# On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations

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# ON THE DIRICHLET PROBLEM FOR SECOND-ORDER ELLIPTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article, we consider the analogue of the Dirichlet problem for second-order elliptic integro-differential equations, which consists in imposing the "boundary conditions" in the whole complementary of the domain. We are looking for conditions on the differential and integral parts of the equation in order to ensure that the Dirichlet boundary condition is satisfied in the classical sense or, in other words, in order that the solution agrees with the Dirichlet data on the boundary of the domain. We also provide a general existence result of a continuous viscosity solution of the nonlocal Dirichlet problem by using Perron's method.

**Keywords:** integro-differential equations, Dirichlet problem, Lévy operators, general nonlocal operators, viscosity solutions

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## Introduction

In this paper, we consider the analogue of the Dirichlet problem for second-order, possibly degenerate and nonlinear, elliptic integro-differential equations. It is well-known that, for these nonlocal equations, the value of the solution has to be prescribed not only on the boundary of the domain but also in its whole complementary (see for instance [11]). Such problems have been already addressed by using different theory: we refer to Bony, Courrège and Priouret [6] for a semi-group approach and to Garroni and Menaldi [11] for a classical PDE approach using Sobolev spaces.

We consider here the viscosity solutions' approach where the Dirichlet boundary condition may be satisfied only in a generalized sense (see for example the User's guide [9]) and the main question we address is: does the solution satisfy the boundary condition in the classical sense? An almost immediate corollary of a positive answer to this question is the existence of a continuous solution for the Dirichlet problem, as we show it.

Let us be more specific now. The problems we look at can be written under the form

$$F\left(x, u, Du, D^2u, \mathcal{I}[u](x)\right) = 0 \qquad \text{in } \Omega\,, \tag{1}$$

$$u = g \qquad \text{on } \Omega^c\,, \tag{2}$$

where $\Omega$ is a $C^2$-open subset of $\mathbb{R}^N$, $\Omega^c$ denotes its complementary and $F : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N\times\mathbb{R} \to \mathbb{R}$ is a continuous function, where $\mathbb{S}^N$ denotes the space of $N\times N$ symmetric matrices. We assume that $F$ satisfies the local and nonlocal degenerate ellipticity condition(s): for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $X, Y \in \mathbb{S}^N$, $l_1, l_2 \in \mathbb{R}$

$$F(x, u, p, X, l_1) \le F(x, u, p, Y, l_2) \quad \text{if } X \ge Y,\ l_1 \ge l_2\,.$$

As we point it out by making such an assumption, the fact that $F(x, u, p, M, l)$ is nonincreasing in $l$ is indeed part of the ellipticity assumption on $F$.

Finally, concerning the nonlocal term, we have typically in mind operators of Lévy's type which, in $\mathbb{R}^N$, have the form

$$\mathcal{I}_L[u](x) = \int_{\mathbb{R}^N} (u(x+z) - u(x) - \nabla u(x) \cdot z\mathbf{1}_B(z)) d\mu_x(z) \tag{3}$$

where the $\mu_x$ are (a priori) singular measures with a singularity at 0, $B$ is the unit ball centered at 0 and $\mathbf{1}_B$ denotes the indicator function of $B$. We always assume that there exists $\bar{c} > 0$ such that, for any $x \in \overline{\Omega}$

$$\int_{|z|<1} |z|^2 d\mu_x(z) \; + \int_{|z|\geq 1} d\mu_x(z) \leq \bar{c} < +\infty \; . \tag{4}$$

This assumption is natural from the probabilistic point of view; a measure satisfying (4) is referred to as a Lévy one.

We also consider Lévy-Itô operators

$$\mathcal{I}_{LI}[u](x) = \int_{\mathbb{R}^N} (u(x + j(x,z)) - u(x) - \nabla u(x) \cdot j(x,z)\mathbf{1}_B(z))\, \mu(dz) \tag{5}$$

where $\mu$ is a Lévy measure (hence it satisfies (4)) and $j(x,z)$ is the size of the jumps at $x$. Such operators enter in the general framework of (3) and the existence of $\mu_x$ is obtained through the representation of integral (5). In order that the operator is well-defined, one assumes

$$|j(x,z)| \leq \bar{c}|z| \quad \text{for some } \bar{c} > 0 \text{ and for any } x, z \in \mathbb{R}^N. \tag{6}$$

In the definition of the Dirichlet problem, these terms have to be slightly modified and we refer the reader to Section 1 for a presentation of these (slight) modifications.

Roughly speaking, our aim is to find general conditions on $F$ and the nonlocal operator $\mathcal{I}$ ensuring that the (continuous) solutions of (1)-(2) (when they exist) satisfy $u = g$ on $\partial\Omega$ and then to show that such solutions *do* exist. In fact, in such problems, it is natural to introduce the function $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \\ g(x) & \text{otherwise} \end{cases}$$

and another way of seeing our task is to say that we are going to investigate the continuity (or discontinuity) of $\tilde{u}$ on $\partial\Omega$. To simplify the notation, we keep writing $u$ instead of $\tilde{u}$ as far as the function defined in the whole space is concerned.

A first model equation is the linear equation

$$-\mathrm{Tr}(a(x)D^2u) - b(x) \cdot Du - \mathcal{I}_L[u](x) + u = f(x) \quad \text{in } \Omega \tag{7}$$

where $a$, $b$, $f$ are continuous functions defined on $\overline{\Omega}$, $a(x)$ being a symmetric nonnegative matrix for any $x \in \overline{\Omega}$.

In the "local" case, *i.e.* when $\mu_x \equiv 0$, it is known that one can solve the Dirichlet problem in a classical way in smooth domains (either by probabilistic or viscosity solutions' methods) if $a$, $b$ and the distance function $d$ to $\partial\Omega$ (which is smooth in a neighborhood of $\partial\Omega$) satisfy: for every $x \in \partial\Omega$,

(i) either $a(x)Dd(x) \neq 0$ *(strong ellipticity in the normal direction)*
(ii) or $\mathrm{Tr}(a(x)D^2d(x)) + b(x) \cdot Dd(x) < 0$ *(curvature and/or transport effects).*

Among all the different proofs of this classical result, the viscosity solutions' proof (*cf.* Barles & Burdeau [3] or Da Lio [10] and references therein for more general results) shows that the effects of the ellipticity and of the drift (together with the curvature ones) are taken into account at

different scales, the ellipticity ones having far stronger importance. One of the main interest of the present work is to analyze the effects of the nonlocal term. We will see that in the case of (1), we face two kinds of "elliptic"-type effects, acting also at different scales.

In order to give a general idea of the results we obtain in the linear case for instance, if one first considers the case of the pure fractional Laplace operator, that is if $d\mu_x(z) = d\mu(z) = dz/|z|^{N+\alpha}$ in (4) with $\alpha \in (0,2)$ and $a \equiv 0$ in (7), we then obtain the following result (see Section 4):

*Let $a \equiv 0$, $b$ and $f$ be continuous. If either (i) $\alpha \geq 1$ or (ii) $b(x) \cdot Dd(x) < 0$ or (iii) $b \equiv 0$, then any solution of (7) takes the boundary data in the continous sense.*

Moreover, if $0 \leq \alpha < 1$, we show that if $b \not\equiv 0$ the boundary data may be lost (see Section 5); hence the corresponding result is somehow optimal.

If more general Lévy's operators are at stake, three important quantities associated with the measures $\mu_x$ play an important role. Precisely, for $0 < \delta < r$ and $x \in \overline{\Omega}$, let us define

$$I^{e,1}_{\delta,r}(x) := \int_{\substack{x+z\notin\Omega \\ \delta\leq|z|\leq r}} d\mu_x(z)\ , \qquad I^{e,2}_{\delta,r}(x) := \int_{\substack{x+z\notin\Omega \\ \delta\leq|z|\leq r}} Dd(x)\cdot z d\mu_x(z)$$

$$I^{3}_{r}(x) := \int_{|z|>r} Dd(x)\cdot z\mathbf{1}_B(z)\, d\mu_x(z)\ .$$

where "e" stands for "exterior" since these terms take into account parts of the integral for $x+z$ outside $\Omega$. We will explain why the proper "structure condition" on the measures $\mu_x$ is the following one.

*There exists a function $H : \mathbb{R}^+ \to \mathbb{R}^+$ and $C > 0$ such that, for any $x \in \overline{\Omega}$, if $\bar{\delta} := \max(d(x), \delta)$*

$$I^{e,1}_{\delta,r}(x) \geq H(\bar{\delta}) \quad \text{and} \quad I^{e,2}_{\delta,r}(x) \geq -C\bar{\delta}H(\bar{\delta}) \quad \text{with} \quad H(\bar{\delta}) \to +\infty \ \text{ as } \bar{\delta} \to 0. \tag{8}$$

Next, a general condition ensuring that the Dirichlet problem for a linear equation is satisfied in the classical sense can therefore be exhibited (see (13)), generalizing the result we gave above in the case of the fractional Laplace operator. We also explain how to reduce the study of $C^2$-domains to domains with flat boundary (see (14)).

Of course, the analysis of the nonlocal term allows us not only to provide results for the above simple linear equation (7) but also for a large class of nonlinear equations under suitable growth conditions with respect to (wrt for short) $Du$.

We conclude this short (and very vague) presentation of the paper by mentioning that, if we are able to analyse the boundary behaviour of the sub and supersolutions in a rather general framework, and in particular for rather general $x$-dependent measures in the nonlocal term, far more restrictive assumptions on $F$ and the nonlocal term are to be imposed in order to prove the existence of a (continuous) solution for the Dirichlet problem. Indeed, existence is obtained by applying Perron's method together with a comparison result for the integro-differential equation, and restrictive assumptions are to be made on the nonlinearity and the singular measure in order to get such a comparison result.

We refer the reader to Ishii [13] or the User's guide [9] for the presentation of Perron's method which extends to the case of nonlocal equations because of the general stability result for integro-differential equations of Bensaoud and Sayah [5] (see also [4]) and to [15, 14, 2, 4] and references therein for comparison results for second order elliptic integro-differential equations.

We point out that other existence results have been obtained by Arisawa [2] by assuming the existence of sub and supersolutions agreeing with the boundary data on $\partial\Omega$ and we also want to mention the study of options with barriers by Cont and Voltchkova [8] which leads in dimension

1 to similar questions but addressed from a probabilistic point of view. Finally, for nonlocal equations associated to bounded measures, Chasseigne [7] shows that the Dirichlet problem can be solved completely although the Dirichlet boundary condition has to be considered in a generalized sense since it does not hold in the classical one.

**Organization of the Paper.** After defining viscosity solutions in Section 1, we provide existence and uniqueness results for Lévy-Itô operators. Section 2 is devoted to the key technical result of this paper. It is shown that, at points where the boundary condition is not satisfied in the classical sense, necessary conditions hold, involving first- and second-order differential terms and nonlocal ones. We next use this technical result to study the behaviour of solutions at the boundary in differents cases (Section 3). The results we give provide sufficient conditions in order that the Dirichlet condition is satisfied in the classical sense. We start with the simplest case, namely a linear equation involving the fractional Laplacian on the half-space and we generalize this result progressively to a large class of Lévy operators on smooth domains. We next treat further examples involving different nonlinearities (Section 4). Eventually, we give in Section 5 an explicit example showing that if the measure is not singular enough near the origin, the behaviour of the solution at the boundary may change if a drift term appears in the equation; we will show that the boundary condition may not be satisfied in the classical sense anymore.

**Notation.** Throughout the paper, $B_r(x) \subset \mathbb{R}^N$ denotes the open ball of radius $r$ centered at $x$. If $x = 0$, we simply write $B_r$. The open unit ball is denoted $B$ and $\mathbf{1}_B$ denotes the indicator function of $B$. When estimating nonlocal terms, we use also the following notation: $A \propto B$ means that $c_1 B \leq A \leq c_2 B$ for two constants $c_1, c_2 > 0$.

Recall that $\Omega$ is assumed to be a $C^2$-domain. This implies in particular that the signed distance function $d$ to $\partial\Omega$, which is nonnegative in $\Omega$ and nonpositive in its complementary, is $C^2$ in a neighbourhood of $\partial\Omega$. Furthermore, $d$ also denotes a $C^2$-function in $\mathbb{R}^N$ which agrees with $d$ in a neighbourhood of $\partial\Omega$.

## 1. Viscosity solutions of the Dirichlet problem

We recall in this section the definition of viscosity sub and supersolutions for the Dirichlet problem and we construct solutions by Perron's method.

### 1.1. Definition of viscosity solutions.

We mentioned in the Introduction that sub and supersolutions are seen as functions defined in $\mathbb{R}^N$ and equal to $g$ on $\Omega^c$ (or less/greater than $g$ on $\Omega^c$).

**Definition 1.** *A usc function $u : \mathbb{R}^N \to \mathbb{R}$ is a* subsolution *of (1)-(2) if, for any test function $\phi \in C^2(\mathbb{R}^N)$, at each maximum point $x_0 \in \overline{\Omega}$ of $u - \phi$ in $B_\delta(x_0)$, we have*

$$\mathrm{E}(u, \phi, x_0) := F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0), I_\delta^1[\phi](x_0) + I_\delta^2[u](x_0)) \leq 0 \quad \text{if } x_0 \in \Omega\,,$$

*or*

$$\min(\mathrm{E}(u, \phi, x_0); u(x_0) - g(x_0)) \leq 0 \quad \text{if } x_0 \in \partial\Omega\,,$$

*where*

$$I_\delta^1[\phi](x_0) = \int_{\{|z|<\delta\}} \{\phi(x_0 + z) - \phi(x_0) - (D\phi(x_0) \cdot z)\mathbf{1}_B(z)\} d\mu_{x_0}(z)$$

$$I_\delta^2[u](x_0) = \int_{\{|z|\geq\delta\}} \{u(x_0 + z) - u(x_0) - (D\phi(x_0) \cdot z)\mathbf{1}_B(z)\} d\mu_{x_0}(z).$$

*A lsc function $v : \mathbb{R}^N \to \mathbb{R}$ is a* supersolution *of (1)-(2) if, for any test function $\phi \in C^2(\mathbb{R}^N)$, at each minimum point $x_0 \in \overline{\Omega}$ of $u - \phi$ in $B_\delta(x_0)$, we have*

$$\mathrm{E}(u, \phi, x_0) := F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0), I_\delta^1[\phi](x_0) + I_\delta^2[u](x_0)) \geq 0 \quad \text{if } x_0 \in \Omega\,,$$

*or*

$$\max(\mathrm{E}(u, \phi, x_0); u(x_0) - g(x_0)) \geq 0 \quad \text{if } x_0 \in \partial\Omega\,.$$

*Finally, a* viscosity solution *of (1)-(2) is a function whose upper and lower semicontinuous envelopes are respectively sub- and supersolution of the problem.*

1.2. **Existence and uniqueness of viscosity solutions.** We now turn to the existence issue for (1)-(2). We provide such a result in the case of Lévy-Ito measures and we do it by assuming $F$ to be defined in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times \mathbb{R}$ (instead of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times \mathbb{R}$) to simplify the exposition.

The assumptions we use are the following ones (see the remark below for some comments).

**(A1)** The measure $\mu(dz)$ and the function $j(x, z)$ satisfy: there exists a constant $\bar{c} > 0$ such that

$$\int_B |j(x,z)|^2 \mu(dz) < +\infty\ ,\ \int_{\mathbb{R}^N \setminus B} \mu(dz) < +\infty\ ,$$

$$\int_{\mathbb{R}^N} |j(x,z) - j(y,z)|^2 \mu(dz) \leq \bar{c}|x-y|^2 \ \text{ and } \ \int_{\mathbb{R}^N \setminus B} |j(x,z) - j(y,z)| \mu(dz) \leq \bar{c}|x-y|$$

**(A2)** There exists $\gamma > 0$ such that for any $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $p \in \mathbb{R}^N$, $X \in \mathbb{S}^N$ and $l \in \mathbb{R}$

$$F(x, u, p, X, l) - F(x, v, p, X, l) \geq \gamma(u - v) \qquad \text{when} \qquad u \geq v.$$

**(A3-1)** For any $R > 0$, there exist moduli of continuity $\omega$ (independent of $R$) and $\omega_R$ such that, for any $|x|, |y| \leq R$, $|v| \leq R$, $l \in \mathbb{R}$ and for any $X, Y \in \mathbb{S}^N$ satisfying

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + r(\beta) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{9}$$

for some $\varepsilon > 0$ and $r(\beta) \to 0$ as $\beta \to 0$, then, if $s(\beta) \to 0$ as $\beta \to 0$, we have

$$F(y, v, \varepsilon^{-1}(x-y), Y, l) - F(x, v, \varepsilon^{-1}(x-y) + s(\beta), X, l) \leq \omega(\beta) + \omega_R(|x-y| + \varepsilon^{-1}|x-y|^2)$$

or

**(A3-2)** For any $R > 0$, $F$ is uniformly continuous on $\mathbb{R}^N \times [-R, R] \times B_R \times D_R \times \mathbb{R}$ where $D_R := \{X \in \mathbb{S}^N;\ |X| \leq R\}$ and there exist a modulus of continuity $\omega_R$ such that, for any $x, y \in \mathbb{R}^d$, $|v| \leq R$, $l \in \mathbb{R}$ and for any $X, Y \in \mathbb{S}^N$ satisfying (9) and $\varepsilon > 0$, we have

$$F(y, v, \varepsilon^{-1}(x-y), Y, l) - F(x, v, \varepsilon^{-1}(x-y), X, l) \leq \omega_R(\varepsilon^{-1}|x-y|^2 + |x-y| + r(\beta))\ .$$

**(A4)** $F(x, u, p, X, l)$ is Lipschitz continuous in $l$, uniformly with respect to all the other variables.

**(A5)** $\sup_{x \in \mathbb{R}^N} |F(x, 0, 0, 0, 0)| < +\infty$.

**(A6)** For any $x \in \partial\Omega$, the infimum limit in (10) is strictly positive and the supremum limit in (11) is strictly negative.

We can now state the existence and uniqueness result.

**Theorem 1.** *Assume that (A1), (A2), (A4), (A5) and (A6) hold. Then there exists a unique bounded, continuous solution of the Dirichlet problem (1)-(2) if, in addition, one of the following set of conditions is fullfilled*

*(i) (A3-1) holds and $g$ is a bounded continuous function in $\mathbb{R}^N$,*

*(ii) (A3-2) holds, $\Omega$ is a bounded open subset of $\mathbb{R}^N$ and $g$ is a bounded uniformly continuous function in $\mathbb{R}^N$.*

*Moreover, in both cases, one has a comparison result for this problem in the class of bounded sub- and supersolutions.*

*Remark* 1. Let us make a few comments about the assumptions. Of course, (A2) is very classical and, unfortunately, the combination of (A3)-(A4) does not allow a very general form and dependence of the nonlocal term: in particular, we do not know how to handle operators of the general form (3).

As far as (A3) is concerned, (A3-1) allows more general dependence wrt $x$ than (A3-2), which, on the contrary, allows more general dependence wrt $p$, and this is reflected in Theorem 1 by the cases (i) and (ii) for reasons which will be clear in the proof (at least, we hope so!).

We refer to [4] for more comments on these assumptions ensuring that a comparison result holds true in the whole space $\mathbb{R}^N$.

*Proof.* We consider two bounded continuous functions $\psi_1, \psi_2 : \mathbb{R}^N \to \mathbb{R}$ such that $\psi_1 \geq \psi_2$ in $\mathbb{R}^N$ and $\psi_1 = \psi_2 = g$ on $\Omega^c$. Notice that they are uniformly continuous in the case of (ii).

We set

$$\tilde{F}(x,u,p,X,l) := \min(u-\psi_2; \max(u-\psi_1; F(x,u,p,X,l)) \, .$$

It is straightforward to show that $\tilde{F}$ still satisfies (A1), (A2), (A3), (A4), (A5) and this is where the difference between (i) and (ii) plays a role: indeed, in the case of (i), $\psi_1, \psi_2$ are just bounded continuous functions and this is consistent with (A3-1), while, in the case of (ii), they are bounded uniformly continuous functions which is consistent with (A3-2).

If $M = \max(||g||_\infty; \gamma^{-1}||F(x,0,0,0,0)||_\infty)$, then $-M$ and $+M$ are respectively sub and supersolution of the equation $\tilde{F} = 0$ in $\mathbb{R}^N$. By applying the Perron's Method (*cf.* [9, 13] for local equations and [1, 5, 12, 4] for nonlocal ones) together with the comparison result of [4] provides the existence of a solution $u$ for this integro-differential equation such that $|u| \leq M$.

In order to prove the existence of a solution for the Dirichlet problem, the idea is to choose an increasing sequence of functions $(\psi_1^\alpha)_\alpha$ and a decreasing sequence of functions $(\psi_2^\alpha)_\alpha$ such that $\psi_1^\alpha(x) \to +\infty$ and $\psi_2^\alpha(x) \to -\infty$ for any $x \in \Omega$. Since the associated solutions $(u^\alpha)_\alpha$ we constructed above are uniformly bounded by $M$, the half-relaxed limit method provides us with an usc subsolution $\overline{u}$ and a lsc supersolution $\underline{u}$ of the Dirichlet problem (in the sense of Definition 1) such that $\overline{u} \geq \underline{u}$.

But, by (A6) and Lemma 1, one has $g \leq \underline{u} \leq \overline{u} \leq g$ on $\Omega^c$ and therefore $\overline{u}, \underline{u}$ are continuous on $\partial\Omega$ with $\underline{u} = \overline{u} = g$ on $\partial\Omega$. Using this property, it is easy to build $\psi_1, \psi_2$ such that $\psi_2 \leq \underline{u}, \overline{u} \leq \psi_1$ on $\overline{\Omega}$ and $\psi_1 = \psi_2 = g$ on $\Omega^c$; moreover, $\psi_1, \psi_2$ can be built in order to be bounded continuous functions in the case of (i) and bounded uniformly continuous functions in the case of (ii).

In both cases, $\overline{u}, \underline{u}$ are respectively viscosity sub and supersolution of the equation of the type $\tilde{F} = 0$ in $\mathbb{R}^N$ associated to $\psi_1, \psi_2$ for which we have a comparison result by the same arguments as above. Therefore $\overline{u} \leq \underline{u}$ in $\mathbb{R}^N$. Finally $u := \overline{u} = \underline{u}$ is the solution of the Dirichlet problem we were looking for and the Dirichlet condition is satisfied in the classical sense.

A similar argument provides the comparison result for the Dirichlet problem. □

## 2. A key technical lemma

Let us now state the main technical result of this article.

**Lemma 1.** *Let $u$ be a bounded usc viscosity subsolution of the Dirichlet problem* (1)-(2) *and $R := ||u||_\infty$. If, for some $x \in \partial\Omega$, we have $u(x) > g(x)$, there then exists $\nu > 0$ such that for any $k_1, k_2 > 0$, $r > 0$ small enough, we have*

$$\liminf_{\substack{y\to x,\, y\in\overline{\Omega}\\ \eta\downarrow 0,\, d(y)\eta^{-1}\to 0}} \left\{ \sup_{0<\delta<r} \inf_{-R\le s\le R} [F(y,s,p_\eta(y),M_\eta(y),I_{\eta,\delta,r})] \right\} \le 0 \,, \tag{10}$$

*where*

$$\begin{aligned} p_\eta(y) &:= k_1\eta^{-1}Dd(y) + o(\eta^{-1}) \,,\\ M_\eta(y) &:= k_1\eta^{-1}D^2d(y) - (k_2\eta^{-2} + o(\eta^{-2}))Dd(y)\otimes Dd(y) + o(\eta^{-1}) \,,\\ I_{\eta,\delta,r}(y) &:= -\nu I^{e,1}_{\delta,r}(y) - (k_1\eta^{-1} + o(\eta^{-1}))I^{e,2}_{\delta,r} - (k_1\eta^{-1} + o(\eta^{-1}))I^3_r(x) + o(\eta^{-1}) \,. \end{aligned}$$

*Let $v$ be a bounded lsc viscosity supersolution of the Dirichlet problem and $R := ||v||_\infty$. If, for some $x \in \partial\Omega$, we have $v(x) < g(x)$, there then exists $\nu > 0$ such that for any $k_1, k_2, r > 0$, we have*

$$\limsup_{\substack{y\to x\, y\in\overline{\Omega}\\ \eta\downarrow 0,\, d(y)\eta^{-1}\to 0}} \left\{ \inf_{0<\delta<r} \sup_{-R\le s\le R} [F(y,s,-p_\eta(y),-M_\eta(y),-I_{\eta,\delta,r})] \right\} \ge 0 \,. \tag{11}$$

*Remark* 2. We have considered in Lemma 1 nonlocal terms written as Lévy's operators but in stochastic control with jump processes (see for instance [16]), one can also consider the so-called Lévy-Ito diffusions whose infinitesimal generators are of the form (5). It is worth pointing out that Lemma 1 can be readily translated for such operators by using appropriate $x$-dependent measures $\mu_x(dz)$ associated with $\mu$ and $j(x,\cdot)$. See below for further details.

*Proof of Lemma 1.* We only provide the proof for the subsolution case, since the supersolution one is analogous.

We set $4\nu := u(x) - g(x) > 0$. Since $g$ is continuous, there exists $r = r(x) > 0$ such that, for all $y \in B(x,2r)\cap\Omega^c$,

$$|g(x) - g(y)| \le \nu \,. \tag{12}$$

Next we introduce the test-function

$$\psi(y) := \frac{\chi(y-x)}{\varepsilon} + \varphi\left(\frac{d(y)}{\eta}\right) \,,$$

where $\chi : \mathbb{R}^N \to \mathbb{R}$ is a smooth bounded function such that $\chi(0) = 0$ and $\chi(y) > 0$ if $y \neq 0$, $\liminf_{|y|\to\infty} \chi(y) > 0$, and $D^2\chi$ is bounded, while $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded smooth function, which is concave on $(0,+\infty)$ and such that $\varphi(0) = 0$, $\varphi'(0) = k_1$, $\varphi''(0) = -k_2$, $||\varphi||_\infty \le \nu$. For the sake of clarity, we have dropped the dependence of $\psi$ in $\varepsilon$ and $\eta$ and we will do so too for the maximum points we consider below. Let us also mention that in all the proof below, the parameters $\varepsilon$ and $\eta$ are chosen such that $\eta \ll \varepsilon \ll 1$. We can think of these conditions as if $\varepsilon$ depends on $\eta$ and $1 \ll \varepsilon^{-1} \ll \eta^{-1}$.

By classical arguments, $y \mapsto u(y) - \psi(y)$ achieves its maximum at a point $\bar{x}$ and since $\varphi$ is bounded, we know that $\dfrac{\chi(\bar{x}-x)}{\varepsilon}$ remains bounded as $\varepsilon \to 0$ and therefore $\bar{x} \to x$. Moreover, since $(u-\psi)(x) \le (u-\psi)(\bar{x})$, we also have

$$g(x) + 4\nu = u(x) \le u(\bar{x}) - \frac{\chi(\bar{x}-x)}{\varepsilon} - \varphi\left(\frac{d(\bar{x})}{\eta}\right) \le u(\bar{x}) + \nu \,,$$

(remember that $\chi \geq 0$ and $|\varphi| \leq \nu$) and therefore, for $\varepsilon$ small enough $\bar{x}$ is necessarily in $\overline{\Omega}$, otherwise the above inequality would contradict (12).

This implies $d(\bar{x}) \geq 0$ and therefore, since the $\chi$-term is nonnegative, we have $\varphi(d(\bar{x})/\eta) \leq u(\bar{x}) - u(x)$ from which we deduce, using the upper semicontinuity of $u$

$$\varphi\left(\frac{d(\bar{x})}{\eta}\right) \to 0 \quad i.e. \quad \frac{d(\bar{x})}{\eta} \to 0 \qquad \text{and} \qquad u(\bar{x}) \to u(x)\,.$$

Since $u(\bar{x}) \to u(x)$, for $\varepsilon$ small enough, we cannot have $u(\bar{x}) \leq g(\bar{x})$ if $\bar{x} \in \partial\Omega$ and therefore, we deduce that the $F$-viscosity inequality holds, namely

$$F(\bar{x}, u(\bar{x}), D\psi(\bar{x}), D^2\psi(\bar{x}), I^1_\delta[\psi](\bar{x}) + I^2_\delta[u](\bar{x})) \leq 0\,,$$

where $0 < \delta < r$ and

$$I^1_\delta[\psi](\bar{x}) = \int_{\{|z|<\delta\}} \{\psi(\bar{x}+z) - \psi(\bar{x}) - (D\psi(\bar{x})\cdot z)\mathbf{1}_B(z)\} d\mu_{\bar{x}}(z)\,,$$
$$I^2_\delta[u](\bar{x}) = \int_{\{|z|\geq\delta\}} \{u(\bar{x}+z) - u(\bar{x}) - (D\psi(\bar{x})\cdot z)\mathbf{1}_B(z)\} d\mu_{\bar{x}}(z)\,.$$

The proof consists now in estimating all the terms of this equation, and in particular the nonlocal ones.

As far as derivatives are concerned, since $D\chi(0) = 0$ and thanks to the properties of $\bar{x}$ and $\varphi$ and the fact that we choose $\eta \ll \varepsilon$, we have

$$\begin{aligned} D\psi(\bar{x}) &= \frac{o(1)}{\varepsilon} + \frac{k_1 + o(1)}{\eta} Dd(\bar{x}) = \frac{k_1}{\eta} Dd(\bar{x}) + \frac{o(1)}{\eta}\,, \\ D^2\psi(\bar{x}) &= \frac{O(1)}{\varepsilon} + \frac{k_1 + o(1)}{\eta} D^2 d(\bar{x}) - \frac{k_2 + o(1)}{\eta^2} Dd(\bar{x}) \otimes Dd(\bar{x}) \\ &= \frac{k_1}{\eta} D^2 d(\bar{x}) - \frac{k_2 + o(1)}{\eta^2} Dd(\bar{x}) \otimes Dd(\bar{x}) + \frac{o(1)}{\eta}\,. \end{aligned}$$

We now turn to nonlocal terms. We have to estimate four terms: the first one is $I^1_\delta[\psi](\bar{x})$ and the other one comes from $I^2_\delta[u](\bar{x})$ which decomposes as the sum of three terms as follows (see Figure 1)

$$\begin{aligned} (A) &= \int_{\{|z|>r\}} \{u(\bar{x}+z) - u(\bar{x}) - (D\psi(\bar{x})\cdot z)\mathbf{1}_B(z)\} d\mu_{\bar{x}}(z) \\ (B) &= \int_{\left\{\substack{\delta\leq|z|\leq r \\ \bar{x}+z\in\Omega}\right\}} \{u(\bar{x}+z) - u(\bar{x}) - (D\psi(\bar{x})\cdot z)\mathbf{1}_B(z)\} d\mu_{\bar{x}}(z) \\ (C) &= \int_{\left\{\substack{\delta\leq|z|\leq r \\ \bar{x}+z\notin\Omega}\right\}} \{g(\bar{x}+z) - u(\bar{x}) - (D\psi(\bar{x})\cdot z)\mathbf{1}_B(z)\} d\mu_{\bar{x}}(z) \end{aligned}$$

Estimate of $I^1_\delta$: recalling that $\varphi$ is concave on $(0, +\infty)$, we have

$$\begin{aligned} I^1_\delta[\psi](\bar{x}) &\leq O(\varepsilon^{-1}) + \frac{1}{\eta}\varphi'\left(\frac{d(\bar{x})}{\eta}\right) \int_{|z|<\delta} (d(\bar{x}+z) - d(\bar{x}) - Dd(\bar{x})\cdot z\mathbf{1}_B(z)) \mu_{\bar{x}}(dz) \\ &= O(\varepsilon^{-1}) + \frac{k_1 + o(1)}{\eta} ||D^2 d||_\infty \int_{|z|<\delta} |z|^2 d\mu_{\bar{x}}(z) = \frac{o(1)}{\eta}\,. \end{aligned}$$

We used the fact that the function $d$ is $C^2$ to get the last line.

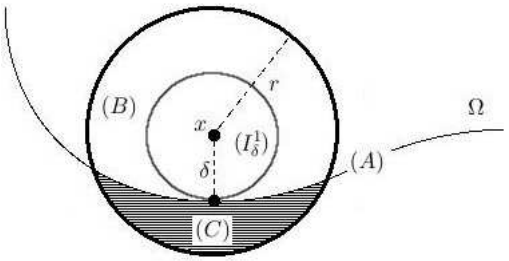


FIGURE 1. Decomposition of the integrals with the natural choice $\delta = d(\bar{x})$ if $d(\bar{x}) \neq 0$

ESTIMATE OF $(A)$: Since $u$ is bounded and $\int_{\mathbb{R}^N \setminus B} d\mu_{\bar{x}}(z) < \infty$, we obtain (recall that $r$ is fixed)

$$(A) \leq O(\varepsilon^{-1}) - \frac{k_1 + o(1)}{\eta} I_r^3(\bar{x})\,.$$

ESTIMATE OF $(B)$: Using the fact that $\bar{x}$ is a global maximum point of $u - \psi$, we have

$$u(\bar{x}+z) - u(\bar{x}) \leq \psi(\bar{x}+z) - \psi(\bar{x})$$

from which we deduce that

$$(B) \leq \int_{\left\{ \begin{smallmatrix} \delta \leq |z| \leq r \\ \bar{x}+z \in \Omega \end{smallmatrix} \right\}} \{\psi(\bar{x}+z) - \psi(\bar{x}) - D\psi(\bar{x}) \cdot z\} d\mu_{\bar{x}}(z)\,.$$

Next arguing as for $I_1^\delta$, we obtain

$$\begin{aligned}
(B) &\leq O(\varepsilon^{-1}) + \frac{k_1 + o(1)}{\eta} \int_{\left\{ \begin{smallmatrix} \delta \leq |z| \leq r \\ \bar{x}+z \in \Omega \end{smallmatrix} \right\}} \{d(\bar{x}+z) - d(\bar{x}) - Dd(\bar{x}) \cdot z\} d\mu_{\bar{x}}(z) \\
&\leq O(\varepsilon^{-1}) + \frac{k_1 + o(1)}{\eta} ||D^2 d||_\infty \int_{\left\{ \begin{smallmatrix} \delta \leq |z| \leq r \\ \bar{x}+z \in \Omega \end{smallmatrix} \right\}} |z|^2 d\mu_{\bar{x}}(z) \\
&= o(\eta^{-1}) + \frac{k_1 + o(1)}{\eta} o_r(1)
\end{aligned}$$

where $o_r(1)$ is a quantity which tends to 0 when $r$ tends to 0.

ESTIMATE OF $(C)$: In this term, we see the jump of $u$ at the boundary. We choose $\varepsilon$ small enough so that $|\bar{x} - x| < r$ and $|u(\bar{x}) - u(x)| \leq \nu$. Using the definition of $r$, this implies, in particular, that

$$\text{for all } z \in B_r, \bar{x} + z \notin \Omega^c, \quad g(\bar{x}+z) - u(\bar{x}) \leq -\nu$$

and we then conclude that, for $\varepsilon$ small enough, we have

$$(C) \leq -\nu I_{\delta,r}^{e,1} - \frac{k_1 + o(1)}{\eta} I_{\delta,r}^{e,2}\,.$$

Gathering all these estimates in the viscosity subsolution's inequality and using the fact that $F$ is nonincreasing with respect to its last argument, we finally have

$$F\left(\bar{x}, u(\bar{x}), p_\eta(\bar{x}), M_\eta(\bar{x}), -\nu I_{\delta,r}^{e,1} - \frac{k_1 + o(1)}{\eta} I_{\delta,r}^{e,2} - \frac{k_1 + o(1)}{\eta} I_r^3(\bar{x}) + \frac{k_1 + o(1)}{\eta} o_r(1) + \frac{o(1)}{\eta}\right) \leq 0\,.$$

Noticing that $|u(\bar{x})| \leq R$ and that this inequality holds true for any $0 < \delta < r$ small enough, we conclude that (10) holds by letting $\eta$ tend to 0. □

## 3. The linear model equation

In this section, we first study the boundary behaviour of a viscosity solution in a model framework by considering (7)-(2) where $\mathcal{I}[u]$ is an operator of Lévy type, *i.e.* given by (3). We first mainly treat the case of the fractional Laplace operator, for which we recall that the Lévy measure is given by

$$d\mu_x(z) = \frac{dz}{|z|^{N+\alpha}}$$

where $0 < \alpha < 2$. We also consider the special case $\alpha = 0$ and we refer to the corresponding integral operator as the "zero-Laplacian". We next try to generalize the results to more general operators.

In order to avoid hiding important ideas below technicalities, we start with the simple case of an equation involving the fractional Laplacian on a flat boundary. We next consider general smooth sets and eventually give general sufficient conditions on a Lévy measure so that the fractional Laplacian can be replaced with a general Lévy operator.

3.1. **Dirichlet problem on a half-space.** Let us first begin by considering the model problem (7) in the half-space $\Omega = \{x_N > 0\}$. We recall that we assume in this case that $a$, $b$, $f$ are continuous functions defined on $\overline{\Omega}$, $a(x)$ being a symmetric nonnegative matrix for any $x \in \overline{\Omega}$.

The question is whether a viscosity sub- or supersolution $u$ of (7)-(2) satisfies the Dirichlet boundary condition in the classical sense or not.

The answer to this question we provide below, as the other similar results we obtain next, relies on the technical lemma we stated in the previous section (Lemma 1). It provides us with a necessary condition in the case where $u \neq g$ at the boundary involving both differential terms (corresponding to $Du$ and $D^2u$) and integral terms.

We only consider the subsolution case since the supersolution one can be treated analogously and leads to the same conditions since the equation is linear. We want to prove that $u \leq g$ on $\partial\Omega$ and we argue by contradiction assuming that it is not true. Hence (10) provides us for $r$ small enough some $y \in B_r(x)$ such that for any $\delta \in (0, r)$

$$\frac{k_2 + o(1)}{\eta^2} a(y)Dd(y) \cdot Dd(y) - \frac{k_1 + o(1)}{\eta}\Big(b(y) \cdot Dd(y) + \mathrm{Tr}(a(y)D^2d(y)) + I_r^3(y)\Big) \\ + \nu I_{\delta,r}^{e,1} + \frac{k_1 + o(1)}{\eta} I_{\delta,r}^{e,2} + \frac{o(1)}{\eta} \leq 0$$

and we try to exhibit a contradiction. In order to do it, we have to use more precise estimates of the terms $I_{\delta,r}^{e,1}$, $I_{\delta,r}^{e,2}$ and $I_r^3$ appearing in the previous inequality and combine them.

We next use Condition (8). We will check below that fractional Laplace operator satisfies it. Using this condition leads to

$$\frac{k_2 + o(1)}{\eta^2} a(y)Dd(y) \cdot Dd(y) - \frac{k_1 + o(1)}{\eta}\Big(b(y) \cdot Dd(y) + \mathrm{Tr}(a(y)D^2d(y)) + I_r^3(y)\Big) \\ + \nu H(\bar{\delta}) - C\frac{k_1 + o(1)}{\eta}\bar{\delta}H(\bar{\delta}) + \frac{k_1 + o(1)}{\eta} o_r(1) + \frac{o(1)}{\eta} \leq 0\,.$$

where, as above, $\bar\delta = \max(d(y), \delta)$. This inequality shows that $y$ cannot be on the boundary: indeed, otherwise we can fix $\eta$ and let $\delta$ tends to 0; we are led to a contradiction because of the behavior of $H$ at 0 imposed by (8).

Since $d(y) > 0$, a natural choice is $\delta = d(y)$ and therefore $\delta = o(\eta)$. It follows that the term $\dfrac{k_1 + o(1)}{\eta}\delta H(\delta)$ is a $o(H(\delta))$ and the final quantity to examine is

$$\frac{k_2 + o(1)}{\eta^2} a(y)Dd(y)\cdot Dd(y) - \frac{k_1 + o(1)}{\eta}\Big(b(y)\cdot Dd(y) + \mathrm{Tr}(a(y)D^2d(y)) + I_r^3(y)\Big) \\ +(\nu - o(1))H(\delta) + \frac{k_1 + o(1)}{\eta}o_r(1) + \frac{o(1)}{\eta} \le 0\,.$$

We next distinguish different cases.

• First, it is clear that if $a(x)Dd(x)\cdot Dd(x) > 0$, then the first term controls all the other terms, except maybe the $H(\delta)$-one but this term has the right sign, and we conclude that (10) cannot hold. Therefore there is no loss of boundary data in this case.

• Next, if $a(x)Dd(x)\cdot Dd(x) = 0$, we drop the first term which has the right sign but an unknown behaviour. If there exists a constant $\tilde C > 0$ such that $\delta H(\delta) \ge \tilde C$, then the $H(\delta)$-term controls all the other terms, again because $\delta = o(\eta)$; and there is no loss of boundary data in this case.

• Finally, if $a(x)Dd(x)\cdot Dd(x) = 0$ and we cannot find $\tilde C > 0$ such that $\delta H(\delta) \ge \tilde C$ for any $\delta > 0$, then we also drop the $H(\delta)$-term (which may not be the leading term) and we obtain

$$-\frac{k_1 + o(1)}{\eta}\Big(b(y)\cdot Dd(y) + \mathrm{Tr}(a(y)D^2d(y)) + I_r^3(y)\Big) + \frac{k_1 + o(1)}{\eta}o_r(1) + \frac{o(1)}{\eta} \le 0\,.$$

Therefore the new condition for no loss of boundary conditions turns out to be

$$b(x)\cdot Dd(x) + \mathrm{Tr}(a(x)D^2d(x)) + I_0^3(x) < 0\,.$$

We use here the fact that, for the fractional Laplacian operator, this case happens only if $\alpha < 1$, since (see below), $H(\delta)$ is of order $\delta^{-\alpha}$. Thus, $z$ is $\mu$-integrable here and therefore $I_0^3$ makes sense. Indeed, such a condition is sufficient by replacing $I_0^3$ with $I_r^3$ for $r$ small enough; in the same way, the $o_r(1)$-term can be absorbed by the first term. Fixing $r$ in that way, the above quantity is indeed positive for $\varepsilon$ and $\eta$ close enough to 0 and there is no loss of boundary data in this case neither.

Summing up the previous discussion, we conclude that there are no loss of boundary condition for the flat boundary if, for any $x \in \partial\Omega$,

$$(13)\quad \delta H(\delta) \ge \tilde C > 0 \ \text{ or } \ a(x)Dd(x)\cdot Dd(x) > 0 \ \text{ or } \ b(x)\cdot Dd(x) + \mathrm{Tr}(a(x)D^2d(x)) + I_0^3(x) < 0$$

where $H$ appears in (8) and the first condition has only to be satisfied for $\delta > 0$ sufficiently small.

3.2. **The fractional Laplacian.** It remains to check the above conditions by computing the function $H$ for which (8) hold. We prove a little bit more precise result.

**Lemma 2.** *For any $0 < \alpha < 2$, as $\delta \to 0$ we have*

$$\int_{\substack{|z|\le r\\ z_N > \delta}} \mu(dz) \propto \delta^{-\alpha} \quad , \quad \int_{\substack{|z|\le r\\ z_N > \delta}} |z|\mu(dz) \propto \delta^{1-\alpha}$$

$$\int_{|z|<\delta} |z|\mu(dz) \propto \delta^{1-\alpha} \quad , \quad \int_{|z|<\delta} |z|^2\mu(dz) \propto \delta^{2-\alpha}$$

*The same results hold if instead of $|z|$ and $|z|^2$, we integrate $|z_N|$ and $|z_N|^2$ respectively. Hence* (8) *is true with $H(\delta) = \delta^{-\alpha}$ and $\delta H(\delta) \geq \tilde{C}$ is equivalent to $\alpha \geq 1$.*

*For $\alpha = 0$, as $\delta \to 0$ we have*

$$\int_{\substack{|z|\leq r\\ z_N>\delta}} \mu(dz) \propto \ln\frac{1}{\delta} \quad , \quad \int_{\substack{|z|\leq r\\ z_N>\delta}} |z|\mu(dz) \propto \delta \ln\frac{1}{\delta}$$
$$\int_{|z|<\delta} |z|\mu(dz) \propto \delta \quad , \quad \int_{|z|<\delta} |z|^2\mu(dz) \propto \delta^2,$$

*and in this case Condition* (8) *holds with $H(\delta) = \ln(1/\delta)$.*

*Proof.* Let us first deal with $\alpha \in (0,2)$. We assume for clarity (and without loss of generality) that $r = 1$. Let us first compute $\int_{|z|\leq 1, z_N>\delta} \mu(dz)$ by writing $z = (z', z_N)$ with $z' \in \mathbb{R}^{N-1}$:

$$\int_{\substack{|z|\leq 1\\ z_N>\delta}} \mu(dz) \quad \propto \quad \int_\delta^1 \int_{B(0,1)} \frac{dz'}{|z'^2 + z_N^2|^{(N+\alpha)/2}} dz_N = \int_\delta^1 |z_N|^{-\alpha-1} G_\alpha(z_N)\, dz_N$$

where $G_\alpha$ is the $(N-1)$-dimensional integral

$$G_\alpha(z_N) = \int_{B\left(0,\sqrt{1/z_N^2-1}\right)} \frac{dy}{(y^2+1)^{(N+\alpha)/2}}\,.$$

Now since $N + \alpha > N - 1$ for any $\alpha > 0$, $G_\alpha(z_N)$ converges as $z_N \to 0$ so that we obtain the first result: as $\delta \to 0$,

$$\int_{\substack{|z|\leq 1\\ z_N>\delta}} \mu(dz) \propto \int_\delta^1 |z_N|^{-\alpha-1} dz_N \propto \delta^{-\alpha}\,.$$

The other estimate on $\{z_N > \delta\}$ is based on the same decomposition: integrating $|z|$ leads to replace $\alpha$ with $\alpha - 1$, which gives easily the second result since we still have $N + (\alpha - 1) > N - 1$ for $\alpha > 0$. Finally the integrals over $\{|z| < \delta\}$ follow from a simple computation that we omit here, and replacing $z$ with $z_N$ in the integrals changes the estimates only up to constants, so that we keep the same behaviour as $\delta \to 0$.

We now turn to the case $\alpha = 0$. We still assume that $r = 1$ for clarity. We write

$$\begin{aligned}\int_{\substack{|z|\leq 1\\ z_N>\delta}} \mu(dz) \quad &\propto \quad \int_\delta^1 \int_{B(0,1)} \frac{dz'}{(z'^2 + z_N^2)^{N/2}} dz_N \\ &\propto \quad \int_\delta^1 z_N^{-1} G_0(z_N) dz_N\end{aligned}$$

where $G_0$ is the $(N-1)$-dimensional integral

$$G_0(z_N) = \int_{B(0,\sqrt{1/z_N^2-1})} \frac{dy}{(y^2+1)^{N/2}}\,.$$

Since $N > N - 1$, $G_0(z_N)$ converges as $z_N \to 0$, so that as $\delta \to 0$,

$$\int_{\substack{|z|\leq 1\\ z_N>\delta}} \mu(dz) \propto \int_\delta^1 z_N^{-1}\, dz_N \propto \ln\frac{1}{\delta}\,.$$

Now integrating $|z|$ leads to considering the same problem without the term $z_N^{-1}$:

$$\int_{\substack{|z|\leq 1\\ z_N>\delta}} \mu(dz) \propto \int_\delta^1 G_0'(z_N) dz'\,,$$

where

$$G_0'(z_N) = \int_{B(0,\sqrt{1/z_N^2-1})} \frac{dy}{(y^2+1)^{N-1}}\,.$$

But here, this last integral does not have a finite limit as $z_N \to 0$ (remember that it is a $(N-1)$-dimensional integral), and in fact

$$G_0'(z_N) \propto \int_{B(0,\sqrt{1/z_N^2-1})} \frac{dt}{t^{N-1}} \propto \ln\frac{1}{z_N}\,.$$

So that finally

$$\int_{\substack{|z|\le 1\\ z_N>\delta}} |z|\mu(dz) \propto \int_\delta^1 \ln\frac{1}{z_N} dz_N \propto \delta \ln\frac{1}{\delta}\,.$$

The two last estimates are straightforward. □

3.3. **Dirichlet problem in a domain.** We would like to be able to get sufficient conditions ensuring that the Dirichlet boundary condition is satisfied in the classical sense in the case of a general smooth domain too. In view of the previous discussion, we see that it is enough to ensure that the measures $\mu_x$ still satisfy Condition (8). Lemma 3 below shows that under some rather natural condition on the measures $\mu_x$, one can pass from a half-space type domain to a general $C^2$-smooth domain.

In order to state the first lemma, let us first introduce some new notation: if $x \in \Omega$, we recall that $d(x)$ is the distance of $x$ to the boundary. If $x$ is close enough to $\partial\Omega$, we denote by $P(x) \in \partial\Omega$ the unique point such that $d(x) = |x - P(x)|$. Finally let $\mathbf{n}_{P(x)} = Dd(x)$ denote the outward unit normal to $\partial\Omega$ at $P(x)$ so that $P(x) - x = d(x)\mathbf{n}_{P(x)}$. We use new coordinates by choosing $x$ as being the origin and we decompose $z \in \mathbb{R}^N$ as follows: $z = z' + z_N\mathbf{n}_{P(x)}$ with $z' \cdot \mathbf{n}_{P(x)} = 0$.

**Lemma 3.** *Let us assume that, for some $r > 0$, the measure $\mu$ has the following property*

$$\forall c > 0, \quad \mu\big(B_r \cap \{|z_N + \delta| < c|z'|^2\}\big) = \mu\big(B_r \cap \{z_N + \delta < 0\}\big) \cdot o(1) \quad \text{as} \quad \delta \to 0\,. \tag{14}$$

*Then, for any $x \in \Omega$ and $r$ small enough, we have as $d(x) \to 0$*

$$\int_{B_r \setminus \Omega_x} \mu(dz) \propto \int_{B_r \cap \{z_N + d(x) < 0\}} \mu(dz)$$

*where $\Omega_x = \Omega - \{x\}$.*

Figure 2 illustrates the condition imposed on the measure: the hatched region is negligeable in front of $B_r \cap \{z_N + d(x) < 0\}$ as $\delta \to 0$, for any $x \in \Omega$. We then shall verify that this condition holds in the case of the fractional Laplacian. With this in hand, all the conclusions we obtained in the case of a half-space remain true in such a general domain $\Omega$.

**Lemma 4.** *In the case of the fractional Laplace operator, $\mu(z) = \frac{1}{|z|^{N+\alpha}}$ with $0 \le \alpha < 2$, property* (14) *is satisfied.*

We now turn to the proofs of these two lemmata.

*Proof of Lemma 3.* Let us choose $d(x)$ small enough and a local chart centered at $x$ and containing $B_r(x)$ (and hence $P(x) \in \partial\Omega$). The boundary may then be locally written as follows: $z_N = \gamma(z')$ with $\gamma(0) = -d(x)$. In particular, $P(x) = (0, -\delta)$. Since $\Omega$ is $C^2$, we have around $P(x)$

$$\gamma(z') = -\delta + O(|z'|^2) \quad \text{hence} \quad |z_N + \delta| \le c|z'|^2\,,\ \forall z \in \partial\Omega$$

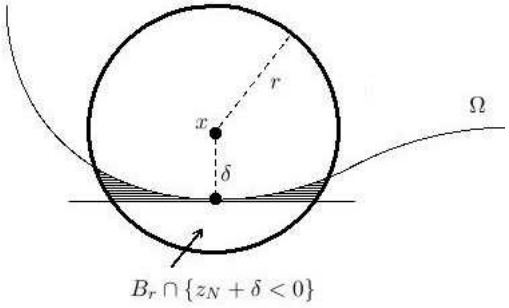


FIGURE 2

for some $c > 0$. Thus,

$$\int_{B_r \setminus \Omega_x} \mu(dz) = I(\delta) + \int_{B_r \cap \{z' > \delta\}} \mu(dz)\,,$$

where

$$I(\delta) = \int_{B_r \cap \{\gamma(z') \le z_N \le -\delta\}} \le \int_{B_r \cap \{|z_N + \delta| < c|z'|^2\}} \mu(dz)\,.$$

Now under the assumption on $\mu$ we made (see (14)), it is clear that $I(\delta)$ is dominated by the second integral as $\delta = d(x) \to 0$ so that the result holds. □

*Proof of Lemma 4.* Let us first treat the case $\alpha \in (0,2)$. Using the fact that for the fractional Laplacian, the integral over $B_r \cap \{z_N > \delta\}$ is of order $\delta^{-\alpha}$ (see Lemma 2), we conclude that we need to prove that $\delta^\alpha I(\delta) = o(1)$ with

$$I(\delta) = \int_{B_r \cap \{|z_N + \delta| < c|\tilde{z}|^2\}} \frac{dz}{|z|^{N+\alpha}}\,.$$

We first write

$$I(\delta) \;=\; \int_{|z'|<r} \int_{\delta - c|z'|^2}^{\delta + c|z'|^2} \frac{dz_N}{(z'^2 + z_N^2)^{(N+\alpha)/2}}\, dz' = \int_{|z'|<r} |z'|^{1-N-\alpha} G_\alpha(\delta, z') dz'$$

where

$$G_\alpha(\delta, z') = \int_{\delta/|z'| - c|z'|}^{\delta/|z'| + c|z'|} \frac{dt}{(1+t^2)^{(N+\alpha)/2}}\,.$$

Now we can estimate $I(\delta)$ as follows:

$$I(\delta) = \int_{|z'| \le \delta^\beta} \{\dots\} + \int_{\delta^\beta \le |z'| \le r} \{\dots\}$$

for $\beta > 0$ to be chosen later. Now consider $z'$ such that $|z'| \le \delta^\beta$. If one chooses $\beta > 1$, then $\frac{\delta}{|z'|} \gg 1$ and $c|z'| \ll 1$. Hence we get for such $z'$

$$\begin{aligned} G_\alpha(\delta, z') \;&\leqslant\; \int_{\delta/|z'| - c|z'|}^{\delta/|z'| + c|z'|} \frac{dt}{t^{N+\alpha}} \\ &\leqslant\; 2c|z'| \times \left(\frac{\delta}{2|z'|}\right)^{-N-\alpha} \\ &\leqslant\; C\delta^{-N-\alpha} |z'|^{N+\alpha+1}\,. \end{aligned}$$

On the other hand for $|z'| > \delta^\beta$, since $(1+t^2)^{(N+\alpha)/2} \geqslant 1$,

$$G_\alpha(\delta, z') \leq 2c|z'|\,.$$

Thus we have finally (remember that the integrals on $z'$ are $(N-1)$-dimensional) for $\alpha \neq 1$

$$\begin{aligned} I(\delta) &\leq C\left(\int_{|z'|<\delta^\beta} |z'|^2\delta^{-N-\alpha}dz' + \int_{\delta^\beta<|z'|<r} |z'|^{2-N-\alpha}dz'\right) \\ &\leq C\left(\delta^{-N-\alpha}\int_0^{\delta^\beta} t^N dt + \int_{\delta^\beta}^r t^{-\alpha}dt\right) \\ &\leq C\delta^{-\alpha}(\delta^{(N+1)\beta-N} + \delta^{\beta(1-\alpha)+\alpha}). \end{aligned}$$

Since $\beta > 1$, we have $(N+1)\beta > N$ so the first term is good. If $\alpha \in [0,1)$, then $\beta(1-\alpha)+\alpha > 0$ and the second term is still good. If $\alpha > 1$, we choose $\beta \in \left(1, \frac{\alpha}{\alpha-1}\right)$. Now if $\alpha = 1$, we have

$$\delta I(\delta) \leq C\left(\delta^{(N+1)\beta-N} + \delta \ln\frac{1}{\delta}\right)$$

and we conclude in the same way (even more easily).

Finally in the case $\alpha = 0$, we get the same estimate (with $\alpha = 0$) which proves that $I(\delta)$ remains bounded as $\delta \to 0$, for any choice of $\beta > 0$. Then since the mass of $\mu$ in $B_r \cap \{z_N > \delta\}$ is of order $\ln(1/\delta)$, we obtain that $\ln(1/\delta)^{-1}I(\delta) = o(1)$.

Thus we conclude that for any $0 \leqslant \alpha < 2$, the measure $d\mu = dz/|z|^{N+\alpha}$ satisfies (14). □

3.4. **General results in the case of a linear equation.** We discussed thoroughly the model linear equation (7) and we would like to state the result we finally proved. We only state it for fractional Laplace operators. Since we also want to include the existence and uniqueness results provided by Theorem 1, we introduce the following conditions on $a$, $b$ and $f$, which ensure, in particular, that (A3-1) holds.

**(HD)** $a = \sigma\sigma^T$ with $\sigma$ bounded and Lipschitz continuous on $\overline{\Omega}$, $b$ is bounded and Lipschitz continuous on $\overline{\Omega}$ and $f$ is bounded and continuous on $\overline{\Omega}$.

We formulate the following result in two parts: in the first one, we provide conditions under which there is no loss of boundary conditions using minimal hypotheses on $a$, $b$ and $f$, while, in the second one, we give a general existence and uniqueness result under more restrictive assumptions on $a$, $b$ and $f$, namely **(HD)**.

**Theorem 2.** *Assume that $a$, $b$, $f$ are continuous functions defined on $\overline{\Omega}$, where $\Omega$ is a $C^2$-domain, $a(x)$ being a symmetric nonnegative matrix for any $x \in \overline{\Omega}$, $d\mu_x(z) = d\mu(z) = dz/|z|^{N+\alpha}$ $(0 \leq \alpha < 2)$ and that $g$ is bounded and continuous on $\Omega^c$. If, at a given point $x \in \partial\Omega$*

$$\text{(15)}\quad \textit{either}\quad a(x)Dd(x)\cdot Dd(x) > 0 \quad \textit{or} \quad \alpha \geq 1 \quad \textit{or} \quad b(x)\cdot Dd(x) + \mathrm{Tr}(a(x)D^2d(x)) < 0$$

*then any usc subsolution $u$ of the Dirichlet problem (7)-(2) (resp. any lsc supersolution $v$), satisfies $u(x) \leq g(x)$ (resp. $v(x) \geq g(x)$).*

*If these conditions hold for any $x \in \partial\Omega$ and if* **(HD)** *holds, there then exists a unique continuous solution of the Dirichlet problem which assumes the boundary condition in the classical sense.*

We point out that Condition (15) is exactly (13): indeed, $I_0^3(x) = 0$ for any $x$ since the measure is radially symmetric ($I_0^3$ is well-defined since this quantity plays a role only if $\alpha < 1$), and $\delta H(\delta) \geq \tilde{C} > 0$ is equivalent to $\alpha \geq 1$.

3.5. **More general operators.** We would like to conclude this section by exhibiting other measures that can be handled with the techniques we presented above to deal with the linear equation. For instance, in mathematical finance [16], tempered $\alpha$-stable Lévy measures in one dimensional space appear in several models. They can be written under the following form

$$\mu(dz) = c_+ \frac{e^{-\gamma_+ z}}{|z|^{1+\alpha}} \mathbf{1}_{(0,+\infty)}(z)dz + c_- \frac{e^{\gamma_- z}}{|z|^{1+\alpha}} \mathbf{1}_{(-\infty,0)}(z)dz.$$

One can check that such a measure satisfies (8) and hence the previous results apply.

If now one considers operators whose singular measure $\mu$ depends on $x$, we see that the only condition that is now required is some continuity. Precisely,

$$\begin{cases} x \mapsto I^{e,1}_{\delta,r}(x) \\ x \mapsto I^{e,2}_{\delta,r} \\ x \mapsto I^{3}_{r}(x) \end{cases} \quad \text{are continuous.} \tag{16}$$

To finish with, let us briefly explain how to handle Lévy-Itô operators of the form (5) by considering proper $x$-dependent measures. Assume for simplicity that $\mu = \frac{dz}{|z|^{N+\alpha}}$. We first need to assume that there exists an inverse map for $j(x,z)$ that has some continuity properties. Precisely assume that there exists $J(x,Z)$ such that

$$j(x, J(x,Z)) = Z \quad J(x, j(x,z)) = z\,.$$

Then the operator (5) can be rewritten as

$$\mathcal{I}_{LI}[u](x) = \int (u(x+Z) - u(x) - \nabla u(x) \cdot Z \mathbf{1}_B(J(x,Z))) \ \frac{[\det D_Z J(x,Z)]}{|J(x,Z)|^{N+\alpha}} \ dZ.$$

If one can ensure next that there exists two constants $c_0, C_0$ such that

$$c_0|Z| \leq |J(x,Z)| \leq C_0|Z|,$$

then we end up with a measure

$$\mu_x(dZ) = \frac{[\det D_Z J(x,Z)]}{|J(x,Z)|^{N+\alpha}} \ dZ \leq C_0^{-N-\alpha} \frac{[\det D_Z J(x,Z)]}{|Z|^{N+\alpha}} \ dZ$$

assuming that $x \mapsto D_Z J(x,Z)$ is continuous is now enough to conclude.

## 4. Two nonlinear examples and the parabolic case

We present in this section a few examples of applications of our techniques to nonlinear equations. We also briefly discuss the parabolic case.

Following the explicit computations of Section 3, we keep working with the fractional Laplace operator and we choose $\delta = d(y)$ in Lemma 1. In particular the measure $\mu$ satisfies (8) for $x \in \overline{\Omega}$ with $H$ given in Lemma 2.

4.1. **Hamilton-Jacobi-Bellman Equations.** Next we turn to the case of Hamilton-Jacobi-Bellman Equations, namely

$$\sup_{\beta\in\mathcal{B}}\Big\{-\mathrm{Tr}(a^\beta(x)D^2u)-b^\beta(x)\cdot Du-\mathcal{I}^\beta[u](x)+u-f^\beta(x)\Big\}=0\quad\text{in }\Omega\,, \tag{17}$$

where $\mathcal{B}$ is a compact metric space, $a^\beta$, $b^\beta$, $f^\beta$ are continuous functions and $\mathcal{I}^\beta$ are nonlocal Lévy-Ito operators associated to $\mu^\beta, j^\beta$ satisfying (4) and (6) with $\bar c>0$ being independent of $\beta$. We assume also the following "equi-integrability property"

*There exists $\mu, j$ satisfying (A1) such that $|j^\beta(x,z)|\le|j(x,z)|$ for any $x,z$ and $\mu^\beta\le\mu$ in the sense that for any $\mu$-integrable, nonnegative function $g$,*

$$\int_{\mathbb{R}^N}g(z)d\mu^\beta(z)\le\int_{\mathbb{R}^N}g(z)d\mu(z).$$

All the above "basic" conditions on $\mathcal{I}^\beta$ will be refered to below as (IP) for "integrability property".

This equation does not enter into the frameworks of Theorem 1 and Lemma 1 since we deal with several nonlocal terms at the same time. But these results extend to this more general case, under suitable assumptions, since the methods to prove them readily apply: indeed, the above Hamilton-Jacobi-Bellman equation is a supremum of a family of linear operators and we can treat it in a similar way as the case of linear equations if we have enough uniformity in the assumptions wrt $\beta$; but, of course, the supremum creates also some dissymmetry.

To be more precise, we introduce the following condition, which again ensures that a (A3-1)-type assumption holds.

**(HD-HJB)** $a^\beta$, $b^\beta$ satisfy **(HD)** with $L^\infty$-bounds and Lipschitz constants being independent of $\beta$, $f^\beta$ is uniformly bounded in $\overline{\Omega}$ and uniformly continuous in compact subsets of $\mathbb{R}^N$, uniformly wrt $\beta$. For any $\beta\in\mathcal{B}$, $\mathcal{I}^\beta$ satisfy (A1) with a constant $\bar c$ which is independent of $\beta$.

Our result is the

**Theorem 3.** *Assume that $\Omega$ is a $C^2$-domain, $a^\beta$, $b^\beta$, $f^\beta$ are continuous functions defined on $\overline{\Omega}$, $g$ is bounded and continuous on $\Omega^c$ and that the nonlocal terms $\mathcal{I}^\beta$ satisfy (IP).*

*If, at $x\in\partial\Omega$, there exists $\beta\in\mathcal{B}$ such that Condition (8) holds with $H^\beta$, $C^\beta$, and if (13) holds, namely if there exists $\tilde C>0$ such that, one of the following conditions holds*

*(i) $a^\beta(x)Dd(x)\cdot Dd(x)>0$,*
*(ii) $\delta H^\beta(\delta)\ge\tilde C$, for $\delta>0$ sufficiently small,*
*(iii) $b^\beta(x)\cdot Dd(x)+\mathrm{Tr}(a^\beta(x)D^2d(x))+I_0^{\beta,3}(x)<0$,*

*then any usc subsolution $u$ of the Dirichlet problem (7)-(2) satisfies $u(x)\le g(x)$.*

*If, at $x\in\partial\Omega$, there exists $\theta>0$ and $\mathcal{B}_1(x),\mathcal{B}_2(x),\mathcal{B}_3(x)$ such that $\mathcal{B}=\mathcal{B}_1(x)\cup\mathcal{B}_2(x)\cup\mathcal{B}_3(x)$ with*

*(i)' for all $\beta\in\mathcal{B}_1(x)$, $a^\beta(x)Dd(x)\cdot Dd(x)>\theta$,*
*(ii)' for all $\beta\in\mathcal{B}_2(x)$, Condition (8) holds for $\mathcal{I}^\beta$ with $H^\beta=H$, $C^\beta=C$ independent of $\beta$ and $\delta H(\delta)\ge\theta$, for $\delta>0$ sufficiently small,*
*(iii)' for all $\beta\in\mathcal{B}_3(x)$, $b^\beta(x)\cdot Dd(x)+\mathrm{Tr}(a^\beta(x)D^2d(x))+I_0^{\beta,3}(x)<-\theta$,*

*then any lsc supersolution $v$ of the Dirichlet problem (7)-(2) satisfies $v(x)\ge g(x)$.*

*If these two conditions hold for any $x \in \partial\Omega$, and if* **(HD-HJB)** *holds, then there exists a unique continuous solution of the Dirichlet problem which assumes the boundary condition in the classical sense.*

4.2. **Fractional Laplace Equations.** Our last stationary example is concerned with the case of fractional Laplace equations involving a nonlinearity in the gradient

$$(-\Delta)^{\alpha/2}u - b(x)\cdot Du + c(x)|Du|^m + u = f(x) \quad \text{in } \Omega\ , \tag{18}$$

where $0 < \alpha < 2$, $b, c$ are continuous functions, $c \geq 0$ on $\overline{\Omega}$ and $m > 0$.

By Section 3, we know how the fractional Laplacian term behaves (see Subsection 3.2). The main interest of this example is to see the interaction of this term with the nonlinear term $c(x)|Du|^m$ and try to figure out what are the "allowed" powers $m$.

Since the term $c(x)|Du|^m$ is nonnegative, the most interesting analysis of loss of boundary condition concern supersolutions. The fractional Laplacian term provides a $\delta^{-\alpha}$ contribution while the linear $b$-term yields $k_1\eta^{-1}$ and the $c$-term yields $c(x)k_1^m\eta^{-m}$. As above the $\delta^{-\alpha}$ contribution dominates if $\alpha \geq 1$ and if $m \leq \alpha$ (recall that $\delta = o(1)\eta$). If $0 < \alpha < 1$, then the linear $b$ term may play the main role (see Section 5) and, if it is the case, then $m < 1$ is the natural condition.

We can formulate the

**Theorem 4.** *Assume that $b$, $c$, $f$ are continuous functions, that $\Omega$ is a $C^2$-domain and $g$ is bounded and continuous on $\Omega^c$. If, at $x \in \partial\Omega$, we have one of the following assumptions*

*(i)* $\alpha \geq 1$
*(ii)* $b(x)\cdot Dd(x) < 0$
*(iii)* $\alpha \geq 0$ *and* $b \equiv 0$

*then any usc subsolution $u$ of the Dirichlet problem* (18)-(2) *satisfies $u(x) \leq g(x)$.*

*If, at $x \in \partial\Omega$, we have one of the following assumptions*

*(i)'* $\alpha \geq 1$ *and* $m \leq \alpha$
*(ii)'* $b(x)\cdot Dd(x) < 0$ *and* $m < 1$
*(iii)'* $\alpha \geq 0$, $m \leq \alpha$ *and* $b \equiv 0$

*then any lsc supersolution $v$ of the Dirichlet problem (18)-(2) satisfies $v(x) \geq g(x)$.*

*If these two sets of conditions hold for any $x \in \partial\Omega$ and if $b, c$ are Lipschitz continuous then there exists a unique continuous solution of the Dirichlet problem which assumes the boundary condition in the classical sense.*

4.3. **Parabolic Equations.** We conclude this section by remarking that, as far as the losses of boundary conditions are concerned, the parabolic case can be treated exactly in the same way and leads exactly to the same conditions. Indeed, if one considers a problem on $\mathcal{O}\times(0,T)$, the parabolic PDE can be seen as a degenerate elliptic PDE in $\Omega = \mathcal{O}\times(0,T)$. The domain $\Omega$ is $C^2$ if $\mathcal{O}$ is $C^2$ and it can easily be seen that the involved quantities are (essentially) the same. The $u_t$ term is treated as a "tangential" derivative and therefore is a $o(\eta^{-1})$-term which, in most cases, does not interfere with the other terms.

## 5. Influence of the Drift Term

In this section we show that under some circumstances, the boundary data may be lost, even if the measure is singular and not integrable near the origin. This phenomenon happens due

to the fact that the influence of the drift term may be greater than that of the nonlocal term. In the case of the fractional Laplace operator, this may happen when $\mu(z) = 1/|z|^{N+\alpha}$ with $0 < \alpha < 1$ (for $1 \leq \alpha < 2$ on the contrary, the boundary data is always taken even if we have a drift term).

We shall construct here an example of such a situation in dimension $N = 1$, for a measure $\mu$ similar to the fractional Laplace:

$$\mu(z) = \frac{1}{|z|^{1+\alpha}} \mathbf{1}_B.$$

Considering the following model equation

$$\mathcal{L}u := -\int \{u(x+z) - u(x)\}\mu(dz) - u_x = f(x) \text{ in } \mathbb{R}_+ \,,$$

we have the

**Proposition 1.** *For any $0 \leq \alpha < 1$, there exists a bounded continuous function $f_\alpha : \mathbb{R}_+ \to \mathbb{R}$ and a solution $u = u_\alpha$ of $\mathcal{L}(u) = f_\alpha$ in $\{x \geq 0\}$, with $u = 0$ in $\{x < 0\}$ such that $u(0) = 1 > 0$.*

*Proof.* Let us consider $k$ exponents $\gamma_i \in (0,1)$, $i = 1, \dots, k$ where $k \geq 1$ is an integer to be fixed therafter, satisfying $\gamma_{i+1} < \gamma_i$, and a function $u$ as follows

$$u(x) = a_1 x^{\gamma_1} + a_2 x^{\gamma_2} + \cdots + a_k x^{\gamma_k} + 1 \text{ for } x \geq 0 \,,$$

and $u = 0$ for $x < 0$. We shall compute the operator applied to $u$ and thus obtain a bounded function $f(x)$ for a suitable choice of the parameters $(\gamma_i)_i$.

For any $\gamma_i > 0$ and $0 \leq x < 1$, we claim that

$$\mathcal{L}(x^{\gamma_i}) = l_i x^{\gamma_i - \alpha} + M_i(x) - \gamma_i x^{\gamma_i - 1}$$

where $l_i$ is a constant and $M_i(x)$ is a continuous function. Now we gather terms and obtain

$$\begin{aligned} \mathcal{L}u(x) &= \sum_{i=1}^{k-1} \left( a_i l_i x^{\gamma_i - \alpha} - a_{i+1}\gamma_{i+1} x^{\gamma_{i+1}-1} \right) + a_k l_k x^{\gamma_k - \alpha} + \left( l_0 x^{-\alpha} - a_1 \gamma_1 x^{\gamma_1 - 1} \right) \\ &\quad + \sum_{i=0}^{k} a_i M_i(x). \end{aligned}$$

We next choose

$$\gamma_i = i(1-\alpha) \quad \text{and} \quad a_1 = \frac{l_0}{\gamma_1} \quad \text{and} \quad a_{i+1} = \frac{a_i l_i}{\gamma_{i+1}}$$

so that, in particular, $\gamma_i - \alpha = \gamma_{i+1} - 1$ and $-\alpha = \gamma_1 - 1$, and we obtain

$$\mathcal{L}u(x) = a_k l_k x^{\gamma_k - \alpha} + \sum_{i=0}^{k} a_i M_i(x).$$

To finish with, choose $k$ large enough such that $\gamma_k \geq \alpha$; precisely, $k \geq \frac{\alpha}{1-\alpha}$. Eventually, we arrive at a bounded right-hand side

$$f(x) = \mathcal{L}(a_1 x^{\gamma_1} + a_2 x^{\gamma_2} + \cdots + a_k x^{\gamma_k} + 1) \,,$$

which implies that we have a solution which does not take the boundary data although $f$ is bounded and continuous up to the boundary. □

**Remarks:**

• Note that if $\alpha$ is very small, that is, if $0 \leq \alpha \leq 1/2$, then the condition $k \geq \alpha/(1-\alpha)$ is fulfilled for any $k \geq 1$. Hence we obtain a counter-example of the form $u(x) = 1 + a \cdot x^{1-\alpha}$.

• In the case of the zero-Laplacian, the counter-example has the form $u(x) = 1 + a \cdot x$, that is, we have an affine function.

• As $\alpha \to 1^-$, $k$ increases and we cannot construct a counter-example when $\alpha = 1$ which would require an infinite number of powers. Indeed, we know that when the measure is singular enough, the boundary data is always taken, even if a drift term is involved in the equation.

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