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Samy Skander Bahoura

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ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up analysis for a Brezis and Merle's problem with Dirichlet condition. As an application we have a proof of a compactness result under Lipschitz condition on the prescribed scalar curvature and a weaker assumption on the regularity of the domain (smooth domain or $C^{2,\alpha}$ domain, $1 \geq \alpha > 0$).

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1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = -(\partial_{11} + \partial_{22})$ on open set Ω of \mathbb{R}^2 with a smooth (or $C^{2,\alpha}$, $\alpha > 0$) boundary.

We consider the following equation:

$$(P) \begin{cases} \Delta u = V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b < +\infty, \quad e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega).$$

We can see in [7] a nice formulation of this problem (P) in the sens of the distributions. This Problem arises from geometrical and physical problems see for example [1, 2, 18, 19]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-19], where one can find some existence and compactness results. In [6] we have the following important Theorem,

Theorem A(Brezis-Merle [6]).*For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to (P) with,*

$$0 < a \leq V_i \leq b < +\infty$$

then it holds,

$$\sup_K u_i \leq c,$$

with c depending on a, b, K and Ω .

One can find in [6] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} , namely, we have:

Theorem B(Brezis-Merle [6]).*For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to the problem (P) with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then it holds;

$$\sup_K u_i \leq c,$$

with c depending on b, C, K and Ω .

We look to the uniform boundedness on all $\bar{\Omega}$ of sequences of solutions of the Problem (P) and when $a = 0$ the boundedness of $\int_{\Omega} e^{u_i}$ is a necessary condition in the problem (P) as showed in [6] by the following counterexample.

Theorem C(Brezis-Merle [6]).*There are two sequences $(u_i)_i$ and $(V_i)_i$ of the problem (P) with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

such that,

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument with the Fatou's lemma and they applied the maximum principle in $W_0^{1,1}(\Omega)$ which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

Remarks: 1) Theorem 1 of [6], can be obtained by the usual maximum principle and Agmon regularity theorem which require C^2 regularity on the domain.

2) The duality Theorem which we use can require C^2 regularity on the domain, see Gilbarg-Trudinger books of the embedding $W_0^{1,q'}$, $q' > 2$ in L^∞ in dimension 2 for exemple).

Note that for the problem (P), by using the Pohozaev identity, we can prove that $\int_{\Omega} e^{u_i}$ is uniformly bounded when $0 < a \leq V_i \leq b < +\infty$ and $\|\nabla V_i\|_{L^\infty} \leq A$ and Ω starshaped, when $a = 0$ and $\nabla \log V_i$ is

uniformly bounded, we can bound uniformly $\int_{\Omega} V_i e^{u_i}$. In [16] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9] Chen-Li have proved that if $a = 0$ and $\nabla \log V_i$ is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [9] Chen-Li have proved that if $a = 0$ and $\int_{\Omega} e^{u_i}$ is uniformly bounded and $\nabla \log V_i$ is uniformly bounded, then we have the compactness result directly. Ma-Wei in [16], extend this result in the case where $a > 0$.

If we assume V more regular, we can have another type of estimates called sup + inf type inequalities. It was proved by Shafrir see [17] that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [10] an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A its Lipschitz constant then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [10]. Also, one can see in [14] an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [15] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8] we have some a priori estimates on the 2 and 3-spheres $\mathbb{S}_2, \mathbb{S}_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [6]) is:

Problem. Suppose that $V_i \rightarrow V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i$. Also, we consider a sequence of solutions (u_i) of (P) relative to (V_i) such that,

$$\int_{\Omega} e^{u_i} dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorems. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b.$$

The condition $V_i \rightarrow V$ in $C^0(\bar{\Omega})$ is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results are:

Theorem 1.1. *Assume that $\max_{\Omega} u_i \rightarrow +\infty$, where (u_i) are solutions of the problem (P) with:*

$$0 \leq V_i \leq b \text{ and } \int_{\Omega} e^{u_i} dx \leq C, \quad \forall i,$$

then, after passing to a subsequence, there is a function u , there is a number $N \in \mathbb{N}$ and there are N points $x_1, \dots, x_N \in \partial\Omega$, such that,

$$\partial_\nu u_i \rightarrow \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi, \text{ in the sens of measures on } \partial\Omega.$$

$$u_i \rightarrow u \text{ in } C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

Theorem 1.2. *Assume that (u_i) are solutions of (P) relative to (V_i) with the following conditions:*

$$0 \leq V_i \leq b, \|\nabla V_i\|_{L^\infty} \leq A \text{ and } \int_{\Omega} e^{u_i} \leq C,$$

we have,

$$\|u_i\|_{L^\infty} \leq c(b, A, C, \Omega).$$

In the previous theorem we have a proof of the global a priori estimate which concern the problem (P). The proof of Chen-Li and Ma-Wei [9,16], use the moving-plane method for the case $\nabla \log V_i$ uniformly bounded (and $C^{2,\alpha}$ domain, $1 \geq \alpha > 0$) and for analytic domain for the case ∇V_i uniformly bounded.

2. PROOF OF THE THEOREMS

Proof of theorem 1.1:

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since $e^{u_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

We denote by $\partial_\nu u_i$ the inner normal derivative. By the maximum principle we have, $\partial_\nu u_i \geq 0$.

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$. We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0, 2)$ or $B(x_0, 2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, because f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = \Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω .

We use the following estimate, see [3, 7, 19],

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou's lemma). Also, V_i weakly converge to a nonnegative function V in L^∞ . The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} \Delta u = Ve^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [6], we have $e^{ku} \in L^1(\Omega)$, $k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

For two vectors v, w of \mathbb{R}^2 we denote by $v \cdot w$ the inner product of v and w .

We can write,

$$\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon. \quad (1)$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C' \epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \rightarrow \mu(\eta_\epsilon) \leq \mu(J'_\epsilon) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$.

Remark: for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

(Proof of this fact; let's consider $d(x, \partial\Omega) = d(x, z_0)$, $z_0 \in \partial\Omega$, this imply that $(d(x, z_0))^2 \leq (d(x, z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z - z_0) \cdot (2x - z - z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0)

With this fact, we can say that $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k, \epsilon_k)), k = 1, \dots, m$, by the area formula the measure of $S \cap B(z_k, \epsilon_k)$ is less than a $k\epsilon$ (a ϵ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point on the boundary).

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx. \quad (5)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q .

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3 (k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant C_1 does not depend on ϵ but on Ω .

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (8)$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use Theorem 1 of [6] to conclude that there is $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_{\Omega} e^{q|u_i - u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$. (We can take, $f(B_{\epsilon^3}(0))$ and we have $B_{k_2\epsilon^3}(x_0) \subset f(B_{\epsilon^3}(0)) \subset B_{k_1\epsilon^3}(x_0)$ for a chart $(f, B_1(0))$ around x_0).

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0$, $q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

By the elliptic estimates (see [13]) $(u_i)_i$ is uniformly bounded in $W^{2, q_1}(V_\epsilon(x_0))$ and also, in $C^1(V_\epsilon(x_0))$. Finally, we have, for some $\epsilon > 0$ small enough,

$$\|u_i\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the sequence $(u_i)_i$ is locally uniformly bounded (in $C^{1, \theta}$, $\theta > 0$) in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

Proof of theorem 1.2:

We know that:

$$u_i \in W^{2, k}(\Omega) \cap C^{1, \epsilon}(\bar{\Omega}).$$

We can do integration by parts. The first Pohozaev identity applied around each blow-up point see for example [16] gives :

$$\int_{\partial\Omega_{x_k}} [(\partial_\nu u_i) \nabla u_i - \frac{1}{2} |\nabla u_i|^2 \nu] dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i} \nu, \quad (10)$$

We use the boundary condition on Ω and the boundedness of u_i and $\partial_j u_i$ outside the x_k , to have:

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \quad (11)$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi dx, \quad \forall \varphi \in L^2(\partial\Omega), \quad (12)$$

Thus, $\alpha_j = 0$, $j = 1, \dots, N$ and (u_i) is uniformly bounded.

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DEPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, 4 PLACE JUSSIEU, 75005, PARIS, FRANCE
AND BEIJING NORMAL UNIVERSITY, BEIJING, CHINA.

E-mail address: samybahoura@yahoo.fr, samybahoura@gmail.com