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DYNAMICAL DIOPHANTINE APPROXIMATION

AI-HUA FAN, JÖRG SCHMELING, AND SERGE TROUBETZKOY

Abstract. Let μ be a Gibbs measure of the doubling map T of the circle. For a μ-generic point x and a given sequence $\{r_n\} \subset \mathbb{R}^+$, consider the intervals $(T^n x - r_n \pmod 1, T^n x + r_n \pmod 1)$. In analogy to the classical Dvoretzky covering of the circle we study the covering properties of this sequence of intervals. This study is closely related to the local entropy function of the Gibbs measure and to hitting times for moving targets. A mass transference principle is obtained for Gibbs measures which are multifractal. Such a principle was shown by Beresnevich and Velani [BV] only for monofractal measures. In the symbolic language we completely describe the combinatorial structure of a typical relatively short sequence, in particular we can describe the occurrence of "atypical" relatively long words. Our results have a direct and deep number-theoretical interpretation via inhomogeneous diadic diophantine approximation by numbers belonging to a given (diadic) diophantine class.

1. Introduction

Let (X, d) be a complete metric space. Given a sequence $\{x_n\}_{n\geq 1}$ of points in X and a sequence $\{r_n\}_{n\geq 1}$ of positive numbers we define

$$\begin{aligned} I(\{x_n\}, \{r_n\}) : &= \overline{\lim_{n\to\infty}} B(x_n, r_n), \\ F(\{x_n\}, \{r_n\}) : &= X \setminus I(\{x_n\}, \{r_n\}) \end{aligned}$$

where $B(x_n, r_n)$ denotes the ball of center x_n with radius r_n. By diophantine approximation we mean the study of the sets $I(\{x_n\}, \{r_n\})$ and $F(\{x_n\}, \{r_n\})$.

Classic diophantine approximation is a special case. Let $X = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle equipped with the metric

$$\|x - y\| = \inf_{k\in\mathbb{Z}} |(x - y) - k|.$$

Let $\{x_n\} = \{n\alpha \pmod 1\}$ be the orbit of the irrational rotation determined by an irrational number α. Then $0 \in I(\{n\alpha\}, \{r_n\})$ means $\|\alpha n\| < r_n$ holds for an infinite number of n's. This is nothing but the homogeneous diophantine approximation of α. More generally $y \in I(\{n\alpha\}, \{r_n\})$ means $\|\alpha n - y\| < r_n$ holds for an infinite number of n's. This is what is called inhomogeneous diophantine approximation. In [FS], based on the results in [ST], both $I(\{n\alpha\}, \{r_n\})$ and $F(\{n\alpha\}, \{r_n\})$ have been analyzed for an irrational number α when $r_n = n^{-\kappa}$. The case for general sequence $\{r_n\}$ has been studied in [FW2].

Another special case is the dynamical Borel-Cantelli lemma or shrinking target problem. Consider a measure preserving map T. A shrinking target is a sequence of balls with decreasing radius and with centers fixed or moving (more generally, other forms than balls are also allowed). The question is to study the set of orbits $T^n x$ (or equivalently of the initial points) which hit the target or equivalently which are well approximated by the target, see for example [HV] and the references therein.

There is another well studied case. Consider an i.i.d. sequence $\{x_n\} \subset \mathbb{S}^1$ uniformly distributed on the unit circle $\mathbb{S}^1$ with respect to Lebesgue measure, a decreasing sequence of positive numbers $\{\ell_n\} \subset \mathbb{R}^+$ and the associated random intervals $(x_n - \ell_n/2 \pmod 1, x_n + \ell_n/2 \pmod 1)$ (i.e. $r_n = \ell_n/2$ in the above terminology). Since $\{x_n\}$ are independent and uniformly distributed, the Borel–Cantelli Lemma assures that almost surely (a.s. for short) we have $I(\{x_n\}, \{r_n\}) = \mathbb{S}^1$ except for a set of null Lebesgue measure, i.e. Lebesgue a.e. point in $\mathbb{S}^1$ is covered infinitely often by the intervals with probability one if and only if $\sum_{n=1}^{\infty} \ell_n = \infty$. Moreover $\sum_{n=1}^{\infty} \ell_n < \infty$ implies that Lebesgue a.e. point in $\mathbb{S}^1$ is covered finitely often with probability one. In 1956, Dvoretzky observed the possibility that *all* points in $\mathbb{S}^1$ are covered infinitely often with probability one for some slowly decreasing sequence $\{\ell_n\}$ [D]. In 1972, Shepp obtained a necessary and sufficient condition for all points in $\mathbb{S}^1$ to be covered infinitely often with probability one [Sch]:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty.$$

This condition is satisfied for example by $\ell_n = \frac{1}{n}$. Important contributions were made by J.P. Kahane, P. Billard, P. Erdós, S. Orey, B. Mandelbrot et al. See Kahane's book [K] for a full history and a complete reference up to 1985 and see [BF, F1, F2, FK, FW1, JS] for more recent developments.

In the present work, we consider the dynamics defined by the angle doubling map on the circle. We shall consider a generic orbit $\{x_n\} = \{T^n x\}$ of this map relative to a Gibbs measure. Recall that the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$ is defined by

$$Ts = 2s \pmod 1.$$

We are interested in the quantity

$$\|T^n x - y\| = \|2^n x - y\| < r_n.$$

This is diadic diophantine approximation, homogeneous in the case $y = 0$ and inhomogeneous in the case $y \neq 0$. The sets $I(\{x_n\}, \{r_n\})$ and $F(\{x_n\}, \{r_n\})$ are respectively the sets of y which are well aproximable or badly approximable with speed r_n. In other words I is the set of points obeying a diophantine equation with speed r_n. Our theorems are similar to Jarnik type results in number theory. For $\kappa > 0$ consider the special

sequence $r_n = \frac{1}{n^\kappa}$. Write

$$J_n^\kappa(s) = (T^n s - r_n \pmod 1, T^n s + r_n \pmod 1).$$

For $s \in \mathbb{S}^1$ let

$$I^\kappa(s) := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} J_n^\kappa(s) = \left\{ t \in \mathbb{S}^1 \, : \, \sum_{n=0}^{\infty} \mathbf{1}_{J_n^\kappa(s)}(t) = \infty \right\},$$

$$F^\kappa(s) := \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} J_n^\kappa(s)^c = \left\{ t \in \mathbb{S}^1 \, : \, \sum_{n=0}^{\infty} \mathbf{1}_{J_n^\kappa(s)}(t) < \infty \right\}.$$

The following decomposition is obvious:

$$\mathbb{S}^1 = F^\kappa(s) \cup I^\kappa(s), \quad F^\kappa(s) \cap I^\kappa(s) = \emptyset.$$

It is easy to see by definition that if the orbit of s is dense, then $I^\kappa(s)$ is a residual set, in particular, $I^\kappa(s) \neq \emptyset$. It is the case for a typical point s relative to an ergodic measure with full support. However, as we will see, it is possible for $F^\kappa(s) = \emptyset$ for typical points. Let ν_ϕ, ν_ψ be two T-invariant probability Gibbs measures on $\mathbb{S}^1$ associated to normalized Hölder potentials ϕ and ψ (i.e. the pressures of ϕ and ψ are equal to zero). The measure ν_ϕ will be used to describe the randomness and the measure ν_ψ to describe sizes of sets.

Let

$$\begin{aligned} \kappa_{\phi,\psi,\mathbb{S}^1} : &= \sup \{ \kappa : \nu_\psi(I^\kappa(s)) = 1 \text{ for } \nu_\phi - a.e. \ s \}, \\ \kappa^F_{\phi,\mathbb{S}^1} : &= \sup \{ \kappa : F^\kappa(s) = \emptyset \text{ for } \nu_\phi - a.e. \ s \}. \end{aligned}$$

We are interested in the following questions:

(**Q1**) *How to determine the critical value $\kappa_{\phi,\psi}$? More precisely when is $I^\kappa(s)$ of full ν_ψ-measure for ν_ϕ-almost every s?*

(**Q2**) *How to determine the critical value $\kappa^F_{\phi,\mathbb{S}^1}$? More precisely when is $I^\kappa(s)$ equal to $\mathbb{S}^1$ for ν_ϕ-almost every s ?*

(**Q3**) *What are the Hausdorff dimensions $\dim_H(F^\kappa(s))$, $\dim_H(I^\kappa(s))$ for ν_ϕ-almost every s ?*

Our answers to these questions are stated in the following theorems. Let

$$\begin{aligned} e^- &= \inf_{\nu:\text{invariant}} \int (-\phi) d\nu, \\ e_{\max} &= \int (-\phi) d\text{Leb}, \\ e^+ &= \sup_{\nu:\text{invariant}} \int (-\phi) d\nu \end{aligned}$$

where e_- and e_+ are respectively the minimal and maximal local entropy of ν_ϕ. Let $E(t)$ be the entropy spectrum of ν_ϕ, which is defined by

$$E(t) = \dim_H \left\{ y : \lim_{r \to o} \frac{\log \nu_\phi((y-r, y+r))}{\log r} = t \right\}.$$

It is well known that $E(t)$ is continuous on $[e^-, e^+]$, strictly concave and real analytic in (e^-, e^+) (see [P]).

Theorem 1.1. *The critical value $\kappa_{\phi,\psi,\mathbb{S}^1}$ satisfies*

$$\kappa_{\phi,\psi,\mathbb{S}^1} = \frac{1}{\int(-\phi)d\nu_\psi}.$$

Notice that the integral $\int(-\phi)d\nu_\psi$ is nothing but the conditional entropy of ν_ϕ relative to ν_ψ. The theorem says that for ν_ϕ-a.e s the set $I^\kappa(s)$ supports the Gibbs measure ν_ψ if κ is small enough so that $\int(-\phi)d\nu_\psi < \frac{1}{\kappa}$. Also notice that for fixed s, the question whether $\nu_\psi(I^\kappa(s)) = 1$ is the shrinking target problem or dynamical Borel-Cantelli lemma (see [HV]).

Theorem 1.2. *The critical value $\kappa^F_{\phi,\mathbb{S}^1}$ satisfies*

$$\kappa^F_{\phi,\mathbb{S}^1} = \frac{1}{e_+}.$$

The theorem says that if κ is so small that $e^+ < \frac{1}{\kappa}$, then $I^\kappa(s) = \mathbb{S}^1$ or equivalently $F^\kappa(s) = \emptyset$ for ν_ϕ-a.e. s. This is the counterpart of the Kahane-Billard-Shepp condition for the random Dvoretzky covering.

Theorem 1.3. *For ν_ϕ-a.e. s we have*

$$\dim_H F^\kappa(s) = \begin{cases} 1 & \text{if } \frac{1}{\kappa} \le e_{\max} \\ E(\frac{1}{\kappa}) & \text{if } \frac{1}{\kappa} > e_{\max} \end{cases}.$$

Theorem 1.4. *For ν_ϕ-a.e. s we have*

$$\dim_H I^\kappa(s) = \begin{cases} \frac{1}{\kappa} & \text{if } \frac{1}{\kappa} \le h_{\nu_\phi} \\ E(\frac{1}{\kappa}) & \text{if } h_{\nu_\phi} < \frac{1}{\kappa} < e_{\max} \\ 1 & \text{if } \frac{1}{\kappa} \ge e_{\max} \end{cases}.$$

We will transfer the problem to a similar one in a symbolic framework. As we shall see, our problem is closely related to hitting times and the later is related to local entropy.

The structure of the article is as follows. We start in section 2 with background on ergodic theory, symbolic dynamics, decay of correlations, and multi-fractal analysis. In this section we prove a "multi-relation" and a variational principal which are essential in the proofs of the main results. In section 3 we transfer the covering problem to the symbolic setting and relate then covering properties to hitting time asymptotic. In section 4 we prove a first simple relation between hitting times and local entropy. This yields the proof of the Ornstein-Weiss return time theorem in the special case of Gibbs measures and also allows us the determine the critical exponent $\kappa_{\phi,\psi}$. For the other exponents more sophisticated estimates are needed. Sections 5 and 6 contain the core estimates on the probabilities of hitting time events. The fundamental tools relating hitting times to the entropy spectrum are developed. In section 7 we study the structure of a short typical sequence.

In particular we make a substantial improvement in the mass transference principle [BV] to multi-fractal Gibbs states. Section 8 contains the results in the symbolic framework for the full shift while section 9 generalizes these results to subshifts of finite type. Finally in section 10 we prove the main theorems by transferring them from the shift space.

2. Background

Convention. All logarithms and exponential functions in this article are taken to base 2. With this convention the notions of entropy and dimension coincide in our setup.

Ergodic theory. We need various standard definitions from ergodic theory: the metric entropy of an invariant measure ν denoted by h_ν, the notion of the Gibbs measure μ_ϕ with respect to a potential ϕ and the topological entropy for non compact sets E denoted by $h_{top}(E)$. The definitions of all these notions can be found in [P].

Symbolic dynamics. We use various standard notions from symbolic dynamics. Let (Σ_2^+, σ) denote the one sided full shift on two symbols $0, 1$. For $y = (y_i)_{i\geq 0} \in \Sigma_2^+$ we denote a cylinder set by

$$C_n(y) := [y_0, y_1, \cdots, y_{n-1}].$$

We will denote the length of the cylinder by $|C_n(y)| = n$. We will denote by

$$\pi(y) = \sum_{i=0}^{\infty} \frac{y_i}{2^{i+1}}$$

the natural projection from Σ_2^+ to $\mathbb{S}^1$. We consider the $\frac{1}{2}$-metric on Σ_2^+, i.e. for $x, y \in \Sigma_2^+$ let $d(x, y) = \frac{1}{2^n}$ where n is the least integer such that $x_n \neq y_n$. The pull back of the circle metric $\rho(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}}$ is almost equivalent in the sense that for $x \in \Sigma_2^+$ the ratio $\operatorname{diam}_\rho(C_n(x))/\operatorname{diam}_d(C_n(x))$ is bounded from below and above uniformly in n and x. Thus Hausdorff dimensions do not change under the projection, for details see [S1]. We denote by $\mu_{\max}$ the measure of maximal entropy for the shift. The projection of $\mu_{\max}$ is the Lebesgue measure on the circle.

2.1. **Fast decay of correlation.**

One of the key tools in our study is fast decay of correlations. This is related to Ruelle's theorem on transfer operators. Recall that for a α-Hölder potential $\phi : \Sigma_2^+ \to \mathbb{R}$, i.e.

$$[\phi]_\alpha := \sup_{x,y} |\phi(x) - \phi(y)|/d(x, y)^\alpha < \infty,$$

the transfer operator associated to ϕ is defined as follows

$$L_\phi f(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y).$$

This operator acts on the space of continuous functions $C(\Sigma_2^+)$ equipped with the supremum norm $\|f\|_\infty$ and on the space of α-Hölder continuous functions $H_\alpha(\Sigma_2)$ equipped with the Hölder norm

$$|||f||| := \|f\|_\infty + [f]_\alpha.$$

The well known Ruelle theorem asserts that [Ru]

(i) The spectral radius $\lambda > 0$ of $L_\phi : H_\alpha \to H_\alpha$ is an eigenvalue with an strictly positive eigenfunction h and there is a probability eigenmeasure ν for the adjoint operator L_ϕ^*, i.e. $L_\phi^*\nu = \lambda\nu$.

(ii) Choose h such that $\langle h, \nu\rangle := \int h d\nu = 1$. There exist constants $c > 0$ and $0 < \beta < 1$ such that for any $f \in H_\alpha$ we have

$$\|\lambda^{-n} L_\phi^n f - \langle f, \nu\rangle h\| \le c\beta^n |||f|||. \tag{2·1}$$

Let $P(\phi) = \log\lambda$ and call it the pressure of ϕ. The measure $\mu := h\nu$, denoted by μ_ϕ, is the so-called Gibbs measure associated to ϕ. Assume that ϕ is normalized, that is to say $\lambda = 1$. The Gibbs measure μ has the Gibbs property: there exists a constant $\gamma > 1$ such that

$$\frac{1}{\gamma} e^{S_n\phi(x)} \le \mu(C_n[x]) \le \gamma e^{S_n\phi(x)} \tag{2·2}$$

holds for all $x \in \Sigma_2$ and all $n \ge 1$ where

$$S_n f(y) := \sum_{j=0}^{n-1} f(\sigma^j y).$$

The Gibbs property (2·2) implies the following quasi-Bernoulli property of μ_ϕ: for any two cylinders A and B we have

$$\frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B) \le \mu_\phi(A \cap \sigma^{-|A|}B) \le \gamma^3\mu_\phi(A)\mu_\phi(B). \tag{2·3}$$

For the first inequality take a point $x \in A \cap \sigma^{-|A|}B$. By using three times the Gibbs property we get

$$\mu_\phi(A \cap \sigma^{-|A|}B) \ge \frac{1}{\gamma} 2^{S_{|A|}\phi(x) + S_{|B|}(\sigma^{|A|}x)} \ge \frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B).$$

This quasi-Bernoulli property can be generalized in the following way.

Theorem 2.1 (Multi-relation)**.** *Let $\mu = \mu_\phi$ be the Gibbs measure associated to a Hölder potential function ϕ. Let $\omega > 1$ be a sufficiently large number. For any cylinder D_0 and any finite number of cylinders $D_1, \dots, D_k$ of length n we have*

$$\gamma^{-3}\left(1 - c\beta^n)\right)^k \le \frac{\mu\left(D_0 \cap \bigcap_{j=1}^k \sigma^{-[n_0 + j(n+d)]} D_j\right)}{\prod_{j=0}^k \mu(D_j)} \le \gamma^3\left(1 + c\beta^n)\right)^k \tag{2·4}$$

where $n_0 \ge |D_0|$ and $d = d(n) : \lfloor \omega n \rfloor$ ($\lfloor a \rfloor$ denoting the integral part of a real number a).

Proof. First remark that

$$D_0 \cap \bigcap_{j=1}^{k} \sigma^{-[n_0+j(n+d)]} D_j = D_0 \cap \sigma^{-|D_0|}\mathcal{B}$$

where

$$\mathcal{B} = \bigcap_{j=1}^{k} \sigma^{-[n_0-|D_0|+j(n+d)]} D_j$$

is a finite union of disjoint cylinders, which we denote by B_i's. Applying the quasi-Bernoulli property (2·3) to $A = D_0$ and $B = B_i$ we get

$$\frac{1}{\gamma^3}\mu_\phi(D_0)\mu_\phi(B_i) \le \mu_\phi(D_0 \cap \sigma^{-|D_0|}B_i) \le \gamma^3\mu_\phi(D_0)\mu_\phi(B_i).$$

Sum over all B_i's and we get

$$\frac{1}{\gamma^3}\mu_\phi(D_0)\mu_\phi(\mathcal{B}) \le \mu_\phi(D_0 \cap \sigma^{-|D_0|}\mathcal{B}) \le \gamma^3\mu_\phi(D_0)\mu_\phi(\mathcal{B}). \tag{2·5}$$

Notice that the invariance of μ_ϕ implies

$$\mu_\phi(\mathcal{B}) = \mu_\phi\left(\bigcap_{j=1}^{k} \sigma^{-[(j-1)(n+d)]} D_j\right).$$

Combining this with the equation (2·5), it suffices to prove

$$(1 - c\beta^n))^k \le \frac{\mu\left(\bigcap_{j=1}^{k} \sigma^{-[(j-1)(n+d)]} D_j\right)}{\prod_{j=1}^{k}\mu(D_j)} \le (1 + c\beta^n))^k. \tag{2·6}$$

Actually we can prove a little more. For simplicity, we will use $\mathbb{E}f$ to denote the integral $\int f d\mu$ and write $\|f\|_1 = \|f\|_{L^1(\mu)}$. From the inequality

$$|\mathbb{E}(f \circ \sigma^n \cdot g)| = |\mathbb{E}(f \cdot L^n g)| \le \|L^n g\|_\infty \|f\|_1$$

(applied to $g - \mathbb{E}g$ and f) and Ruelle's theorem, we deduce that for non-negative Hölder functions g and f we have

$$\left(1 - c\frac{\beta^n \|g - \mathbb{E}g\|}{\mathbb{E}g}\right) \le \frac{\mathbb{E}(f \circ \sigma^n \cdot g)}{\mathbb{E}f\mathbb{E}g} \le \left(1 + c\frac{\beta^n \|g - \mathbb{E}g\|}{\mathbb{E}g}\right).$$

Inductively, for a finite number of functions $g_1, \cdots, g_k \in H_\alpha$ and for integers $0 = n_1 < n_2 < \cdots < n_k$ we have

$$\prod_{j=1}^{k-1}\left(1 - c\frac{\beta^{n_{j+1}-n_j}\|g_j - \mathbb{E}g_j\|}{\mathbb{E}g_j}\right)$$
$$\le \frac{\mathbb{E}\prod_{j=1}^{k} g_j \circ \sigma^{n_j}}{\prod_{j=1}^{k}\mathbb{E}g_j} \le \prod_{j=1}^{k-1}\left(1 + c\frac{\beta^{n_{j+1}-n_j}\|g_j - \mathbb{E}g_j\|}{\mathbb{E}g_j}\right).$$

To get (2·6), we apply these inequalities to characteristic functions of cylinders $g_j = 1_{D_j}$. In fact, since all cylinders D_j have the same length n, we have

$$\|g_j\| = 1 + 2^{\alpha n}, \quad \frac{1}{\mathbb{E}g_j} = \frac{1}{\mu(D_j)} \le \gamma 2^{n \max_x(-\phi(x))}$$

(the inequality is a consequence of the Gibbs property). Take $d := \lfloor \omega n \rfloor$ with a sufficiently large ω so that $\beta^\omega 2^{\alpha + \max(-\phi)} < 1$. Take n_j such that $n_1 = 0$ and $n_{j+1} - n_j = n + d$ for $j \ge 2$ and the equation (2·6) follows. □

We will refer to this inequality as the *multi-relation property* of the Gibbs measure μ_ϕ.

2.2. **Multi-fractal analysis.**

Furthermore we will use various notions from multi-fractal analysis which can also be found in the reference [P]. The notion of Hausdorff dimension of a set will be denoted by $\dim_H$. For a point $y \in \Sigma_2^+$ and an invariant measure ν we denote the *lower local entropy* of ν at y by

$$\underline{h}_\nu(y) := \varliminf_{n\to\infty} -\frac{1}{n} \log \nu(C_n(y)). \tag{2·7}$$

We define the *local entropy* $h_\nu(y)$ if the limit exists. For a function $f : \Sigma_2^+ \to \mathbb{R}$ we denote the ergodic sum by

$$S_m f(y) := \sum_{j=0}^{m-1} f(\sigma^j y).$$

We denote a Gibbs measure with respect to a Hölder potential ϕ by μ_ϕ. Without loss of generality we may assume that the potential is normalized so that its pressure $P(\phi) = 0$. Then

$$\underline{h}_{\mu_\phi}(y) = -\varliminf_{n\to\infty} \frac{1}{n} S_n\phi(y) \tag{2·8}$$

and $h_{\mu_\phi}(y)$ satisfies a similar relation when the limit exists. If ν is an ergodic invariant measure then for ν a.e. y

$$h_{\mu_\phi}(y) = -\int_{\Sigma_2^+} \phi \, d\nu.$$

Furthermore if ν is another Gibbs measure μ_ψ then for μ_ψ a.e. y

$$h_{\mu_\phi}(y) = -P'(\psi + t\phi)|_{t=0}. \tag{2·9}$$

Multi-fractal analysis deals with the study of the entropy spectrum

$$E(t) := E_{-\phi}(t) := h_{\text{top}} \left\{ y \,:\, h_{\mu_\phi}(y) = t \right\}.$$

The following conditional variational is well known ([BSS, FF, FFW]).

Theorem 2.2 (Variational principle I)**.** *Let ϕ be a Hölder function. For any $t \in \mathbb{R}$, we have*

$$E(t) = \sup_{\nu:\ invariant} \left\{ h(\nu) : \int (-\phi) d\nu = t \right\}. \tag{2·10}$$

We also have

$$E(t(q)) = P(q\phi) - qP'(q\phi) = h_{\mu_{-P(q\phi)+q\phi}} \tag{2·11}$$

where $t(q) = -P'(q\phi)$. The range of the function $t(q)$ is an interval $[e^-, e^+]$, possibly degenerate to a singleton.

Let us state some more useful facts concerning the variational principle. The function $t(q)$ is invertible on the interval $[e^-, e^+]$. If t is not in this interval, then there is no point $y \in \Sigma_2^+$ with local entropy equal to t. The entropy $E(t)$ attains its maximum at the value

$$e_{\max} = t(0) = \int_{\Sigma_2^+} (-\phi) d\mu_{\max}.$$

We have $t(q) \le e_{\max}$ if and only if $q \ge 0$. Furthermore

$$e^+ = \max_{\mu:\text{invariant}} \int (-\phi)\, d\mu, \quad e^- = \min_{\mu:\text{invariant}} \int (-\phi)\, d\mu.$$

The entropy spectrum is concave and real analytic in the interval (e^-, e^+). Its graph lies below the diagonal. Moreover the interval $[e^-, e^+]$ is degenerate if and only if ϕ is cohomologous to the constant $-h_{\text{top}}$, i.e. the measure μ_ϕ is the measure of maximal entropy. In the degenerate case we have $e^- = e^+ = h_{\text{top}}$ and $E(h_{\text{top}}) = h_{\text{top}}$. For typical potentials in the sense of Baire, $E(e^-) = E(e^+) = 0$.

We will need the following variational principle.

Theorem 2.3 (Variational principle II)**.** *Let ϕ be a Hölder function. For any $t \in \mathbb{R}$, we have*

$$h_{\text{top}} \left\{ \underline{h}_{\mu_\phi}(y) < t \right\} = h_{\text{top}} \left\{ \overline{h}_{\mu_\phi}(y) < t \right\} = \sup_{s<t} E(s),$$

$$h_{\text{top}} \left\{ \underline{h}_{\mu_\phi}(y) \ge t \right\} = h_{\text{top}} \left\{ \overline{h}_{\mu_\phi}(y) \ge t \right\} = \sup_{s\ge t} E(s).$$

Proof. Let us start with the proof of the first fact. From the trivial fact

$$\left\{ \underline{h}_{\mu_\phi}(y) < t \right\} \supset \left\{ \overline{h}_{\mu_\phi}(y) < t \right\} \supset \bigcup_{s<t} \{ h_{\mu_\phi}(y) = s \},$$

we get immediately the following inequalities

$$h_{\text{top}} \left\{ \underline{h}_{\mu_\phi}(y) < t \right\} \ge h_{\text{top}} \left\{ \overline{h}_{\mu_\phi}(y) < t \right\} \ge \sup_{s<t} E(s).$$

Since $\sup_{t<e_{\max}} E(t) = 1$ the converse inequalities are trivial in the case $t \ge e_{\max}$. It remains to consider the case $t < e_{\max}$. Notice that we have

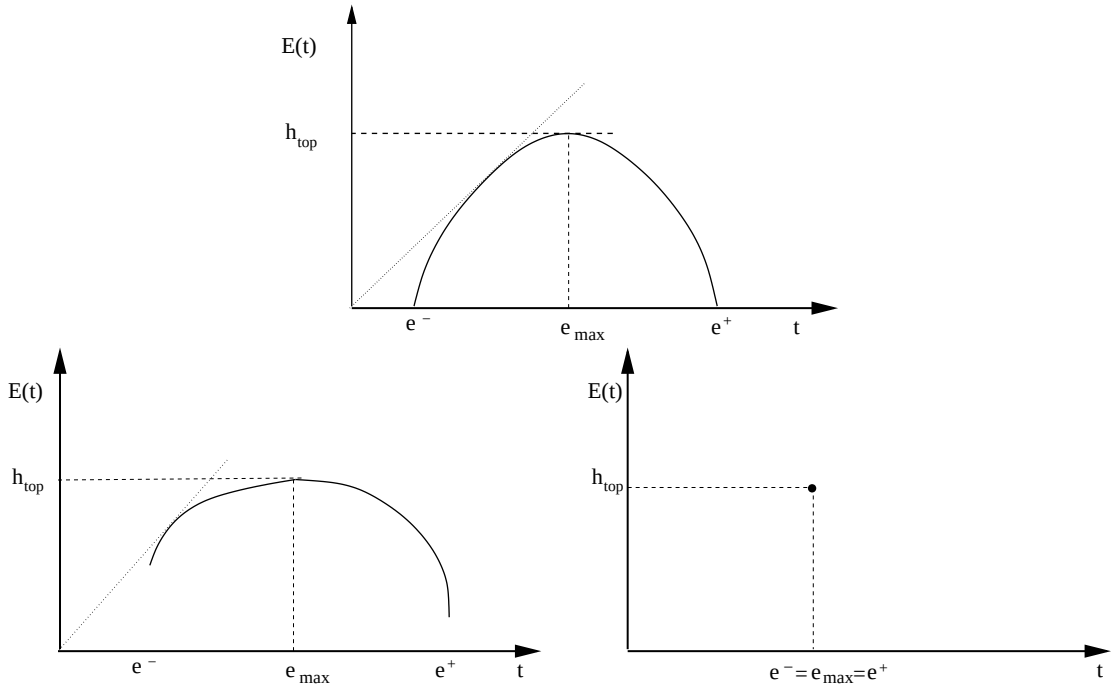

FIGURE 1. The entropy spectrum for typical, nontypical and degenerate potentials.

$E(t) = \sup_{s<t} E(s)$. Also notice that there exists a positive number $q(t) > 0$ such that

$$\min_{q\geq 0}(P(q\phi) + qt) = P(q(t)\phi) + q(t)t = E(t).$$

Now let y be any point such that $\underline{h}_{\mu_\phi}(y) < t$. For $q = q(t) > 0$ we can apply Equation (2·8) to yield

$$\begin{aligned}\underline{h}_{\mu_{-P(q\phi)+q\phi}}(y) &= \varliminf_{n\to\infty} -\frac{1}{n}S_n\big(-P(q\phi) + q\phi\big)(y)\\ &= P(q\phi) + q\left(\varliminf_{n\to\infty} -\frac{1}{n}S_n\phi(y)\right)\\ &\leq P(q\phi) + qt = E(t).\end{aligned}$$

Thus applying the mass distribution principle (see Theorem 7.2 of [P]) yields $h_{\text{top}}\left\{\underline{h}_{\mu_\phi}(y) < t\right\} \leq E(t)$, which completes the proof of the first line.

The second fact may be similarly proved. We just point out the following differences that

$$\{\overline{h}_{\mu_\phi}(y) \geq t\} \supset \left\{\underline{h}_{\mu_\phi}(y) \geq t\right\} \supset \bigcup_{s\geq t}\{h_{\mu_\phi}(y) = s\},$$

and that for $t > e_{\max}$ there exists a negative number $q(t) < 0$ such that $E(t) = P(q(t)\phi) + q(t)t$. □

3. Covering questions are described by hitting times

It is well known that the doubling map is semi-conjugate to the shift map on Σ_2^+. As we shall see, the initial covering questions can be translated into similar questions concerning the shift map and these question are described by the hitting time that we are going to define. We will also see that hitting times are related to local entropy.

For $x \in \Sigma_2^+$ and C a cylinder let

$$\tau(x, C) := \inf\{l \geq 1 : \sigma^l x \in C\}$$

be the *first hitting time* of C by x. For $x, y \in \Sigma_2^+$ let

$$\tau_n(x, y) := \tau(x, C_n(y))$$

$$\alpha(x, y) := \underline{\lim}_{n\to\infty} \frac{1}{n} \log \tau_n(x, y). \tag{3·1}$$

Let

$$\begin{aligned} \mathcal{F}^\kappa(x) &:= \{y \in \Sigma_2^+ : \ y \notin \cap_{N=1}^\infty \cup_{n\geq N} C_{\lfloor \kappa \log n \rfloor}(\sigma^n x)\}, \\ \mathcal{I}^\kappa(x) &:= \{y \in \Sigma_2^+ : \ y \in \cap_{N=1}^\infty \cup_{n\geq N} C_{\lfloor \kappa \log n \rfloor}(\sigma^n x)\}. \end{aligned}$$

We have the following trivial decomposition

$$\Sigma_2^+ = \mathcal{F}^\kappa(x) \cup \mathcal{I}^\kappa(x), \qquad \mathcal{F}^\kappa(x) \cap \mathcal{I}^\kappa(x) = \emptyset.$$

Suppose that μ_ϕ, μ_ψ are σ-invariant probability Gibbs measures on Σ_2^+. Let

$$\begin{aligned} \kappa_{\phi,\psi,\Sigma_2^+} &:= \sup\{\kappa : \mu_\psi(\mathcal{I}^\kappa(x)) = 1 \text{ for } \mu_\phi - a.e.\ x\}, \\ \kappa^F_{\phi,\Sigma_2^+} &:= \sup\{\kappa : \mathcal{F}^\kappa(x) = \emptyset \text{ for } \mu_\phi - a.e.\ x\}. \end{aligned}$$

One of our goals is to determine the values of both critical exponents $\kappa_{\phi,\psi,\Sigma_2^+}$ and $\kappa^F_{\phi,\Sigma_2^+}$ and the other one is to compute the Hausdorff dimensions of $\mathcal{F}^\kappa(x)$ and $\mathcal{I}^\kappa(x)$. Let

$$\mathcal{O}(x) = \{\sigma^n x : n \geq 0\}, \quad \mathcal{O}^+(x) = \mathcal{O}(x) \setminus \{x\}.$$

Lemma 3.1. *There exists an integer $n_0 \geq 1$ such that $y = \sigma^{n_0} x$ (i.e. $y \in \mathcal{O}^+(x)$) if and only if the hitting time sequence $\tau_k(x, y)$ is bounded.*

Proof. If $y = \sigma^{n_0} x$ then it is obvious that $\tau_k(x, y) \leq n_0$ for all k. Conversely, suppose there is a positive constant such that $\tau_k(x, y) \leq K$. Fix an integer $1 \leq t \leq K$ such that $\tau_{k_i}(x, y) = t$ holds for an infinite subsequence k_i. Then $\sigma^t x \in C_{k_i}(y)$ for all i. Letting $i \to \infty$ we get $\sigma^t x = y$. □

Lemma 3.2.

$$\left\{y \in \Sigma_2^+ : \alpha(x, y) > \frac{1}{\kappa}\right\} \subset \mathcal{F}^\kappa(x) \subset \left\{y \in \Sigma_2^+ : \alpha(x, y) \geq \frac{1}{\kappa}\right\} \cup \mathcal{O}^+(x),$$

$$\left\{y \in \Sigma_2^+ : \alpha(x, y) < \frac{1}{\kappa}\right\} \setminus \mathcal{O}^+(x) \subset \mathcal{I}^\kappa(x) \subset \left\{y \in \Sigma_2^+ : \alpha(x, y) \leq \frac{1}{\kappa}\right\}.$$

Proof. The top left and bottom right inclusions imply one another. Let us prove the bottom right inclusion. Suppose $y \in \mathcal{I}^\kappa(x)$. Then $y \in C_{\lfloor \kappa \log n \rfloor}(\sigma^n x)$ or equivalently $\sigma^n x \in C_{\lfloor \kappa \log n \rfloor}(y)$ for infinitely many n. Thus $\tau_{\lfloor \kappa \log n \rfloor}(x,y) \le n$ for infinitely many n, which implies $\alpha(x,y) \le \kappa^{-1}$.

The top right and bottom left inclusions imply one another. So, it remains to prove the bottom left inclusion. Suppose $\alpha := \alpha(x,y) < \kappa^{-1}$ and $y \notin \mathcal{O}^+(x)$. Take $\varepsilon > 0$ such that $\kappa < \frac{1}{\alpha+\varepsilon}$. By the definition of $\alpha := \alpha(x,y)$, there is a subsequence k_i such that $\log \tau_{k_i}(x,y) \le (\alpha+\varepsilon)k_i$, i.e. $k_i \ge \frac{\log \tau_{k_i}(x,y)}{\alpha+\varepsilon}$. The definition of $\tau_{k_i}(x,y)$ implies that

$$\sigma^{\tau_{k_i}} x \in C_{k_i}(y) \subset C_{\left\lfloor \frac{\log \tau_{k_i}}{\alpha+\varepsilon} \right\rfloor}(y) \subset C_{\lfloor \kappa \log \tau_{k_i} \rfloor}(y).$$

Since $y \notin \mathcal{O}^+(x)$ the previous lemma yields that τ_{k_i} is not bounded. Thus $\sigma^n x \in C_{\lfloor \kappa \log n \rfloor}(y)$ or equivalently $y \in C_{\lfloor \kappa \log n \rfloor}(\sigma^n x)$ for infinitely many $n = \tau_{k_i}$.

□

We should point out that points y on the orbit $\mathcal{O}^+(x)$ have the property that $\alpha(x,y) = 0 < 1/\kappa$, but they are not necessarily contained in $\mathcal{I}^\kappa(x)$. For example, if x is an eventually periodic point but not periodic and if y is on the orbit $\mathcal{O}^+(x)$ but not in the cycle of x, then $y \notin \mathcal{I}^\kappa(x)$. However, for μ_ϕ-almost all x, we have the following situation.

Lemma 3.3. *For μ_ϕ a.e. x, we have $\mathcal{O}(x) \subset \mathcal{I}^\kappa(x)$ if $\frac{1}{\kappa} > h_{\mu_\phi}$ and $\mathcal{O}(x) \subset \mathcal{F}^\kappa(x)$ if $\frac{1}{\kappa} < h_{\mu_\phi}$.*

Proof. Let $y \in \mathcal{O}(x)$ where x is not eventually periodic. Then there exists a unique integer $n_0 \ge 0$ such that $y = \sigma^{n_0} x$. Define the hitting time after n_0 by

$$\tau_n^{(n_0)}(x,y) := \inf\{k > n_0 : \sigma^k x \in C_n(y)\} = \tau_n(\sigma^{n_0} x, y) + n_0.$$

Since $y \notin \mathcal{O}^+(\sigma^{n_0} x))$ Lemma 3.1 implies that $\tau_n^{(n_0)}(x,y) \to \infty$ as $n \to \infty$. Let

$$\alpha^{(n_0)}(x,y) = \varliminf_{n\to\infty} \frac{1}{n} \log \tau_n^{(n_0)}(x,y). \tag{3·2}$$

Hence

$$y \in \mathcal{I}^\kappa(x) \ \text{ if } \ \alpha^{(n_0)}(x,y) < \frac{1}{\kappa}, \quad \text{and} \quad y \in \mathcal{F}^\kappa(x) \ \text{ if } \ \alpha^{(n_0)}(x,y) > \frac{1}{\kappa}.$$

Now

$$\alpha^{(n_0)}(x,y) = \alpha(y,y) = \alpha(\sigma^{n_0} x, \sigma^{n_0} x).$$

Thus applying the Ornstein-Weiss return time theorem [OW] yields that $\alpha(x,x) = h_{\mu_\phi}$ for μ_ϕ-a.e. x. Finally the invariance of μ implies that $\alpha(\sigma^n x, \sigma^n x) = h_{\mu_\phi}$ for μ_ϕ a.e. x and for all n. □

4. Hitting time and local entropy: basic relation

As Lemmas 3.2 and 3.3 show, we have to study the hitting time $\alpha(x,y)$ of the Gibbs measure μ_ϕ. We will show that the hitting time is related to the local entropy. Local entropy have been well studied in the literature.

In this section, we start with a basic relation between hitting times and local entropy. This allows us to compute the critical value $\kappa_{\phi,\psi,\Sigma_2^+}$.

Let us first introduce a generalized notion of local entropy. Let (C_n) be a sequence of (arbitrary) cylinders with length $|C_n| = n$. We define the *lower local entropy of the sequence* (C_n) by

$$\underline{h}_{\mu_\phi}(\{C_n\}) := \varliminf_{n\to\infty} -\frac{\log \mu_\phi(C_n)}{n}. \tag{4·1}$$

4.1. **Basic relation.** We have the following basic relation between local entropy and the hitting times.

Theorem 4.1. *Suppose that μ_ϕ is a Gibbs measure associated to a Hölder potential ϕ and that (C_n) is a sequence of (arbitrary) cylinders of length n. Then for μ_ϕ a.e. x we have*

$$\varliminf_{n\to\infty} \frac{\log \tau(x, C_n)}{n} = \underline{h}_{\mu_\phi}(\{C_n\}) \tag{4·2}$$

Proof. A special case of this theorem was proven by Chazottes [C]. The proof follows the idea of Chazottes closely. We include it for completeness.

Let $\tau_n(x) := \tau(x, C_n)$. Note that the Gibbs property implies $\mu_\phi(C_n) \to 0$. Fix $\varepsilon > 0$ and let

$$A_n := \left\{x \in \Sigma_2^+ :\ \tau_n(x)\mu_\phi(C_n) < 2^{-\varepsilon n}\right\},$$
$$B_n := \left\{x \in \Sigma_2^+ :\ \tau_n(x)\mu_\phi(C_n) > 2^{\varepsilon n}\right\}.$$

We will prove that

$$\sum \mu_\phi(A_n \cup B_n) \le \sum \mu_\phi(A_n) + \sum \mu_\phi(B_n) < \infty.$$

Once we have shown this we apply the first part of the Borel-Cantelli lemma to conclude the proof.

First consider the series $\sum \mu_\phi(A_n)$, which is simpler to handle. We have

$$A_n \subset A_n^0 \cup \cdots \cup A_n^m$$

where

$$A_n^i := \{x \in \Sigma_2^+ :\ \sigma^i x \in C_n\}, \quad m = \lfloor 2^{-\varepsilon n}/\mu_\phi(C_n)\rfloor.$$

Since $\mu_\phi(A_n^i) = \mu_\phi(A_n^j) = \mu_\phi(C_n)$, this yields

$$\mu(A_n) \le \left(\frac{2^{-\varepsilon n}}{\mu_\phi(C_n)} + 2\right)\mu_\phi(C_n) \le 2^{-\epsilon n} + 2\mu_\phi(C_n).$$

Now we distinguish two cases: $\underline{h}_{\mu_\phi}(\{C_n\}) > 0$ and $\underline{h}_{\mu_\phi}(\{C_n\}) = 0$. In the first case, $\mu_\phi(C_n)$ decays exponentially fast, so that $\sum \mu_\phi(C_n) < \infty$, then

$\sum \mu_\phi(A_n) < \infty$. In the second case, since $\mu_\phi(C_n) \to 0$, we can find some subsequence n_k such that $\sum_k \mu_\phi(C_{n_k}) < \infty$ so that $\sum_k \mu_\phi(A_{n_k}) < \infty$. So

$$\varliminf_{n\to\infty} \frac{\log \tau(x, C_n)}{n} \le \varliminf_{k\to\infty} \frac{\log \tau(x, C_{n_k})}{n_k} = 0.$$

Now we turn to the analysis of the series $\sum \mu_\phi(B_n)$. Choose a big $\omega > 0$ and $d := d(n) := \lfloor \omega n \rfloor$. Let

$$B_n^i := \{x : \sigma^{i(n+d)} x \notin C_n\}, \quad m := \lfloor 2^{\varepsilon n}/\mu_\phi(C_n)(n+d) \rfloor - 1.$$

Thus

$$B_n \subset B_n^0 \cap \cdots \cap B_n^m = \bigcup_{D_0,\ldots,D_m} D_0 \cap \sigma^{-(n+d)} D_1 \cap \cdots \cap \sigma^{-m(n+d)} D_m$$

where the D_i are cylinders (not necessarily distinct) of length n disjoint from C_n. Thus, by the multi-relation property, we get

$$\begin{aligned}
\mu_\phi(B_n) &\le \sum_{D_0,\ldots,D_m} \mu_\phi(D_0 \cap \sigma^{-(n+d)} D_1 \cap \cdots \cap \sigma^{-m(n+d)} D_m) \\
&\le (1 + c\beta^d)^m \sum_{D_0,\ldots,D_m} \prod_{i=0}^{m} \mu_\phi(\sigma^{-i(n+d)} D_i) \\
&\le [(1 + c\beta^d)(1 - \mu_\phi(C_n))]^{m+1} \\
&\le \left(1 - \frac{\mu_\phi(C_n)}{2}\right)^{m+1} \\
&\le e^{-(m+1)\mu_\phi(C_n)/2} \\
&\le e^{-2^{\epsilon n - 1}/(n+d)}.
\end{aligned}$$

□

Corollary 4.2. *For any $y \in \Sigma_2^+$ and for μ_ϕ a.e. x*

$$\alpha(x, y) = \underline{h}_{\mu_\phi}(y).$$

An application of Fubini's Theorem yields

Corollary 4.3. *Let ν be a probability measure on Σ_2^+. Then for $\mu_\phi \times \nu$ a.e. (x, y) we have*

$$\alpha(x, y) = \underline{h}_{\mu_\phi}(y).$$

The hitting time $\alpha(x, x)$ is what we called the return time. The following result due to Ornstein and Weiss [OW] concerning the return time is well known and holds for all ergodic measures. For Gibbs measures, it can be similarly proved as the above theorem.

Corollary 4.4. *For μ_ϕ a.e. x we have*

$$\alpha(x, x) = \alpha(\sigma^k x, \sigma^k x) = \underline{h}_{\mu_\phi}(x) = h_{\mu_\phi} \qquad (\forall k \ge 1).$$

4.2. **Determination of $\kappa_{\phi,\psi,\Sigma_2^+}$.**

Recall that $-\int \phi d\mu_\psi$ is nothing but the conditional entropy of μ_ϕ relative to μ_ψ. As a direct consequence of Lemma 3.2 and Chazottes' theorem, we get immediately the following critical value.

Theorem 4.5. *Let ϕ and ψ be Hölder functions on Σ_2^+. We have*

$$\kappa_{\phi,\psi} = \frac{1}{-\int_{\Sigma_2} \phi \, d\mu_\psi} = -\frac{1}{\frac{d}{dt}P(\psi + t\phi)|_{t=0}}.$$

Proof. Suppose that μ_ϕ and μ_ψ are ergodic Gibbs measures with $P(\phi) = P(\psi) = 0$. Corollary 4.3 implies that for $\mu_\phi \times \mu_\psi$ a.e. (x, y)

$$\alpha(x, y) = h_{\mu_\phi}(y) = -\int_{\Sigma_2} \phi \, d\mu_\psi = -\frac{d}{dt}P(\psi + t\phi)|_{t=0}.$$

Thus applying Lemma 3.2 yields the assertion of the theorem. □

5. Big hitting probability and Study of $\mathcal{F}^\kappa(x)$

We will give answers to question (**Q2**) and to the part of question (**Q3**) concerning $\mathcal{F}^\kappa(x)$.

5.1. **Big hitting probability.**

Heuristically points of small local entropy (i.e. large "local measure") are hit with big probability. More precisely we have

Lemma 5.1 (Big hitting probability)**.** *Let $K := 2^{hn}$. Fix L cylinders $C_1, \cdots C_L$ of length n satisfying $\mu_\phi(C_i) \geq 2^{-(h-\gamma)n}$. Then*

$$\mu_\phi\{x : \exists C \in \{C_i\} \text{ such that } \tau_n(x, C) > K\} \leq 2^{-\lambda n}$$

for any positive λ for sufficiently large n.

Proof. We have L possibilities for the cylinder C. Let $m := \lfloor K/(1+\omega)n \rfloor - 1$. Fix a choice C from these L cylinders and let $D_0, \ldots, D_m$ denote any cylinders of length n (possibly with repetition), which are disjoint from C. Choose $\omega > 0$ so that $\beta^\omega < 2^{e^+}$. Let $d := d(n) := \lfloor \omega n \rfloor$.

For a fixed C, let G_C be the set of points in Σ_2^+ in which the chosen cylinder C, considered as a word, does not appear up to time K. In particular, it does not appear at times $n + d, \cdots, m(n + d)$. Thus

$$\mu_\phi(G_C) \leq \sum_{D_0,\ldots,D_m} \mu_\phi(D_0 \cap \sigma^{-n+d} D_1 \cap \cdots \cap \sigma^{-m(n+d)} D_m).$$

By the multi-relation property, we get

$$\begin{aligned}\mu_\phi(G_C) &\le (1+c\beta^d)^{m+1} \sum_{D_0,\dots,D_m} \prod_{i=0}^{m} \mu_\phi(\sigma^{-i(n+d)} D_i) \\ &= \left[(1+c\beta^d)(1-\min_{C_i} \mu_\phi(C_i))\right]^m \\ &\le \left(1-\frac{1}{2}\min_{C_i} \mu_\phi(C_i)\right)^m .\end{aligned}$$

Summing over all the $L(\le 2^n)$ possible cylinders C yields

$$\begin{aligned}&\mu_\phi\{x : \exists C \in \{C_i\} \text{ such that } \tau_n(x,C) > K\} \\ &\le \sum_C \mu_\phi(G_C) \\ &\le L\left(1-\frac{1}{2}\min_{C_i} \mu_\phi(C_i))\right)^m \\ &\le L\left(1-\frac{1}{2}\min_{C_i} \mu_\phi(C_i))\right)^{2^{\gamma n}/(\min_{C_i} \mu_\phi(C_i)(1+\omega)n)} \\ &\le const \cdot 2^n \cdot \left(e^{-1/2}\right)^{2^{\gamma n}/(1+\omega)n} \\ &\le 2^{-\lambda n}\end{aligned}$$

for any positive λ and sufficiently large n. □

5.2. **The set of late hits.**

Let us recall that $\{y \in \Sigma_2^+ : \alpha(x,y) \ge t\}$ is random but $\{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) \ge t\}$ is deterministic (i.e. independent of x). The following theorem is deduced from Lemma 5.1 (big hitting probability) and Corollary 4.3 (Ornstein-Weiss type theorem on return times).

Theorem 5.2. *For any $t \ge 0$ and for μ_ϕ a.e. x we have*

$$\{y \in \Sigma_2^+ : \alpha(x,y) \ge t\} \subset \{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) \ge t\}. \tag{5·1}$$

Moreover if ν is any probability measure on Σ_2, then for μ_ϕ a.e. x we have

$$\{y \in \Sigma_2^+ : \alpha(x,y) \ge t\} \overset{\nu}{=} \{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) \ge t\}.$$

Proof. The case $t = 0$ is trivial. Assume $t > 0$. Let

$$H_{\ge t}(x) = \{y \in \Sigma_2^+ : \alpha(x,y) \ge t\}, \quad E_{\ge t} = \{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) \ge t\}.$$

By definition, we have

$$H_{\ge t}(x) = \bigcap_{\varepsilon>0} \varliminf_{n\to\infty} H_{n,\varepsilon}(x)$$

with $H_{n,\varepsilon}(x) = \{y : \tau_n(x,y) \ge 2^{(t-\varepsilon)n}\}$, and

$$E_{\ge t} = \bigcap_{\varepsilon>0} \varliminf_{n\to\infty} E_{n,\varepsilon}$$

with $E_{n,\varepsilon}(x) = \{y : \mu_\phi(C_n(y)) \le 2^{-(t-2\varepsilon)n}\}$. Thus it remains to prove that for μ_ϕ-a.e. x there exists $n(x) > 0$ such that

$$H_{n,\varepsilon}(x) \subset E_{n,\varepsilon} \quad \forall n \ge n(x).$$

Equivalently

$$E^c_{n,\varepsilon} \subset H^c_{n,\varepsilon}(x) \quad \forall n \ge n(x).$$

Notice that $E^c_{n,\varepsilon}$ is the union of all n-cylinders C such that $\mu_\phi(C) > 2^{-(t-2\varepsilon)n}$. Let $\mathcal{C}_{n,\varepsilon}$ be the set of all these cylinders. Applying Lemma 5.1 to $\{C_1, \cdots, C_L\} := \mathcal{C}_{n,\varepsilon}$ leads to

$$\sum_n \mu_\phi\{x \in \Sigma_2 : \exists C \in \mathcal{C}_{n,\varepsilon} \text{ s.t. } \tau_n(x, C) \ge 2^{(t-\varepsilon)n}\} < \infty.$$

So, by the Borel-Cantelli lemma, for μ_ϕ-a.e. x, for large n and for all $C \in \mathcal{C}_{n,\varepsilon}$ we have $\tau_n(x, C) < 2^{(t-\varepsilon)n}$, i.e. $C \subset H^c_{n,\varepsilon}(x)$. This proves the first assertion.

To prove the second assertion, it suffices to show that for μ_ϕ-a.e. x we have

$$\nu\{y \in \Sigma_2 : \underline{h}_{\mu_\phi}(y) \ge t, \alpha(x, y) < t\} = 0.$$

Let

$$E = \{(x, y) : \alpha(x, y) = \underline{h}_{\mu_\phi}(y)\}, \quad E_x = \{y : \alpha(x, y) = \underline{h}_{\mu_\phi}(y)\}.$$

By Corollary 4.3, we have $\mu_\phi \times \nu(E) = 1$. Then Fubini's theorem asserts that for μ_ϕ-a.e. x we have $\nu(E_x) = 1$, i.e.

$$\nu(E^c_x) = \nu\{y : \alpha(x, y) \ne \underline{h}_{\mu_\phi}(y)\} = 0.$$

We conclude by noticing

$$\{y : \underline{h}_{\mu_\phi}(y) \ge t, \alpha(x, y) < t\} \subset E^c_x.$$

□

We should point out that (5·1) is equivalent to

$$\{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) < t\} \subset \{y \in \Sigma_2^+ : \alpha(x, y) < t\}. \tag{5·2}$$

This justifies our heuristics that points of small local entropy are hit early. We point out that the inverse inclusion of (5·2) does not hold. Actually for $t < e^-$, the deterministic set $\{y \in \Sigma_2^+ : \underline{h}_{\mu_\phi}(y) < t\}$ is empty, but if $1/\kappa < t$, the random set $\{y \in \Sigma_2^+ : \alpha(x, y) < t\}$ contains $I^\kappa(x)$ which is a residual set.

5.3. **Computation of** $\dim_H\{y : \alpha(x, y) \ge t\}$ **and** $\dim_H \mathcal{F}^\kappa(x)$**.**

Theorem 5.3. *For μ_ϕ-a.e. x, we have*

$$\dim_H \big\{y : \ \alpha(x, y) \ge t\big\} = \dim_H \big\{y : \ \underline{h}_{\mu_\phi} \ge t\big\}.$$

Proof. By the second variational principle (Theorem 2.3), there exists an $s \geq t$ such that

$$\dim_H \{y : \ \underline{h}_{\mu_\phi} \geq t\} = \dim_H \mu_{-P(q(s)\phi)+q(s)\phi}. \tag{5·3}$$

Applying Corollary 4.3 (with $\nu = \mu_{-P(q(s)\phi)+q(s)\phi}$) implies that

$$\mu_{-P(q(s)\phi)+q(s)\phi}(\{y : \ \underline{h}_{\mu_\phi}(y) = \alpha(x,y) = s\}) = 1 \ \text{ for } \mu_\phi - \text{a.e. } x.$$

It follows that for μ_ϕ-a.e. x we have

$$\begin{aligned}\dim_H \{y : \ \alpha(x,y) \geq t\} &\geq \dim_H \{y : \ \underline{h}_{\mu_\phi}(y) = \alpha(x,y) = s\} \\ &\geq \dim \mu_{-P(q(s)\phi)+q(s)\phi}.\end{aligned}$$

This, together with (5·3), implies

$$\dim_H \left\{y : \ \alpha(x,y) \geq t\right\} \geq \dim_H \left\{y : \ \underline{h}_{\mu_\phi} \geq t\right\} \quad \mu_\phi\text{-a.e.}$$

Now we turn to the reverse inequality. Observe the following decomposition

$$\{y : \alpha(x,y) \geq t\} = \{\alpha(x,y) \geq t, \underline{h}_{\mu_\phi}(y) < t\} \cup \{\alpha(x,y) \geq t, \underline{h}_{\mu_\phi}(y) \geq t\}.$$

Since

$$\dim_H \{\underline{h}_{\mu_\phi}(y) \geq t, \alpha(x,y) \geq t\} \leq \dim_H \{\underline{h}_{\mu_\phi}(y) \geq t\},$$

it suffices to remark that $\{y : \ \underline{h}_{\mu_\phi}(y) < t, \alpha(x,y) \geq t\} = \emptyset$ for μ_ϕ a.e. x. □

By this theorem, Lemmas 3.2 and 3.3, and the second variational principle (Theorem 2.3) we get

Theorem 5.4. *For μ_ϕ-a.e. x we have*

$$\begin{aligned} h_{\text{top}}(\mathcal{F}^\kappa(x)) &= 1 \qquad\qquad \text{for } \frac{1}{\kappa} \leq e_{\max}, \\ h_{\text{top}}(\mathcal{F}^\kappa(x)) &= h_{\mu_{q(\kappa)\phi}} \quad \text{for} \quad e_{\max} \leq \frac{1}{\kappa} < \ e_+ \end{aligned}$$

where $q(\kappa)$ is chosen such that $h_{\mu_\phi}(y) = \frac{1}{\kappa}$ for $\mu_{q(\kappa)\phi}$ a.e. y. We also have

$$\begin{aligned} \mathcal{F}^\kappa(x) &= \emptyset \text{ (or equivalently } \mathcal{I}^\kappa(x) = \mathbb{S}^1) \text{ if } \frac{1}{\kappa} > e_+, \\ \mathcal{F}^\kappa(x) &\neq \emptyset \text{ (or equivalently } \mathcal{I}^\kappa(x) \neq \mathbb{S}^1) \text{ if } \frac{1}{\kappa} < e_+. \end{aligned}$$

Remark that the case $\frac{1}{\kappa} = e^+$ is not covered by the theorem because $E(t)$ is not continuous at $t = e^+$. We have the upper bound $\dim_H \mathcal{F}^{1/e_+} \leq E(e^+)$. A result due to Kahane for the random covering shows that a strict inequality may occur ([K], p.160).

6. Small hitting probability and upper bound of $\dim_H\{y : \alpha(x,y) \le s\}$

6.1. Small hitting probability.

Lemma 6.1 (Small hitting probability)**.** *Let $K := 2^{an}, L := 2^{bn}, N := 2^{cn}$ with $a > 0, b > 0, c > 0$. Fix L different cylinders $C_1, \cdots C_L$ of length n satisfying*

$$\mu_\phi(C_i) \le 2^{-(a+\gamma)n}.$$

Then if $\gamma > \max(b-c, 0)$, for any positive λ and sufficiently large n we have

$$\mu_\phi\{x : \tau_n(x, C_i) \le K \text{ for } N \text{ different cylinders among the } C_i\} \le 2^{-\lambda n}.$$

Proof. Let S be the set in question. That $x \in S$ means there exist times $\ell_1 < \ell_2 < \cdots < \ell_N < K$ and different cylinders $C_{i_1}, C_{i_2}, \cdots, C_{i_N}$ such that

$$\sigma^{\ell_1} x \in C_{i_1}, \quad \sigma^{\ell_2} x \in C_{i_2}, \quad \cdots, \quad \sigma^{\ell_N} x \in C_{i_N}.$$

In this sequence (ℓ_k) of length N there is a subsequence of $N/(3n+d)$ terms, denoted (τ_j) such that $\tau_j - \tau_{j-1} \ge 3n + d$. For example, we may take $\tau_j = \ell_{(3n+d)j}$. Thus $x \in S$ implies

$$\sigma^{\tau_1} x \in C_{j_1}, \quad \sigma^{\tau_2} x \in C_{j_2}, \quad \cdots, \quad \sigma^{\tau_{N'}} x \in C_{j_{N'}}$$

for $N' := N/(3n+d)$ different cylinders taken from the list $C_1, C_2, \cdots, C_L$. Thus to each $x \in S$ we can associate the sequences (τ_j) and (C_{j_k}). Thus

$$x \in C(x) := \bigcap \sigma^{-\tau_i}(C_{j_i})$$

and S is covered by the union of $C(x)$. The multi-relation property implies that the measure of $C(x)$ is bounded by

$$\max_{1 \le i \le L} \mu_\phi(C_i)^{N'} (1 + c\beta^d)^{N'}.$$

Now, we have to estimate the number of different (disjoint) sets $C(x)$. First we have $\binom{L}{N'}$ choices for the N' different cylinders from the list of L words. Then we can choose $\binom{K}{N'}$ places (i.e. we fix the sequence τ_j) to put the chosen words in order to determine $C(x)$. Finally we have $N'!$ ways to arrange words into these N' (now fixed) places.

Thus the measure of the set in question can be majorized by

$$\binom{L}{N'}\binom{K}{N'} \cdot N'! \cdot \max_{C_i} \mu_\phi(C_i)^{N'} \cdot (1 + c\beta^d)^{N'}.$$

This is equal to

$$\frac{L!}{(L-N')!} \cdot \frac{K!}{(K-N')!N'!} \cdot (\max_{C_i} \mu_\phi(C_i))^{N'} \cdot (1 + c\beta^d)^{N'}.$$

Next using the estimates

$$\frac{L!}{(L-N')!} \le L^{N'}, \quad \frac{K!}{(K-N')!N'!} \le const \cdot K^{N'} \cdot \frac{e^{N'}}{N'^{N'}}$$

(the second one is implied by Stirling's formula), we conclude that the measure is majorized by

$$\begin{aligned}
&const \cdot L^{N'} \cdot K^{N'} \cdot e^{N'} \cdot N'^{-N'} \cdot \left(2^{-(a+\gamma)n}\right)^{N'} \cdot (1+c\beta)^{N'} \\
&= const \cdot \left(2^{bn} \cdot 2^{an} \cdot e \cdot 2^{-cn} \cdot 2^{-(a+\gamma)n} \cdot (1+c\beta^d)\right)^{N'} \\
&\leq const \left(e \cdot (1+c\beta^d) \cdot 2^{(b-c-\gamma)n}\right)^{N'}.
\end{aligned}$$

Provided $\gamma > b - c$, this is less that $2^{\lambda n}$ for any positive λ and sufficiently large n. □

6.2. **Upper bound of** $\dim_H\{y : \alpha(x,y) \leq s\}$**.**

Theorem 6.2. *If $h_{\mu_\phi} < s < e_{\max}$ then for μ_ϕ-a.e. x we have*

$$h_{\text{top}}\{y \,:\, \alpha(x,y) \leq s\} \leq E(s). \tag{6·1}$$

If $0 < s \leq h_{\mu_\phi}$ then for all *x we have*

$$h_{\text{top}}\{y \,:\, \alpha(x,y) \leq s\} \leq s. \tag{6·2}$$

Proof. Let

$$\mathcal{A}_x(s) = \{y \,:\, \alpha(x,y) \leq s\}.$$

The case $s \leq h_{\mu_\phi}$ is simple. In fact, if $a > s$, we have

$$\mathcal{A}_x(s) \subset \overline{\lim_{n\to\infty}}\{y \,:\, \tau_n(x,y) \leq 2^{an}\} = \overline{\lim_{n\to\infty}} \bigcup_{k=1}^{2^{an}} C_n(\sigma^k x).$$

Since $C_m(\sigma^k x) \subset C_n(\sigma^k x)$ for $m > n$, we have

$$\mathcal{A}_x(s) \subset \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^{an}} C_n(\sigma^k x).$$

We have $h_{\text{top}}\mathcal{A}_x(s) \leq a$ since $\{C_n(\sigma^k)\}_{1\leq k\leq 2^{an}}$ is a cover of for $\mathcal{A}_x(s)$ by 2^{an} cylinders of length n. We conclude by letting $a \downarrow s$. Remark that $h_{\text{top}}\mathcal{A}_x(s) \leq s$ holds for any non negative s.

We turn to the case $h_{\mu_\phi} < s \leq e_{\max}$. We start with a remark. For $\delta > 0$ and $n \geq 1$ and $0 < h_1 < h_2$, let $\mathfrak{L}_n(h_1,h_2) := \mathfrak{L}_n(h_1,h_2,\delta)$ be the set of cylinders C of length n such that $2^{-(h_2-\delta)n} \leq \mu_\phi(C) \leq 2^{-(h_1+\delta)n}$. Then for n sufficiently large (depending on h_1, h_2 and δ) we have

$$\text{Card}\,\mathfrak{L}_n(h_1,h_2) \leq 2^{nE(h_2)} \quad \text{if} \quad h_2 < e_{\max}$$

$$\text{Card}\,\mathfrak{L}_n(h_1,h_2) \leq 2^{nE(h_1)} \quad \text{if} \quad h_1 > e_{\max}.$$

In fact, assume $h_2 < e_{\max}$ (the other case may be similarly proved). There exists a positive number q such that $E(h_2) = P(q) + h_2 q$. Then

$$2^{-q(h_2-\delta)n}\text{Card}\,\mathfrak{L}_n(h_1,h_2) \leq \sum_{C\in\mathfrak{L}_n(h_1,h_2)} \mu_\phi(C)^q \leq 2^{n(P(q)+q\delta)}.$$

Write

$$\mathcal{A}_x(s) = \left(\mathcal{A}_x(s) \cap \left\{y \,:\, \underline{h}_{\mu_\phi}(y) \le s\right\}\right) \bigcup \left(\mathcal{A}_x(s) \cap \left\{y \,:\, \underline{h}_{\mu_\phi}(y) > s\right\}\right).$$

Since $h_{\text{top}}\{y : \underline{h}_{\mu_\phi}(y) \le s\} \le E(s)$, it suffices to show

$$h_{\text{top}}\left(\mathcal{A}_x(s) \cap \left\{y \,:\, \underline{h}_{\mu_\phi}(y) > s\right\}\right) \le E(s). \tag{6·3}$$

Let

$$\mathcal{H}(h', h'') = \left\{y \,:\, h' \le \underline{h}_{\mu_\phi}(y) \le h''\right\}.$$

If all choices $s < h' < h''$ such that $h'' < e_{\max}$ or $h' > e_{\max}$ the formula

$$h_{\text{top}}\left(\mathcal{A}_x(s) \cap \mathcal{H}(h', h'')\right) \le E(s)$$

holds, then the equation (6·3) also holds.

Let $s < h_1 + \delta < h' < h'' < h_2 - \delta$ with h_1 close to h' and h_2 close to h''. Remark that $y \in \mathcal{H}(h', h'')$ implies that $C_n(y)) \in \mathcal{H}(h_1, h_2)$ for infinitely many n's. In other words

$$\mathcal{H}(h', h'') \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{C \in \mathcal{L}_n(h_1, h_2)} C.$$

That is to say, for any fixed m, $\bigcup_{n \ge m} \mathcal{L}_n(h_1, h_2)$ is a cover of $\mathcal{H}(h', h'')$.

Now we construct a cover of $A_x(s) \cap \mathcal{H}(h', h'')$. For any $s < a < h_1$, let

$$\begin{aligned} \mathcal{L}_n(x; a, h_1, h_2) &= \{C \in \mathfrak{L}_n(h_1, h_2) : \tau(x, C) \le 2^{an}\}, \\ N_n(x; a, h_1, h_2) &= \operatorname{Card} \mathcal{L}_n(x; a, h_1, h_2). \end{aligned}$$

Clearly $\bigcup_{n \ge m} \mathcal{L}_n(x; a, h_1, h_2)$ is a cover of $A_x(s) \cap \mathcal{H}(h', h'')$, because

$$\mathcal{A}_x(s) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{C : \tau(x, C) \le 2^{an}} C.$$

Let $\gamma = h_2 - a$ if $h_2 \le e_{\max}$, or $\gamma = h_1 - a$ if $h_1 > e_{\max}$. Since $E'(t) < 1$ when $t > h_{\mu_\phi}$, we have

$$E(a + \gamma) - E(a) < \gamma, \quad \text{i.e.} \quad E(a + \gamma) - \gamma < E(a).$$

We apply the Small Hitting Probability Lemma to $b = E(a + \gamma)$ and $c = E(a)$ to get

$$\sum_n \mu_\phi\{x : N_n(x; a, h_1, h_2,) > 2^{nE(a)}\} < \infty.$$

By the Borel-Cantelli Lemma, for μ_ϕ-a.e. x, we have $N_n(x; a, h_1, h_2) \le 2^{nE(a)}$ for $n \ge n(x)$. So, if $m \ge n(x)$, for any $\epsilon > 0$ we have

$$\begin{aligned} &\sum_{n \ge m} \sum_{C \in \mathfrak{L}_n(x; a, h_1, h_2)} (\operatorname{diam} C)^{E(a) + \epsilon} \\ \le\ &\sum_{n \ge m} 2^{-n(E(a) + \epsilon)} \cdot 2^{nE(a)} \le \sum_{n \ge m} 2^{-n\epsilon} < \infty. \end{aligned}$$

Since $\bigcup_{n\geq m}\mathcal{L}_n(x;a,h_1,h_2)$ is a cover of $\mathcal{A}_x(s)\cap\mathcal{H}(h',h'')$, we have proved

$$\dim \mathcal{A}_x(s)\cap\mathcal{H}(h',h'')\leq E(a)+\epsilon.$$

We finish the proof by letting first $\varepsilon\downarrow 0$ and then $a\downarrow s$. □

Theorem 6.3. *If $h_{\mu_\phi}<s<e_{\max}$ then for μ_ϕ-a.e. x we have*

$$h_{\text{top}}\{y\,:\,\alpha(x,y)\leq s\}=E(s).$$

Proof. We simply need to prove the reverse inequality of (6·1) in Theorem 6.2. By multi-fractal analysis there is a Gibbs measure with entropy $E(s)$ supported on $\{y:h_{\mu_\phi}(y)=s\}$. Then Corollary 4.3 implies the result. □

For $0<s<h_{\mu_\phi}$, the opposite inequality of (6·2):

$$h_{\text{top}}\{y\,:\,\alpha(x,y)\leq s\}\geq s$$

also holds. But its proof is much more involved. It can not be deduced from the mass transference principle as stated in [BV] since μ_ϕ has nontrivial entropy spectrum. In the next section we make a substantial improvement in the mass transference principle to multi-fractal Gibbs states. In order to prove it, we need to undertake a full investigation of the structure of typical sequences.

7. Typical sequences and Lower bound of $\dim_H\{y:\alpha(x,y)\leq c\}$

Recall that μ_ϕ is a Gibbs measure associated to a normalized Hölder potential ϕ. A cylinder C of length n is said to be a (n,ε)-cylinder if

$$2^{-(h+\varepsilon)n}\leq\mu_\phi(C)\leq 2^{-(h-\varepsilon)n}$$

where $h=h_\phi$ denotes the entropy of μ_ϕ. We denote by $\mathcal{C}_{n,\varepsilon}$ the set of all (n,ε)-cylinders. Sometimes we will say that a (n,ε)-cylinder is a good cylinder or the word determining a (n,ε)-cylinder is a good word. As we shall prove, a relatively short typical word contains plenty of good subwords of a fixed length and they are even different.

The following notations will be used. If C and D are cylinders, we denote by $C\star D$ the cylinder $C\cap\sigma^{-|C|}D$. If we read C and D as words, $C\star D$ is nothing but the concatenation of the words C and D. Let $d\geq 1$ be an integer, by $C\star_d D$ we mean $C\cap\sigma^{-(|C|+d)}D$, i.e.

$$C\star_d D=\bigcup_{G:|G|=d}C\star G\star D.$$

For a set S, $\sharp S$ will denote the cardinality of S.

7.1. Frequency of good words in a typical orbit.

Lemma 7.1. *Let μ_ϕ be a Gibbs measure with entropy $h:=h_{\mu_\phi}>0$. For any $\varepsilon>0$, there exist an integer $n(\varepsilon)\geq 1$ and a Borel set $\mathcal{G}_\varepsilon$ with $\mu_\phi(\mathcal{G}_\varepsilon)>1-\varepsilon$ such that for any $x\in\mathcal{G}_\varepsilon$ and any $n\geq n(\varepsilon)$, the cylinder $C=C_n(x)$ is a (n,ε)-cylinder. Consequently, if $n\geq n(\varepsilon)$, we have*

$$(1-\varepsilon)2^{(h-\varepsilon)n}\leq\sharp\mathcal{C}_{n,\varepsilon}\leq 2^{(h+\varepsilon)n}.$$

Proof. By the Shannon McMillan Breiman theorem, for μ_ϕ-a.e. x we have

$$\lim_{n\to\infty} -\frac{\log \mu_\phi(C_n(x))}{n} = h.$$

Then by Egorov's theorem, there is a number $n(\varepsilon) \geq 1$ such that the set

$$\mathcal{G}_\varepsilon := \left\{ y \in \Sigma_2 \,:\, -\frac{1}{n}\log \mu_\phi(C_n(y)) \in [h-\varepsilon, h+\varepsilon], \quad \forall n > n(\varepsilon) \right\}$$

has measure $\mu_\phi(\mathcal{G}_\varepsilon) > 1-\varepsilon$.

The upper estimate $\sharp\mathcal{C}_{n,\varepsilon} \leq 2^{(h+\varepsilon)n}$ follows from

$$2^{-(h_\mu+\varepsilon)n}\sharp\mathcal{C}_{n,\varepsilon} \leq \sum_{C\in\mathcal{C}_{n,\varepsilon}} \mu_\phi(C) \leq 1.$$

The lower estimate $(1-\varepsilon)2^{(h-\varepsilon)n} \leq \sharp\mathcal{C}_{n,\varepsilon}$ follows from $\mathcal{G}_\varepsilon \subset \bigcup_{C\in\mathcal{C}_{n,\varepsilon}} C$ and

$$1-\varepsilon \leq \mu_\phi(\mathcal{G}_\varepsilon) \leq \sum_{C\in\mathcal{C}_{n,\varepsilon}} \mu_\phi(C) \leq 2^{-(h_\mu-\varepsilon)n}\sharp\mathcal{C}_{n,\varepsilon}.$$

□

We call the set $\mathcal{G}_\varepsilon$ the set of ε-good points. By the definition of $\mathcal{G}_\varepsilon$, we have

$$\mathcal{G}_\varepsilon = \bigcap_{n=n(\varepsilon)}^{\infty} \bigcup_{C\in\mathcal{C}_{n,\varepsilon}} C.$$

Hence it is a G_δ set. We will write it as a decreasing limit of open sets in the following manner

$$\mathcal{G}_\varepsilon = \bigcap_{N=n(\varepsilon)}^{\infty} \bigcap_{n=n(\varepsilon)}^{N} \bigcup_{C\in\mathcal{C}_{n,\varepsilon}} C.$$

This representation of $\mathcal{G}_\varepsilon$ is useful in the proof of the following lemma.

Lemma 7.2. *Let $0 < \varepsilon < 1/2$ and let $L' \geq 1$ be an arbitrary integer. For any cylinder D of length L', we have*

$$\mu_\phi(D \cap \sigma^{-|D|}\mathcal{G}_\varepsilon) \geq \frac{1}{2\gamma^4} 2^{-L'\|\phi\|_\infty}$$

where $\gamma > 1$ is the constant involved in the Gibbs property of μ_ϕ (2·2).

Proof. We first recall the following quasi-Bernoulli property of μ_ϕ (2·3): for any two cylinders A and B we have

$$\mu_\phi(A \cap \sigma^{-|A|}B) \geq \frac{1}{\gamma^3}\mu_\phi(A)\mu_\phi(B).$$

Let us prove the lemma. The set $\mathcal{G}_\varepsilon$ is the decreasing limit of the open sets

$$\mathcal{G}_{N,\varepsilon} = \bigcap_{n=n(\varepsilon)}^{N} \bigcup_{C\in\mathcal{C}_{n,\varepsilon}} C.$$

Observe that $\mathcal{G}_{N,\varepsilon}$ is a union of cylinders of length N. Thus we have

$$\mu_\phi(D \cap \sigma^{-|D|}\mathcal{G}_\varepsilon) = \lim_{N\to\infty} \mu_\phi(D \cap \sigma^{-|D|}\mathcal{G}_{N,\varepsilon}) = \lim_{N\to\infty} \sum_C \mu_\phi(D \cap \sigma^{-|D|}C)$$

where C varies over all N-cylinders contained in $\mathcal{G}_{N,\varepsilon}$. First applying the quasi-Bernoulli property and then using the fact that $\mu_\phi(\mathcal{G}_{N,\varepsilon}) \geq 1-\varepsilon > 1/2$, yields

$$\sum_C \mu_\phi(D \cap \sigma^{-|D|}C) \geq \frac{\mu_\phi(D)}{\gamma^3} \sum_C \mu_\phi(C) = \frac{\mu_\phi(D)}{\gamma^3}\mu_\phi(\mathcal{G}_{N,\varepsilon}) \geq \frac{\mu_\phi(D)}{2\gamma^3}.$$

To conclude, it suffices to remark that

$$\mu_\phi(D) \geq \frac{1}{\gamma} 2^{-|D|\ \|\phi\|_\infty}$$

which is assured by the Gibbs property of μ_ϕ. □

The next theorem essentially says that a typical word of length $2^{cL''}$ contains many good subwords of length n with an arbitrary but fixed prefix D of length L'. We keep the notations $n(\epsilon)$ and $\mathcal{G}_\epsilon$ appearing in Lemma 7.1.

Theorem 7.3. *Let $c > 0$ be fixed. Let $0 < \varepsilon < \min(\frac{1}{2}, c)$, $0 < \eta < \frac{1}{2}$ and $L' \geq 1$. There exist an integer $n(\varepsilon,\eta,L') \geq L' + n(\varepsilon)$ and a Borel set $\mathcal{E}(\varepsilon,\eta,L')$ with $\mu_\phi(\mathcal{E}(\varepsilon,\eta,L')) > 1-\eta$ such that if $x \in \mathcal{E}(\varepsilon,\eta,L')$ and $L'' > n(\varepsilon,\eta,L')$, for each L'-cylinder D there are at least $2^{(c-\varepsilon)L''}$ points of the finite orbit $\sigma^j x$ ($2^{L'}+1 \leq j \leq 2^{cL''}$), which fall into $D \cap \sigma^{-L'}\mathcal{G}_\varepsilon$.*

Proof. Let

$$m(L') := \frac{1}{2\gamma^4} 2^{-L'\|\phi|_\infty}$$

be the lower bound which appeared in the last lemma. For $x \in \Sigma_2$, define

$$n_{D,L',\varepsilon}(x) := \inf\left\{ n \in \mathbb{N} : \frac{1}{N} \sum_{j=2^{L'}+1}^{2^{L'}+N} \mathbf{1}_{D\cap\sigma^{-L'}\mathcal{G}_\varepsilon}(\sigma^j x) > \frac{1}{2} m(L'), \forall N \geq n \right\}$$

and

$$n_{L',\varepsilon}(x) = \max_D n_{D,L',\varepsilon}(x).$$

By Lemma 7.2 and Birkhoff's ergodic theorem we have

$$\mu_\phi(x \in \Sigma_2 : n_{L',\varepsilon}(x) < \infty) = 1.$$

So, for any $\eta > 0$, there exists an integer $\widehat{n}(L',\varepsilon,\eta)$ such that the Borel set

$$\mathcal{E}(L',\varepsilon,\eta) := \{x \in \Sigma_2 : \ n_{L',\varepsilon}(x) \leq \widehat{n}(L',\varepsilon,\eta)\}$$

satisfies

$$\mu_\phi(\mathcal{E}(L',\varepsilon,\eta)) > 1-\eta.$$

Fix $n(L',\varepsilon,\eta) \geq 1$ sufficiently large so that

$$\frac{1}{2} m(L')[2^{\varepsilon n(L',\varepsilon,\eta)} - 2^{L'}] \geq 1,$$

$$n(L',\varepsilon,\eta)-L'\geq n(\varepsilon),$$

$$2^{cn(L',\varepsilon,\eta)}-2^{L'}\geq \widehat{n}(L',\varepsilon,\eta).$$

Assume $x\in\mathcal{E}(L',\varepsilon,\eta)$ and $L''\geq n(L',\varepsilon,\eta)$. Since $N:=2^{cL''}-2^{L'}\geq \widehat{n}(L',\varepsilon,\eta)$, we have

$$\begin{aligned}\sum_{j=2^{L'}+1}^{2^{cL''}}\mathbf{1}_{D\cap\mathcal{G}_\varepsilon}(\sigma^j x) &\geq \frac{1}{2}m(L')[2^{cL''}-2^{L'}]\\ &\geq \frac{1}{2}m(L')[2^{\varepsilon n(L',\varepsilon,\eta)}-2^{L'}]\cdot 2^{(c-\varepsilon)L''}\\ &\geq 2^{(c-\varepsilon)L''}.\end{aligned}$$

□

Let C be a cylinder of length n. If $C_n(\sigma^j x)=C$, we say that the cylinder C is seen in x at time j. Let $\varepsilon>0$, $L'<L''$ and let D be a cylinder of length L'. For any $x\in\Sigma_2$, we define a finite tree, denoted $\mathcal{T}(x,D,L',L'',\varepsilon)$, as follows:

- the nodes of $\mathcal{T}(x,D,L',L'',\varepsilon)$ are all those cylinders $D\star G'$, where G' is a $(\ell-L',\varepsilon)$-cylinder with $L'+n(\varepsilon)\leq\ell\leq L''$, each of which contains at least one $(L'',2\varepsilon)$-cylinder seen in x at a moment between the time $2^{L'}+1$ and the time $2^{cL''}$;
- a ℓ-cylinder $D\star G'\in\mathcal{T}(x,D,L',L'',\varepsilon)$ is the parent of a $(\ell+1)$-cylinder $D\star G''\in\mathcal{T}(x,D,L',L'',\varepsilon)$ if and only if $G''\subset G'$.

Fix $L'<L''$. For $L'+n(\varepsilon)\leq\ell\leq L''$, denote

$$T(x,D,\ell,\varepsilon):=\sharp\{D\star G'\in\mathcal{T}(x,D,L',L'',\varepsilon):|D\star G'|=\ell\}.$$

Theorem 7.3 implies that if L'' satisfies the condition of Theorem 7.3 and if $x\in\mathcal{E}(L',\varepsilon,\eta)$, then in between the times $2^{L'}+1$ and $2^{cL''}$, for each L'-cylinder D we can see at least $2^{(c-\varepsilon)L''}$ cylinders of length L'' in x of the form

$$D\star G'\qquad (G'\in\mathcal{C}_{L''-L',\varepsilon}). \tag{7·1}$$

By the quasi-Bernoulli property (2·3), it is easy to see that if L'' is sufficiently larger than L' then the cylinders $D\star G'$ are good in the sense

$$G:=D\star G'\in\mathcal{C}_{L'',2\varepsilon}. \tag{7·2}$$

Thus we have

$$T(x,D,L'',\varepsilon)\geq 2^{(c-\varepsilon)L''}.$$

Next we will prove that with big probability, for all $L'+n(\varepsilon)\leq\ell\leq L''$

$$T(x,D,\ell,\varepsilon)\geq 2^{(c-2\varepsilon)\ell}.$$

7.2. **Trees associated to a typical orbit.**

Assume that $L'' \geq n(L', \varepsilon, \eta)$. Let $L' + n(\varepsilon) \leq \ell \leq L''$, $x \in \mathcal{E}(L', \varepsilon, \eta)$, and D be a L'-cylinder. By definition $T(x, D, \ell, \varepsilon)$ is the number of *different cylinders* of the form

$$D \star G' \quad \text{with } G' \in \mathcal{C}_{\ell - L', \varepsilon}$$

each of which contains at least one $(L'', 2\varepsilon)$-cylinder belonging to the list $C_{L''}(\sigma^j x)$, $2^{L'} + 1 \leq j \leq 2^{cL''}$.

Theorem 7.4. *There exists $n_0(\varepsilon)$ such that for sufficiently large L'' and for $L' + n_0(\varepsilon) \leq \ell \leq L''$ we have*

$$\mu_\phi \left\{ x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) \leq 2^{(c-2\varepsilon)(\ell - L')} \right\} \leq 2^{-2^{(c-2\varepsilon)L''}}.$$

In the rest of this subsection and the next two subsections we prepare for the proof of this theorem, which will be presented in the subsection 7.5. We need to estimate the measures

$$\mu_\phi \left\{ x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) = K \right\}$$

for $K \leq 2^{(c-2\varepsilon)(\ell - L')}$. We will do that in the following.

For $1 \leq t \leq L'' + d$ (where $d := [\omega L'']$), let

$$\Lambda_t = \left\{ 2^{L'} + k(L'' + d) + t : 0 \leq k \leq \frac{2^{cL''} - 2^{L'}}{L'' + d} \right\}.$$

Fix K cylinders $C_1, \cdots, C_K \in \mathcal{C}_{\ell - L', \varepsilon}$. Let

$$\begin{aligned} &\Upsilon_t(x; C_1, C_2, \cdots, C_K) = \\ &\qquad \sharp \left\{ j \in \Lambda_t : C_{L''}(\sigma^j x) \in \mathcal{C}_{L'', 2\varepsilon} \text{ implies } C_{L''}(\sigma^j x) \subset D \star \widetilde{C} \right\} \end{aligned}$$

where

$$D \star \widetilde{C} := \bigcup_{i=1}^{K} D \star C_i.$$

$T(x, D, \ell, \varepsilon) = K$ means there exist K different $(\ell - L', \varepsilon)$-cylinders, say $C_1, C_2, \cdots, C_K$ such that all $(L'', 2\varepsilon)$-cylinders seen in x in between the times $2^{L'} + 1$ and $2^{cL''}$ are contained in some of the $D \star C_i$'s, i.e. contained in $D \star \widetilde{C}$. On the other hand, by Theorem 7.3, there are at least $2^{(c-\varepsilon)L''}$ of the $(L'', 2\varepsilon)$-cylinders seen in x in between the times $2^{L'} + 1$ and $2^{cL''}$. So, for at least one t the number of the $(L'', 2\varepsilon)$-cylinders seen at moments belonging to Λ_t and contained in $D \star \widetilde{C}$ is at least $\dfrac{2^{(c-\varepsilon)L''}}{L'' + d}$. Thus we get

$$\{ x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) = K \} \subset \bigcup_{t=1}^{L''+d} \bigcup_{C_1, \cdots, C_K} E_t(C_1, \cdots, C_K)$$

where the second union is taken over all possible collections $C_1, \cdots, C_K$ of $(\ell - L', \varepsilon)$-cylinders, and where

$$E_t(C_1, \cdots, C_K) = \left\{ x \in \mathcal{E}(L', \varepsilon, \eta) : \Upsilon_t(x; C_1, C_2, \cdots, C_K) \geq \frac{2^{(c-\varepsilon)L''}}{L'' + d} \right\}.$$

Therefore, using the fact that the number of $(\ell - L', \varepsilon)$-cylinders is at most $2^{(h+\varepsilon)(\ell - L')}$, we have proved

Lemma 7.5.

$$\begin{aligned} & \mu_\phi(x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) = K) \\ \leq \ & (L'' + d) \begin{pmatrix} 2^{(h+\varepsilon)(\ell - L')} \\ K \end{pmatrix} \sup_{t; C_1, \cdots, C_K} \mu_\phi(E_t(C_1, \cdots, C_K)). \end{aligned}$$

7.3. **Generalized quasi Bernoulli property.**

In order to estimate the measure $\mu_\phi(E_t(C_1, \cdots, C_K))$, we need the following generalized quasi Bernoulli property.

Let A be any cylinder and $L \geq 1$ be any integer. For $x \in A$, we define

$$\iota_A(x) = \inf\{|A| + k(L + d(L)) \geq 0 : C_L(\sigma^{|A| + k(L + d(L))} x) \in \mathcal{C}_{L,\varepsilon}\}$$

where $d(L) = \lfloor \omega L \rfloor$ for some big $\omega > 1$ (see Theorem 2.1).

Lemma 7.6 (Generalized quasi Bernoulli property)**.** *Let A be any cylinder, $G \in \mathcal{C}_{L,\varepsilon}$ and ι_A be defined as above. Then*

$$\mu_\phi(x \in A : C_L(\sigma^{\iota_A(x)} x) = G) \leq \frac{\gamma^3}{1 - 2\varepsilon} \mu_\phi(A) \mu_\phi(G).$$

Proof. Notice that

$$\{x \in A : C_L(\sigma^{\iota_A(x)} x) = G\} = \bigcup_{i=0}^{\infty} A_i$$

where

$$A_i = \{x \in A : C_L(\sigma^{\iota_A(x)} x) = G, \iota_A(x) = |A| + i(L + d)\}.$$

For $i = 0$, we have

$$A_0 = A \star G.$$

So, by the Gibbs property (2·2) we get

$$\mu_\phi(A_0) \leq \gamma^3 \mu_\phi(A) \mu_\phi(G).$$

For $i \geq 1$, we have

$$A_i \subset \bigcup_{B_1, \cdots, B_i \notin \mathcal{C}_{L,\varepsilon}} A \star B_1 \star_d \cdots \star_d B_i \star_d G.$$

So, by the multi-relation (2·4) we get

$$\mu_\phi(A_i) \leq \gamma^3 (1 + \beta^d)^i \mu_\phi(A) \mu_\phi(G) \left(\sum_{B \notin \mathcal{C}_{L,\varepsilon}} \mu_\phi(B) \right)^i.$$

Since $\sum_{B\notin\mathcal{C}_{L,\varepsilon}}\mu_\phi(B)\le\mu_\phi(\mathcal{G}^c_{L,\varepsilon})\le\varepsilon$, we get

$$\mu_\phi(A_i)\le\gamma^3(\varepsilon(1+\beta^d))^i\mu_\phi(A)\mu_\phi(G).$$

Thus

$$\begin{aligned}\mu_\phi(x\in A: C_L(\sigma^{\iota_A(x)}x)=G) &\le \gamma^3\mu_\phi(A)\mu_\phi(G)\sum_{i=0}^{\infty}(\varepsilon(1+\beta^d))^i\\ &= \frac{\gamma^3}{1-\varepsilon(1+\beta^d)}\mu_\phi(A)\mu_\phi(G).\end{aligned}$$

We finish the proof by observing that $\beta<1$. □

7.4. **Estimation of $\mu_\phi(E_t(C_1,\cdots,C_K))$.**

Let t be fixed. We define inductively

$$\begin{aligned}\iota_1(x) &= \inf\{j\in\Lambda_t: C_{L''}(\sigma^j x)\in\mathcal{C}_{L'',2\varepsilon}\};\\ \iota_{k+1}(x) &= \inf\{j\in\Lambda_t: j>\iota_k(x); C_{L''}(\sigma^j x)\in\mathcal{C}_{L'',2\varepsilon}\}.\end{aligned}$$

Let

$$\widetilde{n}:=\frac{2^{(c-\varepsilon)L''}}{L''+d}. \tag{7·3}$$

We have

$$\iota_i(x)<\infty \quad \text{if } x\in E_t(C_1,\cdots,C_K), \text{and} \quad \text{if } i\le\widetilde{n}.$$

Then

$$\mu_\phi(E_t(C_1,\cdots,C_K))\le\sum\mu_\phi\left(x:\sigma^{\iota_i(x)}x\in F_i, 1\le\forall i\le\widetilde{n}\right) \tag{7·4}$$

where the sum is taken over all F_i's with the property

$$F_i\in\mathcal{C}_{L'',2\varepsilon},\quad F_i\subset D\star\widetilde{C}\quad(1\le\forall i\le\widetilde{n}).$$

Lemma 7.7. *Let $n\ge1$ and let $F_i\in\mathcal{C}_{L'',2\varepsilon}$ with $1\le i\le n$. We have*

$$\mu_\phi\left(x: C_{L''}(\sigma^{\iota_i(x)}x)=F_i; i=1,2,\cdots,n\right)\le\left(\frac{\gamma^3}{1-4\varepsilon}\right)^n\prod_{i=1}^n\mu_\phi(F_i).$$

Proof. We prove it by induction on n. Let

$$\mathcal{Q}_n=\{x: C_{L''}(\sigma^{\iota_i(x)}x)=F_i; i=1,2,\cdots,n\}.$$

Write

$$\mathcal{Q}_{n+1}=\mathcal{Q}_n\cap\{x: C_{L''}(\sigma^{\iota_{n+1}(x)}x)=F_{n+1}\}.$$

Notice that $\mathcal{Q}_n$ is a disjoint union of cylinders, say

$$\mathcal{Q}_n=\bigcup A_j.$$

Furthermore if $x\in A_j$ we have

$$C_{L''}(\sigma^{\iota_{n+1}(x)}x)=F_{n+1}\Longleftrightarrow C_{L''}(\sigma^{\iota_{A_j}(x)}x)=F_{n+1}.$$

Thus, using the generalized Bernoulli property (Lemma 7.6), we have

$$\begin{aligned}\mu_\phi(\mathcal{Q}_{n+1}) &= \sum_j \mu_\phi(x\in A_j, C_{L''}(\sigma^{\iota_{A_j}(x)}x)=F_{n+1})\\ &\le \frac{\gamma^3}{1-4\varepsilon}\sum_j \mu_\phi(A_j)\mu_\phi(F_{n+1})\\ &= \frac{\gamma^3}{1-4\varepsilon}\mu_\phi(\mathcal{Q}_n)\mu_\phi(F_{n+1}).\end{aligned}$$

□

Lemma 7.8.

$$\mu_\phi(E_t(C_1,\cdots,C_K)) \le \left(2\gamma^6 K 2^{(-h+\varepsilon)(\ell-L')}\right)^{\frac{2^{(c-\varepsilon)L''}}{L''+d}}.$$

Proof. By the last lemma, we have

$$\mu_\phi(E_t(C_1,\cdots,C_K)) \le \left(\frac{\gamma^3}{1-4\varepsilon}\right)^{\widetilde{n}} \sum_{F_1,\cdots,F_{\widetilde{n}}} \prod_{i=1}^{\widetilde{n}} \mu_\phi(F_i)$$

where the sum is taken over all collections $F_1,\ldots,F_n$' consisting of different $(L'',2\varepsilon)$-cylinder contained in $D\star\widetilde{C}$. Recall that $\tilde{n}$ is defined in (7·3).

Since $\mu_\phi(D\star C_i)\le\gamma^3\mu_\phi(D)\mu_\phi(C_i)$ and $\mu_\phi(C_i)\le 2^{(-h+\varepsilon)(\ell-L')}$, we have

$$\sum_{F\in\mathcal{C}_{L'',2\varepsilon},F\subset D\star\widetilde{C}} \mu_\phi(F) \le \mu_\phi(D\star\widetilde{C}) \le K\gamma^3 2^{(-h+\varepsilon)(\ell-L')}.$$

So,

$$\mu_\phi(E_t(C_1,\cdots,C_K)) \le \left(\frac{\gamma^6}{1-4\varepsilon}K2^{(-h+\varepsilon)(\ell-L')}\right)^{\widetilde{n}}.$$

□

7.5. **Number of branches of a tree: Proof of Theorem 7.4.**

By Lemmas 7.5 and 7.8, we have

$$\begin{aligned}&\mu_\phi\Big(x\in\mathcal{E}(L',\varepsilon,\eta): T(x,D,\ell,\varepsilon)=K\Big)\\ &\le (L''+d)\begin{pmatrix}2^{(h+\varepsilon)(\ell-L')}\\ K\end{pmatrix}\left(2\gamma^6K2^{-(h-\varepsilon)(\ell-L')}\right)^{\frac{2^{(c-\varepsilon)L''}}{L''+d}}. \qquad (7\cdot5)\end{aligned}$$

For $K\le 2^{(c-2\varepsilon)(\ell-L')}$ and for $\ell\le L''$, we have on one hand

$$\begin{pmatrix}2^{(h+\varepsilon)(\ell-L')}\\ K\end{pmatrix} \le 2^{(h+\varepsilon)(\ell-L')K} \le 2^{(h+\varepsilon)L''2^{(c-2\varepsilon)L''}}; \qquad (7\cdot6)$$

and on the other hand

$$K2^{-(h-\varepsilon)(\ell-L')} \le 2^{(c-h-\varepsilon)(\ell-L')},$$

which implies that there exists an integer $n_0(\varepsilon)$ such that if $\ell - L' \geq n_0(\varepsilon)$ we have

$$2\gamma^6 K 2^{-(h-\varepsilon)(\ell-L')} \leq \frac{1}{2}, \quad \text{i.e. } 2\gamma^6 2^{-(h-c-\varepsilon)(\ell-L')} \leq \frac{1}{2}. \tag{7·7}$$

So, from (7·5), (7·6) and (7·7) we get

$$\begin{aligned} &\mu_\phi\Big(x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) = K\Big) \\ &\leq \ (L'' + d) \cdot 2^{(h+\varepsilon)L'' 2^{(c-2\varepsilon)L''} - \frac{2^{(c-\varepsilon)L''}}{L''+d}}. \end{aligned} \tag{7·8}$$

Choose L'' sufficiently large so that

$$(h+\varepsilon)L'' 2^{(c-2\varepsilon)L''} \leq \frac{1}{2} \cdot \frac{2^{(c-\varepsilon)L''}}{L''+d}. \tag{7·9}$$

¿From (7·8) and (7·9), we get

$$\mu_\phi(x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) = K) \leq (L'' + d) \cdot 2^{-\frac{2^{(c-\varepsilon)L''}}{2(L''+d)}}.$$

Summing over all $K \leq 2^{(c-2\varepsilon)(\ell-L')}$, we obtain

$$\begin{aligned} &\mu_\phi\left\{x \in \mathcal{E}(L', \varepsilon, \eta) : T(x, D, \ell, \varepsilon) \leq 2^{(c-2\varepsilon)(\ell-L')}\right\} \\ &\leq \ (L'' + d) \cdot 2^{(c-2\varepsilon)(\ell-L')} \cdot 2^{-\frac{2^{(c-\varepsilon)L''}}{2(L''+d)}} \leq 2^{-2^{(c-2\varepsilon)L''}} \end{aligned}$$

for large L'', because $2^{-\frac{2^{(c-\varepsilon)L''}}{2(L''+d)}}$ tends to zero superexponentially fast.

7.6. **The Cantor set and lower bound of** $\dim_H\{y : \alpha(x, y) \leq c\}$**.**

The next theorem is an improvement of the mass transference principle [BV] to the multi-fractal measure μ_ϕ.

Theorem 7.9. (Multi-fractal mass transference principle) *For* $0 < c < h_{\mu_\phi}$, *and for* μ_ϕ*-a.e.* x *we have*

$$h_{\text{top}}\{y : \alpha(x, y) \leq c\} \geq c.$$

Proof. Let $\varepsilon > 0$ be an arbitrary small number. We can find an increasing sequence of integers $(L_k)_{k\geq 0}$ such that

$$L_0 = 0, \quad 2^{-2^{(c-2\varepsilon)L_k}} \leq \frac{\varepsilon}{2^{k+2}}. \tag{7·10}$$

and that for each $k \geq 1$, the couple $(L', L'') = (L_{k-1}, L_k)$ satisfies the condition of Theorem 7.4. Apply Theorem 7.4 to $L' = L_{k-1}, L'' = L_k$ and $\eta = \frac{\varepsilon}{2^{k+1}}$ to get $\mathcal{E}_k(\varepsilon) := \mathcal{E}(L', \varepsilon, \eta)$. It has the properties that

$$\mu_\phi(\mathcal{E}_k(\varepsilon)) > 1 - \frac{\varepsilon}{2^{k+1}}; \tag{7·11}$$

and that there is a subset $\mathcal{E}_k^*(\varepsilon)$ of $\mathcal{E}_k(\varepsilon)$ with

$$\mu_\phi(\mathcal{E}_k(\varepsilon) \setminus \mathcal{E}_k^*(\varepsilon)) < \frac{\varepsilon}{2^{k+1}} \tag{7·12}$$

such that for any $x \in \mathcal{E}_k^*(\varepsilon)$, any L_{k-1}-cylinder D and any $L_{k-1} + n_0(\varepsilon) \leq \ell \leq L_k$ we have

$$T(x, D, \ell, \varepsilon) \geq 2^{(c-2\varepsilon)(\ell - L_{k-1})}.$$

Define

$$\mathcal{E}^*(\varepsilon) = \bigcap_{k=1}^{\infty} \mathcal{E}^*(L_k, \varepsilon).$$

Equations (7·11) and (7·12) imply that $\mu_\phi(\mathcal{E}_k^*(\varepsilon))) \geq 1 - \frac{\varepsilon}{2^k}$ and

$$\mu_\phi(\mathcal{E}^*(\varepsilon)) \geq 1 - \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = 1 - \varepsilon. \tag{7·13}$$

For $x \in \mathcal{E}^*(\varepsilon)$, we have

$$T(x, D, \ell, \varepsilon) \geq 2^{(c-2\varepsilon)(\ell - L_{k-1})} \tag{7·14}$$

for all L_{k-1}-cylinders D and all $L_{k-1} + n_0(\varepsilon) \leq \ell \leq L_k$.

Now, for each $x \in \mathcal{E}^*(\varepsilon)$, we construct a Cantor set as follows.

First step: for $n_0(\varepsilon) \leq \ell \leq L_1$, consider the family $\mathfrak{C}_\ell(x)$ of (ℓ, ε)-cylinders which contain at least one $(L_1, 2\varepsilon)$-cylinder seen in x between the times 1 and 2^{cL_1}. This yields a tree $\mathfrak{T}_{L_1}(x)$ of height L_1. The nodes of the tree $\mathfrak{T}_{L_1}(x)$ are the (ℓ, ε)-cylinders, with $n_0(\varepsilon) \leq \ell \leq L_1$, belonging to $\mathfrak{C}_\ell(x)$. The edges are defined by the containment relation. We will extend this tree inductively.

Second step: Let $k \geq 2$. Suppose that we have constructed a tree $\mathfrak{T}_{L_{k-1}}(x)$ of height L_{k-1}. We will construct a tree of height L_k. Let

$$L' = L_{k-1}, \quad L'' = L_k.$$

Fix a L'-cylinder D seen in x before time $2^{cL'}$, which is the label of a node of the tree $\mathfrak{T}_{L_{k-1}}(x)$ at level L_{k-1}. For $L' + n_0(\varepsilon) \leq \ell \leq L''$, take all (ℓ, ε)-cylinders that contain at least one $(L'', 2\varepsilon)$-cylinder of the form $D \star G$ seen in x between the times $2^{L'} + 1$ and $2^{cL''}$. As before we denote this family by $\mathfrak{C}_\ell(x)$ (both D and G varying). The tree $\mathfrak{T}_{L_k}(x)$ is obtained from $\mathfrak{T}_{L_{k-1}}(x)$ by adding branches to each D. That is to say, by splitting D into (ℓ, ε)-cylinders belonging to $\mathfrak{C}_\ell(x)$.

We define

$$C_\infty(x) = \bigcap_{k=1}^{\infty} \bigcap_{\ell = L_{k-1} + n_0(\varepsilon)}^{L_k} \bigcup_{C \in \mathfrak{C}_\ell(x)} C.$$

We have $C_\infty(x) \subset \{y : \alpha(x, y) \leq c\}$, since for any $y \in C_\infty(x)$ and for all $k \geq 1$

$$y \in \bigcup_{C \in \mathfrak{C}_{L_k}(x)} C,$$

i.e. $y \in C_{L_k}(\sigma^j x)$ for some

$$2^{L_{k-1}} + 1 \leq j \leq 2^{cL_k}.$$

We claim that $\dim_H C_\infty(x) \geq c - 2\varepsilon$. In fact, for $L_{k-1} + n_0(\varepsilon) \leq \ell \leq L_k$, we have

$$\begin{aligned} \log_2 \sharp\mathfrak{C}_\ell(x) &\geq (c-2\varepsilon)(\ell - L_{k-1}) + \sum_{j=1}^{k-1}(c-2\varepsilon)(L_j - L_{j-1}) \\ &\geq (c-2\varepsilon)\ell \end{aligned}$$

Define a probability measure ν on $C_\infty(x)$ by

$$\nu(C) = \frac{1}{\sharp\mathfrak{C}_\ell(x)} \qquad (\forall C \in \mathfrak{C}_\ell(x) \text{ and } l \in \mathbb{N}).$$

It is clear that (note $n(\varepsilon)$ does not depend on L_k)

$$\nu(C) \leq 2^{-(c-2\varepsilon)\ell}.$$

Thus we have proved that with probability bigger than $1-\varepsilon$ we have

$$\dim_H\{y : \alpha(x,y) \leq c\} \geq c - 2\varepsilon.$$

□

Remark: The proofs in this section can be used to obtain a more precise estimate on the growth rate of the tree, however this estimate is not necessary for our purpose. Namely one can show that $L_{k-1} \ll l \leq \frac{c}{h} L_k$ then

$$T(x, D, l, \varepsilon) \geq 2^{(h-3\varepsilon)l}.$$

This implies that the upper box counting dimension of the corresponding Cantor set is $h - 3\varepsilon$ while the lower box dimension equals the Hausdorff dimension equals $c - 2\varepsilon$.

8. Results for the full shift

Our strategy is to prove all the theorems in the symbolic framework and then transfer them to the circle. Let us get together the already obtained results in the symbolic framework.

Lemma 8.1. *For $0 < \kappa < \infty$ we have μ_ϕ-a.e.*

$$\sup\{E(t) : \frac{1}{t} \leq \kappa\} \geq \dim_H \mathcal{F}^\kappa(x) \geq \sup\{E(t) : \frac{1}{t} < \kappa\}.$$

For $\kappa \leq 1/h_{\mu_\phi}$ (i.e. $1/\kappa \geq h_{\mu_\phi}$) we have μ_ϕ-a.e.

$$\sup\{E(t) : \frac{1}{t} \geq \kappa\} \geq \dim_H \mathcal{I}^\kappa(x) \geq \sup\{E(t) : \frac{1}{t} > \kappa\},$$

and for $\kappa > 1/h_{\mu_\phi}$ (i.e. $1/\kappa < h_{\mu_\phi}$) we have μ_ϕ-a.e.

$$\dim_H \mathcal{I}^\kappa(x) = 1/\kappa.$$

Proof. The first line is a consequence of Lemma 3.2, Theorem 5.3 and Theorem 2.3.

The second line is a consequence of Lemma 3.2, Theorem 6.2 and Theorem 2.3.

The third line is a direct consequence of Lemma 3.2, Theorems 6.2 and 7.9. □

Corollary 8.2. *Let $1/\kappa \in (e^-, e^+)$. Then for μ_ϕ a.e. x*

$$\dim_H \mathcal{F}^\kappa(x) = \max_{\nu-ergodic} \{h_\nu \, : \alpha(x,y) \leq \frac{1}{\kappa} \; \nu - a.e.y\}.$$

For $1/\kappa \in (h_{\mu_\phi}, e^+)$ and μ_ϕ a.e. x

$$\dim_H \mathcal{I}^\kappa(x) = \max_{\nu-ergodic} \{h_\nu \, : \alpha(x,y) \geq \frac{1}{\kappa} \; \nu - a.e.y\}.$$

The properties of the entropy spectrum which were stated in the background section immediately imply the following corollary.

Corollary 8.3. *For $1/\kappa \in (e^-, e^+)$ and μ_ϕ a.e. x we have*

$$\sup_{-P'(q)\geq \frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \geq dim_H \mathcal{F}^\kappa(x) \geq \sup_{-P'(q)> \frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \, .$$

For $1/\kappa \in (h_{\mu_\phi}, e^+)$ and μ_ϕ a.e. x we have

$$\sup_{-P'(q)\leq \frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \geq dim_H \mathcal{I}^\kappa(x) \geq \sup_{-P'(q)< \frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \, .$$

If we consider a typical potential, then the function $E(t)$ is continuous on the nontrivial interval (e^-, e^+), equals 0 on the endpoints (see [S2]). Hence the right hand side and left hand side inequalities in Lemma 8.1 and Corollary 8.3 are equal. Since the maximum value of $E(t)$ is attained at the value $t = -\int_{\Sigma_2^+} \phi \, d\mu_{\max}$ and equals $h_{\text{top}}(\Sigma_2^+) = 1$ we have the following corollary.

Corollary 8.4. *For a typical potential and μ_ϕ a.e. x we have*

$$\dim_H \mathcal{F}^\kappa(x) = h_{\text{top}}(\Sigma_2^+) = 1 \quad \textit{for } \kappa \geq \frac{1}{-\int \phi \, d\mu_{\max}.},$$

$$\dim_H \mathcal{I}^\kappa(x) = h_{\text{top}}(\Sigma_2^+) = 1 \quad \textit{for } \kappa \leq \frac{1}{-\int \phi \, d\mu_{\max}.}.$$

Let q_κ be the number such that $P'(q_\kappa \phi) = -\frac{1}{\kappa}$. Then

$$\dim_H \mathcal{F}^\kappa(x) = E\left(\tfrac{1}{\kappa}\right) = P\left(q_\kappa \phi\right) + \tfrac{1}{\kappa} q_\kappa \qquad \textit{for } \kappa < \frac{1}{-\int \phi \, d\mu_{\max}.},$$

$$\dim_H \mathcal{I}^\kappa(x) = E\left(\tfrac{1}{\kappa}\right) = P\left(q_\kappa \phi\right) + \tfrac{1}{\kappa} t q_\kappa \qquad \textit{for } \frac{1}{h_{\mu_\phi}} \geq \kappa > \frac{1}{-\int \phi \, d\mu_{\max}.}.$$

Finally we come to the answer of the symbolic version of question (**Q2**).

Lemma 8.5. *For μ_ϕ a.e. x we have*

$$\mathcal{F}^\kappa(x) = \emptyset \text{ for } \kappa < \frac{1}{e^+} = \frac{1}{\max_{\mu \text{ ergodic}} \int (-\phi)\, d\mu} = \kappa^F_{\phi,\Sigma_2^+}.$$

Proof. ¿From multi-fractal analysis, it is well known that

$$e^+ = \max_\nu \int (-\phi)\, d\nu = \max_{y\in\Sigma_2^+} h_{\mu_\phi}(y).$$

Therefore

$$\mathcal{F}^\kappa(x) \subset \{y : \alpha(x,y) \geq 1/\kappa \text{ and } \underline{h}_{\mu_\phi}(y) \leq e^+ < 1/\kappa\} = \emptyset$$

by Lemma 3.2, Lemma 3.3 and Theorem 5.2. □

Using the techniques developed in the previous sections we can conclude a strong theorem on the structure of typical sequences. The subword structure of a typical sequence up to time L is completely determined by the entropy spectrum of the measure.

Corollary 8.6. *Consider $n \ll L$ sufficiently large, a typical point x and the set of cylinders C_n of length n satisfying $\mu(C_n) \sim 2^{-\beta n}$ which are subwords of the cylinder $C_L(x)$, i.e. the orbit of x hits the cylinder C_n before time L. Then*

$$\sharp\big(C_n\big) \sim \max(0, 2^{\min(E(\beta), E(\beta)-\beta+(\log L)/n)n}).$$

Here $a_n \sim b_n$ means that the ratio a/b is subexponential in n.

9. Extensions to subshifts of finite type

The previous results can be extended in a canonical way to subshifts of finite type: Σ_2^+ is replaced by a subshift space Σ_A and μ_ϕ and μ_ψ by two Gibbs measures of the subsystem $\sigma : \Sigma_A \to \Sigma_A$. Extensions to symbolic spaces of several symbols are also obvious.

Here we consider another kind of extension. Given a compact subset K in Σ_2^+. What can we say about $K \cap \mathcal{I}^\kappa(x)$ and $K \cap \mathcal{F}^\kappa(x)$? We assume that the reference measures μ_ϕ and μ_ψ are Gibbs measure of the *full shift* $\sigma : \Sigma_2^+ \to \Sigma_2^+$. We can answer this question when $K = \Sigma_A$ is a subshift of finite type. The proofs are still slight modifications of those for the full shift, thus we only sketch them briefly here. We will emphasize the differences.

Let $\Sigma_A \subset \{0,1\}^{\mathbb{N}}$ be a subshift of finite type. We are interested in the following two sets:

$$\mathcal{F}^\kappa_A(x) := \mathcal{F}^\kappa(x) \cap \Sigma_A \quad \text{and} \quad \mathcal{I}^\kappa_A(x) := \mathcal{I}^\kappa(x) \cap \Sigma_A.$$

Recall that $\mu_\phi(\Sigma_A) = 0$ if $\Sigma_A \neq \{0,1\}^{\mathbb{N}}$ because Σ_A is a closed invariant set ($\sigma\Sigma_A \subset \Sigma_A$) and μ_ϕ is of full support and ergodic.

The analysis of these sets is related to the determination of the following restricted entropy spectrum: Recall that $-\int \phi d\mu_\psi$ is nothing but the conditional entropy of μ_ϕ relative to μ_ψ. Let

$$E_A(\alpha) := \dim_H\{y \in \Sigma_A : \ h_{\mu_\phi}(y) = \alpha\}.$$

We list some facts concerning $E_A(\alpha)$ which are needed to modify the proofs.

(1) Clearly the restriction $\phi|_{\Sigma_A}$ is a Hölder function.

(2) Let $P_A(\psi)$ be the pressure of a potential $\psi:\ \Sigma_A \to \mathbb{R}$ related to the subsystem $\sigma : \Sigma_A \to \Sigma_A$. Then

$$P_A(\phi|_{\Sigma_A}) \le 0.$$

This a consequence of the variational principle:

$$\begin{aligned} P_A(\phi|_{\Sigma_A}) &= \max_{\mu \text{ inv on } \Sigma_A} (h_\mu + \int_{\Sigma_A} \phi\, d\mu) \\ &\le \max_{\mu \text{ inv on } \Sigma} (h_\mu + \int_{\Sigma} \phi\, d\mu) = P(\phi) = 0. \end{aligned}$$

(3) $\phi_A(x) := \phi|_{\Sigma_A} - P_A(\phi|_{\Sigma_A})$ is normalized in the sense that $P_A(\phi_A) = 0$.

(4) Let μ_{ϕ_A} be the Gibbs measure on Σ_A associated to ϕ_A. It is related to the original Gibbs measure μ_ϕ by

$$\mu_{\phi_A}(C_n(x)) \approx e^{S_n\phi_A(x)} = e^{S_n\phi(x) - nP_A(\phi|_{\Sigma_A})} \approx e^{-nP_A(\phi|_{\Sigma_A}(\phi)}\mu_\phi(C_n(x))$$

for $x \in \Sigma_A$. Here $\approx$ means that the ratio is bounded between two constants independent of n.

(5) Consequently, if one of the local entropies $h_{\mu_{\phi_A}}(x)$ or $h_{\mu_\phi}(x)$ is well defined then both are well defined and we have

$$h_{\mu_{\phi_A}}(x) = h_{\mu_\phi}(x) + P_A(\phi|_{\Sigma_A}),\ x \in \Sigma_A.$$

(6) The following spectrum is well known from multi-fractal analysis

$$\widetilde{E}_A(\beta) := \dim_H\{y \in \Sigma_A :\ h_{\mu_{\phi_A}}(y) = \beta\}.$$

The condition $h_{\mu_{\phi_A}}(y) = \beta$ is equivalent to $\lim_{n\to\infty} n^{-1}(S_n(-\phi_A)(y)) = \beta$.

Now, by (5) and (6), we get that the spectrum $E_A(\cdot)$ is expressed in term of the known spectrum $\widetilde{E}_A(\cdot)$:

$$\begin{aligned} E_A(\alpha) &= \dim_H\{y \in \Sigma_A :\ h_{\mu_\phi}(y) = \alpha\} \\ &= \dim_H\{y \in \Sigma_A :\ h_{\mu_{\phi_A}}(y) = \alpha + P_A(\phi)\} \\ &= \widetilde{E}_A(\alpha + P_A(\phi)). \end{aligned}$$

Furthermore, the set $\{y \in \Sigma_A :\ h_{\mu_\phi}(y) = \alpha\}$ is empty, so $E_A(\alpha) = 0$ unless

$$\tilde{e}_A^- \le \alpha + P_A(\phi) \le \tilde{e}_A^+ \tag{9·1}$$

where $\tilde{e}_A^+, \tilde{e}_A^-$ are respectively the maximal and minimal entropy of $h_{\mu_{\phi_A}}$. That is

$$\begin{aligned} \tilde{e}_A^+ &= \sup_{\text{supp}\mu\subset\Sigma_A} \int (-\phi_A) d\mu = \sup_{\text{supp}\mu\subset\Sigma_A} \int (-\phi) d\mu + P_A(\phi|_{\sigma_A}) \\ \tilde{e}_A^- &= \inf_{\text{supp}\mu\subset\Sigma_A} \int (-\phi_A) d\mu = \inf_{\text{supp}\mu\subset\Sigma_A} \int (-\phi) d\mu + P_A(\phi|_{\Sigma_A}). \end{aligned}$$

Define

$$e_A^- := \inf_{\mathrm{supp}\mu\subset\Sigma_A} \int(-\phi)d\mu, \qquad e_A^+ := \sup_{\mathrm{supp}\mu\subset\Sigma_A} \int(-\phi)d\mu.$$

So, (9·1) is equivalent to

$$e_A^- \le \alpha \le e_A^+.$$

Thus $E_A(\alpha) \le E(\alpha)$ because

$$\begin{aligned}\widetilde{E}_A(\alpha + P_A(\phi)) &= \sup_{\substack{\mathrm{supp}\mu\subset\Sigma_A \\ \int(-\phi_A)d\mu=\alpha+P_A(\phi)}} h_\mu = \sup_{\substack{\mathrm{supp}\mu\subset\Sigma_A \\ \int(-\phi)d\mu=\alpha}} h_\mu \\ &\le \sup_{\int(-\phi)d\nu=\alpha} h_\nu = E(\alpha).\end{aligned}$$

Let $e_A^{\max}$ be the unique value for which $E_A(\alpha)$ attains its maximum (supported by the Parry measure). In particular $E_A(e_A^{\max}) = \dim_H(\Sigma_A)$. Then we can conclude

Theorem 9.1.

$$\dim_H \mathcal{F}_A^\kappa(x) = \begin{cases} \dim_H(\Sigma_A) & \text{if } \frac{1}{\kappa} \le e_A^{\max} \\ E_A(\frac{1}{\kappa}) & \text{if } \frac{1}{\kappa} > e_A^{\max} \end{cases}$$

$$\textit{and} \quad \mathcal{F}_A^\kappa(x) = \emptyset \quad \textit{if} \quad \frac{1}{\kappa} > e_A^+.$$

Theorem 9.2.

$$\dim_H \mathcal{I}_A^\kappa(x) = \begin{cases} \frac{1}{\kappa} + P_A(\phi|_{\Sigma_A}) & \text{if } -P_A(\phi|_{\Sigma_A}) \le \frac{1}{\kappa} \le h_{\mu_{\phi_A}} - P_A(\phi|_{\Sigma_A}) \\ E_A(\frac{1}{\kappa}) & \text{if } h_{\mu_{\phi_A}} - P_A(\phi|_{\Sigma_A}) \le \frac{1}{\kappa} \le e_A^{\max} \\ \dim_H(\Sigma_A) & \text{if } \frac{1}{\kappa} \ge e_A^{\max} \end{cases}$$

$$\textit{and} \quad \mathcal{I}_A^\kappa(x) = \emptyset \quad \textit{if} \quad \frac{1}{\kappa} < -P_A(\phi|_{\Sigma_A}).$$

Remark: Unlike the full shift case $\mathcal{I}_A^\kappa(x)$ is empty for large κ.

Proof. The only statement in the two theorems which differs from the full shift is that $\mathcal{I}_A^\kappa(x)$ may be empty. Fix $\varepsilon > 0$. Let $\frac{1}{\kappa} < -P_A(\phi|_{\Sigma_A}) - \varepsilon$. Then by (9·1) we have

$$h_{\mu_\phi}(y) \ge \tilde{e}_A^- - P_A(\phi|_{\Sigma_A})$$

for all $y \in \Sigma_A$. Then, by Lemma 3.2

$$\begin{aligned}\mathcal{I}_A^\kappa(x) &\subset \left\{y \in \Sigma_A : \ \alpha(x,y) < \frac{1}{\kappa} + \epsilon\right\} \\ &= \{y \in \Sigma_A : \ \alpha(x,y) < \frac{1}{\kappa} + \epsilon, \ h_{\mu_\phi}(y) \ge \tilde{e}_A^- - P_A(\phi|_{\Sigma_A})\} \\ &\subset \{y \in \Sigma_A : \ \alpha(x,y) < -P_A(\phi|_{\Sigma_A}), \ h_{\mu_\phi}(y) \ge \tilde{e}_A^- - P_A(\phi|_{\Sigma_A})\} \\ &\subset \bigcup_{j=0}^{\infty} \{y \in \Sigma_A : \ h_{\mu_\phi}(y) \in \big[j\varepsilon, (j+1)\varepsilon\big) + \tilde{e}_A^- - P_A(\phi|_{\Sigma_A})\}.\end{aligned}$$

Thus, Lemma 6.1 with $K = 2^{\frac{1}{\kappa}n}$, $L = \max(2^{E_A(\tilde{e}_A^- + j\varepsilon)}, 2^{(\tilde{e}_A^- + (j+1)\varepsilon)})$ and $N = 1$ implies that each of the (countably many) sets on the right hand side is empty for μ_ϕ-a.e. x. □

10. Transferring to the circle

In this section we show that the results of the section 8 hold for the doubling map of the circle, i.e. replacing $\mathcal{F}^\kappa(x), \mathcal{I}^\kappa(x)$ by $F^\kappa(s), I^\kappa(s)$. Recall that the projection $\pi : \Sigma \to \mathbb{S}$ was defined in the section 2. For $y \in \Sigma_2$; $y \neq 1^\infty, 0^\infty$ let

$$C_n^*(y) := C_n^-(y) \cup C_n(y) \cup C_n^+(y)$$

where $C_n^-(y)$ denotes the cylinder of length n preceding $C_n(y)$ in the lexicographical order and $C_n^+(y)$ denotes the immediate successor.

Theorem 10.1. *For μ_ϕ a.e. x we have*

$$\dim_H(F^\kappa \pi(x)) = \dim_H \pi(\mathcal{F}^\kappa(x))$$
$$\dim_H(I^\kappa \pi(x)) = \dim_H \pi(\mathcal{I}^\kappa(x)).$$

Proof. For $x \in \Sigma_2$ with $x \neq 1^\infty, 0^\infty$, the projection of each of the cylinders $C_n^-(x)$, $C_n(x)$,$C_n^+(x)$ to $\mathbb{S}^1$ is an interval around $\pi(x)$. Moreover we have

$$\pi(C_{\lfloor \kappa \log n \rfloor + 1}(x)) \subset \left(\pi(x) - \frac{1}{n^\kappa}, \pi(x) + \frac{1}{n^\kappa}\right) \subset \pi(C^*_{\lfloor \kappa \log n \rfloor}(x)). \tag{10·1}$$

Applying the left inclusion, it follows that

$$F^\kappa(\pi(x)) \subset \pi(\mathcal{F}^\kappa(x)).$$

Hence

$$\dim_H(F^\kappa \pi(x)) \leq \dim_H \pi(\mathcal{F}^\kappa(x)),$$

and similarly

$$\dim_H(I^\kappa \pi(x)) \geq \dim_H \pi(\mathcal{I}^\kappa(x)).$$

We turn to the reverse inequalities. For this we define

$$\tau_n^*(x,y) := \inf\{l \geq 1 \,:\, \sigma^l x \in C_n^*(y)\},$$
$$\tau_n^-(x,y) := \inf\{l \geq 1 \,:\, \sigma^l x \in C_n^-(y)\}$$

and

$$\tau_n^+(x,y) := \inf\{l \geq 1 \,:\, \sigma^l x \in C_n^+(y)\}$$

then

$$\tau_n^*(x,y) = \min\{\tau_n^-(x,y), \tau_n(x,y), \tau_n^+(x,y)\}$$

and

$$\alpha^*(x,y) = \min\{\alpha^-(x,y), \alpha(x,y), \alpha^+(x,y)\}$$

where $\alpha^*, \alpha^-, \alpha^+$ are defined in the corresponding way. Therefore in analogy to Lemma 3.2

$$\left\{\pi(y) \,:\, \alpha^*(x,y) > \frac{1}{\kappa}\right\} \subset F^\kappa(\pi(x)) \tag{10·2}$$

and

$$I^{\kappa}(\pi(x)) \subset \left\{\pi(y) \,:\, \alpha^*(x,y) \le \frac{1}{\kappa}\right\}. \tag{10·3}$$

Next we need the following lemma to prove the reverse inequalities.

Lemma 10.2. *For any $x \in \Sigma_2^+$ and ν an ergodic Borel probability measure different from δ_{0^∞} and δ_{1^∞} we have*

$$\alpha^*(x,y) = \alpha(x,y) \qquad \nu - a.e.$$

Proof. We will prove that $\alpha^+(x,y) \ge \alpha(x,y)$ almost everywhere. The proof for $\alpha^-(x,y) \ge \alpha(x,y)$ a.e. is similar. Since

$$\alpha^*(x,y) = \min\{\alpha^-(x,y), \alpha(x,y), \alpha^+(x,y)\},$$

this will imply the lemma.

Fix $\epsilon > 0$. Let $\mathbf{1}_n$ be the characteristic function of the cylinder set consisting of n 1's. Since ν is not concentrated on 1^∞ we can find an n_ϵ sufficiently large that

$$\int \mathbf{1}_n(x)\, d\nu(x) < \epsilon \qquad (\forall n > n_\epsilon).$$

Now let y be a generic point for ν. Then there is an $n_0 = n_0(y) > n_\epsilon$ such that

$$\frac{1}{m} S_m \mathbf{1}_n(y) < \epsilon \qquad (\forall m > n_0).$$

Let us consider the structure of $C_m^+(y)$.

$$\begin{aligned} C_m^+(y) = &\quad [y_1 \cdots y_{m-1} 1] && \text{if } \; y = y_1 \cdots y_{m-1} 0 \cdots \\ C_m^+(y) = &\quad [y_1 \cdots y_{k-1} 100 \cdots 0] && \text{if } \; y = y_1 \cdots y_{k-1} 011 \cdots 1 y_{m+1} \cdots . \end{aligned}$$

It follows that

$$C_m^+(y) \subset C_{k-1}(y)$$

where $k = k(y,m)$ is characterized by $y_k = 0$ and $y_j = 1$ $(\forall k < j \le m\})$. Thus

$$\tau_m^+(x,y) \ge \tau_{k-1}(x,y). \tag{10·4}$$

For a given x, the more 1's at the end of $C_m(y)$ is the only way to enlarge the difference of x's hitting times of $C_m^+(y)$ and $C_m(y)$. Let $n > n_0$, $m > n-l-1$ and assume that we have a block of $n+l$ ones at the end $(l > n)$. Then (10·4) becomes

$$\tau_m^+(x,y) \ge \tau_{m-n-l-1}(x,y).$$

The worst situation is when this block occurs very early. We are going to estimate this first occurrence. First we observe that

$$\epsilon > \frac{1}{m} S_m \mathbf{1}_n(y) \ge \frac{l}{m}.$$

This implies that the first occurrence of the block in question is not earlier than

$$m - n - l - 1 \ge m - 2l > m(1 - 2\epsilon).$$

Therefore

$$\alpha^+(x,y) = \varliminf_{m\to\infty} \frac{\log \tau_m^+(x,y)}{m} \geq \varliminf_{m\to\infty} \frac{\log \tau_{m-n-l-1}(x,y)}{m} \geq (1-2\epsilon)\alpha(x,y).$$

Letting $\epsilon \to 0$ we obtain the result. □

We continue with the proof of the theorem. For any Borel set A we have

$$\dim_H \pi A = h_{\text{top}}(A)$$

since $\operatorname{diam} \pi(C) = 2^{-|C|}$ for any cylinder set C. Thus applying Theorem 5.3 yields

$$\dim_H \pi(\mathcal{F}^\kappa(x)) = h_{\text{top}}(\mathcal{F}^\kappa(x)) = h_{\mu_{q(\kappa)\phi}}.$$

Let $t(\kappa) = q(\kappa)$ if $\frac{1}{\kappa} \geq e_{\max}$ and $t(\kappa) = 0$ otherwise. Suppose $\varepsilon > 0$. By continuity of the multi-fractal spectrum we have

$$\lim_{\varepsilon\to 0} h_{t(\kappa-\varepsilon)\phi} = h_{t(\kappa)\phi}$$

and

$$h_{\mu_{t(\kappa-\varepsilon)\phi}}(y) = \frac{1}{\kappa-\varepsilon} > \frac{1}{\kappa} \quad \mu_{t(\kappa-\varepsilon)\phi}\text{−a.e. } y.$$

By Corollary 4.3 for $\mu_\phi \times \mu_{q(\kappa-\varepsilon)\phi}$ for a.e. (x,y) we have

$$\alpha^*(x,y) = \alpha(x,y) = h_{\mu_\phi}(y) > \frac{1}{\kappa}.$$

Thus $\pi(y) \in F^\kappa(\pi(x))$ for $\mu_{qt\kappa-\varepsilon)\phi}$ a.e. y and $\dim_H F_\kappa \geq h_{\mu_{t(\kappa-\varepsilon)\phi}}$. Taking the limit $\varepsilon \to 0$ shows

$$\dim_H F^\kappa(\pi(x)) \geq h_{\mu_{t(\kappa)\phi}} = \dim_H \pi(\mathcal{F}^\kappa(x)).$$

This completes the proof for the set F^κ.

It remains to show that $\dim_H I^\kappa(\pi(x)) \leq \dim_H \pi(\mathcal{I}^\kappa(x))$. If $\frac{1}{\kappa} \geq e_{\max}$ then this is trivial since $\dim_H \pi(\mathcal{I}^\kappa(x)) = 1$. Observe that for any κ we have $\dim_H I^\kappa \pi(x) \leq \frac{1}{\kappa}$. To see this consider the natural covering $(T^n\pi(x) - \frac{1}{n^\kappa}, T^n\pi(x) + \frac{1}{n^\kappa})$ of $I^\kappa(\pi(x))$. The s-covering sum is $\sum \frac{1}{n^{\kappa s}} < \infty$ if $s > \frac{1}{\kappa}$. Therefore, if $0 < \frac{1}{\kappa} \leq h_{\mu_\phi}$, we have $\dim I^\kappa(\pi(x)) \leq \frac{1}{\kappa} = \dim_H(\mathcal{I}^\kappa(x))$. Finally if $h_{\mu_\phi} \leq \frac{1}{\kappa} < e_{\max}$ the for any Hölder function $\hat{\phi} \in H^\alpha(\mathbb{S}^1)$ let $\phi = \hat{\phi} \circ \pi$. We have $\phi \in H^\alpha(\Sigma_2)$ and $\phi(x_1,\ldots,x_n 01^\infty) = \phi(x_1,\ldots,x_n,10^\infty)$ thus by the Gibbs property we have

$$\lim_{n\to\infty} \frac{\log \mu_\phi(C_n^\pm(x))}{\log \mu_\phi(C_n(x))} = 1.$$

Hence $h^*_{\mu_\phi}(y) = h_{\mu_\phi}(y)$ for all $y \in \Sigma_2$.

Consider the set

$$\begin{aligned} I^\kappa(\pi(x))\backslash\pi(\mathcal{I}^\kappa(x)) &= \{y : \pi(y) \in I^\kappa(\pi(x)), y \in \mathcal{F}^\kappa(x)\} \\ &\subset \{y : \alpha^*(x,y) < \frac{1}{\kappa}, \alpha(x,y) \geq \frac{1}{\kappa}\}. \end{aligned}$$

By Theorem 5.2 for μ_ϕ-a.e. x we have that the last set is contained in

$$\{y : \alpha^*(x,y) < \frac{1}{\kappa}, h_{\mu_\phi}(y) \geq \frac{1}{\kappa}\}.$$

Thus Lemma 6.1 implies that for any $\varepsilon > 0$ there are at most $C(\varepsilon) \cdot 2E(\frac{1}{\kappa})n$ cylinders of length n needed to cover $\{y : \alpha^*(x,y) < \frac{1}{\kappa}, h_{\mu_\phi}(y) \geq \frac{1}{\kappa} + \varepsilon\}$. Hence

$$\dim_H(I^\kappa(\pi(x))\backslash\pi(\mathcal{I}^\kappa(x)) \leq E(\frac{1}{\kappa}) = \dim_H \pi(\mathcal{F}^\kappa(x)).$$

□

Corollary 10.3. *For μ_ϕ a.e. x we have $F^\kappa(\pi(x)) = \emptyset$ if $\frac{1}{\kappa} > e_+$.*

In Theorem 2.3 and Corollary 4.3 we can ignore the delta measure on fixed points since they have zero entropy and therefore do not give any contribution. This transfer procedure allows us to conclude the following Theorems and Corollaries from the analogous results of the section 8. These results contain more information than those stated in the introduction, thus we reformulate them. We set $\nu_\phi = \mu_\phi \circ \pi^{-1}$.

Theorem 10.4. (Theorem 1.1) $\kappa_{\phi,\psi} = \frac{1}{-\int_{\mathbb{S}^1} \phi \, d\nu_\psi} = \frac{1}{h_{\nu_\phi}(y)} = -\frac{1}{\frac{d}{dt}P(\phi+t\psi)|_{t=0}}$.

Lemma 10.5. *For ν_ϕ a.e. s we have*

$$\sup\{E(t) : \frac{1}{t} \leq \kappa\} \geq \dim_H F^\kappa(s) \geq \sup\{E(t) : \frac{1}{t} < \kappa\}.$$

For ν_ϕ a.e. s and $\kappa < 1/h_{\nu_\phi}$ we have

$$\sup\{E(t) : \frac{1}{t} \geq \kappa\} \geq \dim_H I^\kappa(s) \geq \sup\{E(t) : \frac{1}{t} > \kappa\}.$$

Corollary 10.6. *For ν_ϕ a.e. s*

$$\sup_{-P'(q)\geq\frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \geq dim_H F^\kappa(s) \geq \sup_{-P'(q)>\frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \,.$$

For ν_ϕ a.e. s and for $\kappa < 1/h_{\nu_\phi}$

$$\sup_{-P'(q)\leq\frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \geq dim_H I^\kappa(s) \geq \sup_{-P'(q)<\frac{1}{\kappa}} [P(q\phi) - P'(q\phi)q] \,.$$

Corollary 10.7. (Theorems 1.3 and 1.4) *For a typical potential ϕ and ν_ϕ a.e. s we have*

$$\dim_H F^\kappa(s) = \dim_H(\mathbb{S}^1) = h_{\text{top}}(\mathbb{S}^1) = 1 \quad \textit{for } 1/\kappa \leq -\int \phi \, dLeb.,$$

$$\dim_H I^\kappa(s) = \dim_H(\mathbb{S}^1) = h_{\text{top}}(\mathbb{S}^1) = 1 \quad \textit{for } 1/\kappa \geq -\int \phi \, dLeb..$$

Let q_κ be the number such that $P'(q_\kappa\phi) = -\frac{1}{\kappa}$. Then

$$\dim_H F^\kappa(s) = E\left(\tfrac{1}{\kappa}\right) = P\left(q_\kappa\phi\right) + \tfrac{1}{\kappa}q_\kappa \qquad \textit{for } 1/\kappa > -\int \phi\, dLeb.,$$

$$\dim_H I^\kappa(s) = E\left(\tfrac{1}{\kappa}\right) = P\left(q_\kappa\phi\right) + \tfrac{1}{\kappa}q_\kappa \qquad \textit{for } h_{\nu_\phi} \le 1/\kappa < -\int \phi\, dLeb.,$$

$$\dim_H I^\kappa(s) = \tfrac{1}{\kappa} \qquad \textit{for } 1/\kappa < h_{\nu_\phi}.$$

Remark: 1) If $\kappa > 1$ then $\sum l_n < \infty$ and we can not cover Lebesgue almost all points infinitely often no matter which orbit we consider. Thus it is likely that the dimension of $I^\kappa(s)$ is less than 1. In the degenerate case this is clear. To see this in the nondegenerate case note that since the graph of the entropy spectrum is below the diagonal we have $1 = h_{\text{top}} = E(e_{\max}) < e_{\max}$. Therefore the maximum dimension (i.e. 1) is attained for $\kappa < 1$.

2) For a non typical potential we have possibly discontinuities of the function $E(t)$ at $e^\pm$. At these points the upper and lower estimates of Corollary 10.6 do not coincide. This indicates that the question about infinite versus finite covering can not be completely answered in terms of the exponent κ. At this point the answer might depend on a constant c where $l_n = \frac{c}{n^\nu}$. This is in particular the case for the i.i.d. case mentioned in the introduction. The dynamical analog is Lebesgue measure whose entropy spectrum is degenerate. Therefore we can not get any information about the sequence $\frac{c}{n}$ which resembles the i.i.d. case.

Theorem 10.8. (Theorem 1.2) *For ν_ϕ a.e. s we have*

$$F^\kappa(s) = \emptyset \textit{ for } \kappa < \frac{1}{-\inf_{\mu \text{ ergodic}} \int \phi\, d\mu} = \kappa^F_{\phi,\mathbb{S}^1}.$$

These results are summarized in Figure 2.

Remark: The result of Corollary 8.6 can also be transferred to the circle. The interpretation of this result is as follows. The distribution of a typical orbit up to time L is completely determined by the entropy spectrum of the measure.

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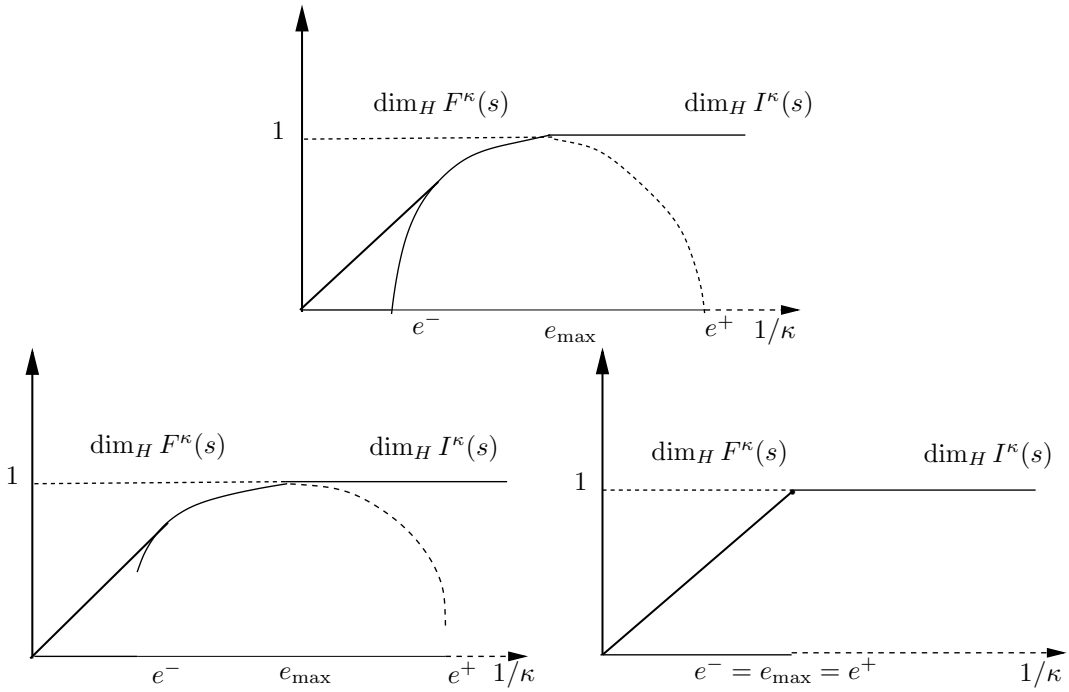

FIGURE 2. The dimension graphs in the typical, nontypical and degenerate cases. The graph of $\dim_H F^\kappa(s)$ is dotted and the graph of $\dim_H I^\kappa(s)$ is solid.

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A. H. Fan: LAMFA, UMR 4160, CNRS, University of Picardie, 33 Rue Saint Leu, 80039 Amiens, France

E-mail address: `ai-hua.fan@u-picardie.fr`

URL: `http://www.mathinfo.u-picardie.fr/fan/`

J. Schmeling: Mathematics Centre for Mathematical Sciences, Lund Institute of Technology, Lund University Box 118 SE-221 00 Lund, Sweden

E-mail address: `joerg@maths.lth.se`

URL: `http://www.maths.lth.se/matematiklth/personal/joerg/`

S. Troubetzkoy: Centre de physique théorique, Fédération de recherches des unités de mathématique de Marseille, Institut de mathématiques de Luminy and, Université de la Méditerrané, Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: `troubetz@iml.univ-mrs.fr`

URL: `http://iml.univ-mrs.fr/~troubetz/`