrow\!\!\!\!\!\rightarrow P$,

$$\alpha \cong \beta \;\Rightarrow\; f \cdot \alpha \cdot h \cong f \cdot \beta \cdot h$$

4. for all of tiling paths $\alpha : f \Longrightarrow f' : M \longrightarrow\!\!\!\!\!\rightarrow N$ and $\beta : g \Longrightarrow g' : N \longrightarrow\!\!\!\!\!\rightarrow P$,

$$(\alpha \cdot g) * (f' \cdot \beta) \cong (f \cdot \beta) * (\alpha \cdot g')$$

Lemma 32 *The equivalence relation $\cong$ defines a 2-category **2-cat**$_\cong(\mathcal{G}, \triangleright)$.*

Proof. The 2-category **2-cat**$_\cong(\mathcal{G}, \triangleright)$ has vertices and paths of $\mathcal{G}$ as objects and morphisms, and equivalence classes modulo $\cong$ of tiling paths as cells. Conditions 1–3. ensure the necessary compositionality properties of **2-cat**$_\cong(\mathcal{G}, \triangleright)$, while condition 4. ensures the so-called interchange law of 2-categories, see [25]. □

Suppose moreover that:

5. for every path $f = u \cdot v$ where u drags the redex v to a redex v_0, and for every standard path g,

$$\forall \alpha, \beta, \qquad \alpha, \beta : f \Longrightarrow g \quad \Rightarrow \quad \alpha \cong \beta$$

6. for every path $f = u \cdot v \cdot w$ where the redex u drags the redex v to a redex v_0, and where the path $u \cdot v$ drags the redex w to a redex w_0, and for every standard path g,

$$\forall \alpha, \beta, \qquad \alpha, \beta : f \Longrightarrow g \quad \Rightarrow \quad \alpha \cong \beta$$

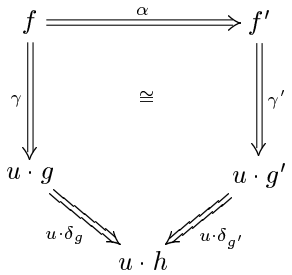
These two additional conditions 5 and 6 regulate the potential *critical pairs* occurring during the 2-dimensional transitions implementing standardization. The lemma below establishes that the two assumptions are sufficient to the purpose.

Lemma 33 *Suppose that the equivalence relation $\cong$ satisfies Conditions 1–6. Then, every standard path $h : M \longrightarrow\!\!\!\!\!\rightarrow N$ is strongly terminal in its connected component in the hom-category **2-cat**$_{\cong}(\mathcal{G}, \rhd)(M, N)$.*

Proof. By induction on the length of $h : M \longrightarrow\!\!\!\!\!\rightarrow N$. Suppose that the property is established for every standard path of length n, and that the path $u \cdot h$ is standard of length $1 + n$. Suppose that f is a path Lévy equivalent to $u \cdot h$. We claim that for every tiling path $\gamma : f \Longrightarrow u \cdot g$ resulting of an extraction $f \searrow_u g$, and for every tiling path $\alpha : f \Longrightarrow f'$ starting from f, there exists a tiling path $\gamma' : f' \Longrightarrow u \cdot g'$ resulting of an extraction $f' \searrow_u g'$, such that

$$\gamma * (u \cdot \delta_g) \;\cong\; \alpha * \gamma' * (u \cdot \delta_{g'}) \quad : f \Longrightarrow u \cdot h \tag{23}$$

where $\delta_g : g \Longrightarrow h$ and $\delta_{g'} : g' \Longrightarrow h$ are arbitrary tiling paths to the terminal object h. To prove the claim, it is sufficient to consider the case when α is a tiling step $(f_1, f_2 \rhd f_2', f_3)$. The general case follows by a straightforward induction on the length of α. So, we want to establish that the diagram below commutes modulo $\cong$ for a tiling step $\alpha = (f_1, f_2 \rhd f_2', f_3) : f \Longrightarrow f'$ and a tiling path $\gamma : f \Longrightarrow u \cdot g$ resulting of an extraction $f \searrow_u g$.



By definition of α, the paths f and f' factor as

$$f = f_1 \cdot f_2 \cdot f_3 \qquad\qquad f' = f_1 \cdot f_2' \cdot f_3.$$

The redex u is extractible from the path $f = f_1 \cdot f_2 \cdot f_3$. One of the three following situations occurs. We say that the redex u is

1. extractible from the component f_1 when the redex u is extractible from the path f_1,
2. extractible from the component f_2 when the redex u is extractible from the path $f_1 \cdot f_2$ but not from the path f_1,
3. extractible from the component f_3 when the redex u is extractible from the path $f_1 \cdot f_2 \cdot f_3$ but not from the path $f_1 \cdot f_2$.

By definition of γ as the tiling path produced by the extraction $f \searrow_u g$, the rewriting paths g and the tiling path γ factor as

$$g = g_1 \cdot g_2 \cdot g_3$$

$$\gamma = (f_1 \cdot f_2 \cdot \gamma_3) * (f_1 \cdot \gamma_2 \cdot g_3) * (\gamma_1 \cdot g_2 \cdot g_3)$$

where the definitions of g_1, g_2, g_3 and $\gamma_1, \gamma_2, \gamma_3$ depend on the component f_1 or f_2 or f_3 from which the redex u is extractible:

1. The redex u is extractible from the component f_1: in that case, $g_3 = f_3$ and $\gamma_3 = \mathrm{id}_{f_3}$, $g_2 = f_2$ and $\gamma_2 = \mathrm{id}_{f_2}$, and $\gamma_1 : f_1 \Longrightarrow u \cdot g_1$ is the result of an extraction $f_1 \searrow_u g_1$,
2. The redex u is extractible from the component f_2: in that case, $g_3 = f_3$ and $\gamma_3 = \mathrm{id}_{f_3}$, $\gamma_2 : f_2 \Longrightarrow u' \cdot g_2$ is the result of an extraction $f_2 \searrow_{u'} g_2$, the path f_1 drags u' to u and $\gamma_1 : f_1 \cdot u' \Longrightarrow u \cdot g_1$ is the result of the extraction $f_1 \cdot u' \searrow_u g_1$,
3. The redex u is extractible from the component f_3: in that case, $\gamma_3 : f_3 \Longrightarrow u'' \cdot g_3$ is the result of an extraction $f_3 \searrow_{u''} g_3$, the path f_2 drags u'' to u' and $\gamma_2 : f_2 \cdot u'' \Longrightarrow u' \cdot g_2$ is the result of the extraction $f_2 \cdot u'' \searrow_{u'} g_2$, the path f_1 drags u' to u and $\gamma_1 : f_1 \cdot u' \Longrightarrow u \cdot g_1$ is the result of the extraction $f_1 \cdot u' \searrow_u g_1$.

The tiling path γ' is defined as

$$\gamma' = (f_1 \cdot f'_2 \cdot \gamma_3) * (f_1 \cdot \gamma'_2 \cdot g_3) * (\gamma_1 \cdot g'_2 \cdot g_3)$$

where the definition of the tiling path γ'_2 is by case analysis.

1. The redex u is extractible from the component f_1: in that case, $g'_2 = f'_2$ and $\gamma'_2 : f'_2 \Longrightarrow g'_2$ is defined as $\mathrm{id}_{f'_2}$. Equivalence (23) follows from induction hypothesis on h, as well as conditions 2, 3 and 4 on the equivalence relation $\cong$.
2. The redex u is extractible from the component f_2: in that case, the path g'_2 and $\gamma'_2 : f'_2 \Longrightarrow u' \cdot g'_2$ are the result of an arbitrary extraction $f'_2 \searrow_{u'} g'_2$. Equivalence (23) follows from the series of equivalence:

$$\begin{array}{ll}
& \gamma * (u \cdot \delta_g) \\
\cong & \gamma * (u \cdot g_1 \cdot (g_2 \Longrightarrow g''_2) \cdot g_3) * (u \cdot \delta_{g_1 \cdot g''_2 \cdot g_3}) & \text{by ind. hyp.} \\
\cong & (f_1 \cdot (\gamma_2 * \eta_2) \cdot g_3) * (\gamma_1 \cdot g''_2 \cdot g_3) * (u \cdot \delta_{g_1 \cdot g''_2 \cdot g_3}) & \text{by cond. 2, 3, 4.} \\
\cong & (f_1 \cdot ((f_2 \rhd f'_2) * \gamma'_2 * \eta'_2) \cdot g_3) * (\gamma_1 \cdot g''_2 \cdot g_3) * (u \cdot \delta_{g_1 \cdot g''_2 \cdot g_3}) & \text{by cond. 5.} \\
\cong & (f_1 \cdot (f_2 \rhd f'_2) \cdot f_3) * (f_1 \cdot (\gamma'_2 * \eta'_2) \cdot g_3) * (\gamma_1 \cdot g''_2 \cdot g_3) * (u \cdot \delta_{g_1 \cdot g''_2 \cdot g_3}) & \\
& & \text{by cond. 2, 3, 4.} \\
\cong & \alpha * (f_1 \cdot \gamma'_2 \cdot g_3) * (\gamma_1 \cdot g'_2 \cdot g_3) * (u \cdot g_1 \cdot \eta'_2 \cdot g_3) * (u \cdot \delta_{g_1 \cdot g''_2 \cdot g_3}) & \text{by cond. 2, 3, 4.} \\
\cong & \alpha * \gamma' * (u \cdot \delta_{g_1 \cdot g'_2 \cdot g_3}) & \text{by ind. hyp.} \\
\cong & \alpha * \gamma' * (u \cdot \delta_{g'})
\end{array}$$

   where g''_2 is a standard path Lévy equivalent to the paths g_2 and g'_2, and where

$$\begin{array}{rcrcl}
\eta_2 : u \cdot g_2 \Longrightarrow u \cdot g''_2 & & \text{and} & & \eta'_2 : u \cdot g'_2 \Longrightarrow u \cdot g''_2 \\
\delta_{g_1 \cdot g'_2 \cdot g_3} : g_1 \cdot g'_2 \cdot g_3 \Longrightarrow h & & \text{and} & & \delta_{g_1 \cdot g''_2 \cdot g_3} : g_1 \cdot g''_2 \cdot g_3 \Longrightarrow h
\end{array}$$

   are arbitrary tiling paths.
3. The redex u is extractible from the component f_3: in that third case, g'_2 and $\gamma'_2 : f'_2 \cdot u'' \Longrightarrow u' \cdot g'_2$ are the result of an arbitrary extraction $f'_2 \cdot u'' \searrow_{u'} g'_2$. Equivalence

(23) follows from the series of equivalence:

$$
\begin{array}{ll}
& \gamma * (u \cdot \delta_g) \\
\cong & \gamma * (u \cdot g_1 \cdot (g_2 \Longrightarrow g_2'') \cdot g_3) * (u \cdot \delta_{g_1 \cdot g_2'' \cdot g_3}) \qquad \text{by ind. hyp.} \\
\cong & (f_1 \cdot f_2 \cdot \gamma_3) * (f_1 \cdot (\gamma_2 * \eta_2) \cdot g_3) * (\gamma_1 \cdot g_2'' \cdot g_3) * (u \cdot \delta_{g_1 \cdot g_2'' \cdot g_3}) \qquad \text{by cond. 2, 3, 4.} \\
\cong & (f_1 \cdot f_2 \cdot \gamma_3) * (f_1 \cdot (((f_2 \rhd f_2') \cdot u'') * \gamma_2' * \eta_2') \cdot g_3) * (\gamma_1 \cdot g_2'' \cdot g_3) * (u \cdot \delta_{g_1 \cdot g_2'' \cdot g_3}) \\
& \qquad \text{by cond 6.} \\
\cong & (f_1 \cdot (f_2 \rhd f_2') \cdot f_3) * (f_1 \cdot f_2' \cdot \gamma_3) * (f_1 \cdot (\gamma_2' * \eta_2') \cdot g_3) * (\gamma_1 \cdot g_2'' \cdot g_3) * (u \cdot \delta_{g_1 \cdot g_2'' \cdot g_3}) \\
& \qquad \text{by cond. 2, 3, 4.} \\
\cong & \alpha * (f_1 \cdot f_2' \cdot \gamma_3) * (f_1 \cdot \gamma_2' \cdot g_3) * (\gamma_1 \cdot g_2' \cdot g_3) * (u \cdot g_1 \cdot \eta_2' \cdot g_3) * (u \cdot \delta_{g_1 \cdot g_2'' \cdot g_3}) \\
& \qquad \text{by cond. 2, 3, 4.} \\
\cong & \alpha * \gamma' * (u \cdot \delta_{g_1 \cdot g_2' \cdot g_3}) \qquad \text{by ind. hyp.} \\
\cong & \alpha * \gamma' * (u \cdot \delta_{g'})
\end{array}
$$

where g_2'' is a standard path Lévy equivalent to g_2 and g_2', and where

$$
\begin{array}{llll}
\eta_2 : u \cdot g_2 \Longrightarrow u \cdot g_2'' & \text{and} & \eta_2' : u \cdot g_2' \Longrightarrow u \cdot g_2'' \\
\delta_{g_1 \cdot g_2' \cdot g_3} : g_1 \cdot g_2' \cdot g_3 \Longrightarrow h & \text{and} & \delta_{g_1 \cdot g_2'' \cdot g_3} : g_1 \cdot g_2'' \cdot g_3 \Longrightarrow h
\end{array}
$$

are arbitrary tiling paths.

This proves our introductory claim. Now, we prove the lemma as follows. Let $\gamma : f \Longrightarrow u \cdot g$ be the result of an arbitrary extraction $f \searrow_u g$. Consider any tiling path α from f to $u \cdot h$. By property (23) proved above, there exists a tiling path γ' such that:

$$\gamma * (u \cdot \delta_g) \;\cong\; \alpha * \gamma' * (u \cdot \delta_h) \quad : f \Longrightarrow u \cdot h$$

In that particular case, as the result of the "empty" extraction $u \cdot h \searrow_u h$, the tiling path γ' is the identity $\mathrm{id}^{u \cdot h} : u \cdot h \Longrightarrow u \cdot h$. Moreover, the tiling path δ_h is the identity $\mathrm{id}^h : h \Longrightarrow h$ by induction hypothesis. It follows that

$$\alpha \quad \cong \quad \gamma * (u \cdot \delta_g)$$

This concludes the proof. □

5.3 The 2-category 2-cat$(\mathcal{G}, \rhd)$

Definition 34 (2-cat$(\mathcal{G}, \rhd)$) *The 2-category* ***2-cat***$(\mathcal{G}, \rhd)$ *is the 2-category* ***2-cat***$_\cong(\mathcal{G}, \rhd)$ *associated to the following equivalence relation on tiling paths:*

$$\alpha \cong \beta \quad \Longleftrightarrow \quad [\alpha] = [\beta]$$

The main goal of the section is to prove Theorem 4.

Lemma 35 *Suppose that $\alpha : f \Longrightarrow g : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a tiling path between the 1-dimensional paths $f = u_1 \cdots u_m$ and $g = v_1 \cdots v_n$. Suppose that w is a redex outgoing from M. The two following assertions are equivalent:*

1. *the path $v_1 \cdots v_{i-1}$ drags the redex v_i to the redex w,*

2. *the index* $[\alpha](i) = j$ *is defined and the path* $u_1 \cdots u_{j-1}$ *drags the redex* u_j *to the redex* w.

Proof. By induction on the length of α. □

Theorem 4 *In the 2-category* ***2-cat***$(\mathcal{G}, \triangleright)$*, every standard path is strongly terminal in its Lévy equivalence class.*

Proof. By Lemma 33, we only need to check conditions 5 and 6 on the equivalence relation $\cong$ on tiling paths $\alpha, \beta : f \Longrightarrow g$ induced by the equality $[\alpha] = [\beta]$. Consider

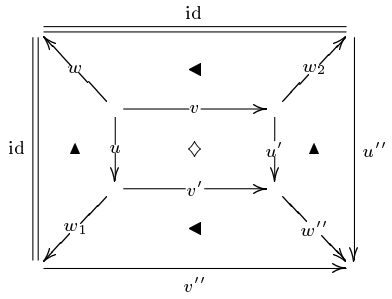
- a path $f = u \cdot v'$ such that u drags v' to a redex v
- or a path $f = u \cdot v' \cdot w''$ such that u drags v' to a redex v, and $u \cdot v'$ drags w'' to a redex w.

Consider two tiling paths $\alpha, \beta : f \Longrightarrow g$ standardizing f into a standard path $g = v_1 \cdots v_n$. Suppose that $[\alpha](i) = j$ for some $i \in [n]$. By Lemma 35($1 \Rightarrow 2$), the path $v_1 \cdots v_{i-1}$ drags the redex v_i to the redex $t = u$ when $j = 1$, to redex $t = v$ when $j = 2$, or to the redex $t = w$ when $j = 3$. Thus, by Lemma 35($1 \Rightarrow 2$), the index $[\beta](i) = k$ is defined and such that the path $u_1 \cdots u_{k-1}$ drags the redex u_k to the redex t. This implies that $j = k$. Applying the argument to every $i \in [n]$, and by symmetry, we deduce that $[\alpha] = [\beta]$. This proves conditions 5 and 6, and we conclude. □

Remark. In the case of the λ-calculus, and more generally in any axiomatic rewriting system derived from an axiomatic nesting system, see Section 6, the partial injection $[\alpha] : [g] \rightharpoonup [f]$ may be replaced by a *total function* $[\alpha] : [g] \twoheadrightarrow [f]$ without breaking Theorem 4. The idea is to replace the partial function $[\alpha]$ associated to an irreversible standardization step α in Definition 30 by the following total function $[\alpha]$:

$$[\alpha] : \quad \begin{cases} k & \mapsto & k & \text{for every } 1 \leq k \leq m \\ m+1 & \mapsto & m+2 & \\ m+1+k & \mapsto & m+1 & \text{for every } 1 \leq k \leq n-1 \\ m+n+k & \mapsto & m+2+k & \text{for every } 1 \leq k \leq p \end{cases}$$

It is not difficult to show that conditions 5 and 6 of Section 5.2 still hold with the new definition — in the case of the λ-calculus or any axiomatic nesting systems. Theorem 4 follows. However, Theorem 4 does not generally hold with the alternative definition. The axiomatic rewriting system



$$\begin{array}{ccc} u'' & = & v'' \\ v \cdot u' & \diamondsuit & u \cdot v' \\ u' \cdot w'' & \blacktriangleright & w_2 \cdot u'' \\ v' \cdot w'' & \blacktriangleright & w_1 \cdot v'' \\ u \cdot w_1 & \blacktriangleright & w \\ v \cdot w_2 & \blacktriangleright & w \end{array}$$

and tiling paths

$$\alpha_1 : u \cdot v' \cdot w'' \overset{1}{\Longrightarrow} v \cdot u' \cdot w'' \overset{1}{\Longrightarrow} v \cdot w_2 \cdot u'' \overset{1}{\Longrightarrow} w \cdot u''$$

$$\alpha_2 : u \cdot v' \cdot w'' \overset{1}{\Longrightarrow} u \cdot w_1 \cdot v'' \overset{1}{\Longrightarrow} w \cdot v'' = w \cdot u''$$

illustrate this point, since both α_1 and α_2 transform the path $u \cdot v' \cdot w''$ to the standard path $w \cdot u'' = w \cdot v''$, but do not define the same *total* functions $[\alpha_1]$ and $[\alpha_2]$, since $[\alpha_1](2) = 1$ and $[\alpha_2](2) = 2$.

6 An Alternative Axiomatics Based on Residuals and Nesting

The 2-dimensional axiomatics formulated in Section 2 is particularly adapted to reason and prove diagrammatically... but it is also far away from common practice, and may be difficult to understand for someone simply interested in checking that the axioms are satisfied by his or her favorite rewriting system. For that reason, we step back (in this section only) to the axiomatics developed in [13] and [27] and based on the trinity of *residuals*, *critical pairs* and *nesting order*. Since the formulation is nearly independent of the remainder of the article, the reader may very well jump this section at a first reading.

The section is organised as follows. Axiomatic nesting system are defined in Section 6.1, and their axioms are formulated in Sections 6.2–6.5. We establish in Section 6.6 that every axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$ defines an axiomatic rewriting system, that is, a 2-dimensional transition system $(\mathcal{G}, \triangleright)$ which satisfies the axioms of Section 2.

Remark: we provide two examples in Section 8

- the argument-nesting λ-calculus,
- the graph of sequentializations of an ordered set X.

which demonstrate that the axiomatics presented in this section is at the same time *strictly more general* than the axiomatics of [13] which inspired it, and *strictly less general* than the 2-dimensional axiomatics formulated in Section 2.

6.1 Axiomatic Nesting Systems

The main definition of the section follows.

Definition 36 *An* Axiomatic Nesting System *is a quadruple* $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$ *consisting of:*

1. *a transition system (or oriented graph)* $\mathcal{G} = (\mathbf{terms}, \mathbf{redexes}, \mathbf{source}, \mathbf{target})$,
2. *for every redex* $u : M \longrightarrow N$, *a binary relation* $[\![u]\!]$ *relating the redexes outgoing from* M *to the redexes outgoing from* N,
3. *for every vertex* M *of* $\mathcal{G}$, *a transitive reflexive antisymmetric relation* $\preceq_M$ *between the redexes outgoing from* M,

4. *for every vertex M of $\mathcal{G}$, a reflexive relation $\uparrow_M$ between the redexes outgoing from M.*

Every nesting system is supposed to satisfy a series of ten (4+2+4) axioms. The first four Axioms **Finite**, **Compat**, **Ancestor**, **Self** state elementary properties of residuals and compatibility. The two next Axioms **FinDev**, **Perm** enforce the well-known property of finite developments, appearing for instance in [32, 18, 20, 3, 27]. The four last Axioms **I**, **II**, **III**, **IV** regulate the properties of the nesting relation vs. the compatibility and residual relations. The ten axioms are called *N-axioms* (N stands for nesting) to distinguish them from the 2-dimensional axioms of Section 2.

6.2 The first N-axioms: Finite, Compat, Ancestor, Self

N-axiom Finite (finite residuals). We ask that a redex $v : M \longrightarrow Q$ has at most a finite number of residuals after a coinitial redex $u : M \longrightarrow P$.

$$\forall u, v \in \mathbf{redexes}, \qquad \text{the set } \{v' \mid v[\![u]\!]v'\} \text{ is finite.}$$

N-axiom Compat (forth compatibility). We ask that two compatible redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$ have compatible residuals u' and v' after a coinitial redex $w : M \longrightarrow N$.

$$\forall u, v, w, u', v' \in \mathbf{redexes}, \quad u[\![w]\!]u' \text{ and } v[\![w]\!]v' \text{ and } u \uparrow v \quad \Rightarrow \quad u' \uparrow v'$$

N-axiom Ancestor (unique ancestor). We ask that two different coinitial redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$ do not have any residual in common after a coinitial redex $w : M \longrightarrow N$.

$$\forall u, v, w, u', v' \in \mathbf{redexes}, \quad u[\![w]\!]u' \text{ and } v[\![w]\!]v' \text{ and } u' = v' \quad \Rightarrow \quad u = v$$

N-axiom Self (self-destruction). We ask that a redex $v : M \longrightarrow Q$ has no residual after itself, or after an incompatible coinitial redex $u : M \longrightarrow P$.

$$\forall u, v \in \mathbf{redexes}, \qquad (u = v \text{ or } \neg(u \uparrow v)) \quad \Rightarrow \quad \{v' \mid v[\![u]\!]v'\} = \emptyset$$

6.3 A few preliminary definitions: multi-redex, development

We need a few preliminary definitions to formulate the N-axioms **FinDev** and **Perm**.

Definition 37 (residual through path) *Given a path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$, the relation $[\![f]\!]$ between the redexes outgoing from M and the redexes outgoing from N, is defined as follows:*

– $[\![f]\!]$ *is the identity relation when* $f = \mathrm{id}_M$,
– $[\![f]\!]$ *is the composite relation* $[\![v_1]\!] \cdots [\![v_n]\!]$ *when* $f = v_1 \cdots v_n$.

Explicitly, for every two redexes u and u',

$$u[\![\mathrm{id}_M]\!]u' \iff u = u'$$

$$u[\![v_1 \cdots v_n]\!]u' \iff \begin{array}{l} \exists u_2, ..., u_{n-1} \in \mathbf{redexes}, \\ u[\![v_1]\!]u_2[\![v_2]\!]u_3 \cdots u_{n-2}[\![v_{n-1}]\!]u_{n-1}[\![v_n]\!]u' \end{array}$$

Definition 38 (multi-redex) *A* multi-redex *in* $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$ *is a pair* (M, U) *consisting of a term* M *and a finite set* U *of pairwise compatible redexes of source* M.

Remark: every redex $u : M \longrightarrow N$ may be identified to the multi-redex $(M, \{u\})$.

Definition 39 (multi-residual) *Suppose that* (M, U) *is a multi-redex and that* v *is a redex compatible with every redex in* U. *The* multi-residual *of* (M, U) *after* v, *notation* $(M, U)[\![v]\!]$, *is the multi-redex* (N, W) *where* $W = \{w \mid u[\![v]\!]w\}$.

Remark: Definition 39 defines a multi-redex (N, W) thanks to the N-axioms **Finite** and **Compat**.

Definition 40 (development) *A* complete development *of a multi-redex* (M, U) *is a path* f *such that:*

– $f = \mathrm{id}_M$ *when* U *is empty,*
– $f = u \cdot g$ *when* $u : M \longrightarrow N$ *is a redex in* U, *and the path* g *is a complete development of the multi-redex* $(M, U)[\![u]\!]$.

A development *of* (M, U) *is a path* $f : M \longrightarrow\!\!\!\!\!\rightarrow P$ *which is prefix of a complete development* $g : M \longrightarrow\!\!\!\!\!\rightarrow N$ *of* (M, U). *Here, we call* f *a prefix of* g *when there exists a path* $h : P \longrightarrow\!\!\!\!\!\rightarrow N$ *such that* $g = f \cdot h$.

We define two notions mentioned informally in Sections 1 and 2, and which appear in the N-axioms **III** and **IV**.

Definition 41 (created redex) *A redex* $u : M \longrightarrow P$ *creates a redex* $v : P \longrightarrow N$, *when there does not exist any redex* w *outgoing from* M, *such that* v *is a residual of* w *after* u.

Definition 42 (disjoint) *Two redexes* u *and* v *are disjoint when* $\neg(u \preceq v)$ *and* $\neg(v \preceq u)$.

6.4 The N-axioms related to finite developement: FinDev and Perm

N-axiom FinDev (finite developments). Let (M, U) be a multi-redex. Then, there does not exist any infinite sequence of redexes

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1} \xrightarrow{u_{n+1}} \cdots$$

such that, for every index n, the path $u_1 \cdots u_n$ is a development of (M, U).
N-axiom Perm (compatible permutation). For every two coinitial, compatible and different redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$, there exists a complete development h_u of $u[\![v]\!]$, and a complete development h_v of $v[\![u]\!]$, such that:

1. the paths h_u and h_v are cofinal,
2. the residual relations $[\![u \cdot h_v]\!]$ and $[\![v \cdot h_u]\!]$ are equal.

6.5 The fundamental N-axioms: I, II, III, IV

N-axiom I (unique residual). We ask that

$$u \uparrow v \text{ and } \neg(v \preceq u) \qquad \Rightarrow \qquad \exists! u',\ u[\![v]\!]u'$$

when u and v are coinitial redexes.
N-axiom II (context-free). Suppose that u, v, w are pairwise compatible redexes, that the redex u' is residual of u after w, and the redex v' residual of v after w. We ask that,

a. $\quad (u \preceq v \Rightarrow u' \preceq v') \text{ or } (w \preceq u \text{ and } w \preceq v)$
b. $\quad (u' \preceq v' \Rightarrow u \preceq v) \text{ or } w \preceq v$

N-axiom III (enclave). Suppose that u and v are two compatible redexes, and that $u \prec v$. Call u' the residual of u after v. We ask that for every redex v' created by v,

$$u' \prec v' \quad \text{or} \quad \neg(u' \uparrow v')$$

N-axiom IV (stability). Suppose that u and v are two compatible disjoint redexes. Call u' the residual of u after v, and v' the residual of v after u. We ask that there exists no triple of redexes (w_1, w_2, w) such that w_1 is a redex created by u, w_2 is a redex created by v, and

$$w_1[\![v']\!]w \quad \text{and} \quad w_2[\![u']\!]w$$

6.6 Every axiomatic nesting system defines an axiomatic rewriting system

Definition 43 *Every axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$ defines a 2-dimensional transition system $(\mathcal{G}, \rhd)$ as follows:*

> *$\rhd$ is the least relation between paths of $\mathcal{G}$ such that $v \cdot h_u \rhd u \cdot h_v$ when*
> *– the paths $u \cdot h_v$ and $v \cdot h_u$ are cofinal, and satisfy $[\![u \cdot h_v]\!] = [\![v \cdot h_u]\!]$,*
> *– u and v are two coinitial redexes outgoing from a term M,*
> *– $u \uparrow v$ and $\neg(v \preceq u)$,*
> *– the path h_u is a complete development of $(M, \{u\})[\![v]\!]$,*
> *– the path h_v is a complete development of $(M, \{v\})[\![u]\!]$.*

Observe that the 2-dimensional transition system $(\mathcal{G}_\lambda, \rhd_{\mathrm{tree}})$ of Section 1.9 is the result of applying Definition 43 to the axiomatic nesting system $(\mathcal{G}_\lambda, [\![-]\!]_\lambda, \preceq_{\mathrm{tree}}, \uparrow_\lambda)$ below:

– $[\![-]\!]_\lambda$ is the usual residual relation between β-redexes in the λ-calculus, as defined in [10, 24, 20, 3],

– $\uparrow_\lambda$ is the compatibility relation between β-redexes, in that case the total relation, indicating that every two coinitial β-redexes are compatible,
– $\preceq_{\mathrm{tree}}$ is the tree-nesting relation between β-redexes, defined in Section 1.9.

The main result of the section (Theorem 5) states that the 2-dimensional transition system $(\mathcal{G}, \triangleright)$ of Definition 43 satisfies the standardization axiomatics of Section 2. Before proving that theorem, we start with five preliminary lemmas.

Lemma 44 *The 2-dimensional transition system $(\mathcal{G}, \triangleright)$ of Definition 43 satisfies Axiom **shape**.*

Proof. Suppose that $f \triangleright g$ is a permutation in $(\mathcal{G}, \triangleright)$. By definition, the two first steps of f and g are different. By the N-axioms **I** and **Self**, the length of the rewriting path f is 2. Axiom **shape** follows. □

Definition 43 exports from axiomatic rewriting systems to axiomatic nesting systems the definitions of standardization preorder $\Longrightarrow$ and Lévy equivalence relation $\equiv$ in Section 1.5, as well as (thanks to Lemma 44) the definitions of extraction and projection in Section 2.5. We prove

Lemma 45 (cube lemma) *Suppose that (M, U) is a multi-redex in an axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$. Then, every two complete developments f and g of (M, U) are Lévy equivalent.*

Proof. By the N-axioms **Finite** and **Compat**, the complete developments of (M, U) ordered by prefix, define a finitely branching tree. The tree is thus finite by König's lemma and N-axiom **FinDev**. We proceed by induction on the length of the longest path of that tree, called the "depth" of (M, U). Suppose that the lemma is established for every multi-redex of depth less than n, and let (M, U) be a multi-redex of depth $n + 1$. Let f and g be two complete developments of (M, U). If one of the two paths f or g is empty, then the set U is empty, and thus the two complete developments f and g are empty: it follows that $f \equiv g$. Otherwise, the two paths f and g factor as $f = u \cdot f'$ and $g = v \cdot g'$ where the redexes u and v are elements of the multi-redex (M, U), the path f' is a complete development of $(M, U)[\![u]\!]$, and the path g' is a complete development of $(M, U)[\![v]\!]$. We proceed by case analysis. Either $u = v$ or $u \neq v$. In the first case, both paths f' and g' are complete developments of the multi-redex $(M, U)[\![u]\!] = (M, U)[\![v]\!]$; the equivalence $f' \equiv g'$ follows from our induction hypothesis applied to the multi-redex $(M, U)[\![u]\!]$, and we conclude that $f \equiv g$. In the second case, when $u \neq v$, it follows from N-axiom **Perm** that there exist two complete developments h_u of $u[\![v]\!]$ and h_v of $v[\![u]\!]$, such that the paths $v \cdot h_u$ and $u \cdot h_v$ are coinitial and cofinal, and induce the same residual relation $[\![u \cdot h_v]\!] = [\![v \cdot h_u]\!]$. Let h be any complete development of the multi-redex $(M, U)[\![u \cdot h_v]\!] = (M, U)[\![v \cdot h_u]\!]$. By definition of a complete development, the path $h_v \cdot h$ is a complete development of $(M, U)[\![u]\!]$, and the path $h_u \cdot h$ is a complete development of $(M, U)[\![v]\!]$. The two equivalence relations $h_v \cdot h \equiv f'$ and $h_u \cdot h \equiv g'$ follow from our induction hypothesis applied to the multi-redexes $(M, U)[\![u]\!]$ and $(M, U)[\![v]\!]$. We conclude that $f \equiv g$ by the series of equivalence:

$$f \;=\; u \cdot f' \;\equiv\; u \cdot h_v \cdot h \;\equiv\; v \cdot h_u \cdot h \;\equiv\; v \cdot g' \;=\; g$$

□

Lemma 46 *Suppose that the path f is a complete development of a multi-redex (M, U) in an axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$. Suppose that a redex u is element of U, and satisfies $\neg(v \preceq u)$ for every redex v in the set $U - \{u\}$. Then, the redex u is extractible from the path f.*

Proof. By induction on the length of the complete development f. The path f is not empty. It thus factors as $f = w \cdot g$, where $w : M \longrightarrow P$ is a redex of U, and g is a complete development of $(M, U)[\![w]\!]$. The lemma is obvious when $u = w$. Otherwise, by hypothesis, $u \uparrow w$ and $\neg(w \preceq u)$. By N-axiom **I**, the redex u has a unique residual after reduction of the redex w. Let us call this redex u'. Let v' denote any redex in $(N, U') = (M, U)[\![w]\!]$ different from the redex u'. We prove that $\neg(v' \preceq u')$. By definition of the redex v', there exists a redex v in U, such that $v[\![w]\!]v'$. Obviously, the redex v is different from the redex u because u' is the unique residual of the redex u after w. It follows from hypothesis on u that $\neg(v \preceq u)$. We apply the N-axiom **II**b. to $\neg(v \preceq u)$ and $\neg(w \preceq u)$ to deduce that $\neg(v' \preceq u')$. We have just proved that $\neg(v' \preceq u')$ for any redex v' in $U' - \{u'\}$. Our induction hypothesis implies then that the redex u' is extractible from the complete development g of (N, U').

To summarize, we know that $u \uparrow w$, that $\neg(w \preceq u)$, and that the unique residual of u after w, denoted u', is extractible from the path g. We claim that it follows from this that the redex u is extractible from the path $w \cdot g$. Indeed, by N-axiom **Perm**, there exists a complete development h_u of the multi-redex $(M, \{u\})[\![v]\!]$ and a complete development h_w of the multi-redex $(M, \{w\})[\![u]\!]$, such that the paths $u \cdot h_w$ and $v \cdot h_u$ are coinitial, cofinal, and induce the same residual relation $[\![u \cdot h_w]\!] = [\![v \cdot h_u]\!]$. Moreover, $h_u = u'$ by N-axioms **I** and **Self**. By definition, $w \cdot u' \rhd u \cdot h_w$. It follows that the redex u is extractible from the path $f = w \cdot g$. This concludes our proof by induction.

□

Lemma 47 *Suppose that $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a complete development of a multi-redex (M, U) in an axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$. Then, every path more standard than f is a complete development of (M, U).*

Proof. Suppose that a complete development of (M, U) factors as

$$M \xrightarrow{f_1} P \xrightarrow{f} Q \xrightarrow{f_2} N$$

and that $f \rhd g$. We show that the path $f_1 \cdot g \cdot f_2$ is also a complete development of (M, U). By definition of a complete development, we may suppose without loss of generality that the path f_1 is empty. By definition of $\rhd$, the paths f and g are two cofinal complete development of a multi-redex $(M, \{u, v\})$, and factor as $f = v \cdot u'$ and $g = u \cdot h_v$ where $\neg(v \preceq u)$, the redex u' is the unique residual of u after v and h_v is a complete development of the residuals of v after u. By definition of a complete development of (M, U), one ancestor of v' before u is element of U. By the N-axiom **Ancestor**, this ancestor is unique, and we already have one candidate: the redex v. We conclude that the redex v is element of U. By definition of $\rhd$, the rewriting paths f and g induce the same residual relation $[\![f]\!] = [\![g]\!]$. We conclude that $f_1 \cdot g \cdot f_2$ is a complete development of the multi-redex (M, U). □

Lemma 48 *Suppose that the rewriting path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a complete development of a multi-redex (M, U) in the axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$. Then,*

- *every redex u extractible from the path f is element of U,*
- *every projection of the path f by extraction of a redex u is a complete development of the multi-redex $(M, U)[\![u]\!]$.*

Proof. Immediate consequence of Lemma 47. □

Theorem 5 *By Definition 43, every axiomatic nesting system $(\mathcal{G}, [\![-]\!], \preceq, \uparrow)$ defines an axiomatic rewriting system $(\mathcal{G}, \rhd)$.*

Proof. We establish that the 2-dimensional transition system $(\mathcal{G}, \rhd)$ satisgies the nine axioms of Section 2.
Axiom 1. Axiom **shape** is established in Lemma 44,
Axiom 2. Axiom **ancestor** follows from N-axiom **Ancestor** and Lemma 45,
Axiom 3. We prove Axiom **reversibility**. Suppose that $f \rhd g \rhd h$. By definition of $\rhd$, there exists five redexes u, v, w, u', v' and a path h' such that $f = u \cdot v'$ and $g = v \cdot u'$ and $h = w \cdot h'$, and $u \uparrow v$ and $v \uparrow w$ and $\neg(u \preceq v)$ and $\neg(v \preceq w)$. By definition of $f \rhd g$, the redex u' is the complete development of the residuals of u after v, thus a fortiori a residual of u after v. By definition of $g \rhd h$, the redex u' is a residual of w after v. The equality $u = w$ follows from N-axiom **Ancestor**. Thus, h' is a complete development of the residuals of v after $u = w$. But, by definition of $f \rhd g$ and N-axiom **I**, the redex v' is the unique residual of v after u. Thus, $h' = v'$ and we conclude Axiom **reversibility** with the equality $h = u \cdot h' = u \cdot v' = f$.
Axiom 4. We prove Axiom **irreversibility**. Suppose that $f \blacktriangleright g$ and $g \Longrightarrow h$. By definition of $f \rhd g$, the paths f and g are complete developments of a multi-redex $(M, \{u, v\})$ with, say, the paths f and g starting by reducing v and u respectively. The nesting relation $u \prec v$ follows easily from $f \blacktriangleright g$. By Lemma 47, and our hypothesis that $g \rhd h$, the path h is a complete development of $(M, \{u, v\})$. We prove that h starts by reducing the redex u. By definition of $g \Longrightarrow h$, there exists a sequence

$$g = h_1 \overset{1}{\Longrightarrow} h_2 \overset{1}{\Longrightarrow} \cdots h_n \overset{1}{\Longrightarrow} h_{n+1} = h$$

of complete development of $(M, \{u, v\})$ and an index $1 \leq i \leq n$ such that h_i starts by reducing the redex u, and h_{i+1} starts by reducing the redex v. This means that h_i and h_{i+1} factor as $h_i = u \cdot w \cdot h'$ and $h_{i+1} = v \cdot h_u \cdot h'$, where $u \cdot w \rhd v \cdot h_u$. This contradicts $u \prec v$. We conclude that the path h starts by reducing u. Obviously, the complete developments f and h are cofinal and induce the same residual relation $[\![f]\!] = [\![h]\!]$. The relation $f \blacktriangleright h$ follows from that and $u \preceq v$. This proves Axiom **irreversibility**.
Axiom 5. We prove Axiom **cube**. Among its hypothesis, we have that $v \cdot u' \rhd u \cdot v_1 \cdots v_n$ and that the redex w_{n+1} is residual of the redex w after the path $u \cdot v_1 \cdots v_n$. By definition of $v \cdot u' \rhd u \cdot v_1 \cdots v_n$, the redex w_{n+1} is also residual of w after the path $v \cdot u'$. By N-axiom **Self**, the redexes u, v, w are pairwise compatible and different. Thus, the pair $(M, \{u, v, w\})$ defines a multi-redex.

We prove that $\neg(u \preceq w)$ and $\neg(v \preceq w)$. The first relation follows from the hypothesis that $u \cdot w_1 \rhd w \cdot h_u$. The second relation is established by case analysis, depending

on whether $u \preceq v$ or $\neg(u \preceq v)$. In the first case, the relation $\neg(v \preceq w)$ holds by transitivity of $\preceq$, because $\neg(u \preceq w)$. In the second case, observe that the permutation $v \cdot u' \rhd u \cdot v_1 \cdots v_n$ is reversible. We write it $v \cdot u' \rhd u \cdot v_1$. The relation $\neg(v_1 \preceq w_1)$ follows from the hypothesis that $v_1 \cdot w_2 \rhd w_1 \cdot h_1$. By $\neg(u \preceq v)$ and $\neg(u \preceq w)$, and N-axiom **II**a. the relation $\neg(v \preceq w)$ follows from $\neg(v_1 \preceq w_1)$.

We have just proved that $\neg(u \preceq w)$ and $\neg(v \preceq w)$. By Lemma 46, the redex w is extractible from the two complete developments $v \cdot u' \cdot w_{n+1}$ and $u \cdot v_1 \cdots v_n \cdot w_{n+1}$ of $(M, \{u, v, w\})$. In particular, there exists a redex w' and two paths h_v and $h_{u'}$ forming permutations $u' \cdot w_{n+1} \rhd w' \cdot h_{u'}$ and $v \cdot w' \rhd w \cdot h_v$. This proves half of Axiom **cube**.

There remains to prove that the paths $h_v \cdot h_{u'}$ and $h_u \cdot h_1 \cdots h_n$ are Lévy equivalent. The two paths are projections by extraction of w of the complete developments $v \cdot u' \cdot w_{n+1}$ and $u \cdot v_1 \cdots v_n \cdot w_{n+1}$ of $(M, \{u, v, w\})$. The Lévy equivalence follows from Lemma 48. This concludes the proof of Axiom **cube**.

Axiom 6. We prove Axiom **enclave**. We recall its hypothesis: the irreversible permutation $v \cdot u' \blacktriangleright u \cdot v_1 \cdots v_n$ and the permutation $u' \cdot w_{n+1} \rhd w' \cdot h_{u'}$. The relations $u \uparrow v$ and $u \prec v$ and $u' \uparrow w'$ and $\neg(u' \preceq w')$ follow from this. By N-axiom **III**, the redex $u : M \longrightarrow N$ does not create the redex w'. Thus, there exists a redex outgoing from M with residual w' after u. This redex is unique by N-axiom **Ancestor**. We call it w.

By definition of $v \cdot u' \rhd u \cdot v_1 \cdots v_n$, the residual relation $w[\![v \cdot u']\!]w_{n+1}$ implies that $w[\![v \cdot v_1 \cdots v_n]\!]w_{n+1}$. It follows from N-axiom **Self** that the three redexes u, v, w are pairwise different and compatible, thus define a multi-redex $(M, \{u, v, w\})$.

We prove that $\neg(u \preceq w)$ and $\neg(v \preceq w)$. The first relation follows from N-axiom **II**a. applied to the relations $\neg(v \preceq u)$ and $\neg(u' \preceq w')$. The second relation follows from transitivity of $\preceq$ and $\neg(u \preceq w)$ and $u \preceq v$.

By Lemma 46, it follows that the redex w is extractible from the complete developments $v \cdot u' \cdot w_{n+1}$ and $u \cdot v_1 \cdots v_n \cdot w_{n+1}$ of the multi-redex $(M, \{u, v, w\})$. Equivalently, both paths $v \cdot u'$ and $u \cdot v_1 \cdots v_n$ drag the redex w_{n+1} to the redex w. This concludes the proof of Axiom **enclave**.

Axiom 7. We prove Axiom **stability**. By definition of $u \cdot v' \rhd v \cdot u' \rhd u \cdot v'$, the two redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$ are compatible, and disjoint. By N-axiom **IV**, either the redex w_1 is not created by u, or the redex w_1 is not created by v.

Suppose for instance that w_2 is not created by v. In that case, there exists a redex w such that $w[\![v]\!]w_2$. Consequently, the redex w_{12} is residual of w after the path $v \cdot u'$. By definition of $u \cdot v' \rhd v \cdot u'$, the redex w_{12} is also residual of w after $u \cdot v'$. Thus, there exists a residual w'_1 of w after u, such that $w'_1[\![v']\!]w_{12}$. The equality $w_1 = w'_1$ follows from $w_1[\![v']\!]w_{12}$ and N-axiom **Ancestor**. We conclude that w_1 is not created by v, and residual of w after u. The case when w_1 is not created by u, is symmetric.

By N-axiom **II**a. and $v[\![u]\!]v'$, $w[\![u]\!]w_1$, the relation $\neg(v \preceq w)$ follows from $\neg(v' \preceq w_1)$ and $\neg(u \preceq v)$. The relation $\neg(u \preceq w)$ holds for symmetric reasons. Axiom **stability** follows easily.

Axiom 8. We prove Axiom **reversible-stability**. By Axiom **stability**, which was established above, applied to the hypothesis of Axiom **reversible-stability**, there exists a redex w such that

- $u \uparrow w$, $\neg(u \preceq w)$, and w_1 is the unique residual of w after v,

– $v \uparrow w$, $\neg(v \preceq w)$, and w_2 is the unique residual of w after u.

We prove that $\neg(w \preceq u)$ and $\neg(w \preceq v)$. Suppose for instance that $w \preceq u$. By N-axiom **II**a. and $u[\![v]\!]u_1$ and $w[\![v]\!]w_2$, the relation $w_2 \preceq u_1$ follows from this and $\neg(v \preceq w)$. This contradicts definition of $w_2 \cdot u_{12} \rhd u_1 \cdot w_{12}$. Thus, $\neg(w \preceq u)$, and symmetrically $\neg(w \preceq v)$. Axiom **reversible-stability** follows from Lemma 46 applied alternatively to extract the redex u from the complete development $w \cdot v_2 \cdot u_{12}$, and the redex v from the complete development $w \cdot u_2 \cdot v_{12}$.

Axiom 9. We prove Axiom **termination** using an argument found in [20]. Suppose that h_1 is a complete development of a multi-redex (M, U). By N-axiom **FinDev** and Lemma 48, there does not exist any infinite sequence of extraction:

$$h_1 \searrow_{u_1} h_2 \searrow_{u_2} \cdots \searrow_{u_{i-1}} h_i \searrow_{u_i} h_{i+1} \cdots$$

where, for every $i \geq 1$, the path h_{i+1} is a projection of the path h_i by extraction of the redex u_i. Now, we prove that there does not exist any infinite sequence

$$f_1 \searrow_{u_1} f_2 \searrow_{u_2} \cdots \searrow_{u_{i-1}} f_i \searrow_{u_i} f_{i+1} \cdots \tag{24}$$

starting from a path $f_1 : M_1 \longrightarrow\!\!\!\!\!\rightarrow N$. We proceed by induction on the length of f_1. Clearly, the property holds when $f_1 = \mathrm{id}_{M_1}$. From now on, we suppose that the path f_1 factors as $f_1 = u \cdot g_1$ composed of a redex u and a path g_1 ¡ of length strictly smaller than the length of f_1. Consider any infinite sequence of the form (24). We prove that, for every index $i \geq 1$, the path f_i factors as $f_i = h_i \cdot g_{\phi(i)}$ where

– h_i is a complete development of the multi-redex (M_i, U_i) defined as:

$$(M_i, U_i) = (M_i, u[\![u_1 \cdots u_{i-1}]\!]) = (M_1, \{u\})[\![u_1]\!] \cdots [\![u_{i-1}]\!]$$

– $\phi(i)$ is an index $1 \leq \phi(i) \leq i$ defining a sequence of extraction starting from g_1:

$$g_1 \searrow_{v_1} g_2 \searrow_{v_2} \cdots \searrow_{v_{\phi(i)-2}} g_{\phi(i)-1} \searrow_{v_{\phi(i)-1}} g_{\phi(i)}$$

for a series of redexes $v_1, \ldots, v_{\phi(i)-1}$.

Suppose that the property holds for a given index $i \geq 1$, and let us prove it for the next index $i+1$. Consider the path $f_i = h_i \cdot g_{\phi(i)}$ and the redex u_i. Either the redex u_i is extractible from h_i, or there exists a redex $v_{\phi(i)}$ extractible from g_i and dragged to u_i by the path h_i. In the first case, we define $\phi(i+1)$ as $\phi(i)$, and conclude that the path f_{i+1} factors as $f_{i+1} = h_{i+1} \cdot g_{\phi(i+1)}$, where h_{i+1} is a projection of h_i by extraction of u_i; here, by Lemma 48, the path h_{i+1} is a complete development of $(M_i, U_i)[\![u_i]\!] = (M_{i+1}, U_{i+1})$ because h_i is a complete development of (M_i, U_i). In the second case, we define $\phi(i+1)$ as $\phi(i)+1$, and observe that the path f_{i+1} factors as $f_{i+1} = h_{i+1} \cdot g_{\phi(i+1)}$, where $h_i \cdot v_{\phi(i)} \searrow_{u_i} h_{i+1}$ and $g_{\phi(i)} \searrow_{v_{\phi(i)}} g_{\phi(i+1)}$; here, by Lemma 48, the path h_{i+1} is a complete development of the multi-redex $(M_i, \{u_i\} \cup U_i)[\![u_i]\!] = (M_{i+1}, U_{i+1})$ because $h_i \cdot v_{\phi(i)}$ is a complete development of the multi-redex $(M_i, \{u_i\} \cup U_i)$. We conclude that the factorization property holds, for every index $i \geq 1$.

The end of the proof follows easily. By induction hypothesis applied to g, there exists an index $j \geq 1$ such that $\phi(j+i) = \phi(j)$, for every index $i \leq j$. Thus, the infinite sequence (24) induces an infinite sequence

$$h_j \searrow_{u_j} h_{j+1} \searrow_{u_{j+1}} \cdots \searrow_{u_{j+i-1}} h_{j+i} \searrow_{u_{j+i}} h_{j+i+1} \cdots$$

from the complete development h_j of (M_j, U_j). This contradicts a preliminary result deduced from N-axiom **FinDev**. It follows that there exists no infinite sequence of the form (24) starting from f. This concludes our reasoning by induction, and establishes Axiom **termination**. □

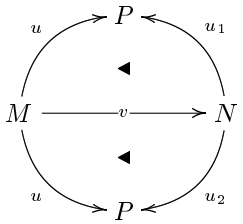
7 Optional Hypothesis on Standardization

7.1 Epimorphisms wrt. $\equiv$

In Lemma 17 of Section 3.1, we establish that every path is epi (=left-cancellable) in the quotient category **2-cat**$(\mathcal{G}, \rhd)/\simeq$. The same epiness property modulo $\equiv$ instead of $\simeq$ has been established in [24, 18, 6] for the λ-calculus and any (left-linear) term rewriting system. Quite interestingly, the redex v and Lévy equivalence

$$M \xrightarrow{v} N \xrightarrow{u_1} P \quad \equiv \quad M \xrightarrow{v} N \xrightarrow{u_2} P$$

in the axiomatic rewriting system

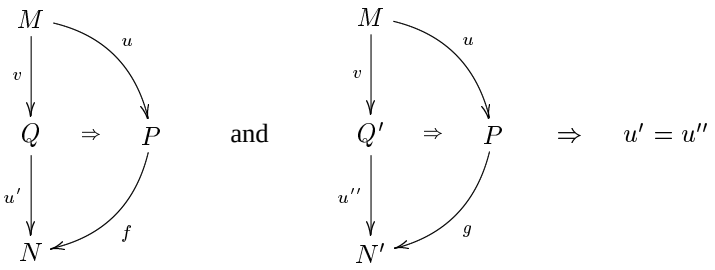


$v \cdot u_1 \blacktriangleright u$
$v \cdot u_2 \blacktriangleright u$

illustrate that the epiness property modulo $\equiv$ does not generalize to axiomatic rewriting systems. However, an additional hypothesis may be added on $(\mathcal{G}, \rhd)$ to ensure epiness of morphisms in the category **2-cat**$(\mathcal{G}, \rhd)/\equiv$.

Optional hypothesis (descendant). Two redexes u' and u'' are equal when they are involved in permutations $v \cdot u' \rhd u \cdot f$ and $v \cdot u'' \rhd u \cdot g$, where u, v are redexes and f, g are paths.

Diagrammatically,



Obviously, hypothesis **descendant** holds in every axiomatic rewriting system derived from an axiomatic nesting system, see Definition 43. Thus, Lemma 49 generalizes the property of epiness modulo $\equiv$ established in [24, 18, 6] for the λ-calculus and term rewriting systems.

Lemma 49 (epi wrt. $\equiv$) *Suppose that $f : M \longrightarrow\!\!\!\!\!\rightarrow P$ and $g_1, g_2 : P \longrightarrow\!\!\!\!\!\rightarrow N$ are three paths in an axiomatic rewriting system $(\mathcal{G}, \rhd)$ and that $(\mathcal{G}, \rhd)$ satisfies hypothesis **descendant**. Then,*

$$f \cdot g_1 \equiv f \cdot g_2 \quad \Rightarrow \quad g_1 \equiv g_2$$

Proof. By induction on the length of the standard path h of $f \cdot g_1$ (and of $f \cdot g_2$.) Let u be the first redex computed in h. We conclude by induction hypothesis when u is extractible from f. Otherwise, there exist a redex v_1 extractible from g_1 and a redex v_2 extractible from g_2, such that f drags v_1 and v_2 to the redex u. By hypothesis **descendant**, the two redexes v_1 and v_2 are the same redex v. We write f', h_1 and h_2 for arbitrary results of the extractions $f \cdot v \setminus_u f'$ and $g_1 \setminus_v h_1$ and $g_2 \setminus_v h_2$. Equivalence $f' \cdot h_1 \equiv f' \cdot h_2$ follows from Lemma 12 (*preservation of extraction*), and definition of u as the first redex of a standard path of $f \cdot g_1$ and $f \cdot g_2$. Equivalence $h_1 \equiv h_2$ follows from this equivalence and our induction hypothesis. The series of equivalence

$$g_1 \equiv v \cdot h_1 \equiv v \cdot h_2 \equiv g_2$$

concludes the proof by induction. □

7.2 Monomorphisms wrt. $\simeq$

A well-known example in [24] shows that β-rewriting paths are not necessarily mono (=right-cancellable) modulo Lévy equivalence $\equiv$. The example is the β-redex w in the Lévy permutation equivalence

$$I(Ia) \xrightarrow{u} Ia \xrightarrow{w} a \qquad \equiv \qquad I(Ia) \xrightarrow{v} Ia \xrightarrow{w} a$$

The example may be adapted to show that β-rewriting paths are not necessarily mono modulo $\simeq$-equivalence in the λ-calculus equipped with the *argument-order* on β-redexes, in the following way:

$$(\lambda x.(\lambda y.y)x)a \xrightarrow{u} (\lambda y.y)a \xrightarrow{w} a \qquad \diamondsuit \qquad (\lambda x.(\lambda y.y)x)a \xrightarrow{v} (\lambda x.x)a \xrightarrow{w} a$$

In contrast, we show that rewriting paths are mono modulo $\simeq$ in every axiomatic rewriting system satisfying the additional property **reversible-shape**. It follows that monoicity modulo $\simeq$ holds in almost every rewriting system, in particular in the λ-calculus equipped with the *tree-order* or the *left-order* on β-redexes, as well as on Petri nets and term rewriting systems.

Optional hypothesis (reversible shape). Two redexes v and v' are different when they are involved in a reversible permutation $u \cdot v \diamondsuit u' \cdot v'$.

Lemma 50 (epi-mono wrt. $\simeq$) *Suppose that $f : M \longrightarrow\!\!\!\!\!\rightarrow P$ and $g_1, g_2 : P \longrightarrow\!\!\!\!\!\rightarrow Q$ and $h : Q \longrightarrow\!\!\!\!\!\rightarrow N$ are four paths in an axiomatic rewriting system $(\mathcal{G}, \triangleright)$ satisfying hypothesis **reversible-shape**. Then,*

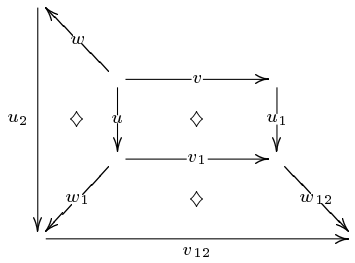
$$f \cdot g_1 \cdot h \simeq f \cdot g_2 \cdot h \quad \Rightarrow \quad g_1 \simeq g_2$$

Proof. Immediate consequence of Lemma 15 for right-cancellation and Lemma 17 for left-cancellation. □

7.3 A simpler structure of starts

The *structure of starts* described in Lemma 16 (Section 3.1) appears to be surprisingly more complicated than the *structure of stops* described in Lemma 15. However, a much simpler characterization of starts is possible in any axiomatic rewriting system $(\mathcal{G}, \triangleright)$ satisfying the additional hypothesis **reversible-cube** formulated below. The new characterization of starts appears in Lemma 51. Note that the property is satisfied by the λ-calculus and more generally by any axiomatic rewriting system derived from an axiomatic nesting system. On the other hand, it is not satisfied by the axiomatic rewriting system defined on order sequentializations, and defined at the endof Section 8.

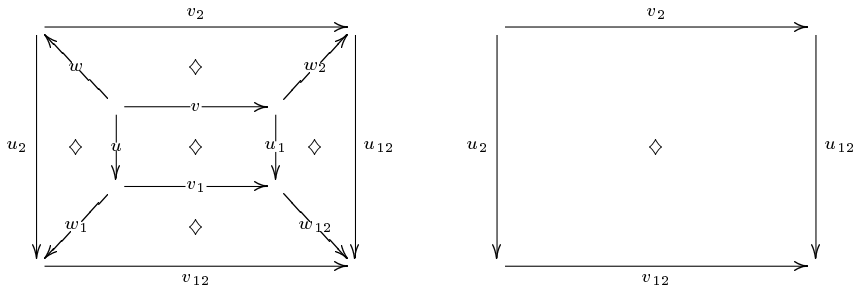
Optional hypothesis (reversible cube). We ask that every diagram



where u, v, u_1, v_1 and $w, w_1, w_{12}, u_2, v_{12}$ are redexes forming the reversible permutations

$$v \cdot u_1 \diamondsuit u \cdot v_1 \qquad u \cdot w_1 \diamondsuit w \cdot u_2 \qquad v_1 \cdot w_{12} \diamondsuit w_1 \cdot v_{12}$$

may be completed as a diagram



where w_2, v_2, u_{12} are three redexes forming reversible permutations

$$v \cdot w_2 \diamondsuit w \cdot v_2 \qquad u_1 \cdot w_{12} \diamondsuit w_2 \cdot u_{12} \qquad v_2 \cdot u_{12} \diamondsuit u_2 \cdot v_{12}$$

Lemma 51 (simpler structure of starts) *Suppose that $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ is a path in an axiomatic rewriting system $(\mathcal{G}, \rhd)$ satisfying hypothesis **reversible-cube**. Then, a redex $u : M \longrightarrow P$ starts the path $u_1 \cdots u_n : M \longrightarrow\!\!\!\!\!\rightarrow N$ if and only there exists an index $1 \leq i \leq n$ and a path $v_1 \cdots v_{i-1}$ such that the path $u_1 \cdots u_{i-1}$ followed by the redex u_i permutes reversibly to the redex u followed by the path $v_1 \cdots v_{i-1}$.*

Proof. Suppose that a path f followed by a redex v permutes reversibly to a redex u followed by a path g. Hypothesis **reversible-cube** implies that for every path $f' \simeq f$, there exists a path $g' \simeq g$ such that the path f' followed by the redex v permutes reversibly to the redex u followed by the path g'. The lemma follows immediately from this, and Lemma 16. □

8 Examples and Open Problems

ASYNCHRONOUS TRANSITION SYSTEMS. Asynchronous transition systems extend both non-deterministic transition systems, and Mazurkiewicz trace languages. They were introduced independently in [4] and [39], see also [33].

An *asynchronous transition system* T is a quintuple $T = (S, i, E, I, \mathrm{Tran})$ where

- S is a set of *states* with *initial state* i,
- E is a set of *events*,
- $\mathrm{Tran} \subset S \times L \times S$ is the *transition relation*,
- $I \subset E \times E$ is an irreflexive, symmetric relation called the *independence relation*.

Every asynchronous transition system is supposed to satisfy four axioms:

1. *parsimony:* $\forall e \in E,\ \exists (s, s') \in S \times S,\ \ (s, e, s') \in \mathrm{Tran}$,
2. *determinacy:* $\forall (s, e, s'), (s, e, s'') \in \mathrm{Tran},\ \ s' = s''$,
3. *independence:* $\forall (s, e_1, s_1), (s, e_2, s_2) \in \mathrm{Tran}$,

$$e_1 I e_2 \ \Rightarrow\ \exists s', (s_1, e_2, s') \in \mathrm{Tran} \text{ and } (s_2, e_1, s') \in \mathrm{Tran}$$

4. *together:* $\forall (s, e_2, s_2), (s_2, e_1, s') \in \mathrm{Tran}$,

$$e_1 I e_2 \ \Rightarrow\ \exists s_1,\ \ (s, e_1, s_1) \in \mathrm{Tran} \text{ and } (s_1, e_2, s') \in \mathrm{Tran}$$

Every asynchronous transition system T defines an axiomatic rewriting system $(\mathcal{G}_T, \rhd_T)$, as follows:

- the graph $\mathcal{G}_T$ has states as vertices and transitions (s, e, s') as arrows,

– two paths f and g are related as $f \rhd_T g$, precisely when there exist four transitions (s, e_1, s_1), (s, e_2, s_2), (s_1, e_2, s'), (s_2, e_1, s') in Tran, such that
  - $f = (s, e_2, s_2) \cdot (s_2, e_1, s')$,
  - $g = (s, e_1, s_1) \cdot (s_1, e_2, s')$,
  - the two events e_1 and e_2 are independent: $e_1 I e_2$.

We check that the standardization axioms hold in $(\mathcal{G}_T, \rhd_T)$. Axiom **shape** follows from anti-reflexivity of the independence relation. Observe that every permutation $f \rhd_T g$ is reversible: it coexists with a permutation $g \rhd_T f$. The three Axioms **irreversibility**, **enclave** and **termination** follow from this, as well as the equivalence between Axiom **stability** and Axiom **reversible-stability**. We establish now the four Axioms **ancestor**, **reversibility**, **cube** and **reversible-stability**. The property (2) of **determinacy** has two remarkable consequences in every asynchronous transition system T:

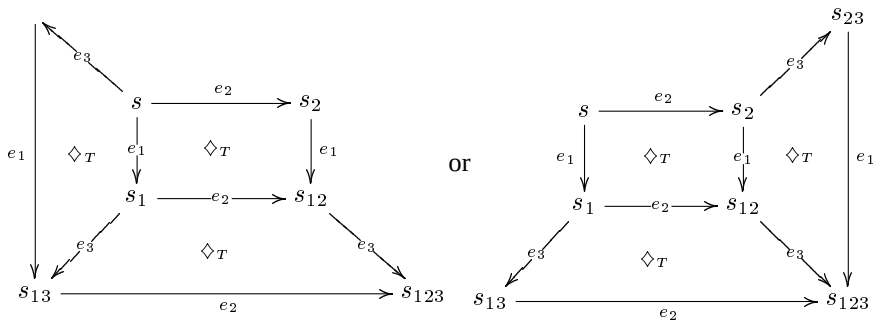
$$f \diamondsuit_T g \quad \text{and} \quad f \diamondsuit_T h \;\Rightarrow\; g = h.$$

$$f \diamondsuit_T g \diamondsuit_T h \;\Rightarrow\; f = h.$$

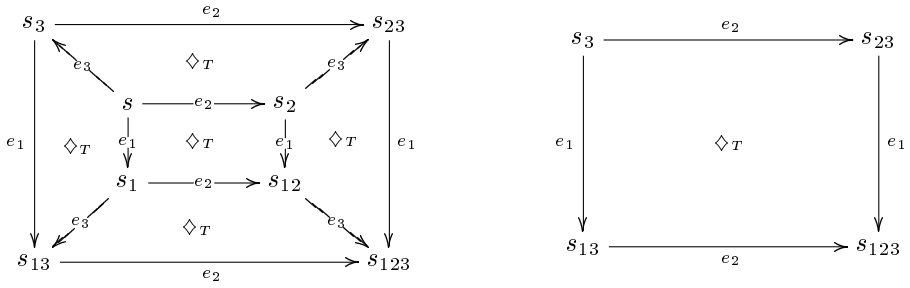
The two Axioms **ancestor** and **reversibility** follow from the first and second assertions, respectively. By definition of the permutation relation $\rhd_T$, the three events e_1, e_2, e_3 are pairwise independent:

$$e_1 I e_2 \qquad e_2 I e_3 \qquad e_1 I e_3.$$

in every diagram



So, it follows from the properties (2) and (4) of **determinacy together** with the properties of the asynchronous transition system T, that the two diagrams above may be completed as:



Axioms **cube** and **reversible-stability** follow immediately. It is also nearly immediate that $(\mathcal{G}_T, \rhd_T)$ enjoys the additional hypothesis **descendant**, **reversible-shape** and **reversible-cube** formulated in Section 7.

Remark: we have just proved the axiomatics (and the additional hypothesis) without ever using properties (1) and (3) of the asynchronous transition system T.

Remark: the standardization theorem is not really informative in $(\mathcal{G}_T, \rhd_T)$ because every permutation being reversible, all paths are standard. However, the axiomatics itself ensures that every asynchronous system satisfies the stability theorem stated in [30] which describes the structure of its successful runs.

PETRI NETS. The theory of Petri nets illustrates nicely the notion of asynchronous transition system. A Petri net is a quintuple $N = (C, j, F, \mathbf{pre}, \mathbf{post})$ where

- C is a set of *conditions*,
- j is a particular marking of N, called the *initial marking*, where a *marking* of N is defined as a multi-set of conditions,
- F is a set of *firings*,
- $\mathbf{pre}, \mathbf{post}$ are two functions associating to every firing $e \in F$ the nonempty markings $\mathbf{pre}(e)$ and $\mathbf{post}(e)$, called respectively the *pre-condition* and *post-condition* of e.

An asynchronous transition system $T_N = (S, i, E, I, \mathrm{Tran})$ is associated to every Petri net N in the following way, see [33]:

- S is the set of markings of N,
- i is the marking $j \in S$,
- E is the set F of firings,
- Tran is the set of triples (p, e, q) such that $p = p_0 \uplus \mathbf{pre}(e)$ and $q = p_0 \uplus \mathbf{post}(e)$ for a marking p_0, where $\uplus$ is the multi-set addition.
- I relates two firings $e_1, e_2 \in F$ precisely when $\mathbf{pre}(e_1) \cap \mathbf{pre}(e_2)$ and $\mathbf{post}(e_1) \cap \mathbf{post}(e_2)$ are empty multi-sets.

The axiomatic rewriting system $(\mathcal{G}_N, \rhd_N)$ associated to the asynchronous transition system T_N may be described directly, as follows. Its transition system $\mathcal{G}_N$ has the markings of N as vertices, and the triples

$$(p_0 \uplus \mathbf{pre}(e), e, p_0 \uplus \mathbf{post}(e)) = (p, e, q)$$

as edges $p \longrightarrow q$. The permutation relation $\rhd_N$ relates two paths $u \cdot v' \rhd v \cdot u'$ precisely when:

1. u and v are edges $u = (p, e_1, p_1)$ and $v = (p, e_2, p_2)$,
2. u' and v' are edges $u' = (p_2, e_1, p')$ and $v' = (p_1, e_2, p')$,
3. $\mathbf{pre}(e_1) \cap \mathbf{pre}(e_2)$ and $\mathbf{post}(e_1) \cap \mathbf{post}(e_2)$ are empty multi-sets.

BUBBLE SORT. The standardization procedure may be viewed as a generalization of the bubble sort algorithm, in which the order is not given *globally* but *locally*. Define $\mathcal{G}$

as the graph with a unique vertex M and, for every natural number $i \in \mathbb{N}$, an edge $[i] : M \longrightarrow M$. Let $\rhd$ be the least relation on paths such that

$$[j] \cdot [i] \rhd [i] \cdot [j]$$

when $i < j$. All the standardization axioms introduced in Section 2 are immediate on $(\mathcal{G}, \rhd)$ — except Axiom **enclave** which follows from the transitivity of the order on natural numbers. The standardization theorem of $(\mathcal{G}, \rhd)$ states that every sequence of natural numbers $[j_1] \cdots [j_k]$ may be reordered by local permutations into an increasing sequence $[i_1] \cdots [i_k]$ — and that this reordering is unique, since all the permutations of $(\mathcal{G}, \rhd)$ are irreversible.

HIERARCHICAL TRANSITION SYSTEMS. Here, we subsume the two previous examples of asynchronous transition systems, and of bubble sort on natural numbers, into what we call a *hierarchical transition system*. The idea is to order events in an asynchronous transition system (typically firings in a Petri net) with a precedence relation $\preceq$ satisfying a *weak transitivity* condition.

A *hierarchical transition system* is a quintuple $T = (S, i, E, \preceq, \mathrm{Tran})$ where

- S is a set of *states* with *initial state* i,
- E is a set of *events*,
- $\mathrm{Tran} \subset S \times L \times S$ is a *transition relation*,
- $\preceq \subset E \times E$ is a reflexive relation called the *precedence relation*.

The independence relation I is defined as

$$eIe' \iff \neg(e \preceq e') \text{ and } \neg(e' \preceq e) \tag{25}$$

The strict precedence relation $\prec$ is defined as

$$e \prec e' \iff e \preceq e' \text{ and } \neg(e' \preceq e)$$

Every hierarchical transition system is supposed to satisfy three axioms:

1. *determinacy:* $\forall (s, e, s'), (s, e, s'') \in \mathrm{Tran}, \quad s' = s''$,
2. *independence:* $\forall (s, e_2, s_2), (s_2, e_1, s') \in \mathrm{Tran}$,

$$\neg(e_2 \preceq e_1) \;\Rightarrow\; \exists s_1, \quad (s, e_1, s_1) \in \mathrm{Tran} \text{ and } (s_1, e_2, s') \in \mathrm{Tran}$$

3. *weak transitivity:* $\forall (e, e', e'') \in E \times E \times E$,

$$e \prec e' \preceq e'' \;\Rightarrow\; e \preceq e''.$$

Hierarchical transition systems extend usual asynchronous transition systems, since every asynchronous transition system $T = (S, i, E, I, \mathrm{Tran})$ may be seen as the hierarchical transition system $V(T) = (S, i, E, \preceq_{V(T)}, \mathrm{Tran})$ with precedence relation $\preceq_{V(T)}$ defined as:

$$\forall (e, e') \in E \times E, \qquad e \preceq_{V(T)} e' \iff \neg(eIe')$$

Here, weak transitivity of $\preceq_{V(T)}$ follows from symmetricity. Now, we associate to every hierarchical transition system $T = (S, i, E, \preceq, \mathrm{Tran})$ the following AxRS $(\mathcal{G}_T, \rhd_T)$:

– whose transition system $\mathcal{G}_T$ has states as vertices and transitions (s, e, s') as arrows,
– whose permutation relation $\rhd_T$ relates two paths f and g as $f \rhd_T g$, precisely when $f = (s, e_2, s_2) \cdot (s_2, e_1, s')$, $g = (s, e_1, s_1) \cdot (s_1, e_2, s')$ and the two events e_1 and e_2 satisfy $\neg(e_2 \preceq e_1)$.

In particular: the permutation $f \rhd_T g$ is reversible iff $e_1 I e_2$ and irreversible iff $e_1 \prec e_2$. We claim that $(\mathcal{G}_T, \rhd_T)$ is an axiomatic rewriting system. All the standardization axioms hold in $(\mathcal{G}_T, \rhd_T)$ for the same reasons as in the case of asynchronous transition systems — except for Axiom **enclave**, which follows from the weak transitivity of the precedence relation $\preceq$.

This enables to state a standardization theorem for every hierarchical transition system T. A particularly interesting case is when the precedence relation $\preceq$ is a partial order. In that case, the standard paths of $(\mathcal{G}_T, \rhd_T)$ may be characterized as the sequences of transition:

$$s_1 \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} s_n$$

in which there exists no pair of indices $1 \leq i < j \leq n$ such that $e_j \prec e_i$ (Hint: use the characterization lemma, Lemma 19). Thus, the standardization theorem states that every sequence of transitions in T

$$s_1 \xrightarrow{e_1} \xrightarrow{e_2} \cdots \xrightarrow{e_{n-1}} s_n$$

may be reorganised, after a series of permutations $\rhd_T$, into such an ordered sequence, and that this sequence is unique, modulo permutation of independent events.

We illustrate our point that weak transitivity of $\preceq$ is necessary to establish standardization. Consider the pseudo hierarchical transition system T with one state s, three events a, b, c, and the following precedence relation $\preceq$

$$a \preceq b \qquad b \preceq c \qquad c \preceq a$$

The relation $\preceq$ is not weakly transitive, and consequently, the uniqueness property fails: the sequence

$$s \xrightarrow{c} s \xrightarrow{b} s \xrightarrow{a} s$$

may be standardized as any of the two transition paths

$$s \xrightarrow{b} s \xrightarrow{c} s \xrightarrow{a} s \qquad \text{and} \qquad s \xrightarrow{c} s \xrightarrow{a} s \xrightarrow{b} s$$

which are not equal modulo permutation of independent events (the independence relation is empty in T.)

ERASING TRANSITION SYSTEMS. We mention only briefly that it is possible to enrich hierarchical transition systems with a notion of *erasure* between events. Start from a hierarchical transition system $(S, i, E, \preceq, \mathrm{Tran})$ and equip it with a binary relation K on events, called the *erasing* relation, chosen among the subrelations of $\prec$. Then, replace property (2) of hierarchical transition systems, by the two axioms:

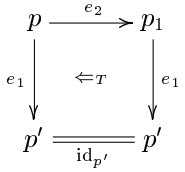
1. *K-erasure:* $\forall (s, e_2, s_2), (s_2, e_1, s') \in \mathrm{Tran}$,

$$e_1 K e_2 \text{ and } \neg(e_2 \preceq e_1) \;\Rightarrow\; (s, e_1, s') \in \mathrm{Tran}$$

2. *K-permutation:* $\forall (s, e_2, s_2), (s_2, e_1, s') \in \mathrm{Tran}$,

$\neg(e_1 K e_2)$ and $\neg(e_2 \preceq e_1) \;\Rightarrow\; \exists s_1, \quad (s, e_1, s_1) \in \mathrm{Tran}$ and $(s_1, e_2, s') \in \mathrm{Tran}$

This defines what we call an *erasing transition system* $T = (S, i, E, \preceq, K, \mathrm{Tran})$. The definition of the AxRS $(\mathcal{G}_T, \rhd_T)$ associated to T proceeds as in the case of hierarchical transition system, except that permutations of the form



are considered when $e_1 K e_2$. The standardization axioms hold in $(\mathcal{G}_T, \rhd_T)$ for the same reasons as in the hierarchical case.

TERM REWRITING SYSTEMS. The reader interested in term rewriting systems will find an introduction to the subject in [21, 19, 2, 11] and a comprehensive study of standardization in [35]. Here, we recall only that

1. a term rewriting system is a pair $\Sigma = (\mathcal{F}, \{\rho_1, ..., \rho_n\})$ where $\mathcal{F}$ is the *signature* of an algebra and every ρ_i is a rewriting rule on this algebra.
2. a *rewriting rule* $\rho : L \to R$ is a pair of open terms of the algebra such that every variable in R also occurs in L,
3. a *redex* in Σ is a quadruple (M, o, ρ, σ) where M is a term, o is an occurrence of M, ρ is a rewriting rule $L \to R$ of the system and σ is a valuation of the variables appearing in L, such the term M decomposes as $M = C[L\sigma]_o$ for some context $C[-]_o$ with unique hole $[-]$ at occurrence o. Notation: we write $u : M \longrightarrow N$ for $N = C[R\sigma]_o$.
4. If the variable x occurs $k \geq 1$ times in L, every redex v in a term $\sigma(x)$ corresponds to k redexes $v_1, ..., v_k$ in the term $M = C[L\sigma]_o$. We say that $u = (M, o, \rho, \sigma)$ *nests* each of the redexes v_i; and that it *nests* the redex v *linearly* when $k = 1$,
5. We say that two redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$ are *disjoint* when their occurrences in M are non comparable w.r.t the prefix order.
6. a rewriting rule $L \to R$ is *left-linear* when L does not contain two occurrences of the same variable. In that case, the only possibility for a redex to nest another redex, is to nest it linearly.

The transition system $\mathcal{G}_\Sigma$ of the rewriting system Σ has the terms M of the algebra as vertices and the redexes $u : M \longrightarrow N$ induced by the system as edges. The relation $\rhd$ on path in $\mathcal{G}_\Sigma$ is the least relation such that:

1. $v \cdot u' \rhd_\Sigma u \cdot v'$ when the redexes $u = (M, o_1, \rho_1, \sigma_1) : M \longrightarrow P$ and $v = (M, o_2, \rho_2, \sigma_2) : M \longrightarrow Q$ are disjoint and $u' = (Q, o_1, \rho_1, \sigma_1)$ and $v' = (P, o_2, \rho_2, \sigma_2)$,
2. $v \cdot u' \rhd_\Sigma u \cdot f$ when $u = (M, o_1, \rho_1, \sigma_1) : M \longrightarrow P$ nests $v = (M, o_1; o, \rho_2, \sigma_2) : M \longrightarrow Q$ linearly, $u' = (Q, o_1, \rho_1, \sigma_1) : Q \longrightarrow N$ and $f : P \longrightarrow N$ is the *complete development* of the copies of v through u (see [21, 18, 27, 22] for a formal definition of complete developments and copies).

In order to prove that $(\mathcal{G}_\Sigma, \rhd_\Sigma)$ satisfies the standardization axioms, we mediate through an axiomatic nesting system $(\mathcal{G}_\Sigma, [\![-]\!]_\Sigma, \preceq_\Sigma, \uparrow_\Sigma)$ and the ten N-axioms of Section 6. Our diagrammatic standardization theorem 2 will generalize the results of [18, 6] to possibly non-left-linear term rewriting systems.

The main point to clarify is: how shall the usual compatibility, nesting and residual relations be extended from left-linear to general term rewriting systems? There is a constraint: that the resulting axiomatic nesting system $(\mathcal{G}_\Sigma, [\![-]\!]_\Sigma, \preceq_\Sigma, \uparrow_\Sigma)$ generates the axiomatic rewriting system $(\mathcal{G}_\Sigma, \rhd_\Sigma)$ defined hereabove. The definition follows immediately. Two coinitial redexes u and v are *compatible*, what we write $u \uparrow_\Sigma v$, when

- the redexes u and v are disjoint,
- or when the redex u nests the redex v linearly,
- or when the redex v nests the redex u linearly.

We define the relation $[\![-]\!]_\Sigma$. When u and v are not compatible, the redex u has simply no residual after v (in particular, $u[\![u]\!]_\Sigma$ is empty). When u and v are compatible, the definition of the residuals of u after v proceeds as in left-linear rewriting systems:

- when the redexes u and $v : M \longrightarrow N$ are disjoint, or when u nests v linearly, then $u = (M, o_1, \rho_1, \sigma_1)$ has the redex $u' = (N, o_1, \rho_1, \sigma'_1)$ with same occurrence in N as residual.
- when the redex $v = (M, o_2, \rho_2 = L \longrightarrow R, \sigma_2)$ nests the redex u linearly, then the redex u has a residual u' after v for each occurrence of the variable x in R — where x is the variable substituted in L by the term $\sigma_2(x)$ containing the redex u.

Finally, we write $u \preceq_\Sigma v$ when the redex u nests the redex v linearly. Obviously, the axiomatic rewriting system $(\mathcal{G}_\Sigma, \rhd_\Sigma)$ derives from the resulting axiomatic rewriting system, by Definition 43. Moreover, each of the ten N-axioms are nearly immediate: N-axioms **Finite**, **Compat**, **Ancestor**, **Self** are obvious, while N-axioms **FinDev** and **Perm** generalize the well-known finite development lemma for left-linear term rewriting systems, established in [18, 20, 3, 27]. The four remaining N-axioms **I**, **II**, **III** and **IV** are also immediate.

Remark: consider the term $F(A, A)$ in the non left-linear rewriting system Σ:

$$F(x, x) \longrightarrow G(x) \qquad\qquad A \longrightarrow B$$

Intuitively, there *should* be a permutation:

$$\begin{array}{ccccc} F(A,A) & & \xrightarrow{\quad A_1 \quad} & & F(B,A) \xrightarrow{\quad A_2 \quad} F(B,B) \\ {\scriptstyle F}\downarrow & & & & \downarrow{\scriptstyle F} \\ G(A) & & \xrightarrow{\qquad\qquad A \qquad\qquad} & & G(B) \end{array} \tag{26}$$

oriented as follows: $A_1 \cdot A_2 \cdot F \Longrightarrow F \cdot A$. However, in our presentation, we replace the permutation by a critical pair (= a hole) between the two redexes $F(A, A) \longrightarrow G(A)$

and $F(A,A) \longrightarrow F(B,A)$. This is one limit of our current axiomatic theory: we do not know how to integrate permutations like (26) in our standardization framework. The 2-categorical approach of Section 5 is likely to provide a solution, at least because it replaces the Axiom **shape** by the more flexible notion of partial injection $[\alpha]$.

λ-CALCULUS [TREE-NESTING ORDER]. We have already established in Section 2, at least informally, that the nine standardization axioms hold for this λ-calculus, and its associated 2-dimensional transition system $(\mathcal{G}_\lambda, \rhd_{\mathrm{tree}})$. It is worth observing that the axiomatic nesting system $(\mathcal{G}_\lambda, [\![-]\!]_\lambda, \preceq_{\mathrm{tree}}, \uparrow_\lambda)$ satisfies moreover the ten N-axioms of Section 6. This follows on one part from traditional results on β-redexes and residuals appearing in [24, 3], and on the other part, from elementary arguments on the dynamics of β-reduction which establish together the N-axioms **I**, **II**, **III** and **IV**. By Theorem 5, this provides another way to prove that $(\mathcal{G}_\lambda, \rhd_{\mathrm{tree}})$ satisfies the 2-dimensional axiomatics of Section 2.

λ-CALCULUS [LEFT ORDER]. It is interesting to examine the reasons why the axiomatic nesting system associated to the λ-calculus and its left-order $\preceq_{\mathrm{left}}$ satisfies the N-axioms formulated in Section 6. Six of the ten N-axioms do not mention the nesting order, and were thus already discussed in the previous paragraph. The four remaining axioms are N-axioms **I**, **II**, **III** and **IV**. The two N-axiom **I** and **IV** are easy to check. N-axiom **IV** for instance follows from the fact that the order $\preceq_{\mathrm{left}}$ is *total*, and thus, that there exists no reversible permutations in the system. The two remaining N-axioms **II** and **III** are less obvious to establish. However, both of them hold inherently for the reason that in a λ-term PQ, no computation in Q may induce (by creation or residual) a β-redex above the λ-term P. This fundamental property of the λ-calculus is precisely the reason for the *left-orientation* of this calculus, discussed at length in the introduction of this article.

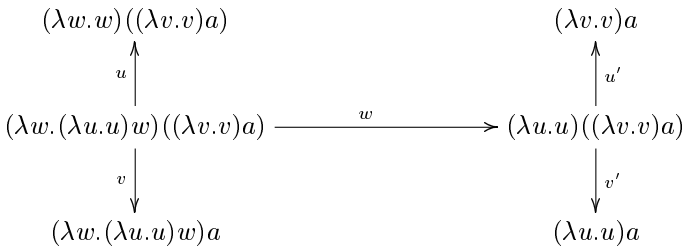
In that specific case, the diagrammatic standardization theorem repeats the traditional *leftmost-outermost* standardization theorem established in [24, 20, 3]. Since there exists no reversible permutation, the equivalence relation $\simeq$ modulo reversible permutation coincides with the equality. This explains why the standard path g of a path f is *unique* in that case — and not just unique *modulo*.

λ-CALCULUS [ARGUMENT ORDER]. In contrast to the two orders $\preceq_{\mathrm{tree}}$ and $\preceq_{\mathrm{left}}$, this particular order on β-redexes does not fall into the scope of our previous axiomatic presented in [13] for the following reason. An axiom requires that whenever two β-redexes u and v have respective residuals u' and v' after β-reduction of a coinitial β-redex w, then:

$$(u' \preceq_{\mathrm{arg}} v' \Rightarrow u \preceq_{\mathrm{arg}} v) \quad \text{or} \quad (w \preceq_{\mathrm{arg}} u \text{ and } w \preceq_{\mathrm{arg}} v). \tag{27}$$

The axiom states that a redex w may only alter the relative positions of redexes u and v when the two redexes are under the redex w. The argument-order $\preceq_{\mathrm{arg}}$ does not satisfy this property in general, typically when the β-redex $w : (\lambda x.M)P$ substitutes its argument P containing the β-redex v inside the argument of a β-redex u in the function

$(\lambda x.M)$. This is illustrated by the three coinitial β-redexes u, v and w:



It is not difficult to see that Property (27) is not satisfied, since:

- the β-redexes u is not in the argument of the β-redex w: thus, $\neg(w \preceq_{\rm arg} u)$.
- the β-redex v is not in the argument of the β-redex u: thus, $\neg(u \preceq_{\rm arg} v)$,
- after β-contraction of the β-redex w, the residual v' of the β-redex v appears in the argument of the residual u' of the β-redex u: thus, $u' \preceq_{\rm arg} v'$.

It took us a lot of time to realize after [13] that Property (27) can be weakened and replaced by the N-axiom **II**b. formulated in Section 6, without breaking the standardization theorem. We recall that the N-axiom **II**b. states that in the earlier situation:

$$(u' \preceq_{\rm arg} v' \Rightarrow u \preceq_{\rm arg} v) \quad \text{or} \quad w \preceq_{\rm arg} v.$$

In other words, it is possible for a redex w above a redex v to position one of its residuals v' under a redex u not nested by the redex w. This is precisely what happens in our example. So, this weaker property and the nine other N-axioms are satisfied by the axiomatic nesting system $(\mathcal{G}_\lambda, \preceq_{\rm arg}, [\![-]\!]_\lambda, \uparrow_\lambda)$. Thus, contrary to what happened in [13], our axiomatics does not discriminate between the three different partial orders $\preceq_{\rm tree}$, $\preceq_{\rm left}$ and $\preceq_{\rm arg}$ on the β-redexes in λ-terms. Consequently, the argument-order $\preceq_{\rm arg}$ induces a well-behaved standardization theorem on the λ-calculus — just like the tree-order $\preceq_{\rm tree}$ and the left-order $\preceq_{\rm left}$.

λ-CALCULUS [CALL-BY-VALUE]. A *value* of the λ-calculus is defined either as a variable or as a λ-term of the form $\lambda x.M$. G. Plotkin introduces in [38] the call-by-value λ-calculus, whose unique β_v-reduction $(\lambda x.M)V \to M[V/x]$ is the β-rule restricted to value arguments V. It is not difficult to show that the λ_v-calculus — interpreted as an axiomatic nesting system — satisfies the ten N-axioms formulated in Section 6. The resulting standardization theorem, which is non-trivial to prove directly on the syntax, leads to Plotkin's formalization of Landin's SECD machine, see [12] for instance.

EXPLICIT SUBSTITUTIONS. The usual β-reduction $(\lambda x.M)P \dashrightarrow M[P/x]$ copies its argument P as many times as the variable x occurs in M. This is fine theoretically, but inefficient if one wants to implement β-reduction in a computer. Thus, in most implementations of the λ-calculus, the argument P is not substituted, but stored in a *closure* and applied only when necessary. Unfortunately, the alternative evaluation mechanism complicates the task of checking the *correctness* of the implementation, by translating it back to the λ-calculus.

$$
\begin{array}{|llll|}
\hline
\multicolumn{4}{|l|}{Beta\ (\lambda a)b \to a[b \cdot id]} \\
 & & & \\
App & (ab)[s] \to a[s]b[s] & VarId & \mathbf{1}[id] \to \mathbf{1} \\
Abs & (\lambda a)[s] \to \lambda(a[\mathbf{1} \cdot (s \circ \uparrow)]) & VarCons & \mathbf{1}[a.s] \to a \\
Clos & a[s][t] \to a[s \circ t] & IdL & id \circ s \to s \\
Map & (a \cdot s) \circ t \to a[t] \cdot (s \circ t) & ShiftId & \uparrow \circ id \to \uparrow \\
Ass & (s_1 \circ s_2) \circ s_3 \to s_1 \circ (s_2 \circ s_3) & ShiftCons & \uparrow \circ (a \cdot s) \to s \\
\hline
\end{array}
$$

Fig. 6. The 11 rules of the $\lambda\sigma$-calculus

So, the $\lambda\sigma$-calculus was introduced in [1] to bridge the λ-calculus and its implementations. In the $\lambda\sigma$-calculus, substitutions are *explicit*, they can be delayed and stored just like closures. This enables to factorize many translations from abstract machines to the λ-calculus, see [15].

$$\text{Abstract Machine} \xrightarrow{\text{translation}} \lambda\sigma\text{-calculus} \xrightarrow{\text{interpretation}} \lambda\text{-calculus}$$

Formally, the $\lambda\sigma$-calculus contains two classes of objects: terms and substitutions. Terms are written in the de Bruijn notation.

$$
\begin{array}{ll}
\textbf{terms} & a ::= \mathbf{1} \mid ab \mid \lambda a \mid a[s] \\
\textbf{substitutions} & s ::= id \mid \uparrow \mid a \cdot s \mid s \circ t
\end{array}
$$

Ten rules (called the σ-rules) describe how substitutions should be delayed, propagated, composed and performed. An eleventh rule of the calculus, the $Beta$ rule, mimicks the β-rule of the λ-calculus, see Figure 6.

This makes the $\lambda\sigma$-calculus a *fibered* rewriting system with underlying *basis* the λ-calculus. The σ-calculus is strongly normalizing and confluent. Thus, every (closed) $\lambda\sigma$-term may be interpreted as the λ-term $\sigma(a)$ obtained by σ-normalization. The fiber F_M indexed by the λ-term M contains all $\lambda\sigma$-terms a interpreted as $\sigma(a) = M$. It is possible to extend the interpretation from terms to computations, and to project every $\lambda\sigma$-rewriting path $a \longrightarrow b$ to a β-rewriting path $\sigma(a) \longrightarrow \sigma(b)$ (modulo equivalence $\simeq$ though). Properties of the interpretation are studied thoroughly in [14, 9, 40, 28].

The $\lambda\sigma$-calculus is kind of hybrid between deterministic and non-deterministic rewriting systems. As a fibered system over the λ-calculus, it satisfies many properties of conflict-free rewriting systems, like confluence. At the same time, with eleven rules and eleven critical pairs (see Figure 7) the $\lambda\sigma$-calculus is an elaborate instance of a calculus with conflicts. Besides, to add some spice, its evaluation mechanism may behave counter-intuitively, as witnessed by the author's non-termination example of a simply-typed $\lambda\sigma$-term, presented in [26].

For all these reasons, the $\lambda\sigma$-calculus has been our training partner since the early days of the axiomatic theory. Many fundamental ideas of the theory (e.g. factorization, stability) originate from the meticulous analysis of its evaluation mechanism. Of course, like every term rewriting system, the $\lambda\sigma$-calculus defines an axiomatic rewriting system. As such, it satisfies the standardization theorem established in the article, as well as the

$App + Beta$	$(\lambda a)[s](b[s])$	$\overset{App}{\leftarrow}$	$((\lambda a)b)[s]$	$\overset{Beta}{\longrightarrow}$	$a[b \cdot id][s]$
$Clos + App$	$(ab)[s \circ t]$	$\overset{Clos}{\leftarrow}$	$(ab)[s][t]$	$\overset{App}{\longrightarrow}$	$(a[s](b[s]))[t]$
$Clos + Abs$	$(\lambda a)[s \circ t]$	$\overset{Clos}{\leftarrow}$	$(\lambda a)[s][t]$	$\overset{Abs}{\longrightarrow}$	$(\lambda(a[\mathbf{1} \cdot s \circ \uparrow]))[t]$
$Clos + VarId$	$\mathbf{1}[id \circ s]$	$\overset{Clos}{\leftarrow}$	$\mathbf{1}[id][s]$	$\overset{VarId}{\longrightarrow}$	$\mathbf{1}[s]$
$Clos + VarCons$	$\mathbf{1}[(a \cdot s) \circ t]$	$\overset{Clos}{\leftarrow}$	$\mathbf{1}[a \cdot s][t]$	$\overset{VarCons}{\longrightarrow}$	$a[t]$
$Clos + Clos$	$a[s][t \circ t']$	$\overset{Clos}{\leftarrow}$	$a[s][t][t']$	$\overset{Clos}{\longrightarrow}$	$a[s \circ t][t']$
$Ass + Map$	$(a \cdot s) \circ (t \circ t')$	$\overset{Ass}{\leftarrow}$	$((a \cdot s) \circ t) \circ t'$	$\overset{Map}{\longrightarrow}$	$(a[t] \cdot s \circ t) \circ t'$
$Ass + IdL$	$id \circ (s \circ t)$	$\overset{Ass}{\leftarrow}$	$(id \circ s) \circ t$	$\overset{IdL}{\longrightarrow}$	$s \circ t$
$Ass + ShiftId$	$\uparrow \circ (id \circ s)$	$\overset{Ass}{\leftarrow}$	$(\uparrow \circ id) \circ s$	$\overset{ShiftId}{\longrightarrow}$	$\uparrow \circ s$
$Ass + ShiftCons$	$\uparrow \circ ((a \cdot s) \circ t)$	$\overset{Ass}{\leftarrow}$	$(\uparrow \circ (a \cdot s)) \circ t$	$\overset{ShiftCons}{\longrightarrow}$	$s \circ t$
$Ass + Ass$	$(s \circ s') \circ (t \circ t')$	$\overset{Ass}{\leftarrow}$	$((s \circ s') \circ t) \circ t'$	$\overset{Ass}{\longrightarrow}$	$(s \circ (s' \circ t)) \circ t'$

Fig. 7. The 11 critical pairs of the $\lambda\sigma$-calculus

factorization and stability theorems established in later articles [29, 30]. We believe that this series of structure theorems play the same regulating role for the $\lambda\sigma$-calculus as the Church-Rosser property plays traditionnaly for the λ-calculus. For instance, we were able to formulate and establish in this way a normalization theorem for the needed strategies of the $\lambda\sigma$-calculus, see [28].

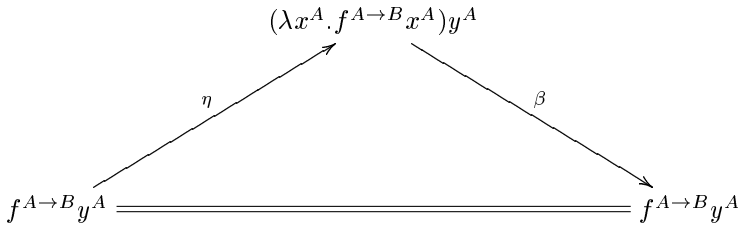
DAGS. The definition of a rewriting system Σ on directed acyclic graphs (dags) may be found in [8]. We interpret any dag rewriting system Σ as the following axiomatic rewriting system $(\mathcal{G}_\Sigma, \rhd_\Sigma)$. The graph $\mathcal{G}_\Sigma$ has dags and redexes of Σ as vertices and edges. Two paths f and g are related as $f \rhd_\Sigma g$ in two cases only:

- the reversible case: $f = v \cdot u'$ and $g = u \cdot v'$, when u and v are different compatible redexes, u' is the unique residual of u after v, and v' is the unique residual of v after u.
- the irreversible case: $f = v \cdot u'$ and $g = u$, when u and v are different compatible redexes, u' is the unique residual of u after v, and v does not have any residual after v, or equivalently, v is erased by u.

The nine standardization axioms are not too difficult to establish on $(\mathcal{G}_\Sigma, \rhd_\Sigma)$ in the same way as for erasing transition systems, considered a few paragraphs above.

Remark: in the case of a non-erasing dag rewriting system Σ, every rewriting path is standard. This indicates that our current axiomatic description of dag rewriting systems is not really satisfactory. Obviously, standardization should consider redex occurrence instead of simply redex erasure. We still do not know how to integrate such considerations in our standardization theory, see the discussion [27]. One solution may be to relax the notion of 2-dimensional normal form (=standard path) in a way similar to B. Hilken when he relaxes the definition of 1-dimensional normal form, in order to characterize the $\beta\eta$-long normal forms of simply-typed λ-calculus, see [16, 28] and the paragraph below.

λ-CALCULUS [ETA-EXPANSION]. B. Hilken considers the following permutation in simply-typed λ-calculus with β-reduction and η-expansion, see [16]:



(28)

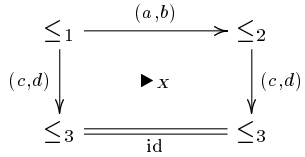
In this way, B. Hilken characterizes the $\beta\eta$-long normal forms as the λ-terms M such that, for every rewriting path $f : M \longrightarrow\!\!\!\!\!\rightarrow N$, there exists a path $g : N \longrightarrow\!\!\!\!\!\rightarrow M$ such that $f \cdot g : M \longrightarrow\!\!\!\!\!\rightarrow M$ is equivalent to $\mathrm{id}_M : M \longrightarrow\!\!\!\!\!\rightarrow M$ modulo permutation. This is one of the most interesting open problems of our Axiomatic Rewriting Theory: despite much effort, we do not know yet how permutations like (28) should be integrated in our diagrammatic theory.

ORDER SEQUENTIALIZATION. Here, we illustrate the fact that axiomatic rewriting systems *strictly* generalize axiomatic nesting systems. We fix a set X, and construct the transition system $\mathcal{G}_X$ as follows:

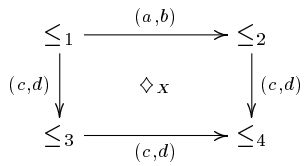
- its vertices are the partial orders on the set X,
- its edges $\leq_1 \!\longrightarrow\! \leq_2$ are the quadruples $(\leq_1, a, b, \leq_2)$ where (a, b) is a pair of incomparable elements in the partial order $(X, \leq_1)$, and the partial order $\leq_2$ is defined as:

$$\leq_2 \;=\; \leq_1 \;\cup\; \{(x, y) \in X \times X \mid x \leq_1 a \text{ and } b \leq_1 y\}$$

The 2-dimensional transition system $(\mathcal{G}_X, \rhd_X)$ is then defined as follows. Its *irreversible* permutations $f \blacktriangleright_X g$ relate two paths

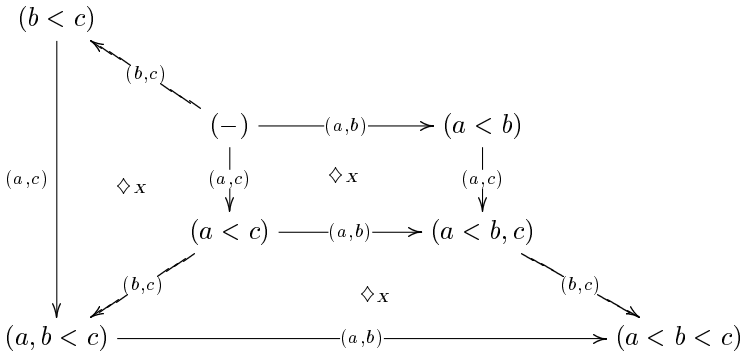


when $c \leq_1 a$ and $b \leq_1 d$. The *reversible* permutation relation $\diamondsuit_X$ relates two paths



when neither ($c \leq_1 a$ and $b \leq_1 d$) nor ($d \leq_1 a$ and $b \leq_1 c$).

It is easy to prove that the 2-dimensional transition system $(\mathcal{G}_X, \triangleright_X)$ defines an axiomatic rewriting system, for every set X. The normal forms of this system are the total orders on X. The interesting point is that the axiomatic rewriting system $(\mathcal{G}_X, \triangleright_X)$ associated to $X = \{a, b, c\}$ does not satisfy Axiom **reversible-cube** formulated in Section 7.3 — and thus, cannot be expressed as an axiomatic nesting system. Indeed, $(\mathcal{G}_X, \triangleright_X)$ contains the diagram



By Lemma 45 and 46 any such diagram may be completed as a reversible cube in an axiomatic rewriting system associated to an axiomatic nesting system. However, this diagram cannot be completed in $(\mathcal{G}_X, \triangleright_X)$.

9 Conclusion

Axiomatic Rewriting Theory is the latest attempt since Abstract Rewriting Theory [32, 17, 21] to describe *uniformly* all existing rewriting systems — from Petri nets to higher-order rewriting systems. The theory uncovers a series of diagrammatic principles underlying the syntactic mechanisms of computation, and reduces in this way the endemic variety of syntax to a uniform geometry of causality. In about a decade, the theory has bridged the gap with category theory and denotational semantics, and solved several difficult problems of Rewriting Theory:

- a normalization theorem for needed strategies in the $\lambda\sigma$-calculus, a λ-calculus with explicit substitutions, has been formulated and established in [28],
- a factorization theorem separating functorially the useful part of a rewriting path from the junk has been established in [29],
- an algebraic characterization of head-reductions in rewriting systems with critical pairs has been formulated in [30]. A syntactic characterization of head-reductions has been also formulated in the case of the $\lambda\sigma$-calculus [28].

This series of results demonstrates that a purely diagrammatic approach to Rewriting Theory is possible and fruitful. It also opens a series of interesting research directions, at the frontier of Rewriting Theory and Higher-Dimensional Categories, see for instance [23] and [31]. More specifically, we would like to capture *properly* the causal principles underlying Rewriting Systems like the λ-calculus with β-reduction and η-expansion, the non left-linear term rewriting systems, or the directed acyclic graph

rewriting systems. We are inclined to think that the diagrammatic language has something singular and innovative to articulate on these traditional topics of Rewriting Theory.

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