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## ABOUT A BREZIS-MERLE PROBLEM IN DIMENSION 2.

SAMY SKANDER BAHOURA

ABSTRACT. On a bounded open set  $\Omega$  of  $\mathbb{R}^2$  with smooth boundary, we consider two sequences of functions  $(u_i)_i$  and  $(V_i)_i$  such that,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega. \end{cases}$$

With,  $0 \leq a \leq V_i \leq b < +\infty$ .

Assume that,

$$\int_{\Omega} e^{u_i} dx \leq C.$$

If,

$$a > 0,$$

or,

$$a = 0 \text{ and } V_i \rightarrow V \text{ in } C^0(\bar{\Omega}).$$

We have,

$$\|u_i\|_{L^\infty(\bar{\Omega})} \leq c(a, b, C, \Omega).$$

### 1. INTRODUCTION AND RESULTS.

We set  $\Delta = -\partial_{11} - \partial_{22}$  the geometric Laplacian on  $\mathbb{R}^2$ .

On an open set  $\Omega$  of  $\mathbb{R}^2$ , with a smooth boundary, we consider the following problem:

$$(P) \quad \begin{cases} \Delta u = V e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function  $V$  is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on  $V$ , for example we suppose  $V \geq 0$  and  $V \in L^p(\Omega)$  or  $V e^u \in L^p(\Omega)$  with  $p \in [1, +\infty]$ .

We can see in [B-M] the two following important Theorems,

**Theorem A (Brezis-Merle).** *If  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the problem (P) with,  $0 \leq V_i \leq b < +\infty$  and  $\int_{\Omega} e^{u_i} dx \leq C$ , then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(a, b, C, K, \Omega)$$

**Theorem B (Brezis-Merle).** *If  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the problem (P) with,  $0 < a \leq V_i \leq b < +\infty$ , then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

If, we assume  $V$  with more regularity, we can have another type of estimates,  $\sup + \inf$ . It was proved, by Shafrir, see [S], that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C(a/b) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [C-L], an explicit value of  $C(a/b) = \sqrt{\frac{a}{b}}$ .

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  the Lipschitz constant, then,  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$ , see Brézis-Li-Shafrir [B-L-S]. This result was extended for Hölderian sequences  $(V_i)_i$  by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [L-S] explicit form,  $(8\pi m, m \in \mathbb{N}^*$  exactly), for the numbers in front of the Dirac masses, when the solutions blow-up.

The questions of Brezis-Merle :

Question 1:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions  $(u_i)_i$  solutions of the problem  $(P)$  relatively to  $(V_i)_i$  with,

$$\begin{cases} 0 < a \leq V_i \leq b < +\infty \\ \int_{\Omega} e^{u_i} dx \leq C'. \end{cases}$$

Is it possible to have,

$$\|u_i\|_{L^\infty(\Omega)} \leq c' = c'(a, b, C', \Omega)?$$

Question 2:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions  $(u_i)_i$  solutions of the problem  $(P)$  relatively to  $(V_i)_i$  with,

$$\begin{cases} 0 \leq V_i \leq b < +\infty \\ V_i \rightarrow V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \leq C'. \end{cases}$$

Is it possible to have,

$$\|u_i\|_{L^\infty(\Omega)} \leq c' = c'(a, b, C', \Omega)?$$

We can find some results which concerning this question. It was proved by Chen-Li, that if we suppose  $a = 0$  and  $\|\nabla V_i\|_{L^\infty(\Omega)} \leq A$ , then, we have a positive answer to the question, see [C-Li].

Also, if we assume  $a > 0$  and  $\|\nabla V_i\|_{L^\infty(\Omega)} \leq A$ , the answer to the question is positive without the assumption on  $e^{u_i}$  was given by Ma-Wei, see [M-W].

Here, we have,

**Theorem 1.** Let us consider three positive numbers  $a, b, C$ , and, two sequences of functions,  $(u_i)_i$  and  $(V_i)_i$ , relatively to the problem (P) with the following conditions:

$$\begin{cases} 0 < a \leq V_i \leq b < +\infty \\ \int_{\Omega} e^{u_i} dx \leq C, \quad \forall i \end{cases}$$

Then, there is a positive constant  $c = c(a, b, C, \Omega)$  such that,

$$\|u_i\|_{L^\infty(\Omega)} \leq c.$$

**Theorem 2.** Let us consider two positive numbers  $a, b$ , and, two sequences of functions,  $(u_i)_i$  and  $(V_i)_i$ , relatively to the problem (P) with the following condition:

$$\begin{cases} 0 \leq V_i \leq b < +\infty \\ V_i \rightarrow V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \leq C. \end{cases}$$

Then, there is a positive constant  $c = c(a, b, C, \Omega)$  such that,

$$\|u_i\|_{L^\infty(\Omega)} \leq c, \quad \forall i.$$

### **Proof of the Theorem 1 and 2.**

#### **Lemma 1:**

There is a subsequence  $(u_j)_j$  of  $(u_i)_i$  and a measurable function  $u$  such that:

- 1)  $u \geq 0$  a.e in  $\Omega$ .
- 2)  $\int_{\Omega} e^u dx < +\infty$ .
- 3)  $u_j \rightarrow u$  a.e in  $\Omega$ .
- 4)  $u_j \rightarrow u$  weakly in  $W_0^{1,q}(\Omega)$  for all  $q \in [1, 2]$ .
- 5)  $u_j \rightarrow u$  in  $L_{loc}^\infty(\Omega)$ .

#### **Proof of the Lemma 1:**

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

We know that,

$$\int_{\Omega} e^{u_i} dx \leq C'.$$

We can use Boccardo-Gallouet Theorem, see [B-G], to have:

$$\|\nabla u_i\|_{L^q(\Omega)} \leq c(q, b, C', \Omega).$$

We use the Rellich-Kondrachov embedding, to extract from  $(u_i)$  a subsequence  $(u_t)$  which converge in some  $L^r(\Omega)$ ,  $r > 1$  to some function  $u$ , we know that, we can have a weakly convergence for a subsequence  $(u_s)$  of  $(u_t)$ . From a strong convergence of  $(u_s)$  in  $L^r(\Omega)$ ,  $r > 1$ , we can extract a subsequence  $(u_m)$  which converge to  $u$  almost everywhere on  $\Omega$ , we can see that  $u \geq 0$  almost everywhere. We can use the Fatou lemma for  $e^{u_m}$  to have  $e^u \in L^1(\Omega)$ .

Now, we can take a subsequence of exponent  $q_n \rightarrow 2$ ,  $q_n < 2$  and, we use the diagonal process, to extract from  $(u_m)$  a subsequence  $(u_n)$  which converge to  $u$  ( $\Omega$  is bounded), weakly

in  $W_0^{1,q}(\Omega)$ ,  $1 \leq q < 2$ , almost everywhere and in some  $L^h$  for some  $h > 1$  large enough. Thus, 1), 2), 3) and 4) are true.

If we use a corollary of Theorem 3 of the Brezis-Merle paper, see [B-M], we can say that  $(u_m)$  is in  $L_{loc}^\infty(\Omega)$ .

First, we consider an exhaustive sequence of compact sets of  $\Omega$ ,  $(K_l)$ , we use the diagonal process, to have a subsequence  $(u_j)$  of  $(u_n)$  which converge in  $L^\infty(K_l)$  to a function  $u_{K_l}$  for every compact  $K_l$ , but  $(u_j)$  converge almost everywhere to  $u$  and we can conclude that  $u_{K_l} = u$  a.e. Thus,  $(u_j)$  converge on all compact set  $K$  of  $\Omega$  to  $u$ . Thus, 5) is true.

Here, we denote  $D(\Omega)$  the set of smooth functions with compact support in  $\Omega$ . We write  $supp(\varphi)$  the support of  $\varphi \in D(\Omega)$ .

### 1) Regularity of $u$ .

#### Case 1: $V_i \rightarrow V$ in $C^0(\bar{\Omega})$

We write:

$$\int_{\Omega} \langle \nabla u_i | \nabla \varphi \rangle dx = \int_{\Omega} V_i e^{u_i} \varphi, \quad \forall \varphi \in D(\Omega).$$

Because  $supp(\varphi) \subset\subset \Omega$ , and  $u_i \rightarrow u$  uniformly on all compact sets of  $\Omega$ , we have:

$$\int_{\Omega} \langle \nabla u | \nabla \varphi \rangle dx = \int_{\Omega} V e^u \varphi dx, \quad \forall \varphi \in D(\Omega).$$

We know that  $u \in W_0^{1,q} \cap L^p(\Omega)$ ,  $\forall p > 1$  and  $\int_{\Omega} e^u dx < +\infty$ . We can use the corollary 1 of [B-M] to have  $\int_{\Omega} e^{ku} dx < +\infty$ , and, by the elliptic regularity, we have  $u \in C^1(\bar{\Omega})$ .

We know that, the regularity of  $(u_i)_i$  and  $(u_i)_i$  imply that on each compact set,  $(\nabla u_i)_i$  converge to a function, but we know that  $(u_i)_i$  converge weakly to  $u$  in  $W_0^{1,q}$ ,  $q > 1$ , by the diagonal process and the uniqueness of the weak limit, we can say that  $(\nabla u_i)_i$  converge uniformly to  $\nabla u$  on each compact set of  $\Omega$ .

#### Case 2: $0 < a < a < V_i < b$

We know that  $(V_i)$  is in  $L^\infty(\Omega)$ , we can use the \*-weak topology and the (\*-weakly compactness) Alaoglu-Banach Theorem to have a subsequence  $(V_i)$  and  $V \in L^\infty(\Omega)$  such that,

$$\int_{\Omega} V_i v dx \rightarrow \int_{\Omega} V v dx, \quad \forall v \in L^1(\Omega).$$

A consequence of the previous result is that  $V \geq a$  almost everywhere in  $\Omega$ . To see this, it is sufficient to consider the functions  $v = 1_{\{g \geq a\}}$  and after  $v = 1_{\{g \leq a\}}$ . Also, we have  $g \leq b$

Let  $\varphi \in D(\Omega)$ , we can use the corollary theorem 3 of [B-M], to have the uniform convergence of  $(u_i)_i$  on every compact, also, we use the same arguments as in the previous case 1, the "limit equation" is :

$$\int_{\Omega} \langle \nabla u | \nabla \varphi \rangle dx = \int_{\Omega} V e^u \varphi, \quad \forall \varphi \in D(\Omega).$$

with  $0 < a \leq V \leq b$ . We obtain the same results as in the previous case 1. The function  $u \in C^1(\bar{\Omega})$ .

### 2) The local convergence 1.

#### Goal

In this part we want to prove that there is a finite number of point  $\bar{x}_1, \dots, \bar{x}_m \in \partial\Omega$ , such that  $u_i$  converge in  $C_{loc}^{1,\theta}(\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\})$  to  $u$ .

We have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C. \quad (*)$$

Then, according to the Riesz Theorem ( see [R]), there is a bounded Radon measure  $\mu$  such that,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi d\sigma \rightarrow \int \varphi d\mu, \quad \forall \varphi \in C(\partial\Omega).$$

We call  $x_0$  a non regular point for the measure  $\mu$  if,

$$\mu(\{x_0\}) \geq 4\pi.$$

According to (\*) there is a finite number of non regular points for  $\mu$ . We denote  $\bar{x}_1, \dots, \bar{x}_m$  all the non regular points.

Without loss of , we can suppose the local piece of curve around a point  $x$  is an interval of type  $[-a, a]$ , also, we dnote  $\mu_L$  the usual measure on  $\partial\Omega$  and we can write  $d\sigma = d\mu_L$ .

Let  $x_0$  be a regular point for  $\mu$ , then,  $\mu(\{x_0\}) < 4\pi$ . Consider the family of arcs  $I_\epsilon = ]x_0 - \epsilon, x_0 + \epsilon[ \in \partial\Omega$ , we have,

$$1_{I_\epsilon}(x) \rightarrow_{\epsilon \rightarrow 0} 1_{\{x_0\}}(x), \quad 1_{I_\epsilon} \leq 1 \text{ and } \partial\Omega \text{ is compact.}$$

We can use the Lebesgue dominated convergence Theorem for  $\mu$  to have:

$$\mu(I_\epsilon) \rightarrow \mu(\{x_0\}) \text{ if } \epsilon \rightarrow 0.$$

We construct cutoff function  $\eta_\epsilon \in D(\partial\Omega)$  on  $I_\delta$ ,  $\delta = \min_{1 \leq i \neq j \leq m} \{d(\bar{x}_i, \bar{x}_j)/2\}$  ( we can use charts), as:

$$\begin{cases} \eta_\epsilon \equiv 1, & \text{on } I_\epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, & \text{outside } I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ \|\nabla \eta_\epsilon\|_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

$$\eta_\epsilon \equiv 1 \text{ on } I_\epsilon \text{ and } \eta_\epsilon \equiv 0 \text{ on } \partial\Omega - I_{2\epsilon}.$$

We extend  $\eta_\epsilon$  to a function  $\tilde{\eta}_\epsilon$  on  $\bar{\Omega}$  as,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0, & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega. \end{cases}$$

**Key 1:** We can extend  $\eta_\epsilon$  to a function  $\bar{\eta}_\epsilon$  explicitly, we take  $\eta_\epsilon$  and we translate it with respect to the normal vector in  $x_0$ , after we regulise it like for  $\eta_\epsilon$  between  $\epsilon$  and  $2\epsilon$  on  $[-a, a] \subset \partial\Omega$ .

We know from strong maximum principal and the elliptic regularity,( because  $\bar{\eta}_\epsilon - \tilde{\eta}_\epsilon \in C_0^1(\bar{\Omega})$  l'ensemble des fonctions  $C^1$  nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \bar{\eta}_\epsilon \leq 1, \\ \|\nabla \bar{\eta}_\epsilon\|_{L^\infty(\bar{\Omega})} \leq \frac{C_1}{\epsilon}, \quad C_1 \text{ depends only on } \Omega \text{ and } x_0. \end{cases}$$

Those estimates are easy obtained, because we use the **Key1**, the functions  $\bar{\eta}_\epsilon$  or constant in  $t$ , because  $\bar{\eta}_\epsilon(t, x) = \eta_\epsilon(x)$  in the most important part of the new subdomain of  $\bar{\Omega}$ , also on the corners where we have the estimate of type  $c/\epsilon$ ,  $c > 0$ .

Now, we write:

$$\Delta[(u_i - u)\eta_\epsilon] = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 \langle \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon \rangle. \quad (**)$$

Here we want to prove that for  $\epsilon > 0$  small enough,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0/2. \quad (1)$$

where  $\epsilon_0 > 0$  small enough.

**Remark:** To obtain our estimate, it is sufficient to choose and to reduce the  $I_{2\epsilon}$ . In fact we want to find  $\tilde{\epsilon} > 0$ , and a rank  $i' = i'(\tilde{\epsilon}) \in \mathbb{N}$  such we have an uniform estimate for the sequence  $(u_i)_i$  on a domain which the boundary in  $\partial\Omega$  is  $I_{\tilde{\epsilon}}$ .

Here we do not search to tend  $\epsilon$  to 0 to have an estimate, but only reduce it to have a local uniform estimate.

Step 1: Estimate of the integral of the first term of the right hand side of (\*\*).

We have,

$$\begin{cases} \Delta u = V e^u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between  $\tilde{\eta}_\epsilon$  and  $u$ , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (*')$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between  $u_i$  and  $\eta_\epsilon$  to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \xrightarrow{i \rightarrow +\infty} \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0 \quad (**')$$

From (\*) et (\*\*') we have for all  $\epsilon > 0$  there is  $i_0 = i_0(\epsilon)$  such that, for  $i \geq i_0$ ,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (**''')$$

**Remark:** In fact, we reduce the interval  $I_\epsilon$  and we conserve the fact that the integral is strictly smaller than  $4\pi$ . The fact that  $\epsilon > 0$  is small for the interval is not a problem, because our goal is to find an  $\epsilon$  for which the estimate of the integral in (\*\*') is strictly smaller than  $4\pi$ .

Step 2: Estimate of integral of the second term of the right hand side of (\*\*).

Let  $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^2\}$ ,  $\epsilon > 0$ . Then, for  $\epsilon$  small enough,  $\Sigma_\epsilon$  is hypersurface.

We can construct another hypersurface, more easily. We know that  $\Omega$  is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the compactness, we can cover the boundary by a finite number of open set of charts, we work locally on open sets of  $\mathbb{R}_+^2$  and we construct step by step a 2-manifold  $\Omega_{\epsilon^2}$  contained in  $\Omega$  with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of  $\Omega - \Omega_{\epsilon^2}$  is  $k_2 \leq \mu_L(\Omega - \Omega_{\epsilon^2}) \leq k_1 \epsilon^2$ .

**Remark:** for the unit ball  $\bar{B}(0, 1)$ , our new manifold is  $\bar{B}(0, 1 - \epsilon^2)$ .

We write,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx = \int_{\Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx + \int_{\Omega - \Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx. \quad (***)$$

Step 2.1: Estimate of  $\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$ .

First, we know from the elliptic estimates that  $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1/\epsilon$ ,  $C_1$  depends on  $\Omega$

We know that  $(|\nabla u_i|)_i$  is bounded in  $L^q$ ,  $1 < q < 2$ , we can extract from this sequence a subsequence which converge weakly to  $h \in L^q$ . But, we know that we have locally the uniform convergence to  $|\nabla u|$ , then,  $h = |\nabla u|$  a.e.

If we take  $f = 1_{\Omega - \Omega_{\epsilon^2}}$ , we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| + \epsilon^2.$$

Then, for  $i \geq i_1(\epsilon)$ ,

$$\int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq \text{mes}(\Omega - \Omega_{\epsilon^2}) \|\nabla u\|_{L^{\infty}} + \epsilon^2 = \epsilon^2 (k_1 \|\nabla u\|_{L^{\infty}} + 1).$$

Thus,

$$\begin{aligned} \int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx &\leq \|\nabla \tilde{\eta}_{\epsilon}\|_{\infty} \left( \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| + |\nabla u| \right) \leq \\ &\leq \epsilon C_1 (k_1 \|\nabla u\|_{L^{\infty}} + 1) + C_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 1) \end{aligned}$$

Finally, we have,

$$\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 1) \quad (\bar{*})$$

The constant  $C_1$  do not depend on  $\epsilon$  but on  $\Omega$ , in its definition.

Step 2.2: Estimate of  $\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$ .

We know that,  $\Omega_{\epsilon} \subset \subset \Omega$ , and  $u_i \rightarrow u$  in  $C^1(\Omega_{\epsilon^2})$

We have,

$$\|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon^2})} \leq \epsilon^2, \text{ for } i \geq i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \| \nabla(u_i - u) \|_{L^{\infty}(\Omega_{\epsilon^2})} \| \nabla \tilde{\eta}_{\epsilon} \|_{L^{\infty}} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For  $\epsilon > 0$ , we have for  $i \in \mathbb{N}$ ,  $i \geq \max\{i_1, i_2, i_3\}$ ,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2) \quad (*''''')$$

From  $(*''''')$  and  $(\bar{*})$ , we have, for  $\epsilon > 0$ , there is  $i_3 = i_3(\epsilon) \in \mathbb{N}$ ,  $i_3 = \max\{i_0, i_1, i_2\}$  such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2 + C) \quad (\hat{*})$$

We choose  $\epsilon > 0$  small enough, we have (1).

We have:



$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$ .

We can use the theorem 1 of [B-M] to conclude that there is  $q > 1$  such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i - u)} dx \leq \int_{\Omega} e^{q(u_i - u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where,  $V_\epsilon(x_0)$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ .

Thus, for each  $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$  there is  $\epsilon_{x_0} > 0, q_{x_0} > 1$  such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \quad \forall i.$$

Now, we consider a cutoff function  $\eta \in C^\infty(\mathbb{R}^2)$  such that:

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write,

$$\Delta(u_i \eta) = V_i e^{u_i} \eta - 2 \langle \nabla u_i, \nabla \eta \rangle + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in  $L^{q_1}(\Omega)$  for  $q_1 = \inf\{q_{x_0}, 2\}$ . Thus, we can use the elliptic estimates to have  $(u_i \eta)_i$  uniformly bounded in  $W^{2, q_1}(\Omega)$  and by the Sobolev embedding, we have  $(u_i \eta)_i$  uniformly bounded in  $C^1(\bar{\Omega})$ , and if we repeat the previous procedure, we can say that  $(u_i \eta)_i$  is uniformly bounded in  $C^{1, \theta}(\bar{\Omega})$  for some  $\theta \in ]0, 1[$ .

Finally, we have, for some  $\epsilon > 0$  small enough,

$$\|u_i\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

Finally we have proved that, there is a finite number of points  $\bar{x}_1, \dots, \bar{x}_m$  such that the sequence  $(u_i)_i$  is uniformly bounded in  $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$ .

We know that, because  $u_i = 0$  on  $\partial\Omega$  and  $\Delta u_i \geq 0$ :

$$\max_{\Omega} u_i = u_i(x_i), \quad x_i \in \Omega.$$

Without loss of generality can assume that  $x_i \rightarrow \bar{x}_1$ .

### 3) Comparison of the measures.

Here we want to compare the usual measure  $d\sigma = d\mu_L$  on  $\partial\Omega$  with the new measure  $\mu$ .

#### Step 1:

Here, we want to prove that:

$$\int_{\partial\Omega} \partial_\nu u \varphi d\sigma + \sum_{k=1}^m \bar{\mu}_k \varphi(\bar{x}_k) = \int_{\partial\Omega} \varphi d\mu, \quad (2)$$

here,  $\bar{\mu}_i = \mu(\{\bar{x}_i\})$ .

*Proof.* It follows from the uniform convergence of our sequence on every compact set (obtained by the diagonal process).

#### Step 2:

Next, to simplify our computations, we assume that the piece of curve of  $\partial\Omega$  are as intervals with the usual Lebesgue measure denoted  $\mu_L$ . We also write  $dx = d\mu_L$  and  $\bar{x}_k = k - 1, k = 1, \dots, m$ .

Without loss of , we can suppose the local piece of curve around 0 is an interval of type  $[-a, a]$ , also, we dnote  $\mu_L$  the usual measure on  $\partial\Omega$  and we can write  $d\sigma = d\mu_L$ .

We want to prove that around every blow-up point  $k \in \{0, \dots, m\}$ ,

$$d\sigma = h_k d\mu \text{ with } h \geq 0, h \in L^1([-a_k, a_k], d\mu) \quad (3)$$

where  $a_k > 0$  is such that,

$$V_i \geq \alpha_k > 0 \text{ on } [-a_k, a_k] \quad (4)$$

**Fundamental remark:** (relation between the fact  $0 \leq a \leq V_i \leq b$  and  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$ ): To prove locally a relation between the two measures, we need a condition as in (4), it is the case when we assume  $0 \leq a \leq V_i \leq b$ . Now we look to the condition,  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$ .

We know that 0 is a blow-up of  $(\partial_\nu u_i)_i$ . We take a continuous function  $\varphi_\epsilon$  with compact support on  $[-a, a]$  such that:

$$0 \leq \varphi_\epsilon \leq 1, \varphi_\epsilon \equiv 1 \text{ on } [-b, b] \text{ with } 2b < a.$$

$\varphi_\epsilon$  vanish outside  $[-2b, 2b]$ .

$$\begin{cases} \Delta \bar{\varphi}_\epsilon = 0 \text{ dans } \Omega \\ \bar{\varphi}_\epsilon = \varphi_\epsilon. \end{cases}$$

Now, we can use the Green formula between  $\bar{\varphi}_\epsilon$  et  $u_i$  to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \varphi_\epsilon d\sigma \rightarrow \mu(\varphi_\epsilon) \geq \mu(\{0\}) = \mu_1 > 0$$

The maximum principle imply that  $0 = \min_{\bar{\Omega}} \bar{\varphi}_\epsilon \leq \bar{\varphi}_\epsilon \leq \bar{\varphi}_\epsilon \leq \max_{\bar{\Omega}} \bar{\varphi}_\epsilon = 1$ .

If  $V(0) = 0$ , thus,

$$\forall \epsilon' > 0 \exists \delta > 0 \text{ such that } 0 \leq V(x) \leq \epsilon' \text{ on } [-\delta, \delta]$$

But,  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$ , then,

$$\forall x \in [-\delta', \delta'] \forall i \geq i_0 \ 0 \leq V_i(x) \leq 2\epsilon'$$

with  $\delta' = \inf\{\delta, b\}$ .

If we use the fact that  $\int_{\Omega} e^{u_i} dx \leq C$ , we have,

$$0 < \mu_1 \leq 2C\epsilon' \forall \epsilon' > 0,$$

It is a contradiction if we take  $\epsilon' \rightarrow 0$ .

Finally,

$$V(0) > 0. \quad (5)$$

The fact that  $V_i \rightarrow V$  in  $C^0(\bar{\Omega})$ , imply that,

$$\exists a', \alpha_0 > 0 \text{ such that } V_i(x) \geq \alpha_0 > 0 \text{ on } [-a', a']$$

Now, we want to prove (3).

we try to prove that the measure  $d\mu$  is absolutely continuous to  $d\sigma$  :

We want to prove,

$$\mu(A) = 0 \Rightarrow \mu_L(A) = 0.$$

Without loss of , we can suppose the piece of curve is an interval  $[-a, a]$ , also, we dnote  $\mu_L$  the usual measure on  $\partial\Omega$  and we can write  $d\sigma = d\mu_L$ .

Let  $A \subset [-a, a]$  such that  $\mu(A) = 0$ . The measure  $\mu$  is regular and we have,

$$0 = \mu(A) = \sup\{\mu(U), U \text{ open set } A \subset U\}.$$

$\forall \epsilon > 0, \exists U_\epsilon$  open set of  $[-a, a]$  such that  $\mu(U_\epsilon) \leq \epsilon$ .

We consider  $\omega_\epsilon \subset U_\epsilon$  such that  $\mu_L(U_\epsilon - \omega_\epsilon) \leq \epsilon$ . (It is possible by an exhaustiv sequence of compact sets of  $U_\epsilon$ ).

We take a continuous function  $\varphi_\epsilon$  with compact support on  $U_\epsilon$  such that:

$$0 \leq \varphi_\epsilon \leq 1, \varphi_\epsilon \equiv 1 \text{ on } \omega_\epsilon.$$

Consider the following system:

$$\begin{cases} \Delta \bar{\varphi}_\epsilon = 0 \text{ dans } \Omega \\ \bar{\varphi}_\epsilon = \varphi_\epsilon. \end{cases}$$

Now, we can use the Green formula between  $\bar{\varphi}_\epsilon$  et  $u_i$  to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx = \int_{\partial\Omega} \partial_{\nu} u_i \varphi_\epsilon d\sigma \rightarrow \mu(\varphi_\epsilon) \leq \mu(U_\epsilon) \leq \epsilon.$$

Let  $G$  be the Green function of the Laplacian on  $\Omega$ . We can write:

$$\bar{\varphi}_\epsilon(x) = \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma.$$

We can write,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq a \int_{\Omega} V_i e^{u_i} \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma dx.$$

Then,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq a \int_{\{x, d(x, \partial\Omega) \geq \alpha_0\}} \int_{\partial\Omega} \partial_{\nu, y} G(x, y) \varphi_\epsilon d\sigma dx.$$

With  $0 < \alpha_0 \leq \frac{1}{2} \sup\{d(x, y), x, y \in \bar{\Omega}\}$ .

We use the definition of the Green function,

$$\Delta G(x, y) = \delta_x, \quad G(x, y) = 0 \text{ on } \partial\Omega.$$

Let,

$$E_1 = \{x, d(x, \partial\Omega) \geq \alpha_0 > 0\} \text{ and } E_2 = \{y, d(y, \partial\Omega) \leq \alpha_0/2\}.$$

We have  $E_1 \cap E_2 = \{\emptyset\}$  and by the strong maximum principle, we have,

$$\partial_{\nu, y} G(x, y) \geq \beta_0 > 0 \quad \forall x \in E_1, y \in E_2.$$

Then,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_\epsilon dx \geq m_0 \mu_L(\omega_\epsilon).$$

We can write,

$$m_0 \mu_L(\omega_\epsilon) = m_0 [\mu_L(U) - \epsilon] \leq \mu(U_\epsilon) + \epsilon \leq 2\epsilon.$$

Then,

$$\mu_L(A) \leq \mu_L(U_\epsilon) \leq \frac{(2 + m_0)\epsilon}{m_0}.$$

Finally,

$$\mu_L(A) = 0.$$

If we use the Radon-Nikodym theorem we obtain (3).

#### 4) The Local convergence 2

Next, to simplify our computations, we assume that the piece of curve of  $\partial\Omega$  are as intervals with the usual Lebesgue measure denoted  $\mu_L$ . We also write  $dx = d\mu_L$  and  $\bar{x}_k = k - 1, k = 1, \dots, m$ .

Without loss of , we can suppose the local piece of curve around  $x_0 = 0$  is an interval of type  $[-a, a]$ , also, we dnote  $\mu_L$  the usual measure on  $\partial\Omega$  and we can write  $d\sigma = d\mu_L$ .

We follow the method of the "local convergence 1", we choose other type of function  $\eta_\epsilon$ .

$$\begin{cases} \eta_\epsilon(x) = \left(\frac{|x|}{\epsilon}\right)^{2/3}, & \text{on } |x| \leq \epsilon, \quad 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 1, & \text{on } [\epsilon, 2\epsilon] \\ \eta_\epsilon = -\frac{x}{\epsilon} + 3, & \text{on } [\epsilon, 2\epsilon] \\ \eta_\epsilon \equiv 0, & \text{outside } [3\epsilon, a], \\ 0 \leq \eta_\epsilon \leq 1, \end{cases}$$

A computation gave:

$$\|\eta'_\epsilon\|_{W^{1,5/2}([-3\epsilon, 3\epsilon])} \leq C_1 \epsilon^{-\alpha}, \quad \alpha > 0 \quad (6)$$

**Remark about the exponent 5/2.** here our functions  $\eta_\epsilon$  are not  $C^1$ , but in  $W^{1,5/2}$  with  $5/2 > 2$ , and when we use the Hölder inequality in

$\int_\Omega < \nabla(u_i - u) | \tilde{\eta}_\epsilon$  we can use many argument to have this quantity small.

We extend  $\eta_\epsilon$  to a function  $\tilde{\eta}_\epsilon$  on  $\bar{\Omega}$  as,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0, & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega. \end{cases}$$

As in "The local convrgence 1" we have the:

**Key 2:** We can extend  $\eta_\epsilon$  to a function  $\tilde{\eta}_\epsilon$  explicitly, we take  $\eta_\epsilon$  and we translate it with respect to the normal vector in  $x_o$ , after we regulise it like for  $\eta_\epsilon$  between  $\epsilon$  and  $2\epsilon$  on  $[-a, a] \subset \partial\Omega$ .

We know from strong maximum principal and the elliptic regularity,( because  $\tilde{\eta}_\epsilon - \tilde{\eta}_\epsilon \in C_0^1(\bar{\Omega})$  l'ensemble des fonctions  $C^1$  nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \tilde{\eta}_\epsilon \leq 1, \\ \|\nabla \tilde{\eta}_\epsilon\|_{L^{1,5/2}(\Omega)} \leq \frac{C_1}{\epsilon^\beta}, \beta > 0 \quad C_1 \text{ depends only on } \Omega \text{ and } x_0 \end{cases} \quad (7).$$

Those estimates are easy obtained, because we use the **Key2**, the functions  $\tilde{\eta}_\epsilon$  or constant in  $t$ , because  $\tilde{\eta}_\epsilon(t, x) = \eta_\epsilon(x)$  in the most important part of the new subdomain of  $\bar{\Omega}$ , also on the corners where we have the estimate of type  $c/\epsilon^s, s > 0, c > 0$ .

Now, we write:

$$\Delta[(u_i - u)\eta_\epsilon] = (V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon - 2 < \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon > . \quad (8)$$

Here we want to prove that for  $\epsilon > 0$  small enough,

$$\int_\Omega |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0/2. \quad (9)$$

where  $\epsilon_0 > 0$  small enough.

**Remark:** To obtain our estimate, it is sufficient to choose and to radius the  $I_{2\epsilon}$ . In fact we want to find  $\tilde{\epsilon} > 0$ , and a rank  $i' = i'(e') \in \mathbb{N}$  such we have an uniform estimate for the sequence  $(u_i)_i$  on a domain which the boundary in  $\partial\Omega$  is  $I_{e'}$ .

Here we do not search to tend  $\epsilon$  to 0 to have an estimate, but only reduce it to have a local uniform estimate.

We can use the (1) and (2) to have,

$$\int_{-3\epsilon}^{3\epsilon} (\partial_\nu u_i) \eta_\epsilon(x) dx \rightarrow \int_{-3\epsilon}^{3\epsilon} \eta_\epsilon(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_\nu u) \eta_\epsilon(x) \leq 6\epsilon \|\partial_\nu u\|_{L^\infty}.$$

Step 1: Estimate of the integral of the first term of the right hand side of (9).

We have,

$$\begin{cases} \Delta u = V e^u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 & \text{in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between  $\tilde{\eta}_\epsilon$  and  $u$ , we obtain,

$$\int_{\Omega} V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon \|\partial_\nu u\|_{L^\infty} = C\epsilon \quad (*')$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega \\ u_i = 0, & \text{on } \partial\Omega, \end{cases}$$

We use the Green formula between  $u_i$  and  $\eta_\epsilon$  to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \xrightarrow{i \rightarrow +\infty} \mu(\eta_\epsilon) = \int_{-3\epsilon}^{3\epsilon} \eta_\epsilon(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_\nu u) \eta_\epsilon(x) \leq 6\epsilon \|\partial_\nu u\|_{L^\infty}. (*'')$$

From  $(*)'$  et  $(*)''$  we have for all  $\epsilon > 0$  there is  $i_0 = i_0(\epsilon)$  such that, for  $i \geq i_0$ ,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \quad (*''')$$

**Remark:** In fact, we reduce the interval  $I_\epsilon$  and we conserve the fact that the integral is strictly smaller than  $4\pi$ . The fact that  $\epsilon > 0$  is small for the interval is not a problem, because our goal is to find an  $\epsilon$  for which the estimate of the integral in  $(**)$  is strictly smaller than  $4\pi$ .

Step 2: Estimate of the integral of the second term of the right hand side of (8).

Let  $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^2\}$ ,  $\epsilon > 0$ . Then, for  $\epsilon$  small enough,  $\Sigma_\epsilon$  is hypersurface.

We can construct another hypersurface, more easily. We know that  $\Omega$  is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the compactness, we can cover the boundary by a finite number of open set of charts, we work locally on open sets of  $\mathbb{R}_+^2$  and we construct step by step a 2-manifold  $\Omega_\epsilon$  contained in  $\Omega$  with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of  $\Omega - \Omega_{\epsilon^2}$  is  $\mu_L(\Omega - \Omega_\epsilon) \leq k_1 \epsilon^\gamma$ .

**Key 3:** The choice of  $\gamma$  is linked to  $\beta$ , we choose it later where appear  $\mu_L(\Omega - \Omega_\epsilon)$  and  $\epsilon^{-\beta}$ .

We write,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx = \int_{\Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx + \int_{\Omega - \Omega_{\epsilon'}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx. \quad (***)$$

Step 2.1: Estimate of  $\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$ .

First, we know from the elliptic estimates that  $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{5/2}} \leq C_1/\epsilon^{\beta}$ ,  $C_1$  depends on  $\Omega$

We know that  $(|\nabla u_i|^{5/3})_i$  is bounded in  $L^{3/2}$ ,  $1 < 3/2 < 2$ , we can extract from this sequence a subsequence which converge weakly to  $h \in L^q$ . But, we know that we have locally the uniform convergence to  $|\nabla u|$ , then,  $h = |\nabla u|^{5/3}$  a.e.

If we take  $f = 1_{\Omega - \Omega_{\epsilon^2}}$ , we have:

$$\text{for } \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, i \geq i_1 \int_{\Omega - \Omega_{\epsilon}} |\nabla u_i|^{5/3} \leq \int_{\Omega - \Omega_{\epsilon}} |\nabla u|^{5/3} + \epsilon^{\gamma} \leq (1 + \|\nabla u\|_{L^{\infty}}^{5/3}) \epsilon^{\gamma}.$$

If we use the Hölder inequality we have,

$$\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \|\nabla \tilde{\eta}_{\epsilon}\|_{L^{5/2}} \times \left[ \int_{\Omega - \Omega_{\epsilon}} (|\nabla u_i| + |\nabla u|)^{5/3} dx \right]^{3/5} \leq$$

The Minkowski give,

$$\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq C_2 \epsilon^{-\beta} \left[ \left( \int_{\Omega - \Omega_{\epsilon}} (|\nabla u_i|^{5/3})^{3/5} + \left( \int_{\Omega - \Omega_{\epsilon}} |\nabla u|^{5/3} dx \right)^{3/5} \right)^{5/3} \right]$$

Thus,

$$\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq C_3 \epsilon^{5/3\gamma - \beta}$$

Il suffit de prendre  $5/3\gamma - \beta > 0$  pour avoir une estimation negligible de l'intégrale  $\int_{\Omega - \Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$

Step 2.2: Estimate of  $\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx$ .

We know that,  $\Omega_{\epsilon} \subset \subset \Omega$ , and  $u_i \rightarrow u$  in  $C^1(\Omega_{\epsilon^2})$

We have,

$$\|\nabla(u_i - u)\|_{L^{\infty}(\Omega_{\epsilon^2})} \leq \epsilon^{\beta+1}, \text{ for } i \geq i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon^2}} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \| \langle \nabla(u_i - u) \rangle \|_{L^{\infty}(\Omega_{\epsilon^2})} \|\nabla \tilde{\eta}_{\epsilon}\|_{L^{\infty}} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For  $\epsilon > 0$ , we have for  $i \in \mathbb{N}$ ,  $i \geq \max\{i_1, i_2, i_3\}$ ,

$$\int_{\Omega} | \langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^{\infty}} + 2) \quad (****)$$

From (\*\*\*\*) and (\*\*\*\*), we have, for  $\epsilon > 0$ , there is  $i_3 = i_3(\epsilon) \in \mathbb{N}$ ,  $i_3 = \max\{i_0, i_1, i_2\}$  such that,

$$\begin{aligned} \int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx &\leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1 \|\nabla u\|_{L^{\infty}} + 2 + C) \quad (\hat{*}) \\ &\leq \epsilon C_1(k_1 \|\nabla u\|_{L^{\infty}} + 1) + C_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| \leq \epsilon C_1(2k_1 \|\nabla u\|_{L^{\infty}} + 1) \end{aligned}$$

Finally, we have,

$$\int_{\Omega - \Omega_{\epsilon_2}} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon \rangle| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 1) \quad (\bar{*})$$

The constant  $C_1$  do not depend on  $\epsilon$  but on  $\Omega$ , in its definition.

For  $\epsilon > 0$ , we have for  $i \in \mathbb{N}$ ,  $i \geq \max\{i_1, i_2\}$ ,

$$\int_{\Omega} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_\epsilon \rangle| dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \quad (*''')$$

From  $(*''')$  and  $(\bar{*})$ , we have, for  $\epsilon > 0$ , there is  $i_3 = i_3(\epsilon) \in \mathbb{N}$ ,  $i_3 = \max\{i_0, i_1, i_2\}$  such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \quad (\hat{*})$$

We choose  $\epsilon > 0$  small enough, we have (1).

We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$ .

We can use the theorem 1 of [B-M] to conclude that there is  $q > 1$  such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i - u)\tilde{\eta}_\epsilon} dx \leq \int_{\Omega} e^{q(u_i - u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where,  $V_\epsilon(x_0) = B(x_0, \epsilon) \cap \Omega$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ .

We have,

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}(u_i - u)\tilde{\eta}_\epsilon} dx \leq C, \quad \forall i.$$

Now, we consider a function  $\eta \in C^\infty(\mathbb{R}^2)$  such that:

$$\Delta(u_i \eta) = V_i e^{u_i} \eta - 2 \langle \nabla u_i | \nabla \eta \rangle + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in  $L^{q_1}(\Omega)$  for  $q_1 = \inf q_{x_0}, 2$ . Thus, we can use the elliptic estimates to have  $(u_i \eta_\epsilon)_i$  uniformly bounded in  $W^{2, q_1}(\Omega)$  and by the Sobolev embedding, we have  $(u_i \eta_\epsilon)_i$  uniformly bounded in  $C^1(\bar{\Omega})$ , and if we repeat the previous procedure, we can say that  $(u_i \eta_\epsilon)_i$  is uniformly bounded in  $C^{1, \theta}(\bar{\Omega})$  for some  $\theta \in ]0, 1[$ .

Finally, we have, for some  $\epsilon > 0$  small enough,

$$\|u_i \eta_\epsilon\|_{C^{1, \theta}[B(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

We know that, because  $u_i = 0$  on  $\partial\Omega$  and  $\Delta u_i \geq 0$ :

$$\max_{\Omega} u_i = u_i(x_i), \quad x_i \in \Omega.$$

Without loss of generality can assume that  $x_i \rightarrow \bar{x}_1$ .

We take the convention  $x_0 = 0$ . We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

With  $\|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0$ .

We can use the theorem 1 of [B-M] to conclude that there is  $q > 1$  such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i-u)\tilde{\eta}_\epsilon} dx \leq \int_{\Omega} e^{q(u_i-u)\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega).$$

where,  $V_\epsilon(x_0)$  is a neighborhood of  $x_0$  in  $\bar{\Omega}$ .

There is  $\epsilon_{x_0} > 0, q_{x_0} > 1$  such that:

$$\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0} u_i \eta_\epsilon} dx \leq C, \quad \forall i.$$

Now, we consider a cutoff function  $\eta \in C_0^\infty(\mathbb{R}^2)$  (smooth functions with compact support) such that:

$$\Delta(u_i \eta) = V_i e^{u_i \eta} - 2 \langle \nabla u_i | \nabla \eta \rangle + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in  $L^{q_1}(\Omega)$  for  $q_1 = \inf\{q_{x_0}, 2\}$ . Thus, we can use the elliptic estimates to have  $(u_i \eta)_i$  uniformly bounded in  $W^{2, q_1}(\Omega)$  and by the Sobolev embedding, we have  $(u_i \eta)_i$  uniformly bounded in  $C^1(\bar{\Omega})$ , and if we repeat the previous procedure, we can say that  $(u_i \eta)_i$  is uniformly bounded in  $C^{1, \theta}(\bar{\Omega})$  for some  $\theta \in ]0, 1[$ .

Finally, we have, for some  $\epsilon > 0$  small enough,

$$\|u_i \tilde{\eta}_\epsilon\|_{C^{1, \theta}[\bar{B}(x_0, \epsilon)]} \leq c_3 \quad \forall i.$$

On  $I_\epsilon = [-\epsilon, \epsilon]$ , because  $u_i = 0$  on  $I_\epsilon$ ,

$$\text{For } y \in \Omega \quad |\nabla(u_i \eta_\epsilon)(y)| = |\nabla u_i(x) \eta_\epsilon(y) + u_i(y) \nabla \eta_\epsilon(y)| \leq c_3$$

If we tend  $y \rightarrow x \in I_\epsilon - \{0\}$ , ( $\tilde{\eta}_\epsilon(x) \rightarrow \eta_\epsilon(x)$ ,  $x \neq 0$  and  $u_i(x) = 0$ ) we obtain,

$$|\nabla u_i(x)| = \partial_\nu u_i(x) \leq c_4 |x|^{-2/3}, \quad \text{on } [-\epsilon, \epsilon] - \{0\}$$

Thus,

$$\int_{-\epsilon}^{\epsilon} [\partial_\nu u_i(x)]^{4/3} dx \leq c_5 \quad \forall i \geq i_4, \quad i_4 \in \mathbb{N}$$

But,

$$\partial_\nu u_i \rightarrow \partial_\nu u, \quad \mu_L.a.e.$$

After a subsequence,  $\partial_\nu u_i$  converge weakly to  $\partial_\nu u$ , thus,

$$\int_{-\epsilon}^{\epsilon} \partial_\nu u_i \varphi dx \rightarrow \int_{-\epsilon}^{\epsilon} \partial_\nu u \varphi, \quad \forall \varphi \in C_c([-\epsilon, \epsilon])$$

Where,  $C_c([-\epsilon, \epsilon])$  is a set of continuous function with compact support.

But, we know,

$$\int_{\partial\Omega} \partial_\nu u_i \varphi dx = \int_{-\epsilon}^{\epsilon} \partial_\nu u_i \varphi dx \rightarrow \int_{-\epsilon}^{\epsilon} \varphi d\mu.$$

Also, we know that,

$$dx = h d\mu,$$

Then,

$$\int_{-\epsilon}^{\epsilon} [(\partial_\nu u)h - 1] d\mu \varphi = 0 dx, \quad \forall \varphi \in C_c([-\epsilon, \epsilon])$$

We conclude that,

$$(\partial_\nu u)h = 1. \quad \mu.a.e$$

It means thta,



$$\mu_1 = 0 \quad \text{and} \quad h(0).$$

It is a contradiction.

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