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FLAT MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

ABSTRACT. Let R be a valuation ring and let Q be its total quotient ring. It is proved that any singly projective (respectively flat) module is finitely projective if and only if Q is maximal (respectively artinian). It is shown that each singly projective module is a content module if and only if any non-unit of R is a zero-divisor and that each singly projective module is locally projective if and only if R is self injective. Moreover, R is maximal if and only if each singly projective module is separable, if and only if any flat content module is locally projective. Necessary and sufficient conditions are given for a valuation ring with non-zero zero-divisors to be strongly coherent or π -coherent.

A complete characterization of semihereditary commutative rings which are π -coherent is given. When R is a commutative ring with a self FP-injective quotient ring Q , it is proved that each flat R -module is finitely projective if and only if Q is perfect.

In this paper, we consider the following properties of modules: P-flatness, flatness, content flatness, local projectivity, finite projectivity and single projectivity. We investigate the relations between these properties when R is a PP-ring or a valuation ring. Garfinkel ([11]), Zimmermann-Huisgen ([22]), and Gruson and Raynaud ([13]) introduced the concepts of locally projective modules and strongly coherent rings and developed important theories on these. The notions of finitely projective modules and π -coherent rings are due to Jones ([15]). An interesting study of finitely projective modules and singly projective modules is also done by Azumaya in [1]. For a module M over a ring R , the following implications always hold:

$$\begin{array}{ccccc}
 M \text{ is projective} & \Rightarrow & M \text{ is locally projective} & \Rightarrow & M \text{ is flat content} \\
 & & \downarrow & & \downarrow \\
 & & M \text{ is finitely projective} & \Rightarrow & M \text{ is flat} \\
 & & \downarrow & & \downarrow \\
 & & M \text{ is singly projective} & \Rightarrow & M \text{ is P-flat,}
 \end{array}$$

but there are not generally reversible. However, if R satisfies an additional condition, we get some equivalences. For instance, in [2], Bass defined a ring R to be right perfect if each flat right module is projective. In [23] it is proved that a ring R is right perfect if and only if each flat right module is locally projective, and if and only if each locally projective right module is projective. If R is a commutative arithmetic ring, i.e. a ring whose lattice of ideals is distributive, then any P-flat module is flat. By [1, Proposition 16], if R is a commutative domain, each P-flat module is singly projective, and by [1, Proposition 18 and 15] any flat left module is finitely projective if R is a commutative arithmetic domain or a left noetherian ring. Consequently, if R is a valuation domain each P-flat module is finitely projective.

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When R is a valuation ring, we prove that this result holds if and only if the ring Q of quotients of R is artinian. Moreover, we show that R is maximal if and only if any singly projective module is separable or any flat content module is locally projective, and that Q is maximal if and only if each singly projective module is finitely projective.

In Section 2, necessary and sufficient conditions are given for a commutative semihereditary ring to be π -coherent. Moreover we characterize commutative PP-rings for which each product of singly projective modules is singly projective.

In the last section we study the valuation rings R for which each product of content (respectively singly, finitely, locally projective) modules is content (respectively singly, finitely, locally projective). The results are similar to those obtained by Zimmermann-Huisgen and Franzen in [8], and by Kemper in [16], when R is a domain. However, each valuation domain is π -coherent but not necessarily strongly coherent. We prove that a valuation ring with non-zero zero-divisors is π -coherent if and only if it is strongly coherent.

1. DEFINITIONS AND PRELIMINARIES

If A is a subset of a ring R , we denote respectively by $\ell(A)$ and $r(A)$ its left annihilator and its right annihilator. Given a ring R and a left R -module M , we say that M is **P-flat** if, for any $(s, x) \in R \times M$ such that $sx = 0$, $x \in r(s)M$. When R is a domain, M is P-flat if and only if it is torsion-free. As in [1], we say that M is **finitely projective** (respectively **singly projective**) if, for any finitely generated (respectively cyclic) submodule N , the inclusion map $N \rightarrow M$ factors through a free module F . A finitely projective module is called f-projective in [15]. As in [22] we say that M is **locally projective** if, for any finitely generated submodule N , there exist a free module F , an homomorphism $\phi : M \rightarrow F$ and an homomorphism $\pi : F \rightarrow M$ such that $\pi(\phi(x)) = x$, $\forall x \in N$. A locally projective module is said to be either a trace module or a universally torsionless module in [11]. Given a ring R , a left R -module M and $x \in M$, the **content ideal** $c(x)$ of x in M , is the intersection of all right ideals A for which $x \in AM$. We say that M is a **content module** if $x \in c(x)M$, $\forall x \in M$.

It is obvious that each locally projective module is finitely projective but the converse doesn't generally hold. For instance, if R is a commutative domain with quotient field $Q \neq R$, then Q is a finitely projective R -module: if N is a finitely generated submodule of Q , there exists $0 \neq s \in R$ such that $sN \subseteq R$, whence the inclusion map $N \rightarrow Q$ factors through R by using the multiplications by s and s^{-1} ; but Q is not locally projective because the only homomorphism from Q into a free R -module is zero.

Proposition 1.1. *Let R be a ring. Then:*

- (1) *Each singly projective left R -module M is P-flat. The converse holds if R is a domain.*
- (2) *Any P-flat cyclic left module is flat.*
- (3) *Each P-flat content left module M is singly projective.*

Proof. (1). Let $0 \neq x \in M$ and $r \in R$ such that $rx = 0$. There exist a free module F and two homomorphisms $\phi : Rx \rightarrow F$ and $\pi : F \rightarrow M$ such that $\pi \circ \phi$ is the inclusion map $Rx \rightarrow M$. Since $r\phi(x) = 0$ and F is free, there exist

$s_1, \dots, s_n \in \mathfrak{r}(r)$ and $y_1, \dots, y_n \in F$ such that $\phi(x) = s_1y_1 + \dots + s_ny_n$. Then $x = s_1\pi(y_1) + \dots + s_n\pi(y_n)$. The last assertion is obvious.

(2). Let C be a cyclic left module generated by x and let A be a right ideal. Then each element of $A \otimes_R C$ is of the form $a \otimes x$ for some $a \in A$. If $ax = 0$ then $\exists b \in \mathfrak{r}(a)$ such that $x = bx$. Therefore $a \otimes x = a \otimes bx = ab \otimes x = 0$. Hence C is flat.

(3). Let $x \in M$. Then, since $x \in \mathfrak{c}(x)M$ there exist $a_1, \dots, a_n \in \mathfrak{c}(x)$ and $x_1, \dots, x_n \in M$ such that $x = a_1x_1 + \dots + a_nx_n$. Let $b \in R$ such that $bx = 0$. Therefore $x \in \mathfrak{r}(b)M$ because M is P-flat. It follows that $\mathfrak{c}(x) \subseteq \mathfrak{r}(b)$. So, if we put $\phi(rx) = (ra_1, \dots, ra_n)$, then ϕ is a well defined homomorphism which factors the inclusion map $Rx \rightarrow M$ through ${}_R R^n$. \square

Theorem 1.2. *A ring R is left perfect if and only if each flat left module is a content module.*

Proof. If R is left perfect then each flat left module is projective. Conversely suppose that each flat left module is a content module. Let $(a_k)_{k \in \mathbb{N}}$ be a family of elements of R , let $(e_k)_{k \in \mathbb{N}}$ be a basis of a free left module F and let G be the submodule of F generated by $\{e_k - a_k e_{k+1} \mid k \in \mathbb{N}\}$. By [2, Lemma 1.1] F/G is flat. We put $z_k = e_k + G$, $\forall k \in \mathbb{N}$. Since F/G is content and $z_k = a_k z_{k+1}$, $\forall k \in \mathbb{N}$, there exist $c \in R$ and $n \in \mathbb{N}$ such that $z_0 = cz_n$ and $\mathfrak{c}(z_0) = cR$. It follows that $cR = ca_n \dots a_p R$, $\forall p > n$. Since $z_0 = a_0 \dots a_{n-1} z_n$, there exists $k > n$ such that $ca_n \dots a_k = a_0 \dots a_k$. Consequently $a_0 \dots a_k R = a_0 \dots a_p R$, $\forall p \geq k$. So, R is left perfect because it satisfies the descending chain condition on principal right ideals by [2, Theorem P]. \square

Given a ring R and a left R -module M , we say that M is **P-injective** if, for any $(s, x) \in R \times M$ such that $\ell(s)x = 0$, $x \in sM$. When R is a domain, M is P-injective if and only if it is divisible. As in [19], we say that M is **finitely injective** (respectively **FP-injective**) if, for any finitely generated submodule A of a (respectively finitely presented) left module B , each homomorphism from A to M extends to B . If M is an R -module, we put $M^* = \text{Hom}_R(M, R)$.

Proposition 1.3. *Let R be a ring. Then:*

- (1) *If R is a P-injective left module then each singly projective left module is P-injective;*
- (2) *If R is a FP-injective left module then each finitely projective left module is FP-injective and a content module;*
- (3) *If R is an injective module then each singly projective module is finitely injective and locally projective.*

Proof. Let M be a left module, F a free left module and $\pi : F \rightarrow M$ an epimorphism.

1. Assume that M is singly projective. Let $x \in M$ and $r \in R$ such that $\ell(r)x = 0$. There exists a homomorphism $\phi : Rx \rightarrow F$ such that $\pi \circ \phi$ is the inclusion map $Rx \rightarrow M$. Since F is P-injective, $\phi(x) = ry$ for some $y \in F$. Then $x = r\pi(y)$.

2. Assume that M is finitely projective. Let L be a finitely generated free left module, let N be a finitely generated submodule of L and let $f : N \rightarrow M$ be a homomorphism. Then $f(N)$ is a finitely generated submodule of M . So, there exists a homomorphism $\phi : f(N) \rightarrow F$ such that $\pi \circ \phi$ is the inclusion map $f(N) \rightarrow M$. Since F is FP-injective, there exists a morphism $g : L \rightarrow F$ such that $\phi \circ f$ is the restriction of g to N . Now it is easy to check that $\pi \circ g$ is the restriction of f to N .

Let $x \in M$. There exists a homomorphism $\phi : Rx \rightarrow F$ such that $\pi \circ \phi$ is the inclusion map $Rx \rightarrow M$. Let $\{e_i \mid i \in I\}$ be a basis of F . There exist a finite subset J of I and a family $(a_i)_{i \in J}$ of elements of R such that $\phi(x) = \sum_{i \in J} a_i e_i$. Let A be the right ideal generated by $(a_i)_{i \in J}$. Then $(0 : x) = (0 : \phi(x)) = \ell(A)$. Let B be a right ideal such that $x \in BM$. Then $x = \sum_{k=1}^p b_k x_k$ where $b_k \in B$ and $x_k \in M$, $\forall k$, $1 \leq k \leq p$. Let N be the submodule of M generated by $\{\pi(e_i) \mid i \in J\} \cup \{x_k \mid 1 \leq k \leq p\}$. Thus there exists a homomorphism $\varphi : N \rightarrow F$ such that $\pi \circ \varphi$ is the inclusion map $N \rightarrow M$. Therefore there exist a finite subset K of I and two families $\{d_{k,j} \mid 1 \leq k \leq p, j \in K\}$ and $\{c_{i,j} \mid (i,j) \in J \times K\}$ of elements of R such that $\varphi(\pi(e_i)) = \sum_{j \in K} c_{i,j} e_j$, $\forall i \in J$ and $\varphi(x_k) = \sum_{j \in K} d_{k,j} e_j$, $\forall k$, $1 \leq k \leq p$. It follows that $\varphi(x) = \sum_{j \in K} (\sum_{i \in J} a_i c_{i,j}) e_j = \sum_{j \in K} (\sum_{k=1}^p b_k d_{k,j}) e_j$. So, $\sum_{i \in J} a_i c_{i,j} = \sum_{k=1}^p b_k d_{k,j}$, $\forall j \in K$. Let A' be the right ideal generated by $\{\sum_{i \in J} a_i c_{i,j} \mid j \in K\}$. Then $A' \subseteq A$ and $A' \subseteq B$. Moreover, $\ell(A) = (0 : x) = (0 : \varphi(x)) = \ell(A')$. By [14, Corollary 2.5] $A = A'$. So, $A \subseteq B$. We conclude that $c(x) = A$ and M is a content module.

3. Let M be a singly projective module and $x \in M$. So, there exists a homomorphism $\phi : Rx \rightarrow F$ such that $\pi \circ \phi$ is the inclusion map $Rx \rightarrow M$. Since F is finitely injective, we can extend ϕ to M . By using a basis of F we deduce that $x = \sum_{k=1}^n \phi_k(x) x_k$ where $\phi_k \in M^*$ and $x_k \in M$, $\forall k$, $1 \leq k \leq n$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.4]. By a similar proof as in (2), we show that M is finitely injective, except that L is not necessarily a finitely generated free module. \square

A short exact sequence of left R -modules $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ is **pure** if it remains exact when tensoring it with any right R -module. We say that N is a pure submodule of M . This property holds if L is flat.

Lemma 1.4. *Let R be a local ring, let P be its maximal ideal and let N be a flat left R -module. Assume that N is generated by a family $(x_i)_{i \in I}$ of elements of N such that $(x_i + PN)_{i \in I}$ is a basis of N/PN . Then N is free.*

Proof. Let $(e_i)_{i \in I}$ be a basis of a free left module F , let $\alpha : F \rightarrow N$ be the homomorphism defined by $\alpha(e_i) = x_i$, $\forall i \in I$ and let L be the kernel of α . It is easy to check that $L \subseteq PF$. Let $y \in L$. We have $y = \sum_{i \in J} a_i e_i$ where J is a finite subset of I and $a_i \in P$, $\forall i \in J$. Since L is a pure submodule of F , $\forall i \in J$ there exists $y_i \in L$ such that $\sum_{i \in J} a_i e_i = \sum_{i \in J} a_i y_i$. We have $y_i = \sum_{j \in J_i} b_{i,j} e_j$ where J_i is a finite subset of I , $b_{i,j} \in P$, $\forall (i,j) \in J \times J_i$. Let $K = J \cup (\cup_{i \in J} J_i)$. If $i \in K \setminus J$ we put $a_i = 0$ and $a_{i,j} = 0$, $\forall j \in K$, and if $j \in K \setminus J_i$ we put $a_{i,j} = 0$ too. We get $\sum_{i \in K} a_i e_i = \sum_{j \in K} (\sum_{i \in K} a_i b_{i,j}) e_j$. It follows that $a_j = \sum_{i \in K} a_i b_{i,j}$. So, if A is the right ideal generated by $\{a_i \mid i \in K\}$, then $A = AP$. By Nakayama lemma $A = 0$, whence $F \cong N$. \square

A left R -module is said to be a **Mittag-Leffler** module if, for each index set Λ , the natural homomorphism $R^\Lambda \otimes_R M \rightarrow M^\Lambda$ is injective. The following lemma is a slight generalization of [6, Proposition 2.3].

Lemma 1.5. *Let R be a subring of a ring S and let M be a flat left R -module. Assume that $S \otimes_R M$ is finitely projective over S . Then M is finitely projective.*

Proof. By [15, Proposition 2.7] a module is finitely projective if and only if it is a flat Mittag-Leffler module. So we do as in the proof of [6, Proposition 2.3]. \square

From this lemma and [15, Proposition 2.7] we deduce the following proposition. We can also see .

Proposition 1.6. *Let R be a subring of a left perfect ring S . Then each flat left R -module is finitely projective.*

Proposition 1.7. *Let R be a commutative ring and let S be a multiplicative subset of R . Then:*

- (1) *For each singly (respectively finitely, locally) projective R -module M , $S^{-1}M$ is singly (respectively finitely, locally) projective over $S^{-1}R$;*
- (2) *Let M be a singly (respectively finitely) projective $S^{-1}R$ -module. If S contains no zero-divisors then M is singly (respectively finitely) projective over R .*

Proof. (1). We assume that $M \neq 0$. Let N be a cyclic (respectively finitely generated) submodule of $S^{-1}M$. Then there exists a cyclic (respectively finitely generated) submodule N' of M such that $S^{-1}N' = N$. There exists a free R -module F , a morphism $\phi : N' \rightarrow F$ and a morphism $\pi : F \rightarrow M$ such that $(\pi \circ \phi)(x) = x$ for each $x \in N'$. It follows that $(S^{-1}\pi \circ S^{-1}\phi)(x) = x$ for each $x \in N$. We get that $S^{-1}M$ is singly (respectively finitely) projective over R . We do a similar proof to show that $S^{-1}M$ is locally projective if M is locally projective.

(2) By Lemma 1.5 M is finitely projective over R if it is finitely projective over $S^{-1}R$. It is easy to check that M is singly projective over R if it is singly projective over $S^{-1}R$. \square

If R is a subring of a ring Q which is either left perfect or left noetherian, then then each flat left R -module is finitely projective by [20, Corollary 7]. We don't know if the converse holds. However we have the following results:

Theorem 1.8. *Let R be a commutative ring with a self FP-injective quotient ring Q . Then each flat R -module is finitely projective if and only if Q is perfect.*

Proof. "Only if" requires a proof. Let M be a flat Q -module. Then M is flat over R and it follows that M is finitely projective over R . By Proposition 1.7(1) $M \cong Q \otimes_R M$ is finitely projective over Q . From Proposition 1.3 we deduce that each flat Q -module is content. We conclude by Theorem 1.2 \square

Theorem 1.9. *Let R be a commutative ring with a Von Neumann regular quotient ring Q . Then the following conditions are equivalent:*

- (1) *Q is semi-simple;*
- (2) *each flat R -module is finitely projective;*
- (3) *each flat R -module is singly projective.*

Proof. (1) \Rightarrow (2) is an immediate consequence of [20, Corollary 7] and (2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1). First we show that each Q -module M is singly projective. Every Q -module M is flat over Q and R . So, M is singly projective over R . It follows that $M \cong Q \otimes_R M$ is singly projective over Q by Proposition 1.7(1). Now let A be an ideal of Q . Since Q/A is singly projective, it is projective. So, Q/A is finitely presented over Q and A is a finitely generated ideal of Q . Hence Q is semi-simple. \square

2. π -COHERENCE AND PP-RINGS

As in [22] we say that a ring R is left **strongly coherent** if each product of locally projective right modules is locally projective and as in [3] R is said to be right **π -coherent** if, for each index set Λ , every finitely generated submodule of R_R^Λ is finitely presented.

Theorem 2.1. *Let R be a commutative ring. Then the following conditions are equivalent:*

- (1) R is π -coherent;
- (2) for each index set Λ , R^Λ is finitely projective;
- (3) each product of finitely projective modules is finitely projective.

Proof. (1) \Rightarrow (2). Let N be a finitely generated submodule of R^Λ . There exist a free module F and an epimorphism π from F into R^Λ . It is obvious that R is coherent. Consequently R^Λ is flat. So $\ker \pi$ is a pure submodule of F . Since N is finitely presented it follows that there exists $\phi : N \rightarrow F$ such that $\pi \circ \phi$ is the inclusion map from N into R^Λ .

(2) \Rightarrow (1). Since R^Λ is flat for each index set Λ , R is coherent. Let Λ be an index set and let N be a finitely generated submodule of R^Λ . The finite projectivity of R^Λ implies that N is isomorphic to a submodule of a free module of finite rank. Hence N is finitely presented.

It is obvious that (3) \Rightarrow (2).

(2) \Rightarrow (3). Let Λ be an index set, let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of finitely projective modules and let N be a finitely generated submodule of $M = \prod_{\lambda \in \Lambda} M_\lambda$. For each $\lambda \in \Lambda$, let N_λ be the image of N by the canonical map $M \rightarrow M_\lambda$. We put $N' = \prod_{\lambda \in \Lambda} N_\lambda$. So, $N \subseteq N' \subseteq M$. For each $\lambda \in \Lambda$ there exists a free module F_λ of finite rank such that the inclusion map $N_\lambda \rightarrow M_\lambda$ factors through F_λ . It follows that the inclusion map $N \rightarrow M$ factors through $\prod_{\lambda \in \Lambda} F_\lambda$ which is isomorphic to $R^{\Lambda'}$ for some index set Λ' . Now the monomorphism $N \rightarrow R^{\Lambda'}$ factors through a free module F . It is easy to conclude that the inclusion map $N \rightarrow M$ factors through F and that M is finitely projective. \square

By using [22, Theorem 4.2] and Proposition 1.3, we deduce the following corollary:

Corollary 2.2. *Every strongly coherent commutative ring R is π -coherent and the converse holds if R is self injective.*

Proposition 2.3. *Let R be a π -coherent commutative ring and let S be a multiplicative subset of R . Assume that S contains no zero-divisors. Then $S^{-1}R$ is π -coherent.*

Proof. Let M be a finitely generated $S^{-1}R$ -module. By [3, Theorem 1] we must prove that $\text{Hom}_{S^{-1}R}(M, S^{-1}R)$ is finitely generated on $S^{-1}R$. There exists a finitely generated R -submodule N of M such that $S^{-1}N \cong M$. The following sequence

$$0 \rightarrow N^* \rightarrow \text{Hom}_R(N, S^{-1}R) \rightarrow \text{Hom}_R(N, S^{-1}R/R)$$

is exact. Since N is finitely generated and $S^{-1}R/R$ is S -torsion, $\text{Hom}_R(N, S^{-1}R/R)$ is S -torsion too. So, $\text{Hom}_{S^{-1}R}(M, S^{-1}R) \cong \text{Hom}_R(N, S^{-1}R) \cong S^{-1}N^*$. By [3, Theorem 1] N^* is finitely generated. Hence $\text{Hom}_{S^{-1}R}(M, S^{-1}R)$ is finitely generated over $S^{-1}R$. \square

Theorem 2.4. *Let R be a commutative semihereditary ring and let Q be its quotient ring. Then the following conditions are equivalent:*

- (1) R is π -coherent;
- (2) Q is self injective;

Moreover, when these conditions are satisfied, each singly projective R -module is finitely projective.

Proof. (1) \Rightarrow (2). By Proposition 2.3 Q is π -coherent. We know that Q is Von Neumann regular. It follows from [18, Theorem 2] that Q is self injective.

(2) \Rightarrow (1). Let $(M_i)_{i \in I}$ be a family of finitely projective R -modules, where I is an index set, and let N be a finitely generated submodule of $\prod_{i \in I} M_i$. Then N is flat. Since N is a submodule of $\prod_{i \in I} Q \otimes_R M_i$, $Q \otimes_R N$ is isomorphic to a finitely generated Q -submodule of $\prod_{i \in I} Q \otimes_R M_i$. It follows that $Q \otimes_R N$ is a projective Q -module. Hence N is projective by [6, Proposition 2.3]. We conclude by Theorem 2.1.

Let M be a singly projective R -module and let N be a finitely generated submodule of M . Then $Q \otimes_R M$ is finitely projective over Q by Propositions 1.7(1) and 1.3. It follows that $Q \otimes_R N$ is projective over Q . Hence N is projective by [6, Proposition 2.3]. \square

Proposition 2.5. *Let R be a Von Neumann regular ring. Then a right R -module is content if and only if it is singly projective.*

Proof. By Proposition 1.1(3) it remains to show that each singly projective right module M is content. Let $m \in M$. Then mR is projective because it is isomorphic to a finitely generated submodule of a free module. So, mR is content. For each left ideal A , $mR \cap MA = mA$ because mR is a pure submodule of M . Hence M is content. \square

A topological space X is said to be **extremally disconnected** if every open set has an open closure. Let R be a ring. We say that R is a right **Baer ring** if for any subset A of R , $r(A)$ is generated by an idempotent. The ring R defined in [22, Example 4.4] is not self injective and satisfies the conditions of the following theorem.

Theorem 2.6. *Let R be a Von Neumann regular ring. Then the following conditions are equivalent:*

- (1) *Each product of singly projective right modules is singly projective;*
- (2) *Each product of content right modules is content;*
- (3) *R_R^R is singly projective;*
- (4) *R_R^R is a content module;*
- (5) *R is a right Baer ring;*
- (6) *The intersection of each family of finitely generated left ideals is finitely generated too;*
- (7) *For each cyclic left module C , C^* is finitely generated.*

Moreover, when R is commutative, these conditions are equivalent to the following: $\text{Spec } R$ is extremally disconnected.

Proof. The conditions (2), (4), (6) are equivalent by [11, Theorem 5.15]. By Proposition 2.5 (4) \Leftrightarrow (3) and (1) \Leftrightarrow (2). It is easy to check that (5) \Leftrightarrow (7).

(3) \Rightarrow (5). Let $A \subseteq R$ and let $x = (a)_{a \in A} \in R_R^A$. So, $r(A) = (0 : x)$. Then xR is projective because it is isomorphic to a submodule of a free module. Thus $r(A) = eR$, where e is an idempotent.

(5) \Rightarrow (1). Let $(M_i)_{i \in I}$ be a family of singly projective right modules and $m = (m_i)_{i \in I}$ be an element of $M = \prod_{i \in I} M_i$. For each $i \in I$, there exists an idempotent e_i such that $(0 : m_i) = e_i R$. Let e be the idempotent which satisfies $eR = r(\{1 - e_i \mid i \in I\})$. Then $eR = (0 : m)$, whence mR is projective.

If R is commutative and reduced, then the closure of $D(A)$, where A is an ideal of R , is $V((0 : A))$. So, $\text{Spec } R$ is extremally disconnected if and only if, for each ideal A there exists an idempotent e such that $V((0 : A)) = V(e)$. This last equality holds if and only if $(0 : A) = Re$ because $(0 : A)$ and Re are semiprime since R is reduced. Consequently $\text{Spec } R$ is extremally disconnected if and only if R is Baer. The proof is now complete. \square

Let R be a ring. We say that R is a right **PP-ring** if any principal right ideal is projective.

Lemma 2.7. *Let R be a right PP-ring. Then each cyclic submodule of a free right module is projective.*

Proof. Let C be a cyclic submodule of a free right module F . We may assume that F is finitely generated by the basis $\{e_1, \dots, e_n\}$. Let $p : F \rightarrow R$ be the homomorphism defined by $p(e_1 r_1 + \dots + e_n r_n) = r_n$ where $r_1, \dots, r_n \in R$. Then $p(C)$ is a principal right ideal. Since $p(C)$ is projective, $C \cong C' \oplus p(C)$ where $C' = C \cap \ker p$. So C' is a cyclic submodule of the free right module generated by $\{e_1, \dots, e_{n-1}\}$. We complete the proof by induction on n . \square

Theorem 2.8. *Let R be a commutative PP-ring and let Q be its quotient ring. Then the following conditions are equivalent:*

- (1) *Each product of singly projective modules is singly projective;*
- (2) *R^R is singly projective;*
- (3) *R is a Baer ring;*
- (4) *Q satisfies the equivalent conditions of Theorem 2.6;*
- (5) *For each cyclic module C , C^* is finitely generated;*
- (6) *$\text{Spec } R$ is extremally disconnected;*
- (7) *$\text{Min } R$ is extremally disconnected.*

Proof. It is obvious that (1) \Rightarrow (2). It is easy to check that (3) \Leftrightarrow (5). We show that (2) \Rightarrow (3) as we proved (3) \Rightarrow (5) in Theorem 2.6, by using Lemma 2.7.

(5) \Rightarrow (4). Let C be a cyclic Q -module. We do as in proof of Proposition 2.3 to show that $\text{Hom}_Q(C, Q)$ is finitely generated over Q .

(4) \Rightarrow (1). Let $(M_i)_{i \in I}$ be a family of singly projective right modules and let N be a cyclic submodule of $M = \prod_{i \in I} M_i$. Since R is PP, N is a P-flat module. By Proposition 1.1 N is flat. We do as in the proof of (2) \Rightarrow (1) of Theorem 2.4 to show that N is projective.

(3) \Leftrightarrow (6) is shown in the proof of Theorem 2.6.

(4) \Leftrightarrow (7) holds because $\text{Spec } Q$ is homeomorphic to $\text{Min } R$. \square

3. FLAT MODULES

Let M be a non-zero module over a commutative ring R . As in [10, p.338] we set:

$$M_{\#} = \{s \in R \mid \exists 0 \neq x \in M \text{ such that } sx = 0\} \quad \text{and} \quad M^{\#} = \{s \in R \mid sM \subset M\}.$$

Then $R \setminus M_{\#}$ and $R \setminus M^{\#}$ are multiplicative subsets of R .

Lemma 3.1. *Let M be a non-zero P -flat R -module over a commutative ring R . Then $M_{\#} \subseteq R_{\#} \cap M^{\#}$.*

Proof. Let $0 \neq s \in M_{\#}$. Then there exists $0 \neq x \in M$ such that $sx = 0$. Since M is P -flat, we have $x \in (0 : s)M$. Hence $(0 : s) \neq 0$ and $s \in R_{\#}$.

Suppose that $M_{\#} \not\subseteq M^{\#}$ and let $s \in M_{\#} \setminus M^{\#}$. Then $\exists 0 \neq x \in M$ such that $sx = 0$. It follows that $x = t_1y_1 + \cdots + t_py_p$ for some $y_1, \dots, y_p \in M$ and $t_1, \dots, t_p \in (0 : s)$. Since $s \notin M^{\#}$ we have $M = sM$. So $y_k = sz_k$ for some $z_k \in M$, $\forall k$, $1 \leq k \leq p$. We get $x = t_1sz_1 + \cdots + t_psz_p = 0$. Whence a contradiction. \square

Now we assume that R is a commutative ring. An R -module M is said to be **uniserial** if its set of submodules is totally ordered by inclusion and R is a **valuation ring** if it is uniserial as R -module. If M is a module over a valuation ring R then $M_{\#}$ and $M^{\#}$ are prime ideals of R . In the sequel, if R is a valuation ring, we denote by P its maximal ideal and we put $Z = R_{\#}$ and $Q = R_Z$. Since each finitely generated ideal of a valuation ring R is principal, it follows that any P -flat R -module is flat.

Proposition 3.2. *Let R be a valuation ring, let M be a flat R -module and let E be its injective hull. Then E is flat.*

Proof. Let $x \in E \setminus M$ and $r \in R$ such that $rx = 0$. There exists $a \in R$ such that $0 \neq ax \in M$. From $ax \neq 0$ and $rx = 0$ we deduce that $r = ac$ for some $c \in R$. Since $cax = 0$ and M is flat we have $ax = by$ for some $y \in M$ and $b \in (0 : c)$. From $bc = 0$ and $ac = r \neq 0$ we get $b = at$ for some $t \in R$. We have $a(x - ty) = 0$. Since $at = b \neq 0$, $(0 : t) \subset Ra$. So $(0 : t) \subseteq (0 : x - ty)$. The injectivity of E implies that there exists $z \in E$ such that $x = t(y + z)$. On the other hand $tr = tac = bc = 0$, so $t \in (0 : r)$. \square

In the sequel, if J is a prime ideal of R we denote by 0_J the kernel of the natural map: $R \rightarrow R_J$.

Proposition 3.3. *Let R be a valuation ring and let M be a non-zero flat R -module. Then:*

- (1) *If $M_{\#} \subset Z$ we have $\text{ann}(M) = 0_{M_{\#}}$ and M is an $R_{M_{\#}}$ -module;*
- (2) *If $M_{\#} = Z$, $\text{ann}(M) = 0$ if $M_Z \neq ZM_Z$ and $\text{ann}(M) = (0 : Z)$ if $M_Z = ZM_Z$. In this last case, M is a Q -module.*

Proof. Observe that the natural map $M \rightarrow M_{M_{\#}}$ is a monomorphism. First we assume that R is self FP-injective and $P = M_{\#}$. So $M^{\#} = P$ by Lemma 3.1. If $M \neq PM$ let $x \in M \setminus PM$. Then $(0 : x) = 0$ else $\exists r \in R$, $r \neq 0$ such that $x \in (0 : r)M \subseteq PM$. If $M = PM$ then P is not finitely generated else $M = pM$, where $P = pR$, and $p \notin M^{\#} = P$. If P is not faithful then $(0 : P) \subseteq \text{ann}(M)$. Thus M is flat over $R/(0 : P)$. So we can replace R with $R/(0 : P)$ and assume that P is faithful. Suppose $\exists 0 \neq r \in P$ such that $rM = 0$. Then $M = (0 : r)M$.

Since $(0 : r) \neq P$, let $t \in P \setminus (0 : r)$. Thus $M = tM$ and $t \notin M^\# = P$. Whence a contradiction. So M is faithful or $\text{ann}(M) = (0 : P)$.

Return to the general case. We put $J = M_\#$.

If $J \subset Z$ then R_J is coherent and self FP-injective by [4, Theorem 11]. In this case JR_J is principal or faithful. So M_J is faithful over R_J , whence $\text{ann}(M) = 0_J$. Let $s \in R \setminus J$. There exists $t \in Zs \setminus J$. It is easy to check that $\forall a \in R$, $(0 : a)$ is also an ideal of Q . On the other hand, $\forall a \in Q$, $Qa = (0 : (0 : a))$ because Q is self FP-injective. It follows that $(0 : s) \subset (0 : t)$. Let $r \in (0 : t) \setminus (0 : s)$. Then $r \in 0_J$. So, $rM = 0$. Hence $M = (0 : r)M = sM$. Therefore the multiplication by s in M is bijective for each $s \in R \setminus J$.

Now suppose that $J = Z$. Since Q is self FP-injective then M is faithful or $\text{ann}(M) = (0 : Z)$. Let $s \in R \setminus Z$. Thus $Z \subset Rs$ and $sZ = Z$. It follows that $ZM_Z = ZM$. So, M is a Q -module if $ZM_Z = M_Z$. \square

When R is a valuation ring, N is a pure submodule of M if $rN = rM \cap N$, $\forall r \in R$.

Proposition 3.4. *Let R be a valuation ring and let M be a non-zero flat R -module such that $M_{M_\#} \neq M_\#M_{M_\#}$. Then M contains a non-zero pure uniserial submodule.*

Proof. Let $J = M_\#$ and $x \in M_J \setminus JM_J$. If $J \subset Z$ then M is a module over $R/0_J$ and $J/0_J$ is the subset of zero-divisors of $R/0_J$. So, after replacing R with $R/0_J$ we may assume that $Z = J$. If $rx = 0$ then $x \in (0 : r)M_Z \subseteq ZM_Z$ if $r \neq 0$. Hence Qx is faithful over Q which is FP-injective. So $V = Qx$ is a pure submodule of M_Z . We put $U = M \cap V$. Thus U is uniserial and $U_Z = V$. Then M/U is a submodule of M_Z/V , and this last module is flat. Let $z \in M/U$ and $0 \neq r \in R$ such that $rz = 0$. Then $z = as^{-1}y$ where $s \notin Z$, $a \in (0 : r) \subseteq Z$ and $y \in M/U$. It follows that $a = bs$ for some $b \in R$ and $sbr = 0$. So $b \in (0 : r)$ and $z = by$. Since M/U is flat, U is a pure submodule of M . \square

Proposition 3.5. *Let R be a valuation ring and let M be a flat R -module. Then M contains a pure free submodule N such that $M/PM \cong N/PN$.*

Proof. Let $(x_i)_{i \in I}$ be a family of elements of M such that $(x_i + PM)_{i \in I}$ is a basis of M/PM over R/P , and let N be the submodule of M generated by this family. If we show that N is a pure submodule of M , we deduce that N is flat. It follows that N is free by Lemma 1.4. Let $x \in M$ and $r \in R$ such that $rx \in N$. Then $rx = \sum_{i \in J} a_i x_i$ where J is a finite subset of I and $a_i \in R$, $\forall i \in J$. Let $a \in R$ such that $Ra = \sum_{i \in J} Ra_i$. It follows that, $\forall i \in J$, there exists $u_i \in R$ such that $a_i = au_i$ and there is at least one $i \in J$ such that u_i is a unit. Suppose that $a \notin Rr$. Thus there exists $c \in P$ such that $r = ac$. We get that $a(\sum_{i \in J} u_i x_i - cx) = 0$. Since M is flat, we deduce that $\sum_{i \in J} u_i x_i \in PM$. This contradicts that $(x_i + PM)_{i \in I}$ is a basis of M/PM over R/P . So, $a \in Rr$. Hence N is a pure submodule. \square

4. SINGLY PROJECTIVE MODULES

Lemma 4.1. *Let R be a valuation ring. Then a non-zero R -module M is singly projective if and only if for each $x \in M$ there exists $y \in M$ such that $(0 : y) = 0$ and $x \in Ry$. Moreover $M_\# = Z$ and $M_Z \neq ZM_Z$.*

Proof. Assume that M is singly projective and let $x \in M$. There exist a free module F , a morphism $\phi : Rx \rightarrow F$ and a morphism $\pi : F \rightarrow M$ such that $(\pi \circ \phi)(x) = x$. Let $(e_i)_{i \in I}$ be a basis of F . Then $\phi(x) = \sum_{i \in J} a_i e_i$ where

J is a finite subset of I and $a_i \in R$, $\forall i \in J$. There exists $a \in R$ such that $\sum_{i \in J} Ra_i = Ra$. Thus, $\forall i \in J$ there exists $u_i \in R$ such $a_i = au_i$. We put $z = \sum_{i \in J} u_i e_i$. Then $\phi(x) = az$. Since there is at least one index $i \in J$ such that u_i is a unit, then $(0 : z) = 0$. It follows that $(0 : \phi(x)) = (0 : a)$. But $(0 : x) = (0 : \phi(x))$ because ϕ is a monomorphism. We have $x = a\pi(z)$. So, by [4, Lemma 2] $(0 : \pi(z)) = a(0 : x) = a(0 : a) = 0$. The converse and the last assertion are obvious. \square

Let R be a valuation ring and let M be a non-zero R -module. A submodule N of M is said to be **pure-essential** if it is a pure submodule and if 0 is the only submodule K satisfying $N \cap K = 0$ and $(N + K)/K$ is a pure submodule of M/K . An R -module E is said to be **pure-injective** if for any pure-exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$, the following sequence is exact:

$$0 \rightarrow \text{Hom}_R(L, E) \rightarrow \text{Hom}_R(M, E) \rightarrow \text{Hom}_R(N, E) \rightarrow 0.$$

We say that E is a **pure-injective hull** of K if E is pure-injective and K is a pure-essential submodule of E . We say that R is **maximal** if every family of cosets $\{a_i + L_i \mid i \in I\}$ with the finite intersection property has a non-empty intersection (here $a_i \in R$, L_i denote ideals of R , and I is an arbitrary index set).

Proposition 4.2. *Let R be a valuation ring and let M be a non-zero R -module. Then the following conditions are equivalent:*

- (1) M is a flat content module;
- (2) M is flat and contains a pure-essential free submodule.

Moreover, if these conditions are satisfied, then any element of M is contained in a pure cyclic free submodule L of M . If R is maximal then M is locally projective.

Proof. (1) \Rightarrow (2). Let $0 \neq x \in M$. Then $x = \sum_{1 \leq i \leq n} a_i x_i$ where $a_i \in c(x)$ and $x_i \in M$, $\forall i$, $1 \leq i \leq n$. Since R is a valuation ring $\exists a \in R$ such that $Ra = Ra_1 + \dots + Ra_n$. So, we get that $c(x) = Ra$ and $x = ay$ for some $y \in M$. Thus $y \notin PM$ else $c(x) \subset Ra$. So $PM \neq M$ and we can apply Proposition 3.5. It remains to show that N is a pure-essential submodule of M . Let $x \in M$ such that $Rx \cap N = 0$ and N is a pure submodule of M/Rx . There exist $b \in R$ and $y \in M \setminus PM$ such that $x = by$. Since $M = N + PM$, we have $y = n + pm$ where $n \in N$, $m \in M$ and $p \in P$. Then $n \notin PN$ and $bpm = -bn + x$. Since N is pure in M/Rx there exist $n' \in N$ and $t \in R$ such that $bpn' = -bn + tx$. We get that $b(n + pn') \in N \cap Rx = 0$. So $b = 0$ because $n + pn' \notin PN$. Hence $x = 0$.

(2) \Rightarrow (1). First we show that M is a content module if each element x of M is of the form $s(y + cz)$, where $s \in R$, $y \in N \setminus PN$, $c \in P$ and $z \in M$. Since N is a pure submodule, $PM \cap N = PN$ whence $y \notin PM$. If $x = stw$ with $t \in P$ and $w \in M$ we get that $s(y + cz - tw) = 0$ whence $y \in PM$ because M is flat. This is a contradiction. Consequently $c(x) = Rs$ and M is content. Now we prove that each element x of M is of the form $s(y + cz)$, where $s \in R$, $y \in N \setminus PN$, $c \in P$ and $z \in M$. If $x \in N$, then we check this property by using a basis of N . Suppose $x \notin N$ and $Rx \cap N \neq 0$. There exists $a \in P$ such that $0 \neq ax \in N$. Since N is pure, there exists $y' \in N$ such that $ax = ay'$. We get $x = y' + bz$ for some $b \in (0 : a)$ and $z \in M$, because M is flat. We have $y' = sy$ with $s \in R$ and $y \in N \setminus PN$. Since $as \neq 0$, $b = sc$ for some $c \in P$. Hence $x = s(y + cz)$. Now suppose that $Rx \cap N = 0$. Since N is pure-essential in M , there exist $r \in R$ and $m \in M$ such that $rm \in N + Rx$ and $rm \notin rN + Rx$. Hence $rm = n + tx$ where $n \in N$ and $t \in R$.

Thus $n = by'$ where $b \in R$ and $y' \in N \setminus PN$. Then $b \notin rR$. So, $r = bc$ for some $c \in P$. We get $bcm = by' + tx$. If $t = bd$ for some $d \in P$ then $b(cm - y' - dx) = 0$. Since M is flat, it follows that $y' \in PM \cap N = PN$. But this is false. So $b = st$ for some $s \in R$. We obtain $t(x + sy' - scm) = 0$. Since M is flat and $tsc \neq 0$ there exists $z \in M$ such that $x = s(-y' + cz)$.

Let $y \in M$. There exists $x \in M \setminus PM$ such that $y \in Rx$. We may assume that $x + PM$ is an element of a basis $(x_i + PM)_{i \in I}$ of M/PM . Then Rx is a summand of the free pure submodule N generated by the family $(x_i)_{i \in I}$.

Assume that R is maximal. Let the notations be as above. By [9, Theorem XI.4.2] each uniserial R -module is pure-injective. So, Rx is a summand of M . Let u be the composition of a projection from M onto Rx with the isomorphism between Rx and R . Thus $u \in M^*$ and $u(x) = 1$. It follows that $y = u(y)x$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.1]. \square

Proposition 4.3. *Let R be a valuation ring such that $Z = P \neq 0$ and let M be a non-zero R -module. Then the following conditions are equivalent:*

- (1) M is singly projective;
- (2) M is a flat content module;
- (3) M is flat and contains an essential free submodule.

Proof. (2) \Rightarrow (1) by Proposition 1.1.

(1) \Rightarrow (2). It remains to show that M is a content module. Let $x \in M$. There exists $y \in M$ and $a \in R$ such that $x = ay$ and $(0 : y) = 0$. Since $Z = P$ then $y \notin PM$. We deduce that $c(x) = Ra$.

(2) \Leftrightarrow (3). By Proposition 4.2 it remains to show that (2) \Rightarrow (3). Let N be a pure-essential free submodule of M . Since R is self FP-injective by [12, Lemma], it follows that N is a pure submodule of each overmodule. So, if K is a submodule of M such that $K \cap N = 0$, then N is a pure submodule of M/K , whence $K = 0$. \square

Corollary 4.4. *Let R be a valuation ring. The following conditions are equivalent:*

- (1) $Z = P$;
- (2) Each singly projective module is a content module.

Proof. It remains to show that (2) \Rightarrow (1). By Proposition 1.7 Q is finitely projective over R . If $R \neq Q$, then Q is not content on R because, $\forall x \in Q \setminus Z$, $c(x) = Z$. So $Z = P$. \square

Corollary 4.5. *Let R be a valuation ring. Then the injective hull of any singly projective module is singly projective too.*

Proof. Let N be a non-zero singly projective module. We denote by E its injective hull. For each $s \in R \setminus Z$ the multiplication by s in N is injective, so the multiplication by s in E is bijective. Hence E is a Q -module which is flat by Proposition 3.2. It is an essential extension of N_Z . From Propositions 4.3 and 1.7(2) we deduce that E is singly projective. \square

Let R be a valuation ring and let M be a non-zero R -module. We say that M is **separable** if any finite subset is contained in a summand which is a finite direct sum of uniserial submodules.

Corollary 4.6. *Let R be a valuation ring. Then any element of a singly projective module M is contained in a pure uniserial submodule L . Moreover, if R is maximal, each singly projective module is separable.*

Proof. By Proposition 4.2 any element of M is contained in a pure cyclic free Q -submodule G of M_Z . We put $L = M \cap G$. As in proof of Proposition 3.4 we show that L is a pure uniserial submodule of M . The first assertion is proved.

Since R is maximal L is pure-injective by [9, Theorem XI.4.2]. So, L is a summand of M . Each summand of M is singly projective. It follows that we can complete the proof by induction on the cardinal of the chosen finite subset of M . \square

Corollary 4.7. *Let R be a valuation ring. Then the following conditions are equivalent:*

- (1) R is self injective;
- (2) Each singly projective module is locally projective;
- (3) $Z = P$ and each singly projective module is finitely projective.

Proof. (1) \Rightarrow (2) by Proposition 1.3.

(2) \Rightarrow (3) follows from [11, Proposition 5.14(4)] and Corollary 4.4.

(3) \Rightarrow (1). By way of contradiction suppose that R is not self injective. Let E be the injective hull of R . By Corollary 4.5 E is singly projective. Let $x \in E \setminus R$ and $M = R + Rx$. Since E is finitely projective, then there exist a finitely generated free module F , a morphism $\phi : M \rightarrow F$ and a morphism $\pi : F \rightarrow E$ such that $(\pi \circ \phi)(y) = y$ for each $y \in M$. Let $\tilde{\phi} : M/R \rightarrow F/\phi(R)$ and $\tilde{\pi} : F/\phi(R) \rightarrow E/R$ be the morphisms induced by ϕ and π . Then $(\tilde{\pi} \circ \tilde{\phi})(y + R) = y + R$ for each $y \in M$. Since $\phi(R)$ is a pure submodule of F , then $F/\phi(R)$ is a finitely generated flat module. Hence $F/\phi(R)$ is free and E/R is singly projective. But $E/R = P(E/R)$ by [4, Lemma 12]. This contradicts that E/R is a flat content module. By Proposition 4.3 we conclude that $E = R$. \square

Corollary 4.8. *Let R be a valuation ring. Then Q is self injective if and only if each singly projective module is finitely projective.*

Proof. By [17, Theorem 2.3] Q is maximal if and only if it is self injective. Suppose that Q is self injective and let M be a singly projective R -module. Then M_Z is locally projective over Q by Proposition 1.7(1) and corollary 4.7. Consequently M is finitely projective by Lemma 1.5.

Conversely let M be a singly projective Q -module. Then M is singly projective over R , whence M is finitely projective over R . It follows that M is finitely projective over Q . From Corollary 4.7 we deduce that Q is self injective. \square

Theorem 4.9. *Let R be a valuation ring. Then the following conditions are equivalent:*

- (1) R is maximal;
- (2) each singly projective R -module is separable;
- (3) each flat content module is locally projective.

Proof. (1) \Rightarrow (2) by Corollary 4.6 and (1) \Rightarrow (3) by Proposition 4.2.

(2) \Rightarrow (1) : let \hat{R} be the pure-injective hull of R . By [5, Proposition 1 and 2] \hat{R} is a flat content module. Consequently 1 belongs to a summand L of \hat{R} which is a finite direct sum of uniserial modules. But, by [7, Proposition 5.3] \hat{R} is indecomposable. Hence \hat{R} is uniserial. Suppose that $R \neq \hat{R}$. Let $x \in \hat{R} \setminus R$. Then there exists $c \in P$ such that $1 = cx$. Since R is pure in \hat{R} we get that $1 \in P$ which is absurd. Consequently, R is a pure-injective module. So, R is maximal by [21, Proposition 9].

(3) \Rightarrow (1) : since \widehat{R} is locally projective then R is a summand of \widehat{R} which is indecomposable. So R is maximal. \square

A submodule N of a module M is said to be **strongly pure** if, $\forall x \in N$ there exists an homomorphism $u : M \rightarrow N$ such that $u(x) = x$. Moreover, if N is pure-essential, we say that M is a **strongly pure-essential extension** of N .

Proposition 4.10. *Let R be a valuation ring and let M be a flat R -module. Then M is locally projective if and only if it is a strongly pure-essential extension of a free module.*

Proof. Let M be a non-zero locally projective R -module. Then M is a flat content module. So M contains a pure-essential free submodule N . Let $x \in N$. There exist $u_1, \dots, u_n \in M^*$ and $y_1, \dots, y_n \in M$ such that $x = \sum_{i=1}^n u_i(x)y_i$. Since N is a pure submodule, y_1, \dots, y_n can be chosen in N . Let $\phi : M \rightarrow N$ be the homomorphism defined by $\phi(z) = \sum_{i=1}^n u_i(z)y_i$. Then $\phi(x) = x$. So, N is a strongly pure submodule of M .

Conversely, assume that M is a strongly pure-essential extension of a free submodule N . Let $x \in M$. As in proof of Proposition 4.2, $x = s(y + cz)$, where $s \in R$, $y \in N \setminus PN$, $c \in P$ and $z \in M$. Since N is strongly pure, there exists a morphism $\phi : M \rightarrow N$ such that $\phi(y) = y$. Let $\{e_i \mid i \in I\}$ be a basis of N . Then $y = \sum_{i \in J} a_i e_i$ where J is a finite subset of I and $a_i \in R$, $\forall i \in J$. Since $y \in N \setminus PN$ there exists $j \in J$ such that $a_j \notin P$. We easily check that $\{y, e_i \mid i \in I, i \neq j\}$ is a basis of N too. Hence Ry is a summand of N . Let u be the composition of ϕ with a projection of N onto Ry and with the isomorphism between Ry and R . Then $u \in M^*$, $u(y) = 1$ and $u(y + cz) = 1 + cu(z) = v$ is a unit. We put $m = v^{-1}(y + cz)$. It follows that $x = u(x)m$. Hence M is locally projective by [11, Theorem 3.2] or [22, Theorem 2.1]. \square

Corollary 4.11. *Let R be a valuation ring and let M be a locally projective R -module. If M/PM is finitely generated then M is free.*

Theorem 4.12. *Let R be a valuation ring. The following conditions are equivalent:*

- (1) Z is nilpotent;
- (2) Q is an artinian ring;
- (3) Each flat R -module is finitely projective;
- (4) Each flat R -module is singly projective.

Proof. (1) \Leftrightarrow (2). If Z is nilpotent then $Z^2 \neq Z$. It follows that Z is finitely generated over Q and it is the only prime ideal of Q . So, Q is artinian. The converse is well known.

(2) \Rightarrow (3) is a consequence of [20, Corollary 7] and it is obvious that (3) \Rightarrow (4).

(4) \Rightarrow (2). First we prove that each flat Q -module is singly projective. By Proposition 4.3 it follows that each flat Q -module is content. We deduce that Q is perfect by Theorem 1.2. We conclude that Q is artinian since Q is a valuation ring. \square

5. STRONGLY COHERENCE OR π -COHERENCE OF VALUATION RINGS.

In this section we study the valuation rings, with non-zero zero-divisors, for which any product of content (respectively singly, finitely, locally projective) modules is content (respectively singly, finitely, locally projective) too.

Theorem 5.1. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

- (1) *Each product of content modules is content;*
- (2) *R^R is a content module;*
- (3) *For each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;*
- (4) *Each ideal is countably generated and $R^{\mathbb{N}}$ is a content module;*
- (5) *The intersection of any non-empty family of principal ideals is finitely generated.*

Proof. The conditions (1), (2) and (5) are equivalent by [11, Theorem 5.15]. By [4, Lemma 29] (3) \Leftrightarrow (5).

(2) \Rightarrow (4). It is obvious that $R^{\mathbb{N}}$ is a content module. Since (2) \Leftrightarrow (3) then P is the only prime ideal. We conclude by [4, Corollary 36].

(4) \Rightarrow (3). Let A be a non-finitely generated ideal. Let $\{a_n \mid n \in \mathbb{N}\}$ be a spanning set of A . Then $x = (a_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$. It follows that $x = ay$ for some $a \in c(x)$ and $y \in R^{\mathbb{N}}$, and $c(x) = Ra$. So, if $y = (b_n)_{n \in \mathbb{N}}$, we easily check that P is generated by $\{b_n \mid n \in \mathbb{N}\}$. Hence $A = aP$. \square

By Proposition 1.1 each valuation domain R verifies the first two conditions of the next theorem.

Theorem 5.2. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

- (1) *Each product of singly projective modules is singly projective.*
- (2) *R^R is singly projective;*
- (3) *C^* is a finitely generated module for each cyclic module C ;*
- (4) *$(0 : A)$ is finitely generated for each proper ideal A ;*
- (5) *P is principal or faithful and for each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;*
- (6) *Each ideal is countably generated and $R^{\mathbb{N}}$ is singly projective;*
- (7) *Each product of flat content modules is flat content;*
- (8) *R^R is a flat content module;*
- (9) *Each ideal is countably generated and $R^{\mathbb{N}}$ is a flat content module;*
- (10) *P is principal or faithful and the intersection of any non-empty family of principal ideals is finitely generated.*

Proof. It is obvious that (1) \Rightarrow (2) and (7) \Rightarrow (8).

(3) \Leftrightarrow (4) because $(0 : A) \cong (R/A)^*$.

(2) \Rightarrow (4). Let A be a proper ideal. Then R^A is singly projective too and $x = (a)_{a \in A}$ is an element of R^A . Therefore x belongs to a cyclic free submodule of R^A by Lemma 4.1. Since R^R is flat, R is coherent by [10, Theorem IV.2.8]. Consequently $(0 : A) = (0 : x)$ is finitely generated.

(4) \Rightarrow (5). Then R is coherent because R is a valuation ring. Since $Z \neq 0$, $Z = P$ by [4, Theorem 10]. If P is not finitely generated then P cannot be an annihilator. So P is faithful. By [12, Lemma 3] and [17, Proposition 1.3], if A is a proper ideal then either $A = (0 : (0 : A))$ or $A = P(0 : (0 : A))$. By (4), $(0 : (0 : A)) = Ra$ for some $a \in P$.

(5) \Rightarrow (1). Let $(M_i)_{i \in I}$ be a family of singly projective modules. Let $x = (x_i)_{i \in I}$ be an element of $\prod_{i \in I} M_i$. Since M_i is singly projective for each $i \in I$ there exist $a_i \in R$ and $y_i \in M_i$ such that $x_i = a_i y_i$ and $(0 : y_i) = 0$. We have either

$\sum_{i \in I} Ra_i = Ra$ or $\sum_{i \in I} Ra_i = Pa$ for some $a \in R$. Then, $\forall i \in I$, $\exists b_i \in R$ such that $a_i = ab_i$. Therefore either $\exists i \in I$ such that b_i is a unit, or $P = \sum_{i \in I} Rb_i$. It follows that $x = ay$ where $y = (b_i y_i)_{i \in I}$. Now it is easy to check that $(0 : y) = 0$.

(6) \Rightarrow (4). Since each ideal is countably generated then so is each submodule of a finitely generated free module. So, the flatness of $R^{\mathbb{N}}$ implies that R is coherent. Let A be a proper ideal generated by $\{a_n \mid n \in \mathbb{N}\}$. Then $x = (a_n)_{n \in \mathbb{N}}$ is an element of $R^{\mathbb{N}}$. Therefore x belongs to a cyclic free submodule of $R^{\mathbb{N}}$ by Lemma 4.1. Consequently $(0 : A) = (0 : x)$ is finitely generated because R is coherent.

(5) \Rightarrow (9). By Theorem 5.1((3) \Leftrightarrow (4)) it remains to show that $R^{\mathbb{N}}$ is flat. This is true because (5) \Rightarrow (1).

(1) \Leftrightarrow (7). Since (1) \Rightarrow (2) or (7) \Rightarrow (8), R is coherent. From $Z \neq 0$ and [4, Theorem 10] it follows that $Z = P$. Now we use Proposition 4.3 to conclude.

(2) \Leftrightarrow (8). Since R^R is flat, R is coherent. We do as above to conclude.

(6) \Leftrightarrow (9). Since each submodule of a free module of finite rank is countably generated, then the flatness of $R^{\mathbb{N}}$ implies that R is coherent. So we conclude as above.

(5) \Leftrightarrow (10) by Theorem 5.1((3) \Leftrightarrow (5)).

The last assertion is already shown. So, the proof is complete. \square

Remark 5.3. When R is a valuation domain, the conditions (5), (7), (8), (9) and (10) are equivalent by [16, Theorem 4] and [4, Corollary 36].

Remark 5.4. If R satisfies the conditions of Theorem 5.1 and if P is not faithful and not finitely generated then R is not coherent and doesn't satisfy the conditions of Theorem 5.2.

By [8, Corollary 3.5] or [16, Theorem 3], a valuation domain R is strongly coherent if and only if either its order group is \mathbb{Z} or if R is maximal and its order group is \mathbb{R} . It is easy to check that each Prüfer domain is π -coherent because it satisfies the fourth condition of the next theorem. When R is a valuation ring with non-zero zero-divisors we get:

Theorem 5.5. *Let R be a valuation ring such that $Z \neq 0$. Then the following conditions are equivalent:*

- (1) R is strongly coherent;
- (2) R is π -coherent;
- (3) R^R is singly projective and separable;
- (4) C^* is a finitely generated module for each finitely generated module C ;
- (5) $(0 : A)$ is finitely generated for each proper ideal A and R is self injective;
- (6) R is maximal, P is principal or faithful and for each ideal A there exists $a \in R$ such that either $A = Ra$ or $A = Pa$;
- (7) Each ideal is countably generated and $R^{\mathbb{N}}$ is singly projective and separable;
- (8) R^R is a separable flat content module;
- (9) Each ideal is countably generated and $R^{\mathbb{N}}$ is a separable flat content module;
- (10) Each product of separable flat content modules is a separable flat content module;
- (11) R is maximal, P is principal or faithful and the intersection of any non-empty family of principal ideals is finitely generated.

Proof. By Theorem 2.1 (1) \Rightarrow (2). It is obvious that (10) \Rightarrow (8). By [3, Theorem 1] (2) \Leftrightarrow (4). By Theorem 5.2, Theorem 4.9 and [17, Theorem 2.3] (5) \Leftrightarrow (6) and (6) \Rightarrow (7). By Theorem 5.2 (6) \Leftrightarrow (11), (7) \Leftrightarrow (9) and (3) \Leftrightarrow (8).

(4) \Rightarrow (6). By Theorem 5.2 R is coherent and self FP-injective and it remains to prove that R is maximal if P is not principal. Let E be the injective hull of R . If $R \neq E$ let $x \in E \setminus R$. Since R is an essential submodule of E , $(R : x) = rP$ for some $r \in R$. Then $(R : rx) = P$. Let M be the submodule of E generated by 1 and rx . We put $N = M/R$. Then $N \cong R/P$. We get that $N^* = 0$ and M^* is isomorphic to a principal ideal of R . Moreover, since $(R : rx) = P$, for each $t \in P$ the multiplication by t in M is a non-zero element of M^* . Since P is faithful we get that $M^* \cong R$. Let $g \in M^*$ such that the restriction of g to R is the identity. For each $p \in P$ we have $pg(rx) = prx$. So $(0 : g(rx) - rx) = P$. Since P is faithful, there is no simple submodule in E . Hence $g(rx) = rx$ but this is not possible because $g(rx) \in R$ and $rx \notin R$. Consequently R is self-injective and maximal.

(2) \Rightarrow (1). Since (2) \Rightarrow (6) R is self injective. We conclude by proposition 1.3.

(3) \Rightarrow (1). Since R^R is singly projective, by Theorem 5.2 R is coherent and self FP-injective. So, if U is a uniserial summand of R^R , then U is singly projective and consequently $U \neq PU$. Let $x \in U \setminus PU$. It is easy to check that $U = Rx$ and that $(0 : x) = 0$. Hence R^R is locally projective and R is strongly coherent.

(7) \Rightarrow (4). Let $F_1 \rightarrow F_0 \rightarrow C \rightarrow 0$ be a free presentation of a finitely generated module C , where F_0 is finitely generated. It follows that F_1 is countably generated. As above we prove that $R^{\mathbb{N}}$ is locally projective. By Theorem 5.2 R is coherent and consequently each finitely generated submodule of $R^{\mathbb{N}}$ is finitely presented. Since $F_1^* \cong R^{\mathbb{N}}$ we easily deduce that C^* is finitely generated.

(1) \Rightarrow (10). Since (1) \Rightarrow (6), R is maximal. We use Theorem 4.9 to conclude. The proof is now complete. \square

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