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ABOUT A BREZIS-MERLE PROBLEM IN DIMENSION 2.

SAMY SKANDER BAHOURA

ABSTRACT. On a bounded open set Ω of \mathbb{R}^2 with smooth boundary, we consider two sequences of functions $(u_i)_i$ and $(V_i)_i$ such that,

$$\begin{cases} \Delta u_i = V_i e^{u_i} \text{ in } \Omega, \\ u_i = 0, \text{ on } \partial \Omega. \end{cases}$$

With, $0 \le a \le V_i \le b < +\infty$.

Assume that,

 $\int_{\Omega} e^{u_i} dx \le C.$

If,

a > 0,

or,

a = 0 and $V_i \to V$ in $C^0(\overline{\Omega})$.

We have,

 $||u_i||_{L^{\infty}(\bar{\Omega})} \le c(a, b, C, \Omega).$

1. INTRODUCTION AND RESULTS.

We set $\Delta = -\partial_{11} - \partial_{22}$ the geometric Laplacian on \mathbb{R}^2 .

On an open set Ω of \mathbb{R}^2 , with a smooth boundary, we consider the following problem:

$$(P) \begin{cases} \Delta u = V e^u \text{ in } \Omega, \\ u = 0 \text{ on } \partial \Omega. \end{cases}$$

The previous equation is called, the Prescribed Scalar Curvature, in relation with conformal change of metrics. The function V is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of the previous type were studied by many authors. We can see in [B-M], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V, for example we suppose $V \ge 0$ and $V \in L^p(\Omega)$ or $Ve^u \in L^p(\Omega)$ with $p \in [1, +\infty]$.

We can see in [B-M] the two following important Theorems,

Theorem A(*Brezis-Merle*).*If* $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 \le V_i \le b < +\infty$ and $\int_{\Omega} e^{u_i} dx \le C$, then, for all compact set K of Ω ,

$$\sup_{K} u_i \le c = c(a, b, C, K, \Omega)$$

Theorem B(*Brezis-Merle*).*If* $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \le V_i \le b < +\infty$, then, for all compact set K of Ω ,

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$$\sup_{K} u_i \le c = c(a, b, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [S], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \le V_i \le b < +\infty$, then we have the following interior estimate:

$$C(a/b)\sup_K u_i + \inf_\Omega u_i \le c = c(a,b,K,\Omega).$$
 We can see in [C-L], an explicit value of $C(a/b) = \sqrt{\frac{a}{b}}.$

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, C(a/b) = 1 and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [B-L-S]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [C-L]. Also, we can see in [L], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [L-S] explicit form, $(8\pi m, m \in \mathbb{N}^* \text{ exactly})$, for the numbers in front of the Dirac masses, when the solutions blow-up.

The questions of Brezis-Merle :

Question 1:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions $(u_i)_i$ solutions of the problem (P) relatively to $(V_i)_i$ with,

$$\begin{cases} 0 < a \le V_i \le b < +\infty \\ \int_{\Omega} e^{u_i} dx \le C'. \end{cases}$$

Is it possible to have,

$$||u_i||_{L^{\infty}(\Omega)} \le c' = c'(a, b, C', \Omega)?$$

Question 2:

In their paper Brezis-Merle, asked the following question, see [B-M]:

Let us consider a sequence of functions $(u_i)_i$ solutions of the problem (P) relatively to $(V_i)_i$ with,

$$\begin{cases} 0 \le V_i \le b < +\infty \\ V_i \to V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \le C'. \end{cases}$$

Is it possible to have,

$$||u_i||_{L^{\infty}(\Omega)} \le c' = c'(a, b, C', \Omega)?$$

We can find some results which concerning this question. It was proved by Chen-Li, that if we suppose a = 0 and $||\nabla V_i||_{L^{\infty}(\Omega)} \leq A$, then, we a have a positive answer to the question, see [C-Li].

Also, if we assume a > 0 and $||\nabla V_i||_{L^{\infty}(\Omega)} \leq A$, the answer to the question is positive without the assumption on e^{u_i} was given by Ma-Wei, see [M-W].

Here, we have,

Theorem 1. Let us consider three positive numbers a, b, C, and, two sequences of functions, $(u_i)_i$ and $(V_i)_i$, relatively to the problem (P) with the following conditions:

$$\begin{cases} 0 < a \le V_i \le b < +\infty \\ \int_{\Omega} e^{u_i} dx \le C, \ \forall \ i \end{cases}$$

Then, there is a positive constant $c = c(a, b, C, \Omega)$ such that,

$$||u_i||_{L^{\infty}(\Omega)} \le c.$$

Theorem 2. Let us consider two positive numbers a, b, and, two sequences of functions, $(u_i)_i$ and $(V_i)_i$, relatively to the problem (P) with the following condition:

$$\begin{cases} 0 \le V_i \le b < +\infty \\ V_i \to V \in C^0(\bar{\Omega}) \\ \int_{\Omega} e^{u_i} dx \le C. \end{cases}$$

Then, there is a positive constant $c = c(a, b, C, \Omega)$ such that,

$$||u_i||_{L^{\infty}(\Omega)} \le c, \ \forall \ i.$$

Proof of the Theorem 1 and 2.

Lemma 1:

There is a subsequence $(u_j)_j$ of $(u_i)_i$ and a measurable function u such that:

1) $u \ge 0$ a.e in Ω . 2) $\int_{\Omega} e^u dx < +\infty$. 3) $u_j \to u$ a.e in Ω . 4) $u_j \to u$ weakly in $W_0^{1,q}(\Omega)$ for all $q \in [1, 2[$. 5) $u_j \to u$ in $L_{loc}^{\infty}(\Omega)$.

Proof of the Lemma 1:

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial \Omega. \end{cases}$$

We know that,

$$\int_{\Omega} e^{u_i} dx \le C'.$$

We can use Boccardo-Gallouet Theorem, see [B-G], to have:

$$||\nabla u_i||_{L^q(\Omega)} \le c(q, b, C', \Omega).$$

We use the Rellich-Kondrachov embedding, to extract from (u_i) a subsequence (u_t) which converge in some $L^r(\Omega), r > 1$ to some function u, we know that, we can have a weakly convergence for a subsequence (u_s) of (u_t) . From a strong convergence of (u_s) in $L^r(\Omega), r > 1$, we can extract a subsequence (u_m) which converge to u almost everywhere on Ω , we can see that $u \ge 0$ almost everywhere. We can use the Fatou lemma for e^{u_m} to have $e^u \in L^1(\Omega)$.

Now, we can take a subsequence of exponent $q_n \rightarrow 2, q_n < 2$ and, we use the diagonal process, to extract from (u_m) a subsequence (u_n) which converge to u (Ω is bounded), weakly

in $W_0^{1,q}(\Omega)$, $1 \le q < 2$, almost everywhere and in some L^h for some h > 1 large enough. Thus, 1), 2), 3) and 4) are true.

If we use a corollary of Theorem 3 of the Brezis-Merle paper, see [B-M], we can say that (u_m) is in $L^{\infty}_{loc}(\Omega)$.

First, we consider an exhaustive sequence of compact sets of Ω , (K_l) , we use the diagonal process, to have a subsequence (u_j) of (u_n) which converge in $L^{\infty}(K_l)$ to a function u_{K_l} for every compact K_l , but (u_j) converge almost everywhere to u and we can conclude that $u_{K_l} = u$ a.e. Thus, (u_j) converge on all compact set K of Ω to u. Thus, 5) is true.

Here, we denote $D(\Omega)$ the set of smooth functions with compact support in Ω . We write $supp(\varphi)$ the support of $\varphi \in D(\Omega)$.

1) Regularity of u.

Case 1: $V_i \to V$ in $C^0(\overline{\Omega})$

We write:

$$\int_{\Omega} <\nabla u_i |\nabla \varphi > dx = \int_{\Omega} V_i e^{u_i} \varphi, \ \forall \ \varphi \in D(\Omega).$$

Because $supp(\varphi) \subset \subset \Omega$, and $u_i \to u$ uniformly on all compact sets of Ω , we have:

$$\int_{\Omega} < \nabla u | \nabla \varphi > dx = \int_{\Omega} V e^u \varphi dx, \ \forall \ \varphi \in D(\Omega).$$

We know that $u \in W_0^{1,q} \cap L^p(\Omega)$, $\forall p > 1$ and $\int_{\Omega} e^u dx < +\infty$. We can use the corollary 1 of [B-M] to have $\int_{\Omega} e^{ku} dx < +\infty$, and, by the elliptic regularity, we have $u \in C^1(\overline{\Omega})$.

We know that, the regularity of $(u_i)_i$ and $(u_i)_i$ imply that on each compact set, $(\nabla u_i)_i$ converge to a function, but we know that $(u_i)_i$ converge weakly to u in $W_0^{1,q}$, q > 1, by the diagonal process and the uniqueness of the weak limit, we can say that $(\nabla u_i)_i$ converge uniformly to ∇u on each compact set of Ω .

Case 2: $0 < a < a < V_i < b$

We know that (V_i) is in $L^{\infty}(\Omega)$, we can use the *-weak topology and the (*-weakly compactness) Alaoglu-Banach Theorem to have a subsequence (V_i) and $V \in L^{\infty}(\Omega)$ such that,

$$\int_{\Omega} V_i v dx \to \int_{\Omega} V v dx, \ \forall \ v \in L^1(\Omega).$$

A consequence of the previous result is that $V \ge a$ almost everywhere in Ω . To see this, it is sufficient to consider the functions $v = 1_{\{g \ge a\}}$ and after $v = 1_{\{g \le a\}}$. Also, we have $g \le b$

Let $\varphi \in D(\Omega)$, we can use the corollary theorem 3 of [B-M], to have the uniform convrgence of $(u_i)_i$ on every compact, also, we use the same arguments as in the previous case 1, the "limit equation" is :

$$\int_{\Omega} < \nabla u | \nabla \varphi > dx = \int_{\Omega} V e^{u} \varphi, \ \forall \ \varphi \in D(\Omega).$$

with $0 < a \le V \le b$. We obtain the same results as in the previous case 1. The function $u \in C^1(\bar{\Omega})$.

2) The local convergence 1.

Goal

In this part we want to prove that there is a finite number of point $\bar{x}_1, \ldots, \bar{x}_m \in \partial\Omega$, such that u_i converge in $C_{loc}^{1,\theta}(\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\})$ to u.

We have,

$$\int_{\partial\Omega} \partial_{\nu} u_i d\sigma \le C. \qquad (*)$$

Then, according to the Riesz Theorem (see [R]), there is a bounded Radon measure μ such that,

$$\int_{\partial\Omega} \partial_{\nu} u_i \varphi d\sigma \to \int_{\partial\Omega} \varphi d\mu, \ \forall \ \varphi \in C(\partial\Omega).$$

We call x_0 a non regular point for the measure μ if,

$$\mu(\{x_0\}) \ge 4\pi.$$

According to (*) there is a finite number of non regular points for μ . We denote $\bar{x}_1, \ldots, \bar{x}_m$ all the non regular points.

Without loss of , we can suppose the local piece of curve around a point x is an interval of type [-a, a], also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

Let x_0 be a regular point for μ , then, $\mu(\{x_0\}) < 4\pi$. Consider the family of arcs $I_{\epsilon} =]x_0 - \epsilon, x_0 + \epsilon [\in \partial\Omega]$, we have,

$$1_{I_{\epsilon}}(x) \to_{\epsilon \to 0} 1_{\{x_0\}}(x), \ 1_{I_{\epsilon}} \leq 1 \text{ and } \partial \Omega \text{ is compact.}$$

We can use the Lebesgue dominated convergence Theorem for μ to have:

$$\mu(I_{\epsilon}) \to \mu(\{x_0\}) \text{ if } \epsilon \to 0.$$

We construct cutoff function $\eta_{\epsilon} \in D(\partial\Omega)$ on I_{δ} , $\delta = \min_{1 \le i \ne j \le m} \{d(\bar{x}_i, \bar{x}_j)/2\}$ (we can use charts), as:

$$\begin{cases} \eta_{\epsilon} \equiv 1, \text{ on } I_{I_{\epsilon}}, \quad 0 < \epsilon < \delta/2, \\ \eta_{\epsilon} \equiv 0, \text{ outside } I_{2\epsilon}, \\ 0 \le \eta_{\epsilon} \le 1, \\ ||\nabla \eta_{\epsilon}||_{L^{\infty}(I_{2\epsilon})} \le \frac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

 $\eta_{\epsilon} \equiv 1 \text{ on } I_{I_{\epsilon}} \text{ and } \eta_{\epsilon} \equiv 0 \text{ on } \partial \Omega - I_{2\epsilon}.$

We extend η_{ϵ} to a function $\tilde{\eta}_{\epsilon}$ on $\bar{\Omega}$ as,

$$\begin{cases} \Delta \tilde{\eta}_{\epsilon} = 0, & \text{in } \Omega \\ \tilde{\eta}_{\epsilon} = \eta_{\epsilon}, & \text{on } \partial \Omega. \end{cases}$$

<u>**Key 1:**</u> We can extend η_{ϵ} to a function $\bar{\eta}_{\epsilon}$ explicitly, we take η_{ϵ} and we translate it with respect to the normal vector in x_o , after we regulise it like for η_{ϵ} between ϵ and 2ϵ on $[-a, a] \subset \partial \Omega$.

We know from strong maximum principal and the elliptic regularity,(because $\bar{\eta}_{\epsilon} - \tilde{\epsilon} \in C_0^1(\Omega)$ l'ensemble des fonctions C^1 nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \tilde{\eta}_{\epsilon} \le 1, \\ ||\nabla \tilde{\eta}_{\epsilon}||_{L^{\infty}(\bar{\Omega})} \le \frac{C_1}{\epsilon}, \ C_1 \text{ depends only on } \Omega \text{ and } x_0. \end{cases}$$

Those estimates are easy obtained, because we use the **Key1**, the functions $\bar{\eta}_{\epsilon}$ or constant in t, because $\bar{\eta}_{\epsilon}(t,x) = \eta_{\epsilon}(x)$ in the most important part of the new subdomain of $\bar{\Omega}$, also on the corners where we have the estimate of type c/ϵ , c > 0.

Now, we write:

$$\Delta[(u_i - u)\eta_{\epsilon}] = (V_i e^{u_i} - V e^u)\tilde{\eta}_{\epsilon} - 2 < \nabla(u_i - u)|\nabla\tilde{\eta}_{\epsilon} > . \quad (**)$$

Here we want to prove that for $\epsilon > 0$ small enough,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \le 4\pi - \epsilon_0/2.$$
 (1)

where $\epsilon_0 > 0$ small enough.

Remark: To ontain our estimate, it is sufficient to choose and to redius the $I_{2\epsilon}$. In fact we want to find $\tilde{\epsilon} > 0$, and a rank $i' = i'(\epsilon') \in \mathbb{N}$ such we have an uniform estimate for the sequence $(u_i)_i$ on a domain which the boundary in $\partial\Omega$ is $I_{\epsilon'}$.

Here we do not search to tend ϵ to 0 to have an estimate, but only reduce it to have a local uniform estimate.

Step 1: Estimate of the integral of the first term of the right hand side of (**).

We have,

$$\begin{cases} \Delta u = V e^u \text{ in } \Omega \\ u = 0, \text{ on } \partial \Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_{\epsilon} = 0 \text{ in } \Omega \\ \tilde{\eta}_{\epsilon} = \eta_{\epsilon}, \text{ on } \partial \Omega_{\epsilon} \end{cases}$$

We use the Green formula between $\tilde{\eta}_{\epsilon}$ and u, we obtain,

$$\int_{\Omega} V e^{u} \tilde{\eta}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u \eta_{\epsilon} \leq 4\epsilon ||\partial_{\nu} u||_{L^{\infty}} = C\epsilon \qquad (*')$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} \text{ in } \Omega\\ u_i = 0, \text{ on } \partial \Omega, \end{cases}$$

We use the Green formula between u_i and η_{ϵ} to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u_i \eta_{\epsilon} d\sigma \to_{i \to +\infty} \mu(\eta_{\epsilon}) \le \mu(I_{2\epsilon}) \le 4\pi - \epsilon_0, \ \epsilon_0 > 0 \qquad (*'')$$

From (*') et (*'') we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \ge i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_{\epsilon}| dx \le 4\pi - \epsilon_0 + C\epsilon \qquad (*''')$$

Remark: In fact, we reduce the interval I_{ϵ} and we conserve the fact that the integral is strictly smaller than 4π . The fact that $\epsilon > 0$ is small for the interval is not a problem, because our goal is to find an ϵ for which the estimate of the integral in (**) is strictly smaller than 4π .

Step 2: Estimate of integral of the second term of the right hand side of (**).

Let $\Sigma_{\epsilon} = \{x \in \Omega, d(x, \partial \Omega) = \epsilon^2\}, \epsilon > 0$. Then, for ϵ small enough, Σ_{ϵ} is hypersurface.

We can construct another hypersurface, more easly. We know that Ω is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the conpactness, we can couver the boundary by a finite number of open set of charts, we work locally on open sets of \mathbb{R}^2_+ and we construct step by step a 2-manifold Ω_{ϵ^2} contained in Ω with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of $\Omega - \Omega_{\epsilon^2}$ is $k_2 \leq \mu_L(\Omega - \Omega_{\epsilon^2}) \leq k_1 \epsilon^2$.

Remark: for the unit ball $\overline{B}(0, 1)$, our new manifold is $\overline{B}(0, 1 - \epsilon^2)$.

We write,

$$\int_{\Omega} |<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx=\int_{\Omega_{\epsilon'}}|<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx+\int_{\Omega-\Omega_{\epsilon'}}<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx. \quad (***)$$

<u>Step 2.1:</u> Estimate of $\int_{\Omega - \Omega_{\epsilon^2}} | < \nabla (u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx.$

First, we know from the elliptic estimates that $||\nabla \tilde{\eta}_{\epsilon}||_{L^{\infty}} \leq C_1/\epsilon$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in L^q , 1 < q < 2, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$, then, $h = |\nabla u|$ a.e.

If we take $f = 1_{\Omega - \Omega_{\epsilon^2}}$, we have:

for
$$\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \ge i_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \le \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| + \epsilon^2$$

Then, for $i \ge i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \le mes(\Omega - \Omega_{\epsilon^2}) ||\nabla u||_{L^{\infty}} + \epsilon^2 = \epsilon^2 (k_1 ||\nabla u||_{L^{\infty}} + 1)$$

Thus,

$$\begin{split} &\int_{\Omega-\Omega_{\epsilon^2}} |<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx\leq ||\nabla\tilde{\eta}_{\epsilon}||_{\infty}\left(\int_{\Omega-\Omega_{\epsilon^2}} |\nabla u_i|+|\nabla u|\right)\leq \\ &\leq \epsilon C_1(k_1||\nabla u||_{L^{\infty}}+1)+C_1\int_{\Omega-\Omega_{\epsilon^2}} |\nabla u|\leq \epsilon C_1(2k_1||\nabla u||_{L^{\infty}}+1) \end{split}$$

Finaly, we have,

$$\int_{\Omega-\Omega_{\epsilon^2}} |<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx\leq\epsilon C_1(2k_1||\nabla u||_{L^{\infty}}+1) \qquad (\bar{*})$$

The constant C_1 do not depend on ϵ but on Ω , in its definition.

<u>Step 2.2:</u> Estimate of $\int_{\Omega_{\epsilon^2}} | < \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx.$

We know that, $\Omega_{\epsilon} \subset \subset \Omega$, and $u_i \to u$ in $C^1(\Omega_{\epsilon^2})$ We have,

$$||\nabla(u_i - u)||_{L^{\infty}(\Omega_{\epsilon^2})}|| \le \epsilon^2, \text{ for } i \ge i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon_2}} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \leq || \langle \nabla(u_i - u) ||_{L^{\infty}(\Omega_{\epsilon^2})} || \nabla \tilde{\eta}_{\epsilon} ||_{L^{\infty}} \leq C_1 \epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \ge \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \le \epsilon C_1(2k_1 || \nabla u ||_{L^{\infty}} + 2) \qquad (*'''')$$

From (*''') and (*'''), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \le 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^{\infty}} + 2 + C) \qquad (\hat{*})$$

We choose $\epsilon > 0$ small enough, we have (1).

We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_{\epsilon}] = g_{i,\epsilon} \text{ in } \Omega, \\ (u_i - u)\tilde{\eta}_{\epsilon} = 0 \text{ on } \partial\Omega. \end{cases}$$

With $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is q > 1 such that:

$$\int_{V_{\epsilon}(x_0)} e^{q(u_i - u)} dx \le \int_{\Omega} e^{q(u_i - u)\tilde{\eta}_{\epsilon}} dx \le C(\epsilon, \Omega).$$

where, $V_{\epsilon}(x_0)$ is a neighborhooh of x_0 in $\overline{\Omega}$.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0}u_i} dx \le C, \quad \forall \ i.$$

Now, we consider a cutoff function $\eta \in C^{\infty}(\mathbb{R}^2)$ such that:

$$\eta \equiv 1$$
 on $B(x_0, \epsilon_{x_0}/2)$ and $\eta \equiv 0$ on $\mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3)$.

We write,

$$\Delta(u_i\eta) = V_i e^{u_i}\eta - 2 < \nabla u_i |\nabla \eta > + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf\{q_{x_0}, 2\}$. Thus, we can use the elliptic estimates to have $(u_i\eta)_i$ uniformly bounded in $W^{2,q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i\eta)_i$ uniformly bounded in $C^1(\overline{\Omega})$, and if we repit the previous procedure, we can say that $(u_i\eta)_i$ is uniformly bounded in $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in]0, 1[$.

Finaly, we have, for some $\epsilon > 0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \le c_3 \quad \forall \quad i.$$

Finaly we have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the squence $(u_i)_i$ is uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

We know that, because $u_i = 0$ on $\partial \Omega$ and $\Delta u_i \ge 0$:

$$\max_{\Omega} u_i = u_i(x_i), \ x_i \in \Omega.$$

Without loss of generality can assume that $x_i \rightarrow \bar{x}_1$.

3) Comparison of the measures.

Here we want to compare the usual measure $d\sigma = d\mu_L$ on $\partial\Omega$ with the new measure μ .

Step 1:

Here, we want to prove that:

$$\int_{\partial\Omega} \partial_{\nu} u\varphi d\sigma + \sum_{k=1}^{m} \bar{\mu}_{k} \varphi(\bar{x}_{k}) = \int_{\partial\Omega} \varphi d\mu, \qquad (2)$$

here, $\bar{\mu}_i = \mu(\{\bar{x}_i\}).$

Proof. It follows from the uniform convergence of our sequence on every compact set(obtianed by the diagonal process).

Step 2:

Next, to simplify our computations, we assume that the piece of curve of $\partial\Omega$ are as intervals with the usual Lebesgue measure denoted μ_L . We also write $dx = d\mu_L$ and $\bar{x}_k = k - 1, k = 1, \ldots, m$.

Without loss of , we can suppose the local piece of curve around 0 is an interval of type [-a, a], also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

We want to prove that around every blow-up point $k \in \{0, \ldots, m\}$,

$$d\sigma = h_k d\mu \text{ with } h \ge 0, \ h \in L^1([-a_k, a_k], d\mu)$$
(3)

where $a_k > 0$ is such that,

$$V_i \ge \alpha_k > 0$$
 on $[-a_k, a_k]$ (4)

<u>Fundamental remark</u>: (relation between the fact $0 \le a \le V_i \le b$ and $V_i \to V$ in $C^0(\overline{\Omega})$): To prove locally a realtion between the two measures, we need a condition as in (4), it is the case when we assume $0 \le a \le V_i \le b$. Now we look to the condition, $V_i \to V$ in $C^0(\overline{\Omega})$.

We know that 0 is a blow-up of $(\partial_{\nu} u_i)_i$. We take a continuous function φ_{ϵ} with compact support on [-a, a] such that:

$$0 \le \varphi_{\epsilon} \le 1, \ \varphi_{\epsilon} \equiv 1 \text{ on } [-b, b] \text{ with } 2b < a.$$

 φ_ϵ vanish outside [-2b,2b].

$$\begin{cases} \Delta \bar{\varphi}_{\epsilon} = 0 \text{ dans } \Omega \\ \bar{\varphi}_{\epsilon} = \varphi_{\epsilon}. \end{cases}$$

Now, we can use the Green formula between $\bar{\varphi}_{\epsilon}$ et u_i to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u_i \varphi_{\epsilon} d\sigma \to \mu(\varphi_{\epsilon}) \ge \mu(\{0\}) = \mu_1 > 0$$

The maximum principle imply that $0 = \min_{\bar{\Omega}} \leq \bar{\varphi}_{\epsilon} \leq \max_{\bar{\Omega}} \bar{\varphi}_{\epsilon} = 1$.

If V(0) = 0, thus,

 $\forall \ \epsilon'>0 \ \exists \ \delta>0 \ \text{such that} \ 0\leq V(x)\leq \epsilon' \ \text{ on } [-\delta,\delta]$ But, $V_i\to V \ \text{in } C^0(\bar\Omega)$, then,

$$\forall x \in [-\delta', \delta'] \ \forall i \ge i_0 \ 0 \le V_i(x) \le 2\epsilon'$$

with $\delta' = \inf{\{\delta, b\}}.$

If we use the fact that $\int_{\Omega} e^{u_i} dx \leq C$, we have,

$$0 < \mu_1 \le 2C\epsilon' \ \forall \ \epsilon' > 0,$$

It is a contradiction if we take $\epsilon' \to 0$.

Finaly,

$$V(0) > 0.$$
 (5)

The fact that $V_i \to V$ in $C^0(\overline{\Omega})$, imply that,

$$\exists a', \alpha_0 > 0$$
 such that $V_i(x) \ge \alpha_0 > 0$ on $[-a', a']$

Now, we want to prove (3).

we try to prove that the measure $d\mu$ is absolutly continous to $d\sigma$: We want to prove,

$$\mu(A) = 0 \Rightarrow \mu_L(A) = 0.$$

Without loss of , we can suppose the piece of curve is an interval [-a, a], also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

Let $A \subset [-a, a]$ such that $\mu(A) = 0$. The measure μ is regular and we have,

$$0 = \mu(A) = \sup\{\mu(U), U \text{ open set } A \subset U\}.$$

 $\forall \ \epsilon > 0, \ \exists \ U_{\epsilon} \text{ open set of } [-a,a] \text{ such that } \mu(U_{\epsilon}) \leq \epsilon.$

We consider $\omega_{\epsilon} \subset U_{\epsilon}$ such that $\mu_L(U_{\epsilon} - \omega_{\epsilon}) \leq \epsilon$. (It is possible by an exhaustiv sequence of compact sets of U_{ϵ}).

We take a continuous function φ_{ϵ} with compact support on U_{ϵ} such that:

$$0 \le \varphi_{\epsilon} \le 1, \ \varphi_{\epsilon} \equiv 1 \text{ on } \omega_{\epsilon}.$$

Consider the following system:

$$\begin{cases} \Delta \bar{\varphi}_{\epsilon} = 0 \, \, \mathrm{dans}\Omega \\ \bar{\varphi}_{\epsilon} = \varphi_{\epsilon}. \end{cases}$$

Now, we can use the Green formula between $\bar{\varphi}_{\epsilon}$ et u_i to obtain,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u_i \varphi_{\epsilon} d\sigma \to \mu(\varphi_{\epsilon}) \le \mu(U_{\epsilon}) \le \epsilon.$$

Let G be the Green function of the Laplacian on Ω . We can write:

$$\bar{\varphi}_{\epsilon}(x) = \int_{\partial\Omega} \partial_{\nu,y} G(x,y) \varphi_{\epsilon} d\sigma.$$

We can write,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_{\epsilon} dx \ge a \int_{\Omega} V_i e^{u_i} \int_{\partial \Omega} \partial_{\nu, y} G(x, y) \varphi_{\epsilon} d\sigma dx.$$

Then,

$$\begin{split} &\int_{\Omega} V_i e^{u_i} \bar{\varphi}_{\epsilon} dx \geq a \int_{\{x, d(x, \partial \Omega) \geq \alpha_0} \int_{\partial \Omega} \partial_{\nu, y} G(x, y) \varphi_{\epsilon} d\sigma dx. \end{split}$$
 With $0 < \alpha_0 \leq \frac{1}{2} \sup\{d(x, y), x, y \in \bar{\Omega}\}. \end{split}$

We use the definition of the Green function,

$$\Delta G(x,y)=\delta_x, \ G(x,y)=0 \ {\rm on} \ \partial \Omega.$$

Let,

$$E_1 = \{x, d(x, \partial \Omega) \ge \alpha_0 > 0\}$$
 and $E_2 = \{y, d(y, \partial \Omega) \le \alpha_0/2\}$.
We have $E_1 \cap E_2 = \{\emptyset\}$ and by the strong maximum principle, we have,

$$\partial_{\nu,y} G(x,y) \ge \beta_0 > 0 \ \forall \ x \in E_1, \ y \in E_2.$$

Then,

$$\int_{\Omega} V_i e^{u_i} \bar{\varphi}_{\epsilon} dx \ge m_0 \mu_L(\omega_{\epsilon}).$$

We can write,

$$m_0\mu_L(\omega_\epsilon) = m_0[\mu_L(U) - \epsilon] \le \mu(U_\epsilon) + \epsilon \le 2\epsilon.$$

Then,

$$\mu_L(A) \le \mu_L(U_\epsilon) \le \frac{(2+m_0)\epsilon}{m_0}.$$

Finaly,

$$\mu_L(A) = 0.$$

If we use the Radon-Nikodym theorem we obtain (3).

4) The Local convergence 2

Next, to simplify our computations, we assume that the piece of curve of $\partial \Omega$ are as intervals with the usual Lebesgue measure denoted μ_L . We also write $dx = d\mu_L$ and $\bar{x}_k = k - 1, k = 1, \ldots, m$.

Without loss of , we can suppose the local piece of curve around $x_0 = 0$ is an interval of type [-a, a], also, we dnote μ_L the usual measure on $\partial\Omega$ and we can write $d\sigma = d\mu_L$.

We follow the method of the "local convergence 1", we choose other type of function η_{ϵ} .

$$\begin{cases} \eta_{\epsilon}(x) = \left(\frac{|x|}{\epsilon}\right)^{2/3}, \text{ on } |x| \le \epsilon, \ 0 < \epsilon < \delta/2, \\ \eta_{\epsilon} \equiv 1, \text{ on } [\epsilon, 2\epsilon] \\ \eta_{\epsilon} = -\frac{x}{\epsilon} + 3, \text{ on } [\epsilon, 2\epsilon] \\ \eta_{\epsilon} \equiv 0, \text{ outside } [3\epsilon, a], \\ 0 \le \eta_{\epsilon} \le 1, \end{cases}$$

A computation gave:

$$\|\eta_{\epsilon}'\|_{W^{1,5/2}([-3\epsilon,3\epsilon])} \le C_1 \epsilon^{-\alpha}, \ \alpha > 0 \qquad (6)$$

<u>Remark about the exponent 5/2</u>, here our functions η_{ϵ} are not C^1 , but in $W^{1,5/2}$ with 5/2 > 2, and when we use the Hölder inequality in

 $\int_{\Omega} < \nabla(u_i - u) | \tilde{\eta}_{\epsilon}$ we can use many argument to have this quantity small.

We extend η_{ϵ} to a function $\tilde{\eta}_{\epsilon}$ on $\bar{\Omega}$ as,

$$\begin{cases} \Delta \tilde{\eta}_{\epsilon} = 0, & \text{in } \Omega \\ \tilde{\eta}_{\epsilon} = \eta_{\epsilon}, & \text{on } \partial \Omega. \end{cases}$$

As in "The local convrgence 1" we have the:

<u>Key 2:</u> We can extend η_{ϵ} to a function $\bar{\eta}_{\epsilon}$ explicitly, we take η_{ϵ} and we translate it with respect to the normal vector in x_o , after we regulise it like for η_{ϵ} between ϵ and 2ϵ on $[-a, a] \subset \partial \Omega$..

We know from strong maximum principal and the elliptic regularity, (because $\bar{\eta}_{\epsilon} - \tilde{\eta}_{\epsilon} \in C_0^1(\bar{\Omega})$ l'ensemble des fonctions C^1 nulle au bord, see [J] and [G-T] for example:

$$\begin{cases} 0 < \tilde{\eta}_{\epsilon} \le 1, \\ ||\nabla \tilde{\eta}_{\epsilon}||_{L^{1,5/2}(\Omega)} \le \frac{C_1}{\epsilon^{\beta}}, \beta > 0 \ C_1 \text{ depends only on } \Omega \text{ and } x_0 \end{cases}$$
(7)

Those estimates are easy obtained, because we use the **Key2**, the functions $\bar{\eta}_{\epsilon}$ or constant in t, because $\bar{\eta}_{\epsilon}(t,x) = \eta_{\epsilon}(x)$ in the most important part of the new subdomain of $\bar{\Omega}$, also on the corners where we have the estimate of type c/ϵ^s , s > 0, c > 0.

Now, we write:

$$\Delta[(u_i - u)\eta_{\epsilon}] = (V_i e^{u_i} - V e^u)\tilde{\eta}_{\epsilon} - 2 < \nabla(u_i - u)|\nabla\tilde{\eta}_{\epsilon} > .$$
(8)

Here we want to prove that for $\epsilon > 0$ small enough,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \le 4\pi - \epsilon_0/2.$$
 (9)

where $\epsilon_0 > 0$ small enough.

Remark: To ontain our estimate, it is sufficient to choose and to redius the $I_{2\epsilon}$. In fact we want to find $\tilde{\epsilon} > 0$, and a rank $i' = i'(\epsilon') \in \mathbb{N}$ such we have an uniform estimate for the sequence $(u_i)_i$ on a domain which the boundary in $\partial\Omega$ is $I_{\epsilon'}$.

Here we do not search to tend ϵ to 0 to have an estimate, but only reduce it to have a local uniform estimate.

We can use the (1) and (2) to have,

$$\int_{-3\epsilon}^{3\epsilon} (\partial_{\nu} u_i) \eta_{\epsilon}(x) dx \to \int_{-3\epsilon}^{3\epsilon} \eta_{\epsilon}(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_{\nu} u) \eta_{\epsilon}(x) \le 6\epsilon ||\partial_{\nu} u||_{L^{\infty}}.$$

<u>Step 1:</u> Estimate of the integral of the first term of the right hand side of (9).

We have,

$$\begin{cases} \Delta u = V e^u \text{ in } \Omega \\ u = 0, \text{ on } \partial \Omega, \end{cases}$$

and,

$$\begin{cases} \Delta \tilde{\eta}_{\epsilon} = 0 \text{ in } \Omega\\ \tilde{\eta}_{\epsilon} = \eta_{\epsilon}, \text{ on } \partial \Omega, \end{cases}$$

We use the Green formula between $\tilde{\eta}_{\epsilon}$ and u, we obtain,

$$\int_{\Omega} V e^{u} \tilde{\eta}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u \eta_{\epsilon} \leq 4\epsilon ||\partial_{\nu} u||_{L^{\infty}} = C\epsilon \qquad (*')$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} \text{ in } \Omega\\ u_i = 0, \text{ on } \partial \Omega, \end{cases}$$

We use the Green formula between u_i and η_{ϵ} to have:

$$\int_{\Omega} V_i e^{u_i} \tilde{\eta}_{\epsilon} dx = \int_{\partial \Omega} \partial_{\nu} u_i \eta_{\epsilon} d\sigma \to_{i \to +\infty} \mu(\eta_{\epsilon}) = \int_{-3\epsilon}^{3\epsilon} \eta_{\epsilon}(x) d\mu = \int_{-3\epsilon}^{3\epsilon} (\partial_{\nu} u) \eta_{\epsilon}(x) \le 6\epsilon ||\partial_{\nu} u||_{L^{\infty}} (*'')$$

From (*') et (*'') we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \ge i_0$,

$$\int_{\Omega} |(V_i e^{u_i} - V e^u) \tilde{\eta}_{\epsilon}| dx \le 4\pi - \epsilon_0 + C\epsilon \qquad (*''')$$

Remark: In fact, we reduce the interval I_{ϵ} and we conserve the fact that the integral is strictly smaller than 4π . The fact that $\epsilon > 0$ is small for the interval is not a problem, because our goal is to find an ϵ for which the estimate of the integral in (**) is strictly smaller than 4π .

<u>Step 2:</u> Estimate of the integral of the second term of the right hand side of (8).

Let $\Sigma_{\epsilon} = \{x \in \Omega, d(x, \partial \Omega) = \epsilon^2\}, \epsilon > 0$. Then, for ϵ small enough, Σ_{ϵ} is hypersurface.

We can construct another hypersurface, more easly. We know that Ω is a regular 2-manifold with boundary, we use the definition of 2-manifolds with boundary, and, around the boundary, after using the conpactness, we can couver the boundary by a finite number of open set of charts, we work locally on open sets of \mathbb{R}^2_+ and we construct step by step a 2-manifold Ω_{ϵ} contained in Ω with a smooth boundary. (The only problem is to have regularity of the boundary between each two charts).

The measure of $\Omega - \Omega_{\epsilon^2}$ is $\mu_L(\Omega - \Omega_{\epsilon}) \leq k_1 \epsilon^{\gamma}$.

<u>Key 3</u>: The choice of γ is linked to β , we choose it later where apear $\mu_L(\Omega - \Omega_{\epsilon})$ and $\epsilon^{-\beta}$.

We write,

$$\int_{\Omega} | < \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx = \int_{\Omega_{\epsilon'}} | < \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx + \int_{\Omega - \Omega_{\epsilon'}} < \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx. \quad (***)$$

<u>Step 2.1</u>: Estimate of $\int_{\Omega-\Omega_2} |\langle \nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}\rangle |dx.$

First, we know from the elliptic estimates that $\|\nabla \tilde{\eta}_{\epsilon}\|_{L^{5/2}} \leq C_1/\epsilon^{\beta}$, C_1 depends on Ω

We know that $(|\nabla u_i|^{5/3})_i$ is bounded in $L^{3/2}$, 1 < 3/2 < 2, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$, then, $h = |\nabla u|^{5/3}$ a.e.

If we take $f = 1_{\Omega - \Omega_{\epsilon_2}}$, we have:

for
$$\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \ge i_1 \int_{\Omega - \Omega_{\epsilon}} |\nabla u_i|^{5/3} \le \int_{\Omega - \Omega_{\epsilon}} |\nabla u|^{5/3} + \epsilon^{\gamma} \le (1 + ||\nabla u||_{L^{\infty}}^{5/3})\epsilon^{\gamma}.$$

If we use the Hölder inequality we have,

$$\int_{\Omega-\Omega_{\epsilon^2}} |<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx\leq ||\nabla\tilde{\eta}_{\epsilon}||_{L^{5/2}}\times \left[\int_{\Omega-\Omega_{\epsilon}} (|\nabla u_i|+|\nabla u|)^{5/3}dx\right]^{3/5}\leq ||\nabla \eta_{\epsilon}||_{L^{5/2}}$$

The Minkowski give,

$$\begin{split} &\int_{\Omega-\Omega_{\epsilon^2}}|<\nabla(u_i-u)|\nabla\tilde{\eta}_{\epsilon}>|dx\leq C_2\epsilon^{-\beta}\left[\left(\int_{\Omega-\Omega_{\epsilon}}(|\nabla u_i|^{5/3}\right)^{3/5}+\left(\int_{\Omega-\Omega_{\epsilon}}|\nabla u|)^{5/3}dx\right)^{3/5}\right]\\ &\text{Thus,} \end{split}$$

$$\int_{\Omega - \Omega_{\epsilon^2}} | < \nabla (u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx \le C_3 \epsilon^{5/3\gamma - \beta}$$

Il suffit de prendre $5/3\gamma - \beta > 0$ pour avoir une estimation negligeable de l'intégrale $\int_{\Omega - \Omega_{-2}} | < 1$
$$\begin{split} \nabla(u_i - u) |\nabla \tilde{\eta}_{\epsilon} > | dx \\ \underline{Step \ 2.2:} \text{ Estimate of } \int_{\Omega_{\epsilon_2}} | < \nabla(u_i - u) |\nabla \tilde{\eta}_{\epsilon} > | dx. \end{split}$$

We know that, $\Omega_{\epsilon} \subset \Omega$, and $u_i \to u$ in $C^1(\Omega_{\epsilon_2})$ We have,

$$||\nabla(u_i - u)||_{L^{\infty}(\Omega_{\epsilon_2})}|| \le \epsilon_{\beta+1}, \text{ for } i \ge i_3 = i_3(\epsilon),$$

We write,

$$\int_{\Omega_{\epsilon_2}} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \le || \langle \nabla(u_i - u) ||_{L^{\infty}(\Omega_{\epsilon_2})} || \nabla \tilde{\eta}_{\epsilon} ||_{L^{\infty}} \le C_1 \epsilon \text{ for } i \ge i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \ge \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \le \epsilon C_1(2k_1 || \nabla u ||_{L^{\infty}} + 2) \qquad (*'''')$$
and (x'''') we have for $\epsilon \ge 0$ there is $i_1 = i_1(\epsilon) \in \mathbb{N}$, $i_2 = \max\{i_1, \dots, i_n\}$

From (*''') and (*'''), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^{\infty}} + 2 + C) \qquad (\hat{*})$$
$$\leq \epsilon C_1(k_1||\nabla u||_{L^{\infty}} + 1) + C_1 \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| \leq \epsilon C_1(2k_1||\nabla u||_{L^{\infty}} + 1)$$

Finaly, we have,

$$\int_{\Omega - \Omega_{\epsilon^2}} | < \nabla (u_i - u) | \nabla \tilde{\eta}_{\epsilon} > | dx \le \epsilon C_1 (2k_1 || \nabla u ||_{L^{\infty}} + 1) \qquad (\bar{*})$$

The constant C_1 do not depend on ϵ but on Ω , in its definition. For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \ge \max\{i_1, i_2\}$,

$$\int_{\Omega} |\langle \nabla(u_i - u) | \nabla \tilde{\eta}_{\epsilon} \rangle | dx \le \epsilon C_1(2k_1 || \nabla u ||_{L^{\infty}} + 2) \qquad (*'''')$$

From (*''') and (*'''), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}$, $i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_{\epsilon}]| dx \le 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^{\infty}} + 2 + C) \qquad (\hat{*})$$

We choose $\epsilon > 0$ small enough, we have (1).

We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_{\epsilon}] = g_{i,\epsilon} \text{ in } \Omega, \\ (u_i - u)\tilde{\eta}_{\epsilon} = 0 \text{ on } \partial\Omega. \end{cases}$$

With $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is q > 1 such that:

$$\int_{V_{\epsilon}(x_{0})} e^{q(u_{i}-u)\eta_{\epsilon}} dx \leq \int_{\Omega} e^{q(u_{i}-u)\tilde{\eta}_{\epsilon}} dx \leq C(\epsilon,\Omega).$$

where, $V_{\epsilon}(x_0) = B(x_0, \epsilon) \cap \Omega$ is a neighborhood of x_0 in $\overline{\Omega}$.

We have,

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0}(u_i-u)\eta_{\epsilon}\epsilon} dx \le C, \quad \forall \ i.$$

Now, we consider a function $\eta \in C^{\infty}(\mathbb{R}^2)$ such that:

$$\Delta(u_i\eta) = V_i e^{u_i}\eta - 2 < \nabla u_i |\nabla \eta > + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf q_{x_0}, 2$. Thus, we can use the elliptic estimates to have $(u_i\eta_{\epsilon})_i$ uniformly bounded in $W^{2,q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i\eta_{\epsilon})_i$ uniformly bounded in $C^1(\overline{\Omega})$, and if we repit the previous procedure, we can say that $(u_i\eta_{\epsilon})_i$ is uniformly bounded in $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in]0, 1[$.

Finaly, we have, for some $\epsilon > 0$ small enough,

 $||u_i\eta_{\epsilon}||_{C^{1,\theta}[B(x_0,\epsilon)]} \le c_3 \quad \forall \ i.$

We know that, because $u_i = 0$ on $\partial \Omega$ and $\Delta u_i \ge 0$:

$$\max_{\Omega} u_i = u_i(x_i), \ x_i \in \Omega$$

Without loss of generality can assume that $x_i \rightarrow \bar{x}_1$.

We take the convention $x_0 = 0$. We have:

$$\begin{cases} \Delta[(u_i - u)\tilde{\eta}_{\epsilon}] = g_{i,\epsilon} \text{ in } \Omega, \\ (u_i - u)\tilde{\eta}_{\epsilon} = 0 \text{ on } \partial\Omega. \end{cases}$$

With $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0$.

We can use the theorem 1 of [B-M] to conclude that there is q > 1 such that:

$$\int_{V_{\epsilon}(x_0)} e^{q(u_i-u)\tilde{\eta}_{\epsilon}} dx \leq \int_{\Omega} e^{q(u_i-u)\tilde{\eta}_{\epsilon}} dx \leq C(\epsilon, \Omega).$$

where, $V_{\epsilon}(x_0)$ is a neighborhooh of x_0 in $\overline{\Omega}$.

There is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0}u_i\eta_\epsilon} dx \le C, \ \forall \ i.$$

Now, we consider a cutoff function $\eta \in C_0^\infty(\mathbb{R}^2)$ (smooth functions with compact support) such that:

$$\Delta(u_i\eta) = V_i e^{u_i}\eta - 2 < \nabla u_i |\nabla \eta > + u_i \Delta \eta.$$

It is easy to see that the right hand side of the previous equation is uniformly in $L^{q_1}(\Omega)$ for $q_1 = \inf\{q_{x_0}, 2\}$. Thus, we can use the elliptic estimates to have $(u_i\eta)_i$ uniformly bounded in $W^{2,q_1}(\Omega)$ and by the Sobolev embedding, we have $(u_i\eta)_i$ uniformly bounded in $C^1(\overline{\Omega})$, and if we repit the previous procedure, we can say that $(u_i\eta)_i$ is uniformly bounded in $C^{1,\theta}(\overline{\Omega})$ for some $\theta \in]0, 1[$.

Finaly, we have, for some $\epsilon > 0$ small enough,

$$\begin{split} ||u_i \tilde{\eta}_{\epsilon}||_{C^{1,\theta}[\bar{B}(x_0,\epsilon)]} &\leq c_3 \ \forall \ i \\ \text{On } I_{\epsilon} = [-\epsilon,\epsilon] \text{, because } u_i = 0 \text{ on } I_{\epsilon} \text{,} \end{split}$$

$$\begin{split} \text{For } y \in \Omega \ |\nabla(u_i\eta_\epsilon)(y)| &= |\nabla u_i(x)\eta_\epsilon(y) + u_i(y)\nabla\eta_\epsilon(y)| \leq c_3 \\ \text{If we tend } y \to x \in I_\epsilon - \{0\}, (\ \tilde{\eta}_\epsilon(x) \to \eta_\epsilon(x), x \neq 0 \text{ and } u_i(x) = 0) \text{ we obtain,} \end{split}$$

$$|\nabla u_i(x)| = \partial_{\nu} u_i(x) \le c_4 |x|^{-2/3}, \ on[-\epsilon,\epsilon] - \{0\}$$

Thus,

$$\int_{-\epsilon}^{\epsilon} [\partial_{\nu} u_i(x)]^{4/3} dx \le c_5 \quad \forall \ i \ge i_4, \ i_4 \in \mathbb{N}$$

But,

$$\partial_{\nu} u_i \to \partial_{\nu} u, \ \mu_L.a.e.$$

After a subsequence, $\partial_{\nu} u_i$ converge weakly to $\partial_{\nu} u$, thus,

$$\int_{-\epsilon}^{\epsilon} \partial_{\nu} u_i \varphi dx \to \int_{-\epsilon}^{\epsilon} \partial_{\nu} u \varphi, \ \forall \varphi \in C_c([-\epsilon,\epsilon])$$

Where, $C_c([-\epsilon, \epsilon])$ is a set of continuous function with compact support. But, we know,

$$\int_{\partial\Omega} \partial_{\nu} u_i \varphi dx = \int_{-\epsilon}^{\epsilon} \partial_{\nu} u_i \varphi dx \to \varphi d\mu.$$

Also, we know that,

$$dx = hd\mu$$
,

Then,

$$\int_{-\epsilon}^{\epsilon} [(\partial_{\nu} u)h - 1] d\mu \varphi = 0 dx, \ \forall \varphi \in C_c([-\epsilon, \epsilon])$$

We conclude that,

$$(\partial_{\nu} u)h = 1. \ \mu.a.e$$

It means thta,

$$\mu_1 = 0$$
 and $h(0)$.

It is a contradiction.

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