



About Brezis-Merle Problem with Lipschitz condition

Samy Skander Bahoura

► To cite this version:

Samy Skander Bahoura. About Brezis-Merle Problem with Lipschitz condition. 2018. hal-00149613v7

HAL Id: hal-00149613

<https://hal.science/hal-00149613v7>

Preprint submitted on 1 Oct 2018 (v7), last revised 3 Aug 2020 (v9)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. We give blow-up analysis for a Brezis and Merle's problem with Dirichlet condition. As an application we have another proof for this Problem under Lipschitz condition.

Mathematics Subject Classification: 35J60 35B45 35B50

Keywords: blow-up, boundary, Dirichlet condition, a priori estimate, Lipschitz condition.

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = -(\partial_{11} + \partial_{22})$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary.

We consider the following equation:

$$(P) \begin{cases} \Delta u = V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b < +\infty, \quad e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega).$$

We can see in [7] a nice formulation of this problem (P) in the sens of the distributions. This Problem arises from geometrical and physical problems see for example [1, 2, 18, 19]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-19], where one can find some existence and compactness results. In [6] we have the following important Theorem,

Theorem A(Brezis-Merle [6]).*For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to (P) with,*

$$0 < a \leq V_i \leq b < +\infty$$

then it holds,

$$\sup_K u_i \leq c,$$

with c depending on a, b, K and Ω .

One can find in [6] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} , namely, we have:

Theorem B(Brezis-Merle [6]). *For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to the problem (P) with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then it holds;

$$\sup_K u_i \leq c,$$

with c depending on b, C, K and Ω .

We look to the uniform boundedness on all $\bar{\Omega}$ of sequences of solutions of the Problem (P) and when $a = 0$ the boundedness of $\int_{\Omega} e^{u_i}$ is a necessary condition in the problem (P) as showed in [6] by the following counterexample.

Theorem C(Brezis-Merle [6]). *There are two sequences $(u_i)_i$ and $(V_i)_i$ of the problem (P) with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

such that,

$$\sup_{\Omega} u_i \rightarrow +\infty.$$

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [6]) obtained by an approximation argument with the Fatou's lemma and they applied the maximum principle in $W_0^{1,1}(\Omega)$ which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [5] for a topic about the weak form of the maximum principle.

Note that for the problem (P), by using the Pohozaev identity, we can prove that $\int_{\Omega} e^{u_i}$ is uniformly bounded when $0 < a \leq V_i \leq b < +\infty$ and $\|\nabla V_i\|_{L^\infty} \leq A$ and Ω starshaped, when $a = 0$ and $\nabla \log V_i$ is uniformly bounded, we can bound uniformly $\int_{\Omega} V_i e^{u_i}$. In [16] Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9] Chen-Li have proved that if $a = 0$ and $\nabla \log V_i$ is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [9] Chen-Li have proved that if $a = 0$ and $\int_{\Omega} e^{u_i}$ is uniformly bounded and $\nabla \log V_i$ is uniformly bounded, then we have the compactness result directly. Ma-Wei in [16], extend this result in the case where $a > 0$.

If we assume V more regular, we can have another type of estimates called sup + inf type inequalities. It was proved by Shafrir see [17] that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [10] an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A its Lipschitz constant then $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$ see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [10]. Also, one can see in [14] an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [15] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8] we have some a priori estimates on the 2 and 3-spheres $\mathbb{S}_2, \mathbb{S}_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [6]) is:

Problem. Suppose that $V_i \rightarrow V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i$. Also, we consider a sequence of solutions (u_i) of (P) relative to (V_i) such that,

$$\int_{\Omega} e^{u_i} dx \leq C,$$

is it possible to have:

$$\|u_i\|_{L^\infty} \leq C?$$

Here, we give a characterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorem ($\nabla \log V_i$ uniformly bounded). Also, we give a new proof of Chen-Li result (when ∇V_i is uniformly bounded, the method of Chen-Li use some local estimates). For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b.$$

The condition $V_i \rightarrow V$ in $C^0(\bar{\Omega})$ is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$\|\nabla V_i\|_{L^\infty} \leq A.$$

Our main results are:

Theorem 1.1. *Assume that $\max_{\Omega} u_i \rightarrow +\infty$, where (u_i) are solutions of the problem (P) with:*

$$0 \leq V_i \leq b \text{ and } \int_{\Omega} e^{u_i} dx \leq C, \quad \forall i,$$

then, after passing to a subsequence, there is a function u , there is a number $N \in \mathbb{N}$ and there are N points $x_1, \dots, x_N \in \partial\Omega$, such that,

$$\partial_{\nu} u_i \rightarrow \partial_{\nu} u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi, \text{ in the sense of measures on } \partial\Omega.$$

$$u_i \rightarrow u \text{ in } C_{loc}^1(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

Theorem 1.2. *Assume that (u_i) are solutions of (P) relative to (V_i) with the following conditions:*

$$0 \leq V_i \leq b, \quad \|\nabla V_i\|_{L^\infty} \leq A \text{ and } \int_{\Omega} e^{u_i} \leq C,$$

we have,

$$\|u_i\|_{L^\infty} \leq c(b, A, C, \Omega).$$

In the previous theorem we have a proof of the global a priori estimate which concern the problem (P). The proof of Chen-Li and Ma-Wei [9,16], use the moving-plane method.

2. PROOF OF THE THEOREMS

Proof of theorem 1.1:

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since $e^{u_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [6]) we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

We denote by $\partial_\nu u_i$ the inner normal derivative. By the maximum principle we have, $\partial_\nu u_i \geq 0$.

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \mu(\varphi), \quad \forall \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0, \epsilon) \cap \partial\Omega$. We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0, 2)$ or $B(x_0, 2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x - x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f, \tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x) - x_0| \leq k_2|x| \leq 1$ for $|x| \leq 1/k_2$ and $|f(x) - x_0| \geq k_1|x| \geq 2$ for $|x| \geq 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } f(I_{(1/k_2)\epsilon}), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{(2/k_1)\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on the connected set } J_\epsilon = f(I_\epsilon), \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(J'_\epsilon)} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \leq C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, because f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} \Delta \bar{\eta}_\epsilon = \Delta \eta_\epsilon & \text{in } \Omega \subset \mathbb{R}^2, \\ \bar{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

and finally we set $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C(\|\eta_\epsilon\|_{L^\infty} + \|\nabla \eta_\epsilon\|_{L^\infty} + \|\Delta \eta_\epsilon\|_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω .

We use the following estimate, see [3, 7, 19],

$$\|\nabla u_i\|_{L^q} \leq C_q, \quad \forall i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou's lemma). Also, V_i weakly converge to a nonnegative function V in L^∞ . The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} \Delta u = V e^u \in L^1(\Omega) & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [6], we have $e^{ku} \in L^1(\Omega)$, $k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

For two vectors v, w of $\mathbb{R}^2$ we denote by $v \cdot w$ the inner product of v and w .

We can write,

$$\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon. \quad (1)$$

We use the interior estimate of Brezis-Merle, see [6],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C' \epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (2)$$

By a chart f (f is smooth and in particular Lipschitzian), we have using the area formula:

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \rightarrow \mu(\eta_\epsilon) \leq \mu(J'_{2\epsilon}) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (4)$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2 \epsilon^3 \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$.

Remark: for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

(Proof of this fact; let's consider $d(x, \partial\Omega) = d(x, z_0)$, $z_0 \in \partial\Omega$, this imply that $(d(x, z_0))^2 \leq (d(x, z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z - z_0) \cdot (2x - z - z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t) - \gamma(t_0)) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0))

With this fact, we can say that $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k, \epsilon_k)), k = 1, \dots, m$, by the area formula the measure of $S \cap B(z_k, \epsilon_k)$ is less than a $k\epsilon$ (a ϵ -rectangle). For the reverse inequality, it is sufficient to consider one chart around one point on the boundary).

We write,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx. \quad (5)$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q .

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_{\Omega} |\nabla u_i| f dx \rightarrow \int_{\Omega} |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1, \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \text{meas}(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3 (k_1 \|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (6)$$

The constant C_1 does not depend on ϵ but on Ω .

Step 2.2: Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$\|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \|\nabla(u_i - u)\|_{L^\infty(\Omega_{\epsilon^3})} \|\nabla \tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (7)$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u) \tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (8)$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} \Delta[(u_i - u) \tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i - u) \tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use Theorem 1 of [6] to conclude that there is $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i - u|} dx \leq \int_{\Omega} e^{q|u_i - u| \tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighborhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$. (We can take $(\Omega - \Omega_{\epsilon^3}) \cap B(x_0, \epsilon^3)$).

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0$, $q_{x_0} > 1$ such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i. \quad (9)$$

By the elliptic estimates (see [13]) $(u_i)_i$ is uniformly bounded in $W^{2, q_1}(V_\epsilon(x_0))$ and also, in $C^1(V_\epsilon(x_0))$. Finally, we have, for some $\epsilon > 0$ small enough,

$$\|u_i\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the sequence $(u_i)_i$ is locally uniformly bounded (in $C^{1, \theta}$, $\theta > 0$) in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

Proof of theorem 1.2:

We know that:

$$u_i \in W^{2, k}(\Omega) \cap C^{1, \epsilon}(\bar{\Omega}).$$

We can do integration by parts. The first Pohozaev identity applied around each blow-up point see for example [16] gives :

$$\int_{\partial\Omega_{x_k}} [(\partial_\nu u_i) \nabla u_i - \frac{1}{2} |\nabla u_i|^2 \nu] dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i} \nu, \quad (10)$$

We use the boundary condition on Ω and the boundedness of u_i and $\partial_j u_i$ outside the x_k , to have:

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \quad (11)$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx \rightarrow \int_{\partial\Omega} \partial_\nu u \varphi dx, \quad \forall \varphi \in L^2(\partial\Omega), \quad (12)$$

Thus, $\alpha_j = 0$, $j = 1, \dots, N$ and (u_i) is uniformly bounded.

ACKNOWLEDGEMENT.

One part of this work was done when the author was in China at Beijing Normal University and other part in France. The work was supported by YY. Li's Grant and Pr Jiguang Bao. The author would like to thank Pr. YY.Li for his support.

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.
- [2] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.
- [3] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. 87 no 1, (1989), 149-169.
- [4] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [5] Brezis. H, Marcus. M, Ponce. A. C. Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations, 55-109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
- [6] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).
- [7] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L^1 . J. Math. Soc. Japan 25 (1973), 565-590.
- [8] Chang, Sun-Yung A, Gursky, Matthew J, Yang, Paul C. Scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205-229.
- [9] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
- [10] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in $\mathbb{R}^2$. Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [11] D.G. De Figueiredo, P.L. Lions, R.D. Nussbaum, A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations, J. Math. Pures et Appl., vol 61, 1982, pp.41-63.
- [12] Ding, W, Jost, J, Li, J, Wang. G. The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. Asian J. Math. 1 (1997), no. 2, 230-248.
- [13] D. Gilbarg, N. S, Trudinger. Elliptic Partial Differential Equations of Second order, Berlin Springer-Verlag.
- [14] YY. Li. Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200,421-444 (1999).
- [15] YY. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.
- [16] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.
- [17] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = V e^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.
- [18] Nagasaki, K, Suzuki,T. Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. Asymptotic Anal. 3 (1990), no. 2, 173-188.
- [19] Tarantello, G. Multiple condensate solutions for the Chern-Simons-Higgs theory. J. Math. Phys. 37 (1996), no. 8, 3769-3796.

DEPARTEMENT DE MATHEMATIQUES, UNIVERSITE PIERRE ET MARIE CURIE, 4 PLACE JUSSIEU, 75005, PARIS, FRANCE
AND BEIJING NORMAL UNIVERSITY, BEIJING CHINA.

E-mail address: samybahoura@yahoo.fr, samybahoura@gmail.com