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To cite this version:

Olivier Couronné. Poisson approximation for large clusters in the supercritical FK model. Markov Processes And Related Fields, 2006, 12 (4), pp.627-643. hal-00149390

HAL Id: hal-00149390

https://hal.science/hal-00149390v1

Submitted on 25 May 2007

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hal-00149390, version 1 - 25 May 2007

POISSON APPROXIMATION FOR LARGE CLUSTERS IN THE SUPERCRITICAL FK MODEL

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13 May 2005

Abstract. Using the Chen-Stein method, we show that the spatial distribution of large finite clusters in the supercritical FK model approximates a Poisson process when the ratio weak mixing property holds.

1. Introduction.

We consider here the behaviour of large finite clusters in the supercritical FK model. In dimension two and more, their typical structure is described by the Wulff shape [4, 5, 6, 8, 9, 10, 11]. An interesting issue is the spatial distribution of these large finite clusters. Because of their rarity, a Poisson process naturally comes to mind. Indeed, we prove that the point process of the mass centers of large finite clusters sharply approximates a Poisson process. Furthermore, considering large finite clusters in a large box such that their mean number is not too large, we observe Wulff droplets distributed according to this Poisson process.

Redig and Hostad have recently studied the law of large finite clusters in a given box [20]. Their aim was different, in that they obtained accurate estimates on the law of the maximal cluster in the box, but the intermediate steps are similar. In the supercritical regime they considered only Bernoulli percolation and not FK percolation.

As in [1, 13, 15, 20], our main result is based on a second moment inequality. We have to control the interaction between two clusters. For this, we suppose that ratio weak mixing holds [2]. This property allows us to apply the Chen-Stein method in order to get the approximation by a Poisson process.

The ratio weak mixing holds in dimension two as soon as dual connectivities are exponentially decreasing [2]. For dimensions at least three, we prove that ratio weak mixing holds for p close enough to 1. Hence our main results are valid in all dimensions for p large enough.

The following section is devoted to the statement of our results. In section 3, we define the FK model. We recall the weak and the ratio weak mixing properties and we state a perturbative mixing result in section 4. Section 5 contains the definition

1991 *Mathematics Subject Classification.* 60K35, 82B20.

Key words and phrases. FK model, ratio weak mixing.

I thank R. Cerf for suggesting the problem and for critically reading the manuscript.

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Typeset by $\mathcal{AMS}$-TEX

of our point process and the description of the Chen-Stein method. The core of the article is section 6, where we study a second moment inequality. In section 7, we deal with the probability of having a large finite cluster with its center at the origin. In section 8, we treat the case of distant clusters and we finish the proof of Theorem 1. The proof of Theorem 3 is done in section 9, and the proof of the perturbative mixing result is done in section 10.

2. Statement of the results. We consider the FK measure Φ on the d-dimensional lattice $\mathbb{Z}^d$ and in the supercritical regime. The point $\widehat{p_c}$ stands for $\widehat{p}_g$ in dimension two, and for p_c^{slab} in dimensions three and more. For $q \geq 1$ we let $\mathcal{U}(q)$ be the set such that there exists a unique FK measure on $\mathbb{Z}^d$ of parameters p and q if p is not in $\mathcal{U}(q)$. By [17] this set is at most countable.

Let Λ be a large box in $\mathbb{Z}^d$. We fix n an integer and we consider the finite clusters of cardinality larger than n. We call them *n-large clusters*. Let C be a finite cluster. The *mass center* of C is

$$M_C = \left\lfloor \frac{1}{|C|} \sum_{x \in C} x \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the site of $\mathbb{Z}^d$ whose coordinates are the integer part of those of x. We define a process X on Λ by

$$X(x) = \begin{cases} 1 \text{ if } x \text{ is the mass center of a } n\text{–large cluster } C \\ 0 \text{ otherwise.} \end{cases}$$

Let λ be the expected number of sites x in Λ such that $X(x) = 1$. We denote by $\mathcal{L}(X)$ the law of a process X. For Y a process on Λ, we let $||\mathcal{L}(X) - \mathcal{L}(Y)||_{TV}$ be the total variation distance between the laws of the processes X and Y [7].

Theorem 1. *Let $q \geq 1$ and $p > \widehat{p_c}$ with $p \notin \mathcal{U}(q)$. Let Φ be the FK measure on $\mathbb{Z}^d$ of parameters p and q. We suppose that Φ is ratio weak mixing. There exists a constant $c > 0$ such that: for any box Λ, letting X be defined as above, and letting Y be a Bernoulli process on Λ with the same one-dimensional marginals as X, we have for n large enough*

$$||\mathcal{L}(X) - \mathcal{L}(Y)||_{TV} \leq \lambda \exp(-cn^{(d-1)/d}).$$

As a corollary, the number of large clusters in Λ is approximated by a Poisson variable.

Corollary 2. *Let Φ be as in Theorem 1. Let N be the number of large finite clusters whose mass centers are in the box Λ. Let Z be a Poisson variable of mean λ, and let $c > 0$ be the same constant as in Theorem 1. Then for any $A \subset \mathbb{Z}^+$ and for n large enough,*

$$|P(N \in A) - P(Z \in A)| \leq \lambda \exp\big(-cn^{(d-1)/d}\big).$$

We provide next a control of the shape of the large finite clusters. Here we consider a sequence of boxes $(\Lambda_n)_n$. If the size of Λ_n is not too large, that is of order less than $\exp(\rho n^{(d-1)/d})$ for a certain constant ρ, then the energy created by

the n–large clusters of Λ_n dominates a term of entropy. In this case we can assert that the shape of these n-large clusters are close to the Wulff shape.

More precisely, let $\mathcal{W}$ be the Wulff crystal, let θ be the density of the infinite cluster, and let $\mathcal{L}^d(\cdot)$ be the Lebesgue measure on $\mathbb{R}^d$. Let

$$W = \frac{1}{\big(\theta \mathcal{L}^d(\mathcal{W})\big)^{1/d}} \mathcal{W}$$

be the renormalized Wulff crystal. For $l > 0$, let $V_\infty(C, l)$ be the neighbourhood of C of width l for the metric $|\cdot|_\infty$. For two sets A and B, the notation $A \triangle B$ stands for the symmetric difference between A and B.

Theorem 3. *Let Φ be as in Theorem 1. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n)/n \to 0$ and $f(n)/\ln n \to \infty$ as n goes to infinity. Let $(\Lambda_n)_n$ be a sequence of boxes in $\mathbb{Z}^d$, and let λ_n be the expected number of mass centers of n–large clusters in Λ_n. For all $\delta > 0$, there exists $c > 0$ such that if* $\limsup 1/n^{(d-1)/d} \ln \lambda_n \leq c$.

$$\limsup_{n\to\infty} \frac{1}{n^{(d-1)/d}} \ln \Phi\Big[\mathcal{L}^d\Big(\big(\bigcup_{\substack{x\in\Lambda_n \\ X(x)=1}} (x+W)\big)\triangle \big(n^{-1} \bigcup_{\substack{C\ n\text{-large} \\ M_C\in\Lambda_n}} V_\infty(C, f(n))\big)\Big) \geq \delta\big|\{x : X(x) = 1\}\big|\Big] < 0.$$

For clarity, we omit the subscript n on X.

Remark : Consider a sequence $(\Lambda_n)_n$ such that $|\Lambda_n| \simeq \exp(\rho n^{(d-1)/d})$ and let $w_1 > 0$ be such that [12]:

$$P(n \leq |C(0)| < \infty) \approx \exp(-w_1 n^{(d-1)/d}).$$

On the one hand we need $\rho \geq w_1$ in order to have some n–large clusters in Λ_n. On the other hand the condition on λ_n in theorem 3 may be rewritten as $\rho \leq c + w_1$.

The ratio weak mixing property is a key hypothesis in our results. The following proposition allow us to apply the three preceding results for p large enough in all dimensions.

Proposition 4. *Let $d \geq 3$ and $q \geq 1$. There exists $p_0 < 1$ such that Φ satisfies the ratio weak mixing property for $p > p_0$.*

3. FK model.

We consider the lattice $\mathbb{Z}^d$ with $d \geq 2$. We turn it into a graph by adding bonds between all pairs x, y of nearest neighbours. We write $\mathbb{E}$ for the set of bonds and we let Ω be the set $\{0,1\}^{\mathbb{E}}$. A *bond configuration* ω is an element of Ω. A bond e is *open* in ω if $\omega(e) = 1$, and *closed* otherwise.

A *path* is a sequence $(x_0, \ldots, x_n)$ of distinct sites such that $\langle x_i, x_{i+1}\rangle$ is a bond for each i, $0 \leq i \leq n-1$. A subset Δ of $\mathbb{Z}^d$ is *connected* if for every x, y in Δ, there exists a path included in Δ connecting x and y. If all bonds of a path are open in ω, we say that the path is *open* in ω. A *cluster* is a connected component in $\mathbb{Z}^d$ when we keep only open bonds. It is usually denoted by C. Let x be a site. We write $C(x)$ for the cluster containing x.

To define the FK measure, we first consider finite volume FK measures. Let Λ be a box included in $\mathbb{Z}^d$. We write $\mathbb{E}(\Lambda)$ for the set of bonds $\langle x, y\rangle$ with $x, y \in \Lambda$. Let $\Omega_\Lambda = \{0,1\}^{\mathbb{E}(\Lambda)}$ be the space of bonds configuration in Λ. Let $\mathcal{F}_\Lambda$ be its σ-field, that is the set of subsets of Ω_Λ. For ω in Ω_Λ, we define $\mathrm{cl}(\omega)$ as the number of clusters of the configuration ω.

For $p \in [0,1]$ and $q \geq 1$, the FK measure in Λ with parameters p, q and free boundary condition is the probability measure on Ω_Λ defined by

$$\forall\, \omega \in \Omega_\Lambda \quad \Phi_\Lambda^{f,p,q}(\omega) = \frac{1}{Z_\Lambda^{f,p,q}} \Big(\prod_{e \in \mathbb{E}(\Lambda)} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big) q^{\mathrm{cl}(\omega)},$$

where $Z_\Lambda^{f,p,q}$ is the appropriate normalization factor.

We also define FK measures for arbitrary boundary conditions. For this, let $\partial\Lambda$ be the *boundary* of Λ,

$$\partial\Lambda = \{x \in \Lambda \text{ such that } \exists\, y \notin \Lambda, \langle x, y\rangle \text{ is a bond}\}.$$

For a partition π of $\partial\Lambda$, a π–cluster is a cluster of Λ when we add open bonds between the pairs of sites that are in the same class of π. Let $\mathrm{cl}_\pi(\omega)$ be the number of π–clusters in ω. To define $\Phi_\Lambda^{\pi,p,q}$ we replace $\mathrm{cl}(\omega)$ by $\mathrm{cl}_\pi(\omega)$ and $Z_\Lambda^{f,p,q}$ by $Z_\Lambda^{\pi,p,q}$ in the above formula.

There exists a countable subset $\mathcal{U}(q)$ in $[0,1]$ such that the following holds. As Λ grows and invades the whole lattice $\mathbb{Z}^d$, the finite volume measures converge weakly toward the same infinite measure $\Phi_\infty^{p,q}$ for all $p \notin \mathcal{U}(q)$ [17]. We will always suppose that this occurs, that is $p \notin \mathcal{U}(q)$. We shall drop the superscript and the subscript on $\Phi_\infty^{p,q}$, and simply write Φ. It is known that the FK measure Φ is translation–invariant.

The measure Φ verify the *finite energy property*: for each p in $(0,1)$, there exists $\delta > 0$ such that for every finite–dimensional cylinders ω_1 and ω_2 that differ by only one bond,

$$\Phi(\omega_1)/\Phi(\omega_2) \geq \delta. \tag{1}$$

The random cluster model has a phase transition. There exists $p_c \in (0,1)$ such that there is no infinite cluster Φ–almost surely if $p < p_c$, and an infinite cluster Φ–almost surely if $p > p_c$. Other critical points have been introduced in order to work with 'fine' properties. In dimension two, we define $\widehat{p}_g$ as the critical point for the exponential decay of dual connectivities, see [14, 17]. In three and more dimensions, let p_c^{slab} be the limit of the critical points for the percolation in slabs [22]. For brevity, $\widehat{p}_c$ will stand for $\widehat{p}_g$ in dimension two, and for p_c^{slab} in dimensions three and more. It is believed that $\widehat{p}_c = p_c$ in all dimensions and for all $q \geq 1$, but in most cases we know only that $\widehat{p}_c \geq p_c$.

We now state Theorem 17 of [12], applied to FK measures.
If $q \geq 1$, $p > \widehat{p}_c$ and $p \notin \mathcal{U}(q)$, there exists $w_1 > 0$ such that

$$\lim \frac{1}{n^{(d-1)/d}} \ln \Phi\big(n \leq |C(0)| < \infty\big) = -w_1, \tag{2}$$

where $C(0)$ is the cluster of the origin.

4. Mixing properties.

Let x and y be two points in $\mathbb{Z}^d$ and let $(x_i)_{i=1}^d$ and $(y_i)_{i=1}^d$ be their coordinates. Write $|x-y|_1 = \sum_{i=1}^d |x_i - y_i|$.

Definition 5. *Following* [3]*, we say that* Φ *satisfies the* weak mixing property *if for some* $c, \mu > 0$*, for all sets* $\Lambda, \Delta \subset \mathbb{Z}^d$*,*

$$\begin{aligned} &\sup\big\{\big|\Phi(E \mid F) - \Phi(E)\big| : E \in \mathcal{F}_\Lambda, F \in \mathcal{F}_\Delta, \Phi(F) > 0\big\} \\ &\qquad \le c \sum_{x\in\Lambda, y\in\Delta} e^{-\mu|x-y|_1}. \end{aligned} \tag{3}$$

Definition 6. *Following* [3]*, we say that* Φ *satisfies the* ratio weak mixing property *for some* $c_1, \mu_1 > 0$*, for all sets* $\Lambda, \Delta \subset \mathbb{Z}^d$*,*

$$\begin{aligned} &\sup\Big\{\Big|\frac{\Phi(E\cap F)}{\Phi(E)\Phi(F)} - 1\Big| : E \in \mathcal{F}_\Lambda, F \in \mathcal{F}_\Delta, \Phi(E)\Phi(F) > 0\Big\} \\ &\qquad \le c_1 \sum_{x\in\Lambda, y\in\Delta} e^{-\mu_1|x-y|_1}, \end{aligned} \tag{4}$$

Roughly speaking, the influence of what happens in Δ on the state of the bonds in Λ decreases exponentially with the distance between Λ and Δ.

In dimension two, the measure Φ is ratio weak mixing as soon as $p > \widehat{p}_c$ [3], but such a result is not available in dimension larger than three. We provide a perturbative mixing result, which is valid for all dimensions larger than three, and which is similar to the weak mixing property.

Lemma 7. *Let* $d \ge 3$ *and* $q \ge 1$*. There exists* $p_1 < 1$ *and* $c > 0$ *such that: for all* $p > p_1$*, all connected sets* Γ, Δ *with* $\Gamma \subset \Delta$*, every boundary conditions* η*,* ξ *on* Δ*, every event* E *supported on* Γ*,*

$$|\Phi_\Delta^{\eta,p,q}(E) - \Phi_\Delta^{\xi,p,q}(E)| \le 2|\partial\Delta| \exp\big(-c \inf\big\{|x-y|_1, x \in \Gamma, y \in \partial\Delta\big\}\big).$$

We are not aware of a particular reference for this result, and we give a sketch of the proof in Section 10.

5. The Chen-Stein method.

From the percolation process, we want to extract a point process describing the occurrence of large finite clusters. For a point x in $\mathbb{R}^d$, let $\lfloor x \rfloor$ denotes the site of $\mathbb{Z}^d$ whose coordinates are the integer parts of those of x. Assume that C is a finite subset of $\mathbb{Z}^d$. Then the *mass center* of C is

$$M_C = \Big\lfloor \frac{1}{|C|} \sum_{x\in C} x \Big\rfloor.$$

Let $n \in \mathbb{N}$. A *n–large cluster* is a finite cluster of cardinality larger than n. Let Λ be a box in $\mathbb{Z}^d$. We define a process X on Λ by

$$X(x) = \begin{cases} 1 \text{ if } x \text{ is the mass center of a } n\text{–large cluster } C \\ 0 \text{ otherwise.} \end{cases}$$

In order to apply the Chen-Stein method, we define for x, y in $\mathbb{Z}^d$,

$$\begin{aligned} p_x &= \Phi(X(x) = 1), \\ p_{xy} &= \Phi\big(\exists\, C, C' \text{ two clusters such that: } C \cap C' = \emptyset, \\ &\qquad n \le |C|, |C'| < \infty, M_C = x \text{ and } M_{C'} = y\big), \end{aligned}$$

and we let $B_x = B(x, n^2)$ be the box centered at x of side length n^2. Let λ be the expected number of sites x in Λ such that $X(x) = 1$. We have $\lambda = \sum_{x\in\Lambda} p_x$ and, because of the translation–invariance of Φ, for each site x in Λ

$$\lambda = |\Lambda| \cdot p_x. \tag{5}$$

We introduce three coefficients b_1, b_2, b_3 by:

$$b_1 = \sum_{x\in\Lambda} \sum_{y\in B_x} p_x p_y,$$

$$b_2 = \sum_{x\in\Lambda} \sum_{y\in B_x\setminus x} p_{xy},$$

$$b_3 = \sum_{x\in\Lambda} E\Big|E\Big(X(x) - p_x | \sigma(X(y), y \notin B_x)\Big)\Big|.$$

Let Z_1 and Z_2 be two Bernoulli processes on Λ. The *total variation distance* between the laws of the processes Z_1 and Z_2 [7] is

$$||\mathcal{L}(Z_1) - \mathcal{L}(Z_2)||_{TV} = 2\sup\big\{\big|P(Z_1 \in A) - P(Z_2 \in A)\big|, A \text{ subset of } \{0,1\}^\Lambda\big\}.$$

Let Y be a Bernoulli process on Λ such that the $Y(x)$'s are iid and

$$P(Y(x) = 1) = p_x.$$

The Chen-Stein method provides a control of the total variation distance between X and Y in terms of the b_i's. Indeed we apply Theorem 3 of [7] to obtain that

$$||\mathcal{L}(X) - \mathcal{L}(Y)||_{TV} \le 2(2b_1 + 2b_2 + b_3) + 4\sum_{x\in\Lambda} p_x^2. \tag{6}$$

To prove Theorem 1, we shall provide an upper bound on each term b_i. The ratio weak mixing property is essential to our proof of the bound of b_2. Nevertheless, we believe that one can prove the following inequality, without any mixing assumption:

$$\Phi\big[n \le C(x) < \infty, n \le C(y) < \infty, C(x) \cap C(y) = \emptyset\big] \le \Phi(2n \le C(0) < \infty). \tag{7}$$

Let us give now an upper bound on p_x. By [16], there exists a constant $c > 0$ such that:

$$\Phi(n \le |C(0)| < \infty) \le \exp\big(-cn^{(d-1)/d}\big).$$

But

$$\begin{aligned} p_x &\le \sum_{k\ge n} \Phi\big(\exists\, C, |C| = k, M_C = x\big) \\ &\le \sum_{k\ge n} \sum_{y\in B(x,2k)} \Phi\big(|C(y)| = k\big) \\ &\le \sum_{k\ge n} (2k)^d \exp\big(-ck^{(d-1)/d}\big). \end{aligned}$$

Hence there exists a constant $c > 0$ such that for n large enough

$$p_x \le \exp(-cn^{(d-1)/d}). \tag{8}$$

6. Second moment inequality. In this section we bound the term p_{xy} with the help of the ratio weak mixing property. First we introduce a local version of p_{xy}. We define $\widetilde{p}_{xy}$ by

$$\begin{aligned} \widetilde{p}_{xy} = \Phi\big(&\exists\, C, C' \text{ two clusters such that} \\ &n \le |C| < n^2, n \le |C'| < n^2, M_C = x, \text{ and } M_{C'} = y\big). \end{aligned}$$

The distance between two sets Γ and $\Delta \subset \mathbb{Z}^d$ is

$$d(\Gamma, \Delta) = \inf\{|x - y|_1, x \text{ in } \Gamma, y \text{ in } \Delta\},$$

and it is the length of the shortest path connecting Γ to Δ.

We divide the term $\widetilde{p}_{xy}$ into two parts. Let μ_1 be the constant appearing in the definition of the ratio weak mixing property and let $K > 5/\mu_1$. We define $\widetilde{p}^c_{xy}$ by

$$\begin{aligned} \widetilde{p}^c_{xy} = \Phi\big(&\exists\, C, C' \text{ two clusters such that } d(C, C') \le K \ln n, \\ &n \le |C| < n^2, n \le |C'| < n^2, M_C = x, \text{ and } M_{C'} = y\big). \end{aligned}$$

We define also $\widetilde{p}^d_{xy}$ by

$$\begin{aligned} \widetilde{p}_{xy} = \Phi\big(&\exists\, C, C' \text{ two clusters such that } d(C, C') > K \ln n, \\ &n \le |C| < n^2, n \le |C'| < n^2, M_C = x, \text{ and } M_{C'} = y\big). \end{aligned}$$

The superscripts c and d stand for *close* and *distant*. So $\widetilde{p}_{xy} = \widetilde{p}^c_{xy} + \widetilde{p}^d_{xy}$ and we study separately these two terms.

First we focus on $\widetilde{p}^d_{xy}$. We have

$$\widetilde{p}^d_{xy} \le \sum_{C,C' \text{ distant}} \Phi(C \text{ and } C' \text{ are clusters}),$$

where the sum is over the couples (C, C') of connected subsets of $\mathbb{Z}^d$ such that

$$\begin{aligned} &n \le |C| < n^2,\ n \le |C'| < n^2, \\ &\qquad M_C = x, M_{C'} = y, \text{ and } d(C, C') > K \ln n. \end{aligned}$$

Let c_1, μ_1 be the constants appearing in the definition of the ratio weak mixing property. Let (C, C') be a couple appearing in the sum above. We have

$$\sum_{u \in C, v \in C'} e^{-\mu_1 |u-v|} \leq n^4 \exp(-\mu_1 K \ln n),$$

so for n large enough

$$c_1 \sum_{u \in C, v \in C'} e^{-\mu_1 |u-v|} \leq 1.$$

So for n large enough

$$\Phi(C \text{ and } C' \text{ are clusters}) \leq 2\Phi(C \text{ is a cluster}) \cdot \Phi(C' \text{ is a cluster}),$$

by the ratio weak mixing property (4). Hence there exists $c > 0$ such that for n large enough

$$\begin{aligned} \widetilde{p}^d_{xy} &\leq \sum_{u \in B(x,2n^2), v \in B(y,2n^2)} 2\Phi(n \leq |C(u)| < \infty) \cdot \Phi(n \leq |C(v)| < \infty) \\ &\leq \exp(-cn^{(d-1)/d}). \end{aligned} \tag{9}$$

Now we consider p^c_{xy}. We have

$$\widetilde{p}^c_{xy} \leq \sum_{C, C' \text{ close}} \Phi(C \text{ and } C' \text{ are clusters}),$$

where the sum is over the couples (C, C') of subsets of $\mathbb{Z}^d$ such that

$$\begin{aligned} n \leq |C| < n^2, n \leq |C'| < n^2, \\ M_C = x, M_{C'} = y, \text{ and } d(C, C') \leq K \ln n. \end{aligned}$$

For n large enough, the event $\{C \text{ and } C' \text{ are clusters}\}$ is $\mathcal{F}_{B(x,3n^2)}$-measurable. So we only consider bond configurations in $B(x, 3n^2)$.

We give a deterministic total order on the pairs (u, v) of $\mathbb{Z}^d$ in such a way that if $|u_1 - v_1|_1 < |u_2 - v_2|_1$, then $(u_1, v_1) < (u_2, v_2)$. Let (C, C') be a pair of sets appearing in the above sum. Take a configuration ω in $B(x, 3n^2)$ such that C and C' are clusters in ω. We change the configuration ω as follows.

To start with, we take the pair (u, v) such that $u \in C$, $v \in C'$ and (u, v) is the first such pair for the order above. For $0 \leq i \leq d$, we define t_i the point whose $d - i$ first coordinates are equal to those of u, and the others are equal to those of v. Hence $t_0 = u$, $t_d = v$, and t_i and t_{i+1} differ by only one coordinate. We consider the shortest path $(u_0, \ldots, u_k)$ connecting u to v through the t_i's. It is composed of the segments $[t_i, t_{i+1}]$ for $0 \leq i \leq d - 1$.

We open all the bonds $\langle u_i, u_{i+1} \rangle$ for $i = 0 \ldots k - 1$. In the same time, we close all the bonds incident to u_i for $i = 1 \ldots k - 1$ distinct from the previous bonds $\langle u_j, u_{j+1} \rangle$. Let $\widetilde{\omega}$ be the new configuration in $B(x, 3n^2)$. We denote by $\widetilde{C}$ the set $C \cup C' \cup \{u_i\}_{i=1}^{k-1}$. By construction, $\widetilde{C}$ is a cluster in $\widetilde{\omega}$. We have

$$2n \leq \widetilde{C} < 4n + K \ln n.$$

The number of bonds we have changed is bounded by $2dK \ln n$. By the finite energy property (1):

$$\Phi(\widetilde{\omega}) \geq n^{2dK \ln \delta} \Phi(\omega),$$

for a certain constant δ in $(0,1)$.

Now we control the number of antecedents by our transformation. Take a configuration $\widetilde{\omega}$ of $B(x, 3n^2)$. To get an antecedent of $\widetilde{\omega}$, we have to

(a) choose two sites u, v in $B(x, 3n^2)$, with $|u - v|_1 \leq K \ln n$

(b) take the path connecting u to v along the coordinate axis

(c) choose the state of the bonds that have an endpoint on this path.

In step (a), we have less than $(3n^2)^d (2K \ln n)^d$ choices. In step (b) we have just one choice. In step (c) the number of choices is bounded by $2^{2dK \ln n}$. Hence for n large enough the number of antecedents of $\widetilde{\omega}$ is bounded by n^{4dK}.

Finally,

$$\sum_{C, C' \text{ close}} \Phi(C \text{ and } C' \text{ are clusters}) \leq n^{4dK} \cdot n^{2dK \ln \delta} \sum_{\widetilde{C}} \Phi(\widetilde{C} \text{ is a cluster}),$$

where the sum is over connected subsets $\widetilde{C}$ of $\mathbb{Z}^d$ such that $2n \leq |\widetilde{C}| < 5n$ and $\widetilde{C}$ is contained in $B(x, 3n^2)$. This sum is bounded by

$$|B(x, 3n^2)| \cdot \Phi(2n \leq |C(0)| < 5n).$$

Thus by (2), there exists $c_2 > w_1$ such that for n large enough,

$$\widetilde{p}^c_{xy} \leq \exp(-c_2 n^{(d-1)/d}). \tag{10}$$

To conclude, remark that

$$p_{xy} - \widetilde{p}_{xy} \leq \Phi\big(\exists\, C \text{ a cluster such that } n^2 \leq |C| < \infty, M_C = x\big).$$

By (8), there exists c such that for n large enough the difference between p_{xy} and $\widetilde{p}_{xy}$ is bounded by $\exp(-cn^{2(d-1)/d})$. So by (9) there exists $c > 0$ such that $p_{xy} \leq \widetilde{p}^c_{xy} + \exp(-cn)$. Since in (10) the constant c_2 is strictly larger than w_1, there exists $c_3 > w_1$ such that for n large enough

$$p_{xy} \leq \exp(-c_3 n^{(d-1)/d}). \tag{11}$$

7. A control of p_x. We compare p_x and $\Phi(n \leq |C(0)| < \infty)$.

Lemma 8. *If $q \geq 1$, $p > \widehat{p}_c$, and $p \notin \mathcal{U}(q)$, then*

$$\lim \frac{1}{n^{(d-1)/d}} \ln p_x = -w_1.$$

We note that in [20], the authors take the left endpoints of the clusters instead of the mass center and get the same limit.

Proof of Lemma 8. We begin with a lower bound for p_x. We recall that for all x in $\mathbb{Z}^d$, $p_x = \Phi(X(0) = 1)$. Let $\alpha > 1$. Because of (2), we have

$$\lim \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < \infty) = \lim \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < n^\alpha).$$

Then

$$\begin{aligned}\Phi(n \leq |C(0)| < n^\alpha) &\leq \sum_{x \in B(0,n^\alpha)} \Phi(n \leq |C(0)| < n^\alpha, M_C = x) \\ &\leq |B(0,n^\alpha)| \Phi(X(0) = 1).\end{aligned}$$

We give next an upper bound:

$$\begin{aligned}\Phi(X(0) = 1) &= \Phi(\exists C \text{ a cluster}, M_C = 0, n \leq |C| < n^\alpha) \\ &\qquad + \Phi(\exists C \text{ a cluster}, M_C = 0, n^\alpha \leq |C| < \infty) \\ &\leq \sum_{x \in B(0,n^\alpha)} \Phi(n \leq |C(x)| < \infty) \\ &\qquad + \sum_{k \geq n^\alpha} \Phi\big(\exists C \text{ a cluster}, |C| = k, C \cap B(0,2k) \neq \emptyset\big) \\ &\leq |B(0,n^\alpha)| \Phi(n \leq |C(x)| < \infty) + \sum_{k \geq n^\alpha} |B(0,2k)| \Phi(|C(0)| = k).\end{aligned}$$

Finally, we use the limit (2) to get

$$\lim \frac{1}{n^{(d-1)/d}} \ln p_x = \lim \frac{1}{n^{(d-1)/d}} \ln \Phi(n \leq |C(0)| < \infty) = -w_1. \quad \square$$

8. Proof of Theorem 1.

We recall that Λ is a box and λ is the expected number of the mass centers in Λ of n–large clusters. We write $\mathcal{F}_\Lambda^{B_x}$ for the σ–field $\mathcal{F}_{\Lambda \setminus B_x}$. First, we bound the term

$$E\big|E\big(X(x) - p_x | \mathcal{F}_\Lambda^{B_x}\big)\big|.$$

Let $\widetilde{X}(x)$ be equal to 1 if x is the mass center of a cluster C, with C such that $n \leq |C| < n^2/4$, and equal to 0 otherwise. Let $\widetilde{p}_x = \Phi(\widetilde{X}(x))$. We have

$$\begin{aligned}E\big|E\big(X(x) - p_x | \mathcal{F}_\Lambda^{B_x}\big)\big| &\leq E\big|E\big(X(x) - \widetilde{X}(x) | \mathcal{F}_\Lambda^{B_x}\big)\big| \\ &\quad + E\big|E\big(\widetilde{X}(x) - \widetilde{p}_x | \mathcal{F}_\Lambda^{B_x}\big)\big| + E\big|E\big(\widetilde{p}_x - p_x | \mathcal{F}_\Lambda^{B_x}\big)\big|.\end{aligned} \tag{12}$$

Since the quantity $X(x) - \widetilde{X}(x)$ is always positive,

$$\begin{aligned}E\big|E\big(X(x) - \widetilde{X}(x) | \mathcal{F}_\Lambda^{B_x}\big)\big| &= E\big[E\big(X(x) - \widetilde{X}(x) | \mathcal{F}_\Lambda^{B_x}\big)\big] \\ &= p_x - \widetilde{p}_x.\end{aligned}$$

We have also

$$E\big|E\big(\widetilde{p}_x - p_x | \mathcal{F}_\Lambda^{B_x}\big)\big| = p_x - \widetilde{p}_x.$$

But

$$p_x - \widetilde{p}_x = \Phi(\exists\, C \text{ a cluster}, n^2/4 \le |C| < \infty, M_C = x),$$

so by (8) there exists $c > 0$ such that $p_x - \widetilde{p}_x \le \exp(-cn^2)$.

The variable $\widetilde{X}(x)$ is $\mathcal{F}_{B(x,n^2/4)}$-measurable. The distance between $B(x, n^2/4)$ and the complementary region of B_x is of order n^2. If Φ is weak mixing, or by lemma 7 if p is close enough to 1, there exists a constant $c > 0$ such that for n large enough

$$E\big|E\big(\widetilde{X}(x) - \widetilde{p}_x|\mathcal{F}_\Lambda^{B_x}\big)\big| \le \exp(-cn^2).$$

Putting together the estimates of the three terms on the right-hand side of (12), we conclude that there exists $c > 0$ such that for n large enough

$$E\big|E\big(X(x) - p_x|\mathcal{F}_\Lambda^{B_x}\big)\big| \le \exp(-cn^2). \tag{13}$$

Now observe that $|\Lambda| = \lambda p_x^{-1}$. Using inequality (11) and the limit of Lemma 8, there exists $c > 0$ such that

$$b_2 \le \lambda p_x^{-1} \exp\big(-c_3 n^{(d-1)/d}\big) \le \lambda \exp\big(-cn^{(d-1)/d}\big).$$

Because of (13), there exists $c > 0$, $c' > 0$ such that

$$b_3 \le \lambda p_x^{-1} \exp(-cn^2) \le \lambda \exp(-c'n^2).$$

The term b_1 is controlled by Lemma 8. We apply finally the Chen-Stein inequality (6) to obtain Theorem 1. □

9. Proof of Theorem 3.

The Wulff crystal is the typical shape of a large finite cluster in the supercritical regime. The crystal is built on a surface tension τ. The surface tension is a function from $\mathbb{S}^{d-1}$, the $(d-1)$–dimensional unit sphere of $\mathbb{R}^d$, to $\mathbb{R}^+$. It controls the exponential decay of the probability for having a large separating surface in a certain direction, with all bonds closed. We refer the reader to [9, 12] for an extended survey of this function.

In the regime $p > \widehat{p}_c$ and $p \notin \mathcal{U}(q)$, the surface tension is positive, continuous, and satisfies the weak simplex inequality. We denote by $\mathcal{W}$ the Wulff shape associated to τ,

$$\mathcal{W} = \{x \in \mathbb{R}^d, x.u \le \tau(u) \text{ for all } u \text{ in } \mathbb{S}^{d-1}\}.$$

The Wulff shape is a main ingredient in the proof of (2).

Let $\theta = \Phi(0 \leftrightarrow \infty)$ be the density of the infinite cluster. Let $f : \mathbb{N} \to \mathbb{N}$, such that $f(n)/n \to 0$ and $f(n)/\ln n \to \infty$ as n goes to infinity. Let x and y be two points of $\mathbb{R}^d$, and let $(x_i)_{i=1}^d$ and $(y_i)_{i=1}^d$ be their coordinates. We write $|x-y|_\infty = \max_{1\le i\le d} |x_i - y_i|$. We define a neighbourhood of a cluster C by

$$V_\infty(C, f(n)) = \{x \in \mathbb{R}^d, \exists\, y \in C, |x-y|_\infty \le f(n)\}.$$

Let $(\Lambda_n)_n$ be a sequence of boxes in $\mathbb{Z}^d$, and let λ_n be the expected number of mass centers of n–large clusters in Λ_n. In Theorem 3, we consider the event

$$\Big\{\mathcal{L}^d\Big(\bigcup_{\substack{x\in\Lambda_n\\ X(x)=1}}(x+\theta\mathcal{L}^d(\mathcal{W})^{-1/d}\mathcal{W})\triangle \tag{14}$$
$$n^{-1}\bigcup_{\substack{C\ n\text{–large}\\ M_C\in\Lambda_n}}V_\infty(C,f(n))\Big)\geq\delta\big|\{x:X(x)=1\}\big|\Big\}.$$

It is included in the event

$$\Big\{\text{there exists } C \text{ a } n\text{–large cluster such that } M_C\in\Lambda_n,$$
$$\mathcal{L}^d\Big(\big(M_C+\theta\mathcal{L}^d(\mathcal{W})^{-1/d}\mathcal{W}\big)\triangle\big(n^{-1}V_\infty(C,f(n))\big)\Big)\geq\delta\Big\}.$$

Taking the logarithm of its probability and dividing by $n^{(d-1)/d}$, we may show that for n large it is equivalent to the logarithm divided by $n^{(d-1)/d}$ of the following quantity:

$$\lambda_n\Phi\Big[\mathcal{L}^d\Big(\big(M_{C(0)}+\theta\mathcal{L}^d(\mathcal{W})^{-1/d}\mathcal{W}\big)\triangle\big(n^{-1}V_\infty(C(0),f(n))\big)\Big)\geq\delta\big|n\leq|C(0)|<\infty\Big].$$

By [9, 12], there exists $c>0$ such that if

$$\limsup 1/n^{(d-1)/d}\ln\lambda_n\leq c,$$

then the inequality in Theorem 3 holds. □

10. A perturbative mixing result.

First we prove lemma 7, following the proof of the uniqueness of the FK measure for p close enough to 1 in [18]. The difference is that we consider not just one but two independent FK measures. The idea of using two independent copies of a measure comes from [19]. Then the proof of proposition 4 follows.

Proof of lemma 7.

Let Δ be a connected subset of $\mathbb{Z}^d$. There is a partial order $\preceq$ in Ω_Δ given by $\omega\preceq\omega'$ if and only if $\omega(e)\leq\omega'(e)$ for every bond e. A function $f:\Omega_\Delta\to\mathbb{R}$ is called *increasing* if $f(\omega)\leq f(\omega')$ whenever $\omega\preceq\omega^{'}$. An event is an element of Ω_Δ. An event is called increasing if its characteristic function is increasing. For a pair of probability measures μ and ν on $(\Omega_\Delta,\mathcal{F}_\Delta)$, we say that μ *(stochastically) dominates* ν if for any $\mathcal{F}_\Delta$-measurable increasing function f the expectations satisfy $\mu(f)\geq\nu(f)$ and we denote it by $\mu\succeq\nu$. Let P_p be the Bernoulli bond–percolation measure on $\mathbb{Z}^d$ of parameter p. The FK measures on Δ dominate stochastically a certain Bernoulli measure restricted on $\mathbb{E}(\Delta)$:

$$\Phi_\Delta^{\eta,p,q}\succeq P_{p/[p+q(1-p)]}\big|_{\mathbb{E}(\Delta)}. \tag{15}$$

For $(\omega_1, \omega_2) \in \Omega^2$, we call a site x *white* if $\omega_1(e)\omega_2(e) = 1$ for all bond e incident with x, and *black* otherwise. We define a new graph structure on $\mathbb{Z}^d$. Take two sites x and y and label x_i, y_i their coordinates. If $\max_{i=1\ldots d} |x_i - y_i| = 1$, then $\langle x, y\rangle$ is a $\star$-*bond* and y is a $\star$-*neighbour* of x. A $\star$-*path* is a sequence $(x_0, ..., x_n)$ of distinct sites such that $\langle x_i, x_{i+1}\rangle$ is a $\star$-bond for $0 \leq i \leq n-1$.

For any set V of sites, the *black cluster* $B(V)$ is the union of V together with the set of all x_0 for which there exists a $\star$-path $x_0, \ldots, x_n$ such that $x_n \in V$ and $x_0, \ldots, x_{n-1}$ are all black. Let Γ, Δ be two connected sets with $\Gamma \subset \Delta$. The 'interior boundary' $D(B(\partial\Delta))$ of $B(\partial\Delta)$ is the set of sites x satisfying:

(a) $x \notin B(\partial\Delta)$

(b) there is a $\star$-neighbour of x in $B(\partial\Delta)$

(c) there exists a path from x to Γ that does not use a site in $B(\partial\Delta)$.

Let I be the set of sites x_0 for which there exists a path $x_0, \ldots, x_n$ with $x_n \in \Gamma$, $x_i \notin B(\partial\Delta)$ for all i, see figure 1.

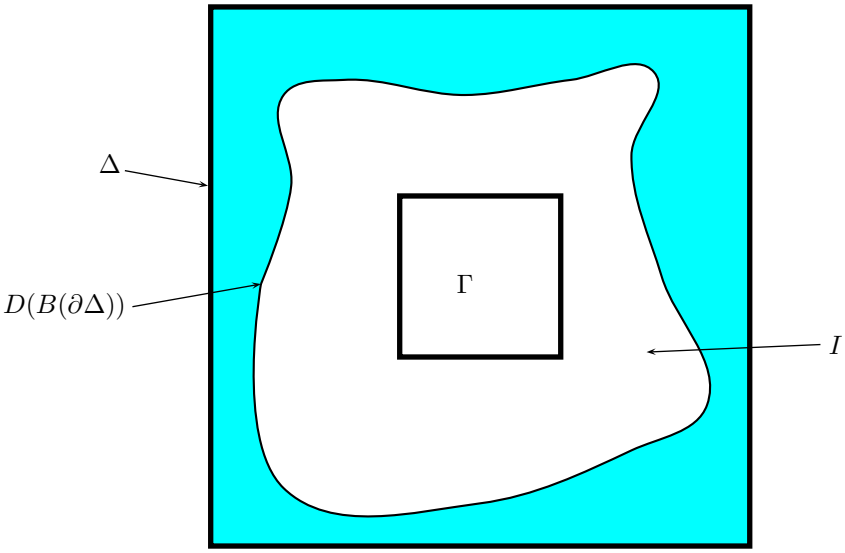

figure 1: The set I inside Δ

Let

$$K_{\Gamma,\Delta} = \big\{\big(B(\partial\Delta) \cup D(B(\partial\Delta))\big) \cap \Gamma = \emptyset\big\}.$$

If $K_{\Gamma,\Delta}$ occurs, we have the following facts:

(a) $D(B(\partial\Delta))$ is connected

(b) every site in $D(B(\partial\Delta))$ is white

(c) $D(B(\partial\Delta))$ is measurable with respect to the colours of sites in $\mathbb{Z}^d \setminus I$

(d) each site in ∂I is adjacent to some site of $D(B(\partial\Delta))$.

These claims have been established in the proof of Theorem 5.3 in [18].

Pick η, ξ two boundary conditions of Δ. For brevity let $\mathcal{P} = \Phi_\Delta^{\eta,p,q} \times \Phi_\Delta^{\xi,p,q}$. We shall write X, Y for the two projections from $\Omega_\Delta \times \Omega_\Delta$ to Ω_Δ. Then for any

$E \in \mathcal{F}_\Gamma$, we have by the claims above

$$\mathcal{P}(X \in E, K_{\Gamma,\Delta}) = \mathcal{P}(Y \in E, K_{\Gamma,\Delta}) = \mathcal{P}(\Phi_I^{w,p,q}(E) 1_{K_{\Gamma,\Delta}}).$$

Hence

$$|\Phi_\Delta^{\eta,p,q}(E) - \Phi_\Delta^{\xi,p,q}(E)| \leq \big(1 - \mathcal{P}(K_{\Gamma,\Delta})\big).$$

Because of inequality (15) and by the stochastic domination result in [21], the process of black sites is stochastically dominated by a Bernoulli site–percolation process whose parameter is independent of Γ, Δ, η, ξ and decreases to 0 as p goes to 1. There exists $p_1 < 1$ such that this Bernoulli process is subcritical for the $\star$-graph structure of $\mathbb{Z}^d$ and for $p \geq p_1$. Hence there exists $c > 0$ such that for $p > p_1$, for all Γ, Δ, η, ξ,

$$\mathcal{P}(K_{\Gamma,\Delta}) \geq 1 - |\partial\Delta| \exp\big(- c\, d(\Gamma, \partial\Delta)\big). \qquad \square$$

Proof of proposition 4. The domination inequality (15) implies that for p large enough, the mesures $\Phi_\Delta^{\eta,p,q}$ have exponentially bounded controlling regions in the terminology of [2]. Thus by theorem 3.3 of [2], lemma 7 implies the ratio weak mixing property for the mesures $\Phi_\Delta^{\eta,p,q}$. $\square$

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