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▶ **To cite this version:**

Denis Feyel, Arnaud de La Pradelle, Gabriel Mokobodzki. A non commutative sewing lemma. 2007. hal-00151183

HAL Id: hal-00151183

https://hal.science/hal-00151183v1

Preprint submitted on 1 Jun 2007

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hal-00151183, version 1 - 1 Jun 2007

4 juin 2007

A NON-COMMUTATIVE SEWING LEMMA

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Key Words : Curvilinear Integrals, Rough Paths, Stochastic Integrals.

AMS 2000 Subject classification : Primary 26B35, 60H05.

Abstract A non-commutative version of the sewing lemma [1] is proved, with some applications.

Introduction

In a preceding paper [1] we proved a sewing lemma which was a key result for the study of Hölder continuous functions. In this paper we give a non-commutative version of this lemma.

In the first section we recall the commutative version, and give some applications (Young integral and stochastic integral).

In the second section we prove the non-commutative version. This last result has interesting applications : an extension of the so-called integral product, a simple case of the semigroup Trotter type formula, and a sharpening of the Lyons theorem about multiplicative functionals [3,4,5].

Note that we replaced the Hölder modulus of continuity t^α by a more general modulus $V(t)$.

This paper was elaborated with the regretted G. Mokobodzki. The writing has only been done after his death.

I. The additive sewing lemma

1 Definition : *We say that a function $V(t)$ defined on $[0,T[$ is a control function if it is non decreasing, $V(0)=0$ and $\sum_{n\geq 1} V(1/n)<\infty$.*

As easily seen, this is equivalent to the property

$$\overline{V}(t)=\sum_{n\geq 0} 2^n V(t.2^{-n})<\infty$$

for every $t\geq 0$. For example, t^α and $t/(\log t^{-1})^\alpha$ with $\alpha>1$ are control functions.

Observe that we have

$$\overline{V}(t)=V(t)+\cdots+2^n V(t.2^{-n})+2^{n+1}\overline{V}(t.2^{-n-1})$$

from which follows that $\operatorname*{Lim}_{t\to 0}\overline{V}(t)/t=0$.

2 Theorem : *Consider a continuous function $\mu(a,b)$ defined for $0\leq a\leq b<T$ satisfying the relation*

$$|\mu(a,b)-\mu(a,c)-\mu(c,b)|\leq V(b-a)$$

for every $c\in[a,b]$, where V is a control function. Then there exists a unique function $\varphi(t)$ on $[0,T[$, up to an additive constant, such that

$$|\varphi(b)-\varphi(a)-\mu(a,b)|\leq \overline{V}(b-a)$$

Proof : Put $\mu'(a,b)=\mu(a,c)+\mu(c,b)$ for $c=(a+b)/2$, and $\mu^{(n+1)}=\mu^{(n)\prime}$. We easily get for $n\geq 0$

$$|\mu^{(n)}(a,b)-\mu^{(n+1)}(a,b)|\leq 2^n V(2^{-n}|b-a|)$$

so that the series $\sum_{n\geq 0}|\mu^{(n)}(a,b)-\mu^{(n+1)}(a,b)|\leq \overline{V}(b-a)$ converges, and the sequence $\mu^{(n)}(a,b)$ converges to a limit $u(a,b)$. For $c=(a+b)/2$ we have $\mu^{(n+1)}(a,b)=\mu^{(n)}(a,c)+\mu^{(n)}(c,b)$ which implies

$$u(a,b)=u(a,c)+u(c,b)$$

We say that u is midpoint-additive.

Now, we prove that u is the unique midpoint-additive function with the inequality $|u(a,b)-\mu(a,b)|\leq \mathrm{Cst}\,\overline{V}(b-a)$. Indeed if we have another one v, we get

$$|v(a,b)-u(a,b)|\leq K.\overline{V}(b-a)$$

and by induction $|v(a,b)-u(a,b)|\leq 2^n K.\overline{V}[2^{-n}(b-a)]$ which vanishes as $n\to\infty$ as mentioned above. Let k be an integer $k\geq 3$, and take the function

$$w(a,b)=\sum_{i=0}^{k-1} u(t_i,t_{i+1})$$

with $t_i = a + i.(b-a)/k$. It follows that w also is midpoint-additive, and satisfies

$$|w(a,b) - \mu(a,b)| \leq \mathrm{Cst}_k \overline{V}(b-a)$$

hence we have $w = u$, that is u is in fact rationally-additive. As μ is continuous, then so also is u, as the defining series converges uniformly for $0 \leq a \leq b < T$. Then u is additive, and it suffices to put $\varphi(t) = u(0,t)$. $\square$

3 Proposition : *(Riemann sums) Let $\sigma = \{t_i\}$ some finite subdivision of $[a,b]$. Put $\delta = \mathrm{Sup}_i |t_{i+1} - t_i|$. Then*

$$\operatorname*{Lim}_{\delta \to 0} \sum_i \mu(t_i, t_{i+1}) = \varphi(b) - \varphi(a)$$

Proof : We have

$$\varphi(b) - \varphi(a) - \sum_i \mu(t_i, t_{i+1}) = \sum_i [\varphi(t_{i+1}) - \varphi(t_i) - \mu(t_i, t_{i+1})]$$

$$\left| \varphi(b) - \varphi(a) - \sum_i \mu(t_i, t_{i+1}) \right| \leq \sum_i \overline{V}(t_{i+1} - t_i)) \leq \varepsilon \sum_i (t_{i+1} - t_i) = (b-a)\varepsilon$$

since $\overline{V}(\delta)/\delta \leq \varepsilon$ as $\delta \to 0$.

4 Remarks : a) In fact the result holds even in the case of discontinuous μ, as proved in the appendix.

b) The result obviously extends to Banach spaces valued functions μ.

In the case $V(t) = t^\alpha$ with $\alpha > 1$, we get $\overline{V}(t) = \dfrac{2^\alpha t^\alpha}{2^\alpha - 2}$.

5 Example : *The Young integral*

Take $V(t) = t^{2\alpha}$ with $\alpha > 1/2$. If x and y are two α-Hölder continuous functions on $[0,1]$, put

$$\mu(a,b) = x_a(y_b - y_a)$$

We get

$$\mu(a,b) - \mu(a,c) - \mu(c,b) = -(x_c - x_a)(y_b - y_c)$$

so that

$$|\mu(a,b) - \mu(a,c) - \mu(c,b)| \leq \|x\|_\alpha \|y\|_\alpha |b-a|^{2\alpha}$$

where $\|x\|_\alpha$ is the norm in the space $\mathcal{C}^\alpha$. Let φ be the function of theorem 2, put

$$\int_a^b x_t \, dy_t = \varphi(b) - \varphi(a)$$

This is a Young integral (cf. also [7]).

6 Remark : *We could take $x \in \mathcal{C}^\alpha$, $y \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$.*

7 Example : *The stochastic integral*

Let X_t be the standard $\mathbb{R}^m$-valued Brownian motion. As is well known, $t \to X_t$ is $\mathcal{C}^{1/2}$ with values in L^2. Let f be a tensor-valued $\mathcal{C}^2$-function with bounded derivatives on $\mathbb{R}^m$. Put

$$\mu(a,b) = f(X_a) \otimes (X_b - X_a) + \nabla f(X_a) \otimes \int_a^b (X_t - X_a) \otimes dX_t$$

where the last integral is taken in the Ito or in the Stratonovitch sense. By straightforward computations, we get

$$N_2[\mu(a,b) - \mu(a,c) - \mu(c,b)] \leq K.\|\nabla f\|_{\mathcal{C}^1} |b-a|^{3/2}$$

By the additive sewing lemma, there exists a unique L^2-valued function $\varphi(t)$ such that $N_2[\varphi(b) - \varphi(a) - \mu(a,b)] \leq \mathrm{Cst}\,.|b-a|^{3/2}$ (the control function is $V(t) = t^{3/2}$). It is easily seen that

$$\varphi(b) - \varphi(a) = \int_a^b f(X_t) \otimes dX_t$$

in the Ito or in the Stratonovitch sense.

Observe that as the stochastic integral $\int_a^b X_t \otimes dX_t$ has $\mathcal{C}^\alpha$-trajectories almost surely for $1/3 < \alpha < 1/2$, analoguous computations as above yield $\mathcal{C}^\alpha$-trajectories for $\int_a^b f(X_t) \otimes dX_t$ on the same set of paths as $\int_a^b X_t \otimes dX_t$.

8 Remark : *For the FBM with $\alpha > 1/4$, the reader is referred to our previous paper [1].*

II. The multiplicative sewing lemma

Here we need a strong notion of control function

9 Definition : *We say that a function $V(t)$ defined on $[0,T[$ is a strong control function if it is a control function and there exists a $\theta > 2$ such that for every t*

$$\overline{V}(t) = \sum_{n \geq 0} \theta^n V(t.2^{-n}) < \infty$$

We consider an associative monoide $\mathcal{M}$ with a unit element I, and we assume that $\mathcal{M}$ is complete under a distance d satisfying

$$d(xz, yz) \leq |z|\, d(x,y), \qquad d(zx, zy) \leq |z|\, d(x,y)$$

for every $x, y, z \in \mathcal{M}$, where $z \to |z|$ is a Lipschitz function on $\mathcal{M}$ with $|I| = 1$.

Let $\mu(a, b)$ be an $\mathcal{M}$-valued function defined for $0 \leq a \leq b < T$. We assume that μ is continuous, that $\mu(a, a) = I$ for every a, and that for every $a \leq c \leq b$ we have

$$d(\mu(a, b), \mu(a, c)\mu(c, b)) \leq V(b - a) \tag{1}$$

We say that an $\mathcal{M}$-valued $u(a, b)$ is multiplicative if $u(a, b) = u(a, c)u(c, b)$ for every $a \leq c \leq b$.

10 Theorem : *There exists a unique multiplicative function u such that $d(\mu(a, b), u(a, b)) \leq \mathrm{Cst}\, \overline{V}(b - a)$ for every $a \leq b$.*

Proof : Put $\mu_0 = \mu$ and by induction

$$\mu_{n+1}(a, b) = \mu_n(a, c)\mu_n(c, b) \quad \text{where} \quad c = (a + b)/2$$

$$h_n(t) = \operatorname*{Sup}_{b-a \leq t} |\mu_n(a, b)|, \qquad U_n(t) = \operatorname*{Sup}_{b-a \leq t} d(\mu_{n+1}(a, b), \mu_n(a, b))$$

The functions h_n and U_n continuous non decreasing with $h_n(0) = 1$ and $U_n(0) = 0$. Let κ be the Lipschitz constant of $z \to |z|$. One has

$$h_{n+1}(t) \leq h_n(t) + \kappa U_n(t) \leq h_0(t) + \kappa U_0(t) + \cdots + \kappa U_n(t)$$

$$U_{n+1}(t) \leq 2h_{n+1}(t/2)U_n(t/2) \tag{2}$$

Let $\tau > 0$ be such that $h_0(\tau) + \kappa\overline{V}(\tau) \leq \theta/2$. Assume that $U_i(t) \leq \theta^i V(t/2^i)$ for $t \leq \tau$ and $i \leq n$. One has $h_{n+1}(t) \leq \theta/2$, then

$$U_{n+1}(t) \leq \theta U_n(t/2) \leq \theta^{n+1} V(t/2^{n+1})$$

for $t \leq \tau$ and every n by induction.

Hence for $t \leq \tau$ the series $U_n(t)$ converges, so that the sequence $h_n(\tau)$ is bounded. By inequality (2) the series $U_n(2\tau)$ converges, and the sequence $h_n(2\tau)$ is bounded. From one step to the other we see that the sequence h_n is locally bounded, and that the series U_n converges locally uniformly on $[0, T[$. It follows that the sequence $\mu_n(a, b)$ converges locally uniformly to a continuous function $u(a, b)$ which is midpoint-multiplicative, that is $u(a, b) = u(a, c)u(c, b)$ for $c = (a + b)/2$. One has $d(u, \mu) \leq \mathrm{Cst}\, \overline{V}$.

Next we prove the unicity of u. Let v be a function with the same properties as u. Put $K(t) = \operatorname*{Sup}_{b-a \leq t} [u(a, b), v(a, b)]$. Let $\tau_1 > 0$ be such that $K(\tau_1) \leq \theta/2$. One has $d(u(a, b), v(a, b)) \leq k\overline{V}(b - a)$ with some constant k, then $d(u(a, b), v(a, b)) \leq 2K(t/2)k\overline{V}(t/2) \leq k\theta\overline{V}(t/2)$ for $b - a \leq t \leq \tau_1$, and by induction $d(u(a, b), v(a, b)) \leq k\theta^n\overline{V}(t/2^n)$. It follows that $u(a, b) = v(a, b)$ for $b - a \leq \tau_1$. This equality extends to every $b - a$ by midpoint-multiplicativity.

Finally we prove that u is multiplicative. We argue as in the additive case, and we put for an integer m

$$w(a,b) = \prod_{i=0}^{m-1} u(t_i, t_{i+1})$$

where $t_i = a + i.(b-a)/m$. For simplicity we limit ourselves to the case $m = 3$, that is

$$w(a,b) = u(a,c')u(c',c'')u(c'',b)$$

with $c' = a+(b-a)/3$, $c'' = a+2(b-a)/3$. Observe that w is obviously midpoint-multiplicative. Take $a \le b \le T_0 < T$, we get successively with a constant k which can be changed from line to line

$$\begin{aligned}
d(w(a,b),\mu(a,b)) \le & k\overline{V}(b-a) + d(w(a,b),\mu(a,c')\mu(c',b)) \\
\le & k\overline{V}(b-a) + d(u(a,c')u(c',c'')u(c'',b), u(a,c')\mu(c',b)) \\
& \qquad + d(\mu(u(a,c')\mu(c',b), \mu(a,c')\mu(c',b)) \\
\le & k\overline{V}(b-a) + kd(u(c',c'')u(c'',b),\mu(c',b)) + kd(\mu(a,c'),\mu(a,c')) \\
\le & k\overline{V}(b-a) + kd(u(c',c'')u(c'',b),\mu(c',b)) \\
\le & k\overline{V}(b-a) + kd(u(c',c'')u(c'',b), u(c',c'')\mu(c'',b)) \\
& \qquad + kd(u(c',c'')\mu(c'',b), \mu(c',c'')\mu(c'',b)) \\
\le & k\overline{V}(b-a) + kd(u(c'',b),\mu(c'',b)) + kd(u(c',c''),\mu(c',c'')) \\
\le & k\overline{V}(b-a)
\end{aligned}$$

By the second step of the proof, we get $w = u$. The same proof extends to every m, so that u is in fact rationally multiplicative. As u is continuous, it is multiplicative. $\square$

11 Example : *The integral product*

Let $t \to A_t$ a $\mathcal{C}^\alpha$ function with values in a Banach algebra $\mathcal{A}$ with a unit I. Put $A_{ab} = A_b - A_a$ and

$$\mu(a,b) = I + A_{ab}$$

We get

$$\mu(a,b) - \mu(a,c)\mu(c,b) = -A_{ac}A_{cb}$$

Suppose that $\alpha > 1/2$, then the multiplicative sewing lemma applies with the obvious distance, and there exists a unique multiplicative function $u(a,b)$ with values in $\mathcal{A}$ such that

$$|u(a,b) - \mu(a,b)| \le \mathrm{Cst}\,|b-a|^{2\alpha}$$

We get the same $u(a,b)$ by taking $\mu(a,b) = \mathrm{e}^{A_{ab}}$. A good notation for $u(a,b)$ is

$$u(a,b) = \prod_a^b (I + dA_t) = \prod_a^b \mathrm{e}^{dA_t}$$

12 Theorem : *Put $H_t = u(0,t)$. Then this is the solution of the EDO*

$$H_t = I + \int_0^t H_s \, dA_s$$

Proof : We have only to verify that $|u(0,b) - u(0,a) - u(0,a)A_{ab}| \leq \mathrm{Cst}\, |b-a|^{2\alpha}$. The first member is worth

$$u(0,a)[u(a,b) - I - A_{ab}] = u(0,a)[u(a,b) - \mu(a,b)]$$

so that we are done.

13 Example : *A Trotter type formula*

Let $t \to A_t$ and $t \to B_t$ as in the previous paragraph, and put

$$\mu(a,b) = [I + A_{ab}][I + B_{ab}]$$

It is straightforward to verify the good inequality

$$|\mu(a,b) - \mu(a,c)\mu(c,b)| \leq \mathrm{Cst}\, |b-a|^{2\alpha}$$

so that we get a multiplicative $u(a,b)$ such that $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\, |b-a|^{2\alpha}$ or equivalently

$$|u(a,b) - I - A_{ab} - B_{ab}| \leq \mathrm{Cst}\, |b-a|^{2\alpha}$$

$$u(a,b) = \prod_a^b (I + dA_t + dB_t) = \prod_a^b \mathrm{e}^{dA_t}\, \mathrm{e}^{dB_t}$$

We then have

$$\mathrm{e}^{A_{ab}+B_{ab}} = \operatorname*{Lim}_{n\to\infty} \prod_{i=1}^n \mathrm{e}^{A_{t_i t_{i+1}}}\, \mathrm{e}^{B_{t_i t_{i+1}}}$$

for $t_{i+1} - t_i = (b-a)/2^n$.

Particularly we can take $A_t = tA$ and $B_t = tB$ with $\alpha = 1$, this yields

$$\mathrm{e}^{A+B} = \operatorname*{Lim}_{n\to\infty} \prod_{i=1}^n \mathrm{e}^{A/2^n}\, \mathrm{e}^{B/2^n}$$

14 Example : *Extending the Lyons theorem*

Let $\mathcal{A}$ be a Banach algebra with a unit I. Take $\mu(a,b)$ of the form

$$\mu(a,b) = \sum_{k=0}^n \lambda^k A_{ab}^{(k)}$$

where $A_{ab}^{(k)} \in \mathcal{A}$, λ *is a real parameter. We have*

$$\mu(a,c)\mu(c,b) = \sum_{k=0}^{n} \lambda^k B_{acb}^{(k)} + \sum_{k=n+1}^{2n} \lambda^k C_{acb}^{(k)}$$

Following [5], we suppose the algebraic hypothesis for $k \leq n$

$$A_{ab}^{(k)} = \sum_{i=0}^{k} A_{ac}^{(i)} A_{cb}^{(k-i)} \tag{3}$$

that is

$$\mu(a,c)\mu(c,b) = \mu(a,b) + \sum_{k=n+1}^{2n} \lambda^k C_{acb}^{(k)}$$

15 Theorem : *Under the condition (3) and the inequality*

$$|A_{ab}^{(k)}| \leq M|b-a|^{k\alpha}$$

for every $k \leq n$, *where* $\alpha > 1/(n+1)$, *there exists a unique multiplicative function* $u(a,b)$ *such that*

$$|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\, |b-a|^{(n+1)\alpha}$$

Moreover we have

$$u(a,b) = \sum_{k=0}^{n} \lambda^k A_{ab}^{(k)} + \sum_{n+1}^{\infty} \lambda^k B_{ab}^{(k)} \tag{4}$$

where the series is normally convergent for every λ.

Proof : The only problem is to prove formula (4), that is to prove that u is the sum of its Taylor expansion with respect to λ. In the case where $\mathcal{A}$ is a complex Banach algebra, the proof of the multiplicative sewing lemma yields a sequence of holomorphic functions which converges uniformly with respect to λ in every compact set of $\mathbb{C}$. Hence $u(a,b)$ in holomorphic in $\lambda \in \mathbb{C}$. If $\mathcal{A}$ is only a real Banach algebra, we get a sequence of holomorphic functions with values in the complexified Banach space of $\mathcal{A}$, and the result follows. It remains to observe that the $n+1$ first terms of the Taylor expansion are the same for every function of the sequence converging to u.

Application to the Lyons theorem : Let E be a Banach space. Denote $E^n = E^{\otimes n}$. For every $k \leq n$, let $(a,b) \to X_{ab}^{(k)}$ an E^k-valued function such that

$$X_{ab}^{(k)} = \sum_{i=0}^{k} X_{ac}^{(i)} \otimes X_{cb}^{(k-i)}$$

for $a \leq c \leq b$. Suppose that every E^n has a cross-norm such that

$$\|u \otimes v\|_{n+m} \leq \|u\|_n \|v\|_m$$

for every $u \in E^n$, $v \in E^m$. Suppose that $\alpha > 1/(n+1)$, and that we have for $k \leq n$

$$\|X_{ab}^{(k)}\|_k \leq M.|b-a|^{k\alpha}$$

Let $\mathcal{A}$ be the completed tensor algebra under the norm

$$\|t\| = \sum_{n \geq 0} \|t_n\|_n$$

This is a Banach algebra. The previous theorem applies, so that there exists a unique $(a,b) \to Y_{ab}^{(k)}$ for every k such that $Y^{(k)} = X^{(k)}$ for $k \leq n$,

$$Y_{ab}^{(k)} = \sum_{i=0}^{k} Y_{ac}^{(i)} \otimes Y_{cb}^{(k-i)}$$

for every k and every $a \leq c \leq b$, and

$$\sum_{k \geq n+1} \|Y_{ab}^{(k)}\|_k \leq \mathrm{Cst}\, |b-a|^{(n+1)\alpha}$$

16 Remark : This theorem sharpens the theorem 3.2.1 of [5].

Some estimations

We return to formula (4) of theorem 9

$$u(a,b) = \sum_{k=0}^{N} \lambda^k A_{ab}^{(k)} + \sum_{N+1}^{\infty} \lambda^k B_{ab}^{(k)}$$

for $N = \mathrm{Ent}(1/\alpha)$, and we put $B_{ab}^{(k)} = A_{ab}^{(k)}$ for simplification, so that we have

$$u(a,b) = \sum_{k=0}^{\infty} \lambda^k A_{ab}^{(k)}$$

There exist best constants K_n such that $|A_{ab}^{(n)}| \leq K_n |b-a|^{n\alpha}$. We have

$$A_{ab}^{(n+1)} = A_{ac}^{(n+1)} + A_{cb}^{(n+1)} + \sum_{k=1}^{n} A_{ac}^{(k)} A_{cb}^{(n-k+1)}$$

By taking $c = (a+b)/2$ we get

$$|A_{ab}^{(n+1)}| \leq 2^{-(n+1)\alpha} \left[2K_{n+1} + \sum_{k=1}^{n} K_k K_{n-k+1} \right] |b-a|^{(n+1)\alpha}$$

and then

$$(2^{(n+1)\alpha} - 2) K_{n+1} \leq \sum_{k=1}^{n} K_k K_{n-k+1}$$

Let $0 < \beta < \alpha$, and introduce the entire function

$$\mathrm{e}(x) = \mathrm{e}_\beta(x) = \sum_{n \geq 0} \frac{x^n}{n!^\beta} \qquad \Rightarrow \qquad \mathrm{e}(x)^2 = \sum_{n \geq 0} E_{n,\beta} \frac{x^n}{n!^\beta}$$

where

$$E_{n,\beta} = \sum_{k=0}^{n} \left[C_n^k \right]^\beta \leq 2^{n\beta}(n+1)$$

There exist $c \geq 0$ and $x > 0$ such that for $1 \leq m \leq N$

$$K_m \leq c.x^m / m!^\beta \tag{5}$$

Hence we have for $n \geq N$

$$(2^{(n+1)\alpha} - 2) K_{n+1} \leq c^2 x^{n+1} \sum_{k=1}^{n} (k!)^{-\beta} (n-k+1)^{-\beta} \leq c^2 x^{n+1} [(n+1)!]^{-\beta} A_{n+1,\beta}$$

In order that (5) holds for every n, it suffices that

$$\frac{1}{c} \geq \operatorname*{Sup}_{n > N} \frac{A_{n+1,\beta}}{2^{(n+1)\alpha} - 2}$$

which is possible since the fraction in the second hand member shrinks to 0 as $n \to \infty$.

17 Corollary : *Put* $c' = \mathrm{Max}(c, 1)$, *we have*

$$|u(a,b)| \leq c' \, \mathrm{e}_\beta(|\lambda| x |b-a|^\alpha)$$

18 Remarks : a) Note that for $\alpha = 1$ one can take $\beta = \alpha = 1$ so that we recover the classical inequality.

b) For $\beta < 1$, the function $\mathrm{e}_\beta(x)$ increases faster than the exponential function (*cf.* Schwartz [6] for $\beta = 1/2$).

c) there are some analoguous computations in Gubinelli [2].

Appendix : the discontinuous case

As announced in Remark 4a), we extend the additive sewing lemma in the case where μ is discontinuous. We go back to the proof of the lemma : we get a unique function $u(a,b)$ which is rationally additive and such that $|u(a,b) - \mu(a,b)| \leq \mathrm{Cst}\,\overline{V}(b-a)$. Put

$$v_n(a,b) = u(a_n,b_n) - u(a_n,a) + u(b_n,b)$$

where $a_n \leq a$ and $b_n \leq b$ are the classical dyadic approximations of a and b. It is straightforward to verify that v_n is additive for every $a \leq c \leq b$. Besides, we have

$$|v_n(a,b) - u(a,b)| \leq 2\overline{V}(b-a) + 2\overline{V}(b_n - a_n) + \overline{V}(a-a_n) + \overline{V}(b-b_n)$$

so that the sequence $v_n(a,b) - u(a,b)$ is bounded. Let $v(a,b)$ be the limit of $v_n(a,b)$ according to an ultrafilter $\mathcal{U} \to \infty$. We first have $v(a,b) = v(a,c) + v(c,b)$ for every $a \leq c \leq b$. Then we get

$$|v(a,b) - \mu(a,b)| \leq 3\overline{V}(b-a) + 2\operatorname*{Lim}_{\mathcal{U}} \overline{V}(b_n - a_n) \leq 5\overline{V}(2(b-a))$$

As $V(2t)$ is also a control function for μ, v is the unique additive function such that $|v(a,b) - \mu(a,b)| \leq 5\overline{V}(2(b-a))$, which implies that $v = u$. Hence u is completely additive. $\square$

Here we point out the important fact that the result also holds if μ takes values in a Banach space B. Indeed, the proof is exactly the same, the last limit according to $\mathcal{U}$ must be taken in the bidual B'' with the topology $\sigma(B'', B')$.

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