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An hp –DISCONTINUOUS GALERKIN METHOD FOR THE TIME-DEPENDENT MAXWELL’S EQUATIONS

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Abstract. One discontinuous Galerkin method for the numerical approximation for the time-dependant Maxwell’s equations in “stable medium” with supraconductive boundary, is introduced and analysed. its hp –analysis is carried out and error estimates that are optimal in the meshsize h and slightly suboptimal in the approximation degree p are obtained.

1 Introduction

The problem considered for the most of this Note is the initial-boundary value problem derived from Maxwell’s equations in “stable medium” with supraconductive boundary

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + c^2 \nabla \times (\nabla \times u) &= f, & \nabla \cdot u &= 0 & \text{in } \Omega \times I; & (1) \\ n \times u(x, t) &= 0 & \text{on } \partial\Omega \times I, & u(x, 0) &= u_0(x), & \frac{\partial u}{\partial t}(x, 0) &= u_1(x) & \text{on } \Omega. & (2) \end{aligned}$$

Here Ω is a convex polyhedron included in \mathbb{R}^3 , $I = [0, t^*] \subset \mathbb{R}$, u_0 and u_1 are in $H_0(\nabla \times, \Omega) \cap H(\nabla \cdot 0, \Omega)$ and f is defined on $\Omega \times I$. Physically, u is the electric field, f is related to a current density. $\mu_0 \varepsilon_0 c^2 = 1$ where $\mu_0 \approx 4\pi 10^{-7}$ H.m⁻¹ and $\varepsilon_0 \approx (36\pi 10^9)^{-1}$ F.m⁻¹ are the magnetic permeability and the electric permittivity in vacuum, respectively. If we assume that the domain Ω is “stable medium” with supraconductive boundary and if u is the exact solution of Maxwell problem then u and $\nabla \times u$ belong to $H^1(\Omega)^3$. For the notations, if I is an interval, X is one function space and ϕ is a function on $\Omega \times I$ then $\|\phi\|_{L^p(I, X)}$ denote the norm in $L^p(I)$ of the function $t \rightarrow \|\phi(\cdot, t)\|_X$. $L^p(X)$ is short for $L^p(I, X)$. Let Π_h be a partition into tetrahedra for Ω and consider the same spaces and notations in [2].

Finite element spaces: Let $p = (p_K)_{K \in \Pi_h}$ be a degree vector that assigns to each element $K \in \Pi_h$ a polynomial approximation order $p_K \geq 1$. The generic hp –finite element space of piecewise polynomials is given by $S^p(\Pi_h) := \{u \in L^2(\Omega) : u|_K \in S^{p_K}(K) \ \forall K \in \Pi_h\}$ where $S^{p_K}(K)$ is the space of real polynomials of degree at most p_K in K . We also set $\Sigma_h := S^p(\Pi_h)^3$.

Now, fix a face $e \subset F_h$ and define the local parameters h, p by $h := \min(h_K, h_{K'})$, $p = \max(p_K, p_{K'})$ in the case of interior faces and $h := h_K$, $p = p_K$ in the case of boundary faces [1].

2 Formulation for the Maxwell problem

In order to derive a weak formulation of (1)-(2), we note that formulas (1)-(2) in [2] implies for any u with $\nabla \times u \in H(\nabla \times, \Omega)$

$$c^2(\nabla \times (\nabla \times u), v) = c^2(\nabla \times u, \nabla \times v) + a(u, v)$$

where we have denoted by

$$a(u, v) = c^2 \langle n \times (\nabla \times u), v \rangle - c^2 \sum_{e \in F_h^I} \langle [v]_T, \{\nabla \times u\} \rangle_e .$$

Now, we introduce the penalty term via the form

$$J_0(u, v) = J(u, v) + J^\sigma(u, v) - a(v, u) \text{ with } J(u, v) = (\nabla \cdot u, \nabla \cdot v)$$

and

$$J^\sigma(u, v) = \sum_{e \in F_h^I} \langle \sigma [u]_N, [v]_N \rangle_e + \sum_{e \in F_h} \langle \sigma [u]_T, [v]_T \rangle_e \quad u, v \in H^1(\nabla \times, \Pi_h)^3$$

where $\sigma := \kappa p^2/h$ is a stabilization parameter and κ is a constant supposed ≥ 1 . We also define

$$A(u, v) = c^2(\nabla \times u, \nabla \times v) + a(v, u) - a(u, v) + J(u, v) \text{ and } B(u, v) = A(u, v) + J^\sigma(u, v).$$

2.1 Properties of the bilinear form

2.1.1 Mesh-dependant norm

We now, introduce norm associated with the bilinear form B and set for $u \in H^1(\nabla \times, \Pi_h)$

$$\begin{aligned} \|u\|_h^2 = & \|u\|^2 + \|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \left\| \frac{1}{\sqrt{\sigma}} \langle \nabla \times u \rangle \right\|_{0, F_h}^2 \\ & + \|\sqrt{\sigma} [u]_N\|_{0, F_h^I}^2 + \|\sqrt{\sigma} [u]_T\|_{0, F_h}^2. \end{aligned}$$

We start by studying the continuity of the bilinear forms introduced above. We have :

Proposition 2.1 $\forall v, u \in H^1(\nabla \times, \Pi_h)$ there exists a constant C independent of h and p such that

$$|A(u, v)| \leq C \|u\|_h \|v\|_h \text{ and } |J^\sigma(u, v)| \leq C \|u\|_h \|v\|_h.$$

Proof: The proof is easily deduced from the definition of A , J^σ , $\|\cdot\|_h$ and the Cauchy-Schwarz inequality.

In order to study the coercivity of the bilinear form B , we start by introducing the following inequality of Poincaré-Friedrichs type valid for $u \in H^1(\Pi_h)^3$.

Lemma 2.1 Let $u \in H^1(\Pi_h)^3$. Then there exists C independent of h and p such that

$$\|u\|^2 \leq C (\|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \in F_h} \|\sqrt{\sigma} [u]_T\|_{0, e}^2 + \sum_{e \in F_h^I} \|\sqrt{\sigma} [u]_N\|_{0, e}^2)$$

Proof: The proof follows immediately from Lemma 3.1 in [2] since $\kappa p^2 \geq 1$.

Now, the following coercivity result holds.

Proposition 2.2 *There exists two constants $\alpha > 0$ and $\tilde{C} > 0$ independent of h and p such that*

$$B(v, v) \geq \alpha \|v\|_h^2 + \tilde{C} J^\sigma(v, v) \quad \forall v \in \Sigma_h.$$

Proof: The proof is easily deduced from the Poincaré inequality given in the previous Lemma. Indeed, $\forall v \in \Sigma_h$ and $\forall \alpha > 0$ we have

$$\begin{aligned} B(v, v) - \alpha \|v\|_h^2 &= (1 - \alpha)A(v, v) + (1 - \alpha)J^\sigma(v, v) - \alpha \|v\|_h^2 \\ &\geq (1 - \alpha)A(v, v) + (1 - \alpha)J^\sigma(v, v) - \alpha CA(v, v) - \alpha CJ^\sigma(v, v) \\ &\geq (1 - \alpha - \alpha C)(A(v, v) + J^\sigma(v, v)). \end{aligned}$$

Now, the following hp -approximation result to interpolate scalar function holds (see [1]).

Proposition 2.3 *Let $K \in \Pi_h$ and suppose that $u \in H^{t_K}(K)$, $t_K \geq 1$. Then there exists a sequence of polynomials $\pi_{p_K}^{h_K}(u) \in S^{p_K}(K)$, $p_K = 1, 2, \dots$ satisfying, $\forall 0 \leq q \leq t_K$*

$$\|u - \pi_{p_K}^{h_K}(u)\|_{q,K} \leq C \frac{h_K^{\min(p_K+1, t_K) - q}}{p_K^{t_K - q}} \|u\|_{t_K, K} \quad \text{and} \quad \|u - \pi_{p_K}^{h_K}(u)\|_{0, \partial K} \leq C \frac{h_K^{\min(p_K+1, t_K) - \frac{1}{2}}}{p_K^{t_K - \frac{1}{2}}} \|u\|_{t_K, K}.$$

The constant C is independent of u , h_K and p_K , but depends on the shape regularity of the mesh.

In order to interpolate vector function, we define

Definition 2.1 *For $u = (u_1, u_2, u_3)$ we define*

$\mathbf{\Pi}_p^h : H^t(\nabla \times, \Pi_h) \longrightarrow \Sigma_h$ by $\mathbf{\Pi}_p^h(u) = (\pi_p^h(u_1), \pi_p^h(u_2), \pi_p^h(u_3))$ with π_p^h is defined by $\pi_p^h(u)|_K = \pi_{p_K}^{h_K}(u|_K)$ where $\pi_{p_K}^{h_K}$ is given by the previous Proposition.

2.2 Model problem

The interior penalty finite element approximation to u is to find $U : I \longrightarrow \Sigma_h$ such that

$$(U_{tt}, v) + B(U, v) = (f, v) \quad \forall v \in \Sigma_h, \quad U(0) = \mathbf{\Pi}_p^h(u_0), \quad U_t(0) = \mathbf{\Pi}_p^h(u_1). \quad (3)$$

Upon choice of a basis for Σ_h and the data f , (3) determines U as the only solution to an initial value problem for a linear system of ordinary differential equations. Note that, if u is the exact solution of (1)-(2), then u satisfies the first equation in (3) and thus the problem is consistent.

We now analyse the proposed procedure by the method of energy estimates.

2.2.1 A priori error estimate

In this Section, u denotes the exact solution of (1)-(2) and U the discrete solution of (3). C is generic constant independent of h and p which takes different values at the different places and depends on α , \tilde{C} the coercivity constants of the form B , t^* and Ω .

Let $\zeta = U - u$, then ζ satisfies

$$(\zeta_{tt}, v) + B(\zeta, v) = 0 \quad \forall v \in \Sigma_h.$$

Decompose ζ as $\mu - \nu$ where $\mu = \mathbf{\Pi}_p^h(u) - u$ and $\nu = \mathbf{\Pi}_p^h(u) - U$. Thus

$$(\nu_{tt}, v) + B(\nu, v) = (\mu_{tt}, v) + B(\mu, v) \quad \forall v \in \Sigma_h.$$

Since $\nu_t(t) \in \Sigma_h$, we can set $v = \nu_t(t)$, obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nu_t(t)\|^2 + \frac{1}{2} \frac{d}{dt} B(\nu(t), \nu(t)) &= (\mu_{tt}(t), \nu_t(t)) + B(\mu(t), \nu_t(t)) \\ &\leq \frac{1}{2} \|\mu_{tt}(t)\|^2 + \frac{1}{2} \|\nu_t(t)\|^2 + B(\mu(t), \nu_t(t)). \end{aligned}$$

So

$$\frac{d}{dt} \|\nu_t(t)\|^2 + \frac{d}{dt} B(\nu(t), \nu(t)) \leq \|\mu_{tt}(t)\|^2 + \|\nu_t(t)\|^2 + 2B(\mu(t), \nu_t(t)).$$

Since $\nu_t(0) = \nu(0) = 0$, integration over $[0, t] \subset I$, yields

$$\|\nu_t(t)\|^2 + B(\nu(t), \nu(t)) \leq \|\mu_{tt}\|_{L^2(L^2)}^2 + \int_0^t \|\nu_t(t)\|^2 dt + 2 \int_0^t B(\mu(t), \nu_t(t)) dt.$$

The final term may be integrated by parts in time. Hence,

$$2 \int_0^t B(\mu(t), \nu_t(t)) dt \leq 2|B(\mu(t), \nu(t))| + 2 \int_0^t |B(\mu_t(t), \nu(t))| dt.$$

Therefore, we can apply the coercivity and continuity of B to get

$$\begin{aligned} &\|\nu_t(t)\|^2 + \alpha \|\nu(t)\|_h^2 + \tilde{C} J^\sigma(\nu(t), \nu(t)) \\ &\leq \|\mu_{tt}\|_{L^2(L^2)}^2 + \int_0^t \|\nu_t(t)\|^2 dt + C \|\nu(t)\|_h \|\mu(t)\|_h + 2 \int_0^t |B(\mu_t(t), \nu(t))| dt \\ &\leq \|\mu_{tt}\|_{L^2(L^2)}^2 + \int_0^t \|\nu_t(t)\|^2 dt + C \|\mu(t)\|_h^2 + \frac{\alpha}{2} \|\nu(t)\|_h^2 + C \int_0^t (\|\mu_t(t)\|_h^2 + \|\nu(t)\|_h^2) dt \\ &\leq C \left(\|\mu_{tt}\|_{L^2(L^2)}^2 + \sup_{t \in I} \|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt \right) + \frac{\alpha}{2} \|\nu(t)\|_h^2 + C \int_0^t (\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2) dt. \end{aligned}$$

In particular,

$$\begin{aligned} &\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 \\ &\leq C \left(\|\mu_{tt}\|_{L^2(L^2)}^2 + \sup_{t \in I} \|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt \right) + C \int_0^t (\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2) dt. \end{aligned}$$

As this holds for all $t \in I$, Gronwall's Lemma implies that

$$\|\nu_t(t)\|^2 + \|\nu(t)\|_h^2 \leq C \left(\|\mu_{tt}\|_{L^2(L^2)}^2 + \sup_{t \in I} \|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt \right).$$

Since $\zeta = \mu - \nu$,

$$\|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 \leq C \left(\|\mu_{tt}\|_{L^2(L^2)}^2 + \sup_{t \in I} \|\mu(t)\|_h^2 + \int_0^{t^*} \|\mu_t(t)\|_h^2 dt + \|\mu_t\|_{L^\infty(L^2)}^2 \right).$$

Thus, error bounds for the finite element approximation to the true solution reduce to the error bounds for the piecewise polynomial interpolant. Thus, we start by estimating $\|u - \Pi_p^h(u)\|_h$ where Π_p^h is defined after Proposition 2.4. By using Proposition 2.4 and the definition of $\|\cdot\|_h$, we obtain the following estimates

$$\|u - \Pi_p^h(u)\|_h^2 \leq C \sum_{K \in \Pi_h} \frac{h_K^{2\mu_K - 2}}{p_K^{2t_K - 3}} \|u\|_{t_K, K}^2 \quad \text{and} \quad \|u - \pi_{p_K}^{h_K}(u)\|_{q, K} \leq C \frac{h_K^{\mu_K - q}}{p_K^{t_K - q}} \|u\|_{t_K, K} \quad \forall 0 \leq q \leq t_K.$$

By using the previous estimates, we can get the following result

Proposition 2.4 Let $\mu_K = \min(p_K + 1, t_K)$ and u be the exact solution of (1)-(2). Suppose that $u|_K \in C^2(I, H^{t_K}(K)^3)$, $\forall K \in \Pi_h$ with $t_K \geq 2$. Let U the discrete solution of (3). Then, the error $\zeta = U - u$ satisfies

$$\begin{aligned} & \|\zeta_t(t)\|^2 + \|\zeta(t)\|_h^2 \\ & \leq C \sum_{K \in \Pi_h} \frac{h_K^{2\mu_K - 2}}{p_K^{2t_K - 3}} \left(\|u_{tt}\|_{L^2(H^{t_K}(K)^3)}^2 + \|u\|_{L^\infty(H^{t_K}(K)^3)}^2 + \|u_t\|_{L^2(H^{t_K}(K)^3)}^2 + \|u_t\|_{L^\infty(H^{t_K}(K)^3)}^2 \right). \end{aligned}$$

References

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