About Brezis-Merle Problem in dimension 2.

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ABOUT BREZIS-MERLE PROBLEM WITH LIPSCHITZ CONDITION.

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ABSTRACT. We give a blow-up analysis for Brezis-Merle problem with Dirichlet condition. As an application, we have another proof for Brezis-Merle Problem with Lipschitz condition.

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1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = -(\partial_{11} + \partial_{22})$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary.

We consider the following equation:

$$(P)\begin{cases} \Delta u = Ve^u & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \quad u = 0 & \text{in } \ \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \leq V \leq b < +\infty, \ e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega).$$

We can see in [6] a nice formulation to this problem (P) in the sens of the distributions. This Problem arises in geometrical and physical problems, see for example [1, 2, 16, 17]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-18], where one can find some existence and compactness results. In [5] we have the following important Theorem,

Theorem A_(Brezis-Merle [5]).For_ $(u_i)_i$ _and_ $(V_i)_i$ _two sequences of functions relative to_ (P) _with,_

$$0 < a \leq V_i \leq b < +\infty$$

then it holds,

$$\sup_K u_i \leq c,$$

with c _depending on_ a, b, K _and_ Ω.

One can find in [5] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}, namely, we have:

Theorem B(*Brezis-Merle [5]).For* $(u_i)_i$ *and* $(V_i)_i$ *two sequences of functions relative to the problem* (P) *with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,$$

then it holds;

$$\sup_K u_i \leq c,$$

with c *depending on* b, C, K *and* Ω.

When $a = 0$, the boundedness of $\int_\Omega e^{u_i}$ is a necessary condition to work on the problem (P) as showed in [5] by the following counterexample.

Theorem C(*Brezis-Merle [5]).There are two sequences* $(u_i)_i$ *and* $(V_i)_i$ *of the problem* (P) *with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,$$

such that,

$$\sup_\Omega u_i \to +\infty.$$

Note that for the problem (P), by using the Pohozaev identity, we can prove that $\int_\Omega e^{u_i}$ is uniformly bounded when $0 < a \leq V_i \leq b < +\infty$ and $||\nabla V_i||_{L^\infty} \leq A$ and Ω starshaped, when $a = 0$ and $\nabla \log V_i$ is uniformly bounded, we can bound uniformly $\int_\Omega V_i e^{u_i}$. In [14], Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [8], Chen-Li have proved that if $a = 0$ and $\nabla \log V_i$ is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [8], Chen-Li have proved that if $a = 0$ and $\int_\Omega e^{u_i}$ is uniformly bounded and $\nabla \log V_i$ is uniformly bounded, then we have the compactness result directly. Ma-Wei in [14], extend this result in the case where $a > 0$.

If we assume V more regular, we can have another type of estimates, a $\sup + \inf$ type inequalities. It was proved by Shafrir see [15], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right)\sup_K u_i + \inf_\Omega u_i \leq c = c(a,b,K,\Omega).$$

One can see in [9] an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality, see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a,b,A,K,\Omega)$, see Brezis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [9]. Also, one can see in [12], an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [13] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, in [18], we have refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

In [7], we have some a priori estimates on the 2 and 3-spheres $\mathbb{S}_2$, $\mathbb{S}_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem with Lipschitz condition.

The Brezis-Merle Problem (see [5]) is:

Problem. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i \leq b$ for some positive constant b. Also, we consider a sequence of solutions (u_i) of (P) relative to (V_i) such that,

$$\int_\Omega e^{u_i} dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C = C(b, C, V, \Omega)?$$

Here, we give a caracterization of the behavior of the blow-up points on the boundary and also, in particular we extend Chen-Li theorem ($\nabla \log V_i$ uniformly bounded). Also, we give a new proof of Chen-Li result (when ∇V_i is uniformly bounded, the method of Chen-Li use some local estimates). For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary, but for the proof of the compactness for the Brezis-Merle problem we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

We have the following caracterization of the behavior of the blow-up points on the boundary.

Theorem 1.1. *Assume that* $\max_\Omega u_i \to +\infty$, *where* (u_i) *are solutions of the problem* (P) *with:*

$$0 \leq V_i \leq b \ \text{ and } \ \int_\Omega e^{u_i} dx \leq C, \ \forall \ i,$$

then, after passing to a subsequence, there is a finction u, *there is a number* $N \in \mathbb{N}$ *and* N *points* $x_1, \ldots, x_N \in \partial\Omega$, *such that,*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j}, \ \ \alpha_j \geq 4\pi, \ \text{weakly in the sens of measure } L^1(\partial\Omega).$$

$$u_i \to u \ \text{ in } \ C^1_{loc}(\bar\Omega - \{x_1, \ldots, x_N\}).$$

In the following theorem, we have a proof of the global a priori estimate which concern the problem (P). The proof of Chen-Li and Ma-Wei [8,14], use the moving-plane method.

Theorem 1.2. *Assume that* (u_i) *are solutions of* (P) *relative to* (V_i) *with the following conditions:*

$$||\nabla V_i||_{L^\infty} \leq A \ \text{ and } \ \int_\Omega e^{u_i} \leq C,$$

we have,

$$||u_i||_{L^\infty} \leq c(b, A, C, \Omega),$$

2. Proof of the theorems

<u>Proof of theorem 1.1:</u>

We have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

Without loss of generality, we can assume that $\partial_\nu u_i > 0$. Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \to \mu(\varphi), \ \ \forall \ \varphi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. Without loss of generality, we can assume that the following curve, $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$ is an interval.(In this case, it is more simple to construct the following test function η_ϵ). We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{ on } \ I_\epsilon, \ \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{ outside } \ I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \quad \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \ \partial\Omega. \end{cases}$$

We use the following estimate, see [3, 6, 17],

$$||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} \Delta u = V e^u \in L^1(\Omega) & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \quad u = 0 & \text{in } \ \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [5], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

We can write,

$$\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 < \nabla(u_i - u)|\nabla \tilde{\eta}_\epsilon > . \tag{1}$$

We use the interior esimate of Brezis-Merle, see [5],

Step 1: Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u, we obtain,

$$\int_\Omega V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon ||\partial_\nu u||_{L^\infty} = C\epsilon \tag{2}$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_\Omega V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \;\; \epsilon_0 > 0 \tag{3}$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_\Omega |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \tag{4}$$

Step 2: Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \leq \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$.

Remark: for the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1-\epsilon^3)$.

We write,

$$\int_\Omega | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx = \int_{\Omega_{\epsilon^3}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx + \int_{\Omega - \Omega_{\epsilon^3}} < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx. \tag{5}$$

Step 2.1: Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} | < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > |dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_\Omega |\nabla u_i| f dx \to \int_\Omega |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \;\exists\; i_1 = i_1(\epsilon) \in \mathbb{N}, \;\; i \geq i_1, \;\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq mes(\Omega-\Omega_{\epsilon^3})||\nabla u||_{L^\infty} + \epsilon^3 = \epsilon^3(k_1||\nabla u||_{L^\infty}+1).$$

Thus, we obtain,

$$\int_{\Omega-\Omega_{\epsilon^3}} | < \nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon > |dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty}+1) \tag{6}$$

The constant C_1 does not depend on ϵ but on Ω.

<u>*Step 2.2:*</u> Estimate of $\int_{\Omega_{\epsilon^3}} | < \nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon > |dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon 3}} | < \nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon > |dx \leq ||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^3})}||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1\epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_\Omega | < \nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon > |dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty}+2) \tag{7}$$

From (4) and (7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_\Omega |\Delta[(u_i-u)\tilde{\eta}_\epsilon]|dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^\infty}+2+C) \tag{8}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases} \Delta[(u_i-u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i-u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use Theorem 1 of [5] to conclude that there is $q > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i-u)}dx \leq \int_\Omega e^{q(u_i-u)\tilde{\eta}_\epsilon}dx \leq C(\epsilon, \Omega).$$

where, $V_\epsilon(x_0)$ is a neighberhooh of x_0 in $\bar{\Omega}$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0}u_i}dx \leq C, \;\; \forall \;\; i. \tag{9}$$

By the elliptic estimates, $(u_i\eta)_i$ is uniformly bounded in $W^{2,q_1}(\Omega)$ and also, in $C^1(\bar{\Omega})$. (Here, η is a cutuf function.)

Finaly, we have, for some $\epsilon > 0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \;\; \forall \;\; i.$$

We have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

Proof of theorem 1.2:

The first Pohozaev identity applied around each blow-up point, see for example [14], gives :

$$\int_{\partial\Omega_{x_k}} [(\partial_\nu u_i)\nabla u_i - \frac{1}{2}||\nabla u_i|^2\nu]dx = \int_{\Omega_{x_k}} \nabla V_i e^{u_i} - \int_{\partial\Omega_{x_k}} V_i e^{u_i}\nu, \tag{10}$$

We use the boundary condition on Ω and the boundedness of u_i and $\partial_j u_i$ outside the x_k, to have:

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega). \tag{11}$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx \to \int_{\partial\Omega} \partial_\nu u \varphi dx, \;\; \forall \;\; \varphi \in L^2(\partial\Omega), \tag{12}$$

Thus, $\alpha_j = 0$, $j = 1, \ldots, N$ and (u_i) is uniformly bounded.

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