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# ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up analysis for a Brezis and Merle's problem with Dirichlet condition. As an application we have a proof of a compactness result under Lipschitz condition on the prescribed scalar curvature and a weaker assumption on the regularity of the domain (smooth domain or  $C^{2,\alpha}$  domain,  $1 \geq \alpha > 0$ ).

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## 1. INTRODUCTION AND MAIN RESULTS

We set  $\Delta = -(\partial_{11} + \partial_{22})$  on open set  $\Omega$  of  $\mathbb{R}^2$  with a smooth (or  $C^{2,\alpha}$ ,  $\alpha > 0$ ) boundary.

We consider the following equation:

$$(P) \begin{cases} \Delta u = V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b < +\infty, \quad e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega).$$

We can see in [7] a nice formulation of this problem (P) in the sense of the distributions. This Problem arises from geometrical and physical problems see for example [1, 2, 18, 19]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-19], where one can find some existence and compactness results. In [6] we have the following important Theorem,

**Theorem A**(Brezis-Merle [6]).*For  $(u_i)_i$  and  $(V_i)_i$  two sequences of functions relative to (P) with,*

$$0 < a \leq V_i \leq b < +\infty$$

*then, for all compact subset  $K$  of  $\Omega$  it holds,*

$$\sup_K u_i \leq c,$$

*with  $c$  depending on  $a, b, K$  and  $\Omega$ .*

One can find in [6] an interior estimate if we assume  $a = 0$ , but we need an assumption on the integral of  $e^{u_i}$ , namely, we have:

**Theorem B**(Brezis-Merle [6]).For  $(u_i)_i$  and  $(V_i)_i$  two sequences of functions relative to the problem (P) with,

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact subset  $K$  of  $\Omega$  it holds;

$$\sup_K u_i \leq c,$$

with  $c$  depending on  $b, C, K$  and  $\Omega$ .

We look to the uniform boundedness on all  $\bar{\Omega}$  of sequences of solutions of the Problem (P). Remark that, when  $a = 0$  the boundedness of  $\int_{\Omega} e^{u_i}$  is a necessary condition in the problem (P) as showed in [6] by the following counterexample.

**Theorem C**(Brezis-Merle [6]).There are two sequences  $(u_i)_i$  and  $(V_i)_i$  of the problem (P) with,

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

such that,

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument with the Fatou's lemma and they applied the maximum principle in  $W_0^{1,1}(\Omega)$  which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

**Remarks:** 1) Theorem 1 of [6], can be obtained by the usual maximum principle and Agmon regularity theorem which require  $C^2$  regularity on the domain.

2) The duality Theorem which we use require  $C^2$  regularity on the domain, see Gilbarg-Trudinger books.

Note that for the problem (P), by using the Pohozaev identity, we can prove that  $\int_{\Omega} e^{u_i}$  is uniformly bounded when  $0 < a \leq V_i \leq b < +\infty$  and  $\|\nabla V_i\|_{L^\infty} \leq A$  and  $\Omega$  starshaped, when  $a = 0$  and  $\nabla \log V_i$  is uniformly bounded, we can bound uniformly  $\int_{\Omega} V_i e^{u_i}$ . In [16] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9] Chen-Li have proved that if  $a = 0$  and  $\nabla \log V_i$  is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [9] Chen-Li have proved that if  $a = 0$  and  $\int_{\Omega} e^{u_i}$  is uniformly bounded and  $\nabla \log V_i$  is uniformly bounded, then we have the compactness result directly. Ma-Wei in [16], extend this result in the case where  $a > 0$ .

If we assume  $V$  more regular, we can have another type of estimates called sup + inf type inequalities. It was proved by Shafrir see [17] that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous

equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [10] an explicit value of  $C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}}$ . In his proof Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  its Lipschitz constant then  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$  see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences  $(V_i)_i$  by Chen-Lin, see [10]. Also, one can see in [14] an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [15] explicit form,  $(8\pi m, m \in \mathbb{N}^*$  exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8] we have some a priori estimates on the 2 and 3-spheres  $\mathbb{S}_2, \mathbb{S}_3$ .

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [6]) is:

**Problem.** Suppose that  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$  with  $0 \leq V_i$ . Also, we consider a sequence of solutions  $(u_i)$  of  $(P)$  relative to  $(V_i)$  such that,

$$\int_{\Omega} e^{u_i} dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorems, indeed, the result of Chen-Li holds for analytic domains and our result holds for smooth of  $C^{2,\alpha}$  domains. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b.$$

The condition  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$  is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results are:

**Theorem 1.1.** Assume that  $\max_{\Omega} u_i \rightarrow +\infty$ , where  $(u_i)$  are solutions of the problem  $(P)$  with:

$$0 \leq V_i \leq b \text{ and } \int_{\Omega} e^{u_i} dx \leq C, \quad \forall i,$$

then, after passing to a subsequence, there is a function  $u$ , there is a number  $N \in \mathbb{N}$  and there are  $N$  points  $x_1, \dots, x_N \in \partial\Omega$ , such that,

$$\partial_{\nu} u_i \rightarrow \partial_{\nu} u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi, \text{ in the sense of measures on } \partial\Omega.$$

$$u_i \rightarrow u \text{ in } C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

**Theorem 1.2.** Assume that  $(u_i)$  are solutions of  $(P)$  relative to  $(V_i)$  with the following conditions:

$$0 \leq V_i \leq b, \quad \|\nabla V_i\|_{L^{\infty}} \leq A \text{ and } \int_{\Omega} e^{u_i} \leq C,$$

we have,

$$\|u_i\|_{L^{\infty}} \leq c(b, A, C, \Omega).$$

In the previous theorem we have a proof of the global a priori estimate which concern the problem  $(P)$ . The proof of Chen-Li and Ma-Wei [9,16], use the moving-plane method for the case  $\nabla \log V_i$  uniformly bounded near the boundary (and  $C^{2,\alpha}$  domain,  $1 \geq \alpha > 0$ ) and for analytic domain for the case  $\nabla V_i$  uniformly bounded.

To prove Theorem 1.2, we argue by contradiction and use Theorem 1.1.

## 2. PROOF OF THE THEOREMS

### Proof of theorem 1.1:

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since  $e^{u_i} \in L^1(\Omega)$  by the corollary 1 of Brezis-Merle's paper (see [6]) we have  $e^{u_i} \in L^k(\Omega)$  for all  $k > 2$  and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

We denote by  $\partial_{\nu} u_i$  the inner normal derivative. By the maximum principle we have,  $\partial_{\nu} u_i \geq 0$ .

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_{\nu} u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure  $\mu$  such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an  $x_0 \in \partial\Omega$  such that,  $\mu(x_0) < 4\pi$ . For  $\epsilon > 0$  small enough set  $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$ . We choose a function  $\eta_\epsilon$  such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a  $\tilde{\eta}_\epsilon$  such that,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

**Remark:** We use the following steps in the construction of  $\eta_\epsilon$ :

We take a cutoff function  $\eta_0$  in  $B(0, 2)$  or  $B(x_0, 2)$ :

1- We set  $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$  in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart  $(f, \tilde{\Omega} = f(B_r(0)))$ , for  $r > 0$  small enough and  $f(0) = x_0$  and we take  $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$  to have connected sets  $I_\epsilon$  and we take  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ . Because  $f, f^{-1}$  are Lipschitz,  $|f(x) - x_0| \leq k_2|x| \leq 1$  for  $|x| \leq 1/k_2$  and  $|f(x) - x_0| \geq k_1|x| \geq 2$  for  $|x| \geq 2/k_1 > 1/k_2$ , the support of  $\eta$  is in  $I_{(2/k_1)\epsilon}$ .

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take:  $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$  and  $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$ , we extend it by 0 outside  $f(B_1(0))$ . We have  $f(B_1(0)) = D_1(x_0)$ ,  $f(B_\epsilon(0)) = D_\epsilon(x_0)$  and  $f(B_\epsilon^+) = D_\epsilon^+(x_0)$  with  $f$  and  $f^{-1}$  smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And,  $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$ , because  $f$  is Lipschitz. Here  $H_1$  is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = \Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set  $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$ . Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with  $C_1$  depends on  $\Omega$ .

We use the following estimate, see [3, 7, 19],

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that,  $(u_i)$  converge weakly in  $W_0^{1,q}(\Omega)$ , almost everywhere to a function  $u \geq 0$  and  $\int_\Omega e^u < +\infty$  (by Fatou's lemma). Also,  $V_i$  weakly converge to a nonnegative function  $V$  in  $L^\infty$ . The function  $u$  is in  $W_0^{1,q}(\Omega)$  solution of :

$$\begin{cases} \Delta u = V e^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [6], we have  $e^{ku} \in L^1(\Omega)$ ,  $k > 1$ . By the elliptic estimates, we have  $u \in C^1(\bar{\Omega})$ .

For two vectors  $v, w$  of  $\mathbb{R}^2$  we denote by  $v \cdot w$  the inner product of  $v$  and  $w$ .

We can write,

$$\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon. \quad (1)$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between  $\tilde{\eta}_\epsilon$  and  $u$ , we obtain,

$$\int_\Omega V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C' \epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between  $u_i$  and  $\tilde{\eta}_\epsilon$  to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_{\epsilon} dx = \int_{\partial\Omega} \partial_{\nu} u_i \eta_{\epsilon} d\sigma \rightarrow \mu(\eta_{\epsilon}) \leq \mu(J'_{\epsilon}) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all  $\epsilon > 0$  there is  $i_0 = i_0(\epsilon)$  such that, for  $i \geq i_0$ ,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_{\epsilon}| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

**Step 2:** Estimate of integral of the second term of the right hand side of (1).

Let  $\Sigma_{\epsilon} = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$  and  $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$ ,  $\epsilon > 0$ . Then, for  $\epsilon$  small enough,  $\Sigma_{\epsilon}$  is a manifold.

The measure of  $\Omega - \Omega_{\epsilon^3}$  is  $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$ . Here  $\mu_L$  is the Lebesgue measure.

**Remark:** for the unit ball  $\bar{B}(0, 1)$ , our new manifold is  $\bar{B}(0, 1 - \epsilon^3)$ .

( Proof of this fact; let's consider  $d(x, \partial\Omega) = d(x, z_0)$ ,  $z_0 \in \partial\Omega$ , this imply that  $(d(x, z_0))^2 \leq (d(x, z))^2$  for all  $z \in \partial\Omega$  which it is equivalent to  $(z - z_0) \cdot (2x - z - z_0) \leq 0$  for all  $z \in \partial\Omega$ , let's consider a chart around  $z_0$  and  $\gamma(t)$  a curve in  $\partial\Omega$ , we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$  and it is clear that,  $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$ , which imply that  $x = z_0 - s\nu_0$  where  $\nu_0$  is the outward normal of  $\partial\Omega$  at  $z_0$ )

With this fact, we can say that  $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$ . It is sufficient to work on  $\partial\Omega$ . Let's consider a charts  $(z, D = B(z, 4\epsilon_z), \gamma_z)$  with  $z \in \partial\Omega$  such that  $\cup_z B(z, \epsilon_z)$  is cover of  $\partial\Omega$ . One can extract a finite cover  $(B(z_k, \epsilon_k)), k = 1, \dots, m$ , by the area formula the measure of  $S \cap B(z_k, \epsilon_k)$  is less than a  $k\epsilon$  (a  $\epsilon$ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point on the boundary).

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| dx. \quad (5)$$

**Step 2.1:** Estimate of  $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_{\epsilon}| dx$ .

First, we know from the elliptic estimates that  $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1/\epsilon^2$ ,  $C_1$  depends on  $\Omega$

We know that  $(|\nabla u_i|)_i$  is bounded in  $L^q, 1 < q < 2$ , we can extract from this sequence a subsequence which converge weakly to  $h \in L^q$ . But, we know that we have locally the uniform convergence to  $|\nabla u|$  (by Brezis-Merle's theorem), then,  $h = |\nabla u|$  a.e. Let  $q'$  be the conjugate of  $q$ .

We have,  $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take  $f = 1_{\Omega - \Omega_{\epsilon^3}}$ , we have:



$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \quad i \geq i_1, \quad \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for  $i \geq i_1(\epsilon)$ ,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant  $C_1$  does not depend on  $\epsilon$  but on  $\Omega$ .

Step 2.2: Estimate of  $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$ .

We know that,  $\Omega_\epsilon \subset \subset \Omega$ , and (because of Brezis-Merle's interior estimates)  $u_i \rightarrow u$  in  $C^1(\Omega_{\epsilon^3})$ . We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \quad \text{for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \quad \text{for } i \geq i_3,$$

For  $\epsilon > 0$ , we have for  $i \in \mathbb{N}$ ,  $i \geq \max\{i_1, i_2, i_3\}$ ,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for  $\epsilon > 0$ , there is  $i_3 = i_3(\epsilon) \in \mathbb{N}$ ,  $i_3 = \max\{i_0, i_1, i_2\}$  such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (8)$$

We choose  $\epsilon > 0$  small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$ .

We can use Theorem 1 of [6] to conclude that there is  $q \geq \tilde{q} > 1$  such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_{\Omega} e^{q|u_i-u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where,  $V_\epsilon(x_0)$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ . Here we have used that in a neighborhood of  $x_0$  by the elliptic estimates,  $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$ . (We can take,  $f(B_{\epsilon^3}(0))$  and we have  $B_{k_2\epsilon^3}(x_0) \subset f(B_{\epsilon^3}(0)) \subset B_{k_1\epsilon^3}(x_0)$  for a chart  $(f, B_1(0))$  around  $x_0$ ).

Thus, for each  $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$  there is  $\epsilon_{x_0} > 0, q_{x_0} > 1$  such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

By the elliptic estimates (see [13])  $(u_i)_i$  is uniformly bounded in  $W^{2, q_1}(V_\epsilon(x_0))$  and also, in  $C^1(V_\epsilon(x_0))$ . Finally, we have, for some  $\epsilon > 0$  small enough,

$$\|u_i\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We have proved that, there is a finite number of points  $\bar{x}_1, \dots, \bar{x}_m$  such that the sequence  $(u_i)_i$  is locally uniformly bounded (in  $C^{1, \theta}, \theta > 0$ ) in  $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$ .

*Proof of theorem 1.2:*

We know that:

$$u_i \in W^{2, k}(\Omega) \cap C^{1, \epsilon}(\bar{\Omega}).$$

We can do integration by parts. The first Pohozaev identity applied around each blow-up point see for example [16] gives :

$$\int_{\partial\Omega_{x_k}} [(\partial_\nu u_i) \nabla u_i - \frac{1}{2} \|\nabla u_i\|^2 \nu] dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i} \nu, \quad (10)$$

Here  $\Omega_{x_k}$  is a neighborhood of  $x_k$  on which we can use the integration by part obtained by a chart around  $x_k$ .

We use the boundary condition on  $\Omega$  and the boundedness of  $u_i$  and  $\partial_j u_i$  outside the  $x_k$ , to have:

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \quad (11)$$

Thus we can use the weak convergence in  $L^2(\partial\Omega)$  to have a subsequence  $\partial_\nu u_i$ , such that:

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi dx, \quad \forall \varphi \in L^2(\partial\Omega), \quad (12)$$

Thus,  $\alpha_j = 0, j = 1, \dots, N$  and  $(u_i)$  is uniformly bounded.

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