



Sub-gaussian measures and associated semilinear problems

Pierre Fougeres, Cyril Roberto, Boguslaw Zegarlinski

► To cite this version:

Pierre Fougeres, Cyril Roberto, Boguslaw Zegarlinski. Sub-gaussian measures and associated semilinear problems. 2007. hal-00149360v1

HAL Id: hal-00149360

<https://hal.science/hal-00149360v1>

Preprint submitted on 25 May 2007 (v1), last revised 24 Jul 2010 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Sub-gaussian measures and associated semilinear problems*

P. Fougères, C. Roberto and B. Zegarliński

Équipe Modal'X, Université Paris X

Université de Marne la Vallée

Imperial College London

April 19, 2007

Abstract

We study the existence, smoothing properties and the long time behaviour for a class of nonlinear Cauchy problems in infinite dimensions under the assumption of F-Sobolev inequalities.

Mathematics Subject Classification: 35J60, 60E15, 26D10.

Keywords: F-Sobolev inequalities, Semilinear parabolic problems, Infinite dimensions.

1 Markovian Semilinear Cauchy Problems: Introduction.

In the bulk of this work we consider the following formal Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(t) &= Lu(t) + \lambda u(t) G\left(\frac{u^2(t)}{\mu(u(t)^2)}\right) \\ u(0) &= f. \end{cases} \quad (\text{MCP})$$

where L is a (linear) Markov generator and G is a certain nonlinearity, (vanishing at one), to be specified later and μ is a probability measure. In the next paragraph we are going to explain what is needed to understand the meaning of this equation. Let us nevertheless note here that under our hypothesis, constants are global solutions of (MCP) and positivity of the initial data results with positive solutions which supremum norms are bounded by those at time zero. This partially justifies to call it a *Markovian Cauchy Problem*.

Our analysis is carried out in suitable functional spaces (involving a probability measures) which have no polynomial volume growth. This is

*Supported by EPSRC GR/R90994/01 & ADERMIP'06

necessary as we are in particular interested in infinite dimensional problems. In such situation the Sobolev inequality which provides a cornerstone for classical PDE analysis cannot be satisfied and we have to rely on weaker coercive inequalities which survive the infinite dimensional limit and are of the following form

$$\mu \left(g^2 F \left(\frac{g^2}{\mu g^2} \right) \right) \leq c \mu |\nabla g|^2$$

with a constant $c \in (0, \infty)$ independent of a function g and the right hand side involving the quadratic form of the elliptic operator L . Inequalities of this type, called later on F-Sobolev inequalities, have been recently studied in [BCR04], [BR03], [RZ05], [LZ], (see also references therein), for probability measures with tails decaying more slowly than the Gaussian ones but faster than exponentially.

In our setup the linear operator L is monotone in the usual sense while the nonlinear part may work to an opposite effect. Our study determines how large the coupling constant $\lambda > 0$ can be, so that the system is still stable in the sense of existence, uniqueness, smoothing properties and the ergodic long time behaviour of a (weak) solution.

We note that the linear semigroup corresponding to L is hypercontractive in an appropriate family of Orlicz spaces; in fact as shown in [BCR04], (generalising the celebrated result of Gross [Gross]), such hypercontractivity is equivalent to F-Sobolev inequality. Under suitable conditions, we show that the C_0 , positivity and unit preserving, semigroup obtained as the solution of the (MCP) is hypercontractive in the appropriate family of Orlicz spaces. The key ingredients in our programme are provided by the F-Sobolev inequality and the fact that quantity on its left hand side has similar properties to the relative entropy.

In recent years an extensive effort was made to understand better the coercive inequalities in infinite dimensional functional spaces, (see e.g. [GZ03], [BCR04], [RZ05], [LZ] and references therein). This provides a basis and a part of motivation to study nonlinear problems. One may hope that the study in this direction may in the future shed also some light/or provide a complementary systematic understanding for a class of problems in infinite dimensions for which some understanding was achieved in the past (as e.g. problems from mathematical physics). This work is also partially motivated by [FZ04] where certain preliminary results were obtained for the case when logarithmic Sobolev inequalities are true.

The organisation of our paper is as follows. In section 2 we introduce the general setting and describe in detail conditions imposed on the linear and nonlinear operators appearing in our problem.

In section 3 we prove the existence and uniqueness of the weak solution of (MCP). In short our strategy is as follows. We first consider a mollified problem with initial data in $L_2(\mu)$ defined by smoothing the nonlinear part

with the linear semigroup P_ε , $\varepsilon > 0$, generated by L . Then under the assumption that the coupling constant $\lambda > 0$ is sufficiently small, we employ the F-Sobolev inequality to prove the existence and uniqueness of the mollified problem via a nonlinear iteration scheme. The estimates and technique developed there help us later to remove the mollification and to demonstrate that in the limit $\varepsilon \rightarrow 0$ we obtain a unique solution of our original problem. The essential part of the analysis which allows us to arrive to that conclusion is based on the fact that for initial data from a suitable Orlicz space (dense in $L_2(\mu)$) the solution lives within a much finer space.

In section 4 we show that the solution of (MCP) defines a C_0 -semigroup which preserves positivity and is contractive in L_∞ norm, (while $L_2(\mu)$ contractivity was already proven in section 3). Moreover we demonstrate that the solution decays exponentially to a constant in $L_2(\mu)$ space and consequently the time average of the solution converges almost everywhere to that constant.

In section 5 we prove that the semigroup is uniformly hypercontractive in certain family of Orlicz norms, i.e. hypercontractive in the corresponding metrics; (as we are dealing with nonlinear semigroup hypercontractivity in the norms is in general a weaker property).

In section 6 we consider briefly the corresponding local problem (in which normalisation by mean value with the measure μ is not present). The analysis here is entirely based on smoothing properties of the linear semigroup generated by L which follows directly from corresponding F -Sobolev inequality. Therefore it allows us to consider essentially weaker nonlinearities than the ones considered earlier for (MCP).

Finally in section 7 we demonstrate that the coercive inequalities which formed a basis for our studies hold true in a large class of infinite dimensional models, and the Appendix collects all the definitions and properties we need on Orlicz spaces.

2 General setting and meaning of the equation

Analysis of problem (MCP) will be developed in a setting on which PDE methods may be extended. The linear operator L involved in (MCP) is a Markov generator given by a Dirichlet structure, that is a measure space $(\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mu)$ and a (conservative) Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on \mathbb{M} (see [FOT], [MZR]). Analytical regularity of this Dirichlet structure will be given by a functional inequality (replacing classical Sobolev inequalities used in PDE problems) which makes sense in the case when μ is a probability measure and with which the nonlinear perturbation is highly related. Usual Gelfand triple for Sobolev spaces $\dot{W}^{1,2}(\Omega) \subset L^2(\Omega) \subset W^{-1,2}(\Omega)$ (see [Wlo87]) has to be replaced by $\mathcal{D}(\mathcal{E}) \subset L^2(\mu) \subset \mathcal{D}'(\mathcal{E})$, where the domain $\mathcal{D}(\mathcal{E})$ is equipped with Hilbertian structure $\langle \cdot, \cdot \rangle_{\mathcal{E}} \equiv \langle \cdot, \cdot \rangle_{L^2} + \mathcal{E}(\cdot, \cdot)$. To recall this and also

to make the reading easier for most, we'll often use notation $W^{1,2}(\mathcal{E})$ for the domain $\mathcal{D}(\mathcal{E})$, $W^{-1,2}(\mathcal{E}) = \mathcal{D}'(\mathcal{E})$ for its dual, and $\|\cdot\|_{W^{1,2}(\mathcal{E})}$ or even $\|\cdot\|_{W^{1,2}}$ for the corresponding norms. We'll also write abusively $\mathcal{E}(f, f) \equiv \mu(|\nabla f|^2) = \int |\nabla f|^2 d\mu$, for natural reasons (see below).

As mentioned in the introduction, our approach of existence for (MCP) rely on considering first a mollified problem and then removing the mollification thanks to regularity results for the mollified solutions in a suitable Orlicz space. (MCP) may be stated and weak solutions may be considered for general Dirichlet forms defining the linearity. And, as long as mollified problem is only under consideration, existence and uniqueness will follow from our approach. Nevertheless, to be able to prove regularity and remove the mollification, we'll assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a *diffusion* Dirichlet form.

In a finite dimensional setting, that is when \mathbb{M} is a locally compact topological space, this diffusion property is satisfied by local (conservative) regular Dirichlet forms. Following Beurling-Deny formula (see [FOT]), this means that \mathcal{E} has neither jumping nor killing part. Recall that when \mathbb{M} is a domain $\Omega \subset \mathbb{R}^d$ (and $\text{supp}(\mu) = \Omega$) such forms (possessing $C_c^\infty(\Omega)$ as a core) may be expressed on compactly supported smooth functions as

$$\mathcal{E}(f) \equiv \mathcal{E}(f, f) = \sum_{i,j=1}^d \int_{\Omega} \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) \nu_{i,j}(dx)$$

with some Radon measures $\nu_{i,j}$. This emphasizes the links with PDE problems (see also [Dav95]).

In infinite dimensional analysis, regularity has to be replaced by quasi-regularity (see [MZR]). Our approach could be developed in this generality. Nevertheless for simplicity reasons, we will restrict ourselves to two particular cases : either \mathcal{E} is a local regular Dirichlet form on a locally compact space or it is given by the following infinite dimensional models coming from interacting spins systems.

Gibbs measures on infinite product of manifolds and generalized Sobolev space. Let $\mathbb{M} = \prod_{i \in \mathcal{R}} M_i$ be an infinite product of Riemannian manifolds (M_i, g_i) , where \mathcal{R} is a countable set (an infinite graph).

Given $z \in M_i$ and $\mathbf{x} = (x_i)_{i \in \mathcal{R}} \in \mathbb{M}$ we define $z \bullet_i \mathbf{x} \equiv \{(z \bullet_i \mathbf{x})_k \equiv \delta_{ik}z + (1 - \delta_{ik})x_k : k \in \mathcal{R}\}$. We say that a function f on \mathbb{M} is cylindrically smooth if f is localized on some finite subset $\Lambda \subset \mathcal{R}$ (we write $\Lambda \Subset \mathcal{R}$) and is smooth when considered as a function on $M_\Lambda = \prod_{i \in \Lambda} M_i$. For such cylindrically smooth functions, we consider the following quadratic operator, called the square field operator,

$$|\nabla f|^2 = \sum_{i \in \mathcal{R}} |\nabla_i f|_i^2$$

where for each site $i \in \mathcal{R}$, $|\nabla_i f|_i(\mathbf{x}) \equiv |\nabla_i f_i(\cdot|\mathbf{x})|(x_i)$ is the length of the usual gradient ∇_i for the metric g_i at x_i of the function $M_i \ni z \mapsto f_i(\cdot|\mathbf{x})(z) \equiv f(\{z \bullet_i \mathbf{x}\})$.

Let μ be a probability measure on \mathbb{M} . For an integrable (or nonnegative) function f we will use the notation $\mu f \equiv \mu(f) \equiv \int f d\mu$. For (compactly supported) cylindrically smooth functions, $\mu(|\nabla f|^2)$ makes sense. Actually, provided μ is a Gibbs measure, this can be defined on a wider class of functions on \mathbb{M} generalizing the Sobolev space $W^{1,2}$.

Briefly speaking, a Gibbs measure is defined as follows. A specification is a family $\mu_\Lambda^\xi(dx_\Lambda)$, $\Lambda \Subset \mathcal{R}$ and $\xi \in \mathbb{M}$, of absolutely continuous probability kernels on $M_\Lambda \equiv \prod_{i \in \Lambda} M_i$, that we extend to kernels E_Λ^ξ on \mathbb{M} by taking product with $\otimes_{i \notin \Lambda} \delta_{\xi_i}$. These kernels are supposed to satisfy compatibility conditions (see section 7 or [GZ03] and references therein) making them possible candidates for being versions of laws (w.r.t. a probability measure μ on \mathbb{M}) conditionally to $\pi_{\mathcal{R} \setminus \Lambda}(\xi) \equiv (\xi_i)_{i \in \mathcal{R} \setminus \Lambda}$. Measures μ on \mathbb{M} for which this holds are called Gibbs measures and can be multiple in general. They are characterized by the Dobrushin-Lanford-Ruelle (DLR) conditions $\mu = \mu E_\Lambda$ (when acting on bounded measurable functions).

Let μ be a fixed Gibbs measure. The generalized Sobolev space $W^{1,2}(\mu)$ can be defined as the space of functions $f \in \mathbb{L}^2(\mu)$ such that, for any $i \in \mathcal{R}$, $|\nabla_i f_i(\cdot|\xi)|$ in the sense of distributions in M_i belongs to $\mathbb{L}^2(M_i, \mu_{\{i\}}^\xi)$ and one has

$$\mathcal{E}(f, f) \equiv \int_{\mathbb{M}} \sum_{i \in \mathcal{R}} \mu_{\{i\}}^\xi \left(|\nabla_i f_i(\cdot|\xi)|^2 \right) \mu(d\xi) < \infty.$$

We will formally write $\mathcal{E}(f, f) = \mu(|\nabla f|^2)$ and $\mathcal{E}(f, g) = \mu(\nabla f \cdot \nabla g)$ for the associated bilinear form. One can check that this coincides with the similar quantity for cylindrically smooth compactly supported functions.

We will assume that \mathcal{E} is closed on the domain $W^{1,2}(\mathcal{E}) \equiv W^{1,2}(\mu)$ so that it is a diffusion Dirichlet form. Then the corresponding Markov generator L is well defined selfadjoint operator on the corresponding domain $\mathcal{D}(L) \subset \mathbb{L}_2(\mu)$. The \mathcal{C}_0 semigroup generated by L is denoted by $P_t \equiv e^{tL}$. It follows from our setup that for any $\varepsilon > 0$ and $f \in \mathbb{L}_2(\mu)$ we have $P_\varepsilon f \in \mathcal{D}(L)$ and $t \mapsto P_t P_\varepsilon f$ is differentiable. Moreover $P_\varepsilon f \rightarrow f$ in $\mathbb{L}_2(\mu)$ as $\varepsilon \rightarrow 0$.

Coercive inequalities as analytic regularity of Dirichlet Structures.

To ensure a nice behaviour of the Dirichlet structure, as well as to specify later the admissible nonlinearity, we introduce the following requirements.

Condition (C1) : In all what follows $F : [0, \infty) \rightarrow \mathbb{R}^+$ denotes a non decreasing \mathcal{C}^2 function. We assume that there exist constants $\theta > \bar{\theta} \geq 1$ and $\bar{B}, K > 0$ such that

$$(\mathbf{C1}) \quad \begin{cases} (i) & F \equiv 0 \text{ on } [0, \bar{\theta}], \\ (ii) & F \text{ is concave on } [\theta, +\infty), \\ (iii) & \forall x \geq 0, \quad xF'(x) \leq \bar{B} \text{ and } F(x) \leq Kx. \end{cases}$$

We assume also that (μ, \mathcal{E}) satisfies **F-Sobolev** inequality, that is that there exists a constant $c_F \in (0, +\infty)$ such that

$$\int f^2 F \left(\frac{f^2}{\mu(f^2)} \right) d\mu \leq c_F \int |\nabla f|^2 d\mu. \quad (\mathbf{FS})$$

for any sufficiently smooth function f . In this case we will use a shorthand notation $\mu \in \mathbf{FS}(c_F)$.

Note that since $F(1) = 0$ such inequality is tight in the sense that both sides are zero for constant functions.

Lemma 1 (Generalized Relative Entropy Inequality).

Suppose a function F satisfies condition **(C1)**. Then for any $x, y \geq 0$, $xF(y) \leq xF(x) + By$, for $B = \max(\bar{B}, K\theta)$.

Therefore for any probability measure μ , and any $f, g \in \mathbb{L}^2(\mu)$,

$$\int f^2 F \left(\frac{g^2}{\mu(g^2)} \right) d\mu \leq \int f^2 F \left(\frac{f^2}{\mu(f^2)} \right) d\mu + B\mu(f^2). \quad (\mathbf{GREI})$$

Proof. As F is non decreasing, this is trivial if $x \geq y$. This is also trivial if $y \leq \bar{\theta}$. So assume $x \leq y$ and $y \geq \bar{\theta}$. Now, if $x \geq \theta$, then

$$x(F(y) - F(x)) = x \frac{F(y) - F(x)}{y - x} (y - x) \leq xF'(x)y \leq \bar{B}y.$$

And if on the contrary $x \leq \theta$, $xF(y) \leq \theta F(y) \leq \theta Ky$. This ends the proof of the bound $xF(y) \leq xF(x) + By$ from which inequality **(GREI)** easily follows. \square

Lemma 2. Assume F satisfies condition **(C1)**. Then

$$\mu \left(f^2 F \left(\frac{f^2}{\mu f^2} \right) \right) \leq \mu \left(\tilde{f}^2 F \left(\frac{\tilde{f}^2}{\mu \tilde{f}^2} \right) \right) + C\mu(\tilde{f}^2)$$

where $\tilde{f} = f - \mu f$ and $C \equiv 4\bar{B} + B$.

Proof. The proof is similar to the one of the Rothaus' inequality when $F = \log$. It is sufficient to consider the case when $\mu f > 0$. Setting $\hat{f} \equiv \frac{\tilde{f}}{\|f\|_2}$, it suffices to prove that for any $t \in \left[0, \frac{\|\tilde{f}\|_2}{\mu f}\right]$,

$$A(t) \equiv \mu \left((1 + t\hat{f})^2 F \left(\frac{(1 + t\hat{f})^2}{1 + t^2} \right) \right) \leq t^2 \mu \left(\hat{f}^2 F(\hat{f}^2) \right) + Ct^2$$

To this end note that $A(0) = 0$ and $A'(0) = 0$, while by concavity of F and Lemma 1

$$\begin{aligned}
A''(t) &= 2\mu \left(\hat{f}^2 F \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \right) \\
&\quad + 2\mu \left(\hat{f}(1+t\hat{f}) F' \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \cdot \left[\frac{2\hat{f}(1+t\hat{f})}{1+t^2} - \frac{(1+t\hat{f})^2 2t}{(1+t^2)^2} \right] \right) \\
&\quad + \mu \left((1+t\hat{f})^2 F'' \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \cdot \left[\frac{2\hat{f}(1+t\hat{f})}{1+t^2} - \frac{(1+t\hat{f})^2 2t}{(1+t^2)^2} \right]^2 \right) \\
&\leq 2\mu \left(\hat{f}^2 F \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \right) \\
&\quad + 4\mu \left(\hat{f} F' \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \cdot \left[\frac{\hat{f}(1+t\hat{f})^2}{1+t^2} - \frac{(1+t\hat{f})^3 2t}{(1+t^2)^2} \right] \right) \\
&\leq 2\mu \left(\hat{f}^2 F \left(\frac{(1+t\hat{f})^2}{1+t^2} \right) \right) + 8\bar{B} \mu(\hat{f}^2) \leq 2\mu \left(\hat{f}^2 F \left(\frac{\hat{f}^2}{\mu \hat{f}^2} \right) \right) + [8\bar{B} + 2B]
\end{aligned}$$

□

Example 3. We introduce now an example the reader may keep in mind as a guideline. Fix $\theta > 2$, $\alpha \in (1, 2]$ and consider a function $F_\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}$

$$x \mapsto F_\alpha(x) = \begin{cases} 0 & \text{if } x \in [0, \theta] \\ (\log(x))^\beta - (\log \theta)^\beta & \text{if } x \geq \theta. \end{cases}$$

where $\beta \equiv 2(1 - \frac{1}{\alpha}) \in (0, 1)$. Note that F_α is continuous, but not \mathcal{C}^2 . To deal with differentiability at $x = \theta$ we introduce a \mathcal{C}^∞ non-negative function g with compact support in $[-1, 0]$ and such that $\int g(y) dy = 1$. For $\varepsilon > 0$, define $g_\varepsilon(x) = \frac{1}{\varepsilon} g(\frac{x}{\varepsilon})$. Note that $F(x) \equiv F_\alpha * g_\varepsilon(x) := \int F_\alpha(x-y) g_\varepsilon(y) dy$ is a \mathcal{C}^∞ function vanishing on $[0, \theta - \varepsilon]$. Fix $\varepsilon \leq 1$ and set $\bar{\theta} = \theta - \varepsilon > 1$.

One can see that F satisfies the Assumptions (i) and (ii) described above. On the other hand a simple computation gives that for any $\varepsilon > 0$, any $x \geq 0$,

$$x(F_\alpha * g_\varepsilon)'(x) \leq \beta(\log \theta)^{\beta-1} \leq \beta(1/2)^{\beta-1} \leq 1. \quad (1)$$

Moreover we have for any $x \geq 0$,

$$F_\alpha(x) \leq F(x) \leq F_\alpha(x + \varepsilon) \leq F_\alpha(x) + \log \left(\frac{\max(\theta, e) + \varepsilon}{\theta} \right). \quad (2)$$

The last inequality comes from the fact that for $x \geq e$, $\log(\varepsilon + x)^\beta - \log(x)^\beta \leq \log(\varepsilon + x) - \log(x) \leq \log([\theta + \varepsilon]/\theta)$. While for $\theta \leq x \leq e$, we have $\log(\varepsilon + x)^\beta - \log(x)^\beta \leq \log(\varepsilon + e)^\beta - \log(\theta)^\beta \leq \log(\varepsilon + e) - \log(\theta)$.

In particular, since $\log(\varepsilon + x)^\beta \leq x$ when $x \geq 1$, we have $F(x) \leq x$ for any $x \geq 0$.

Thus F satisfies Assumption (iii) with $\bar{B} = 1$ and $K = 1$. Note that $B = \theta$. Finally we remark that for F_α and a measure $d\mu_\alpha \equiv \exp\{-|x|^\alpha\}dx/Z_\alpha$, the inequality **(FS)** is true, [BR03]. Hence, using the far right inequality in (2) together with the Lemma 2, we conclude that corresponding coercive inequality is satisfied also with the function F (possibly with a different constant).

Nonlinearity. The nonlinear part in equation (MCP) is described by a function G . We assume that G satisfies the following properties.

Condition (C2) : With F satisfying condition **(C1)**, we assume that

$$G = F + \mathcal{J}$$

with \mathcal{J} such that

- $\mathcal{J} : [0, \infty) \rightarrow \mathbb{R}^-$ is a bounded \mathcal{C}^1 function.
- $\mathcal{J}(1) = 0$
- $\sup x|\mathcal{J}'(x)| < \infty$.

Note that $G(0)$ is well defined and G is Lipschitz at 0 (for a non-Lipschitz at 0 example, see [FZ04]).

Since $G(1) = 0$, constants are global solutions of the corresponding parabolic problem. Clearly $G \leq F$.

Under these hypothesis, $\tilde{B} \equiv \sup_{x \geq 0} x|G'(x)| < \infty$.

Convexity assumption.

Condition (C3) : Let F satisfy condition **(C1)**. We assume furthermore that $x F(x)$ is convex.

Under condition **(C3)** (and **(C1)**), one may prove that, for any $q \geq 0$, the function $\Psi_q(x)$ given by $\Psi_q(x) \equiv x e^{q F(x)}$ is a Young function so that $\Phi_q(x) \equiv \Psi_q(x^2)$ is a Nice Young function (or N -function as called in [RR91]). The associated Orlicz space satisfies $\mathbb{L}^{\Phi_q}(\mu) \subset \mathbb{L}^2(\mu)$ with continuous embedding. Such Orlicz spaces will appear later when more regularity on the initial value will be assumed.

Without further mention we will assume that conditions **(C1)** – **(C3)** are satisfied throughout the rest of the paper. Most of our results are valid under only these hypothesis. Note nevertheless that the argument we give in section 3.1.1 to complete proof of existence assume more on F . In particular, we will sometimes use the following additional condition.

Condition (C4) : There exists a constant $0 \leq R < \infty$ such that, for any $a, b \in (0, \infty)$,

$$F(ab) \leq F(a) + F(b) + R.$$

An immediate consequence of condition (C4) is the striking sub-multiplicativity property for the N -function Φ_q :

$$\Phi_q(xy) \leq e^{Rq} \Phi_q(x) \Phi_q(y). \quad (3)$$

Example 3 continued. The function F described in Example 3 satisfies condition (C4), and also (C3) provided $\theta \geq e$.

Let us prove first that (C4) occurs. To do so first note that, for any $a \geq 0$ and $b \geq 1$,

$$F_\alpha(ab) \leq F_\alpha(a) + (\log b)^\beta. \quad (4)$$

This comes from the computations detailed below. If $0 < a \leq \theta$ and $ab \geq \theta > 1$, then

$$\begin{aligned} F_\alpha(ab) &= (\log(ab))^\beta - (\log \theta)^\beta \\ &\leq (\log(\max(a, 1)) + \log b)^\beta - (\log \theta)^\beta \\ &\leq (\log b)^\beta + (\log \max(a, 1))^\beta - (\log \theta)^\beta \leq (\log b)^\beta. \end{aligned}$$

In the case $a \geq \theta$, $F_\alpha(ab) = (\log a + \log b)^\beta - (\log \theta)^\beta \leq (\log b)^\beta + (\log a)^\beta - (\log \theta)^\beta = F_\alpha(a) + (\log b)^\beta$.

Now, inequality (4) also holds for F . Indeed, if $a \geq 0$ and $b \geq 1$,

$$\begin{aligned} F(ab) &= \int_{-\varepsilon}^0 F_\alpha\left(b\left(a - \frac{y}{b}\right)\right) g_\varepsilon(y) dy \leq (\log b)^\beta + \int_{-\varepsilon}^0 F_\alpha\left(a - \frac{y}{b}\right) g_\varepsilon(y) dy \\ &\leq (\log b)^\beta + \int_{-\varepsilon}^0 F_\alpha(a - y) g_\varepsilon(y) dy = F(a) + (\log b)^\beta \end{aligned}$$

thanks to (4), (remind that $\int_{-\varepsilon}^0 g_\varepsilon(y) dy = 1$), and $a - \frac{y}{b} \leq a - y$ when $y \leq 0$. As a consequence, (C4) holds true with $R = (\log \theta)^\beta$ as, for any $b > 0$, $(\log b)^\beta \leq F_\alpha(b) + (\log \theta)^\beta \leq F(b) + (\log \theta)^\beta$.

Now, let us focus on condition (C3). One has $F'(x) = F_\alpha * g'_\varepsilon(x) = \int_{-\varepsilon}^{x-\theta} F_\alpha(x-y) g'_\varepsilon(y) dy$ so that, after simple integration by parts, one gets also $F' = F'_\alpha * g_\varepsilon$ with the function $(L^1_{\text{loc}}$ but discontinuous at θ)

$$F'_\alpha(x) \stackrel{\text{def.}}{=} \begin{cases} \frac{\beta}{x} (\log x)^{\beta-1} & \text{if } x \geq \theta \\ 0 & x < \theta \end{cases}.$$

Note that there is no boundary terms as $F_\alpha(\theta) = 0$ and $g_\varepsilon(-\varepsilon) = 0$.

As a consequence, $F'' = F'_\alpha * g'_\varepsilon$ and a further integration by parts leads to

$$F''(x) = F'_\alpha(\theta) g_\varepsilon(x - \theta) + F''_\alpha * g_\varepsilon(x)$$

with the L^1_{loc} function

$$F''_{\alpha}(x) \stackrel{\text{def.}}{=} \begin{cases} \frac{\beta}{x^2} (\log x)^{\beta-2} [\beta - 1 - \log x] & \text{if } x \geq \theta \\ 0 & x < \theta \end{cases}.$$

Hence,

$$\begin{aligned} (x F(x))'' &= x F'_{\alpha}(\theta) g_{\varepsilon}(x - \theta) + \int_{-\varepsilon}^{\min(x-\theta, 0)} y F''_{\alpha}(x - y) g_{\varepsilon}(y) dy \\ &\quad + \int_{-\varepsilon}^{\min(x-\theta, 0)} [2F'_{\alpha}(x - y) + (x - y) F''_{\alpha}(x - y)] g_{\varepsilon}(y) dy. \end{aligned}$$

Note first that, for any $z \geq \theta$,

$$2F'_{\alpha}(z) + z F''_{\alpha}(z) = \frac{\beta}{z} (\log z)^{\beta-2} [\beta - 1 + \log z] \geq 0$$

provided $\theta \geq e^{1-\beta}$ and secondly that, for $y \leq \min(x - \theta, 0)$, $y F''_{\alpha}(x - y) \geq 0$. Hence each term of the above sum expressing $(x F(x))''$ is nonnegative.

3 Existence problem

To prove the existence of a weak solution for Cauchy problem (MCP) we implement a constructive nonlinear approximation procedure.

3.1 Weak solutions and preliminary regularity result

Given $T \in (0, \infty)$, define $\mathbb{H}_{T,+}(\mathcal{E}) \equiv \mathbb{L}^2([0, T], W^{1,2}(\mathcal{E}))$, as a Banach space of (classes of) functions $v : [0, T] \times \mathbb{M} \rightarrow \mathbb{R}$, such that

$$\|v\|_{\mathbb{H}_{T,+}}^2 \equiv \int_0^T ds \mu v^2 + \int_0^T ds \mu |\nabla v|^2 < \infty,$$

By $\mathbb{H}_{T,-}(\mu) \equiv \mathbb{L}^2([0, T], W^{-1,2}(\mathcal{E}))$, we'll denote the dual space of $\mathbb{H}_{T,+}(\mathcal{E})$.

Let $\mathcal{A} : \mathbb{H}_{T,+}(\mathcal{E}) \rightarrow \mathbb{H}_{T,-}(\mu)$ be an abstract nonlinear operator. We say that a function $u \in \mathbb{H}_{T,+}(\mathcal{E})$ is a weak solution (on $[0, T]$) of the following Cauchy problem

$$\begin{cases} \partial_t u &= Lu + \mathcal{A}(u) \\ u|_{t=0} &= f \end{cases} \quad (5)$$

with $f \in \mathbb{L}^2(\mu)$, if and only if

1. $u \in C([0, T], \mathbb{L}^2(\mu))$,

2. For any $v \in \mathcal{C}^\infty([0, T]; W^{1,2}(\mathcal{E}))$ and any $t \in [0, T]$, we have

$$\begin{aligned} \int_0^t \mu(u(s) \partial_s v(s)) ds &= \mu u(t) v(t) - \mu f v(0) + \int_0^t \mu \nabla u(s) \cdot \nabla v(s) ds \\ &\quad - \int_0^t \langle \mathcal{A}(u)(s), v(s) \rangle_{W^{-1,2}, W^{1,2}} ds, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{W^{-1,2}, W^{1,2}}$ stands for the duality bracket.

Recall from classical results (see [Eva98] for instance) that a function u satisfying condition 2. admits a weak derivative $\partial_t u \in \mathbb{L}^1([0, T], W^{-1,2}(\mathcal{E}))$ which belongs in fact to $\mathbb{L}^2([0, T], W^{-1,2}(\mathcal{E}))$, and so $u \in C((0, T), \mathbb{L}^2(\mu))$. A weak solution satisfies additionally that $u(t)$ goes to f in \mathbb{L}^2 as t goes to 0. Note also that condition 2. may be extended by density to any $v \in \mathbb{L}^2([0, T], W^{1,2}(\mathcal{E})) \cap W^{1,2}((0, T), \mathbb{L}^2(\mu))$.

Later on in this paper we discuss a situation when the operator \mathcal{A} is given by $\mathcal{A}(u) \equiv \lambda \mathcal{V}(u)$ with a parameter $\lambda \in \mathbb{R}$ and

$$\mathcal{V}(u)(s) \equiv \mathbb{V}(u(s)) \equiv u(s) G(\sigma^2(u(s)))$$

where $\sigma(u) \equiv \frac{u}{(\mu(u^2))^{1/2}}$ for $u \in \mathbb{L}^2(\mu)$, $u \neq 0$, and $\sigma(0) \equiv 0$.

With this definition of \mathcal{A} , it makes sense to consider weak solutions for Cauchy problem (5) thanks to the following basic regularity result.

Lemma 4. Regularity for the nonlinear operator.

Suppose $\mu \in \mathbf{FS}(c_F)$. Then, for any $u \in \mathbb{H}_{T,+}(\mathcal{E})$, $\mathcal{V}(u) \in \mathbb{H}_{T,-}(\mu)$. Moreover, $\mathcal{V} : \mathbb{H}_{T,+}(\mathcal{E}) \rightarrow \mathbb{H}_{T,-}(\mu)$ is Lipschitz continuous.

Proof. Suppose u and g in $W^{1,2}(\mathcal{E})$. Then one has

$$\begin{aligned} |\mu(g u G(\sigma^2(u)))| &\leq \|\mathcal{J}\|_\infty \mu(|g| |u|) + \mu(|g| |u| F(\sigma^2(u))) \\ &\leq \|\mathcal{J}\|_\infty \|u\|_2 \|g\|_2 + (\mu(g^2 F(\sigma^2(u))))^{\frac{1}{2}} (\mu(u^2 F(\sigma^2(u))))^{\frac{1}{2}} \end{aligned}$$

where we used jointly that F is non negative and Cauchy-Schwarz inequality. Making use of (GREI) and then of \mathbf{F} -Sobolev inequality (1), one gets

$$\mu(g^2 F(\sigma^2(u))) \leq \mu(g^2 F(\sigma^2(g))) + B\mu g^2 \leq \max(c_F, B) \|g\|_{W^{1,2}}^2$$

so that finally

$$|\mu(g u G(\sigma^2(u)))| \leq (\|\mathcal{J}\|_\infty + \max(c_F, B)) \|g\|_{W^{1,2}} \|u\|_{W^{1,2}}.$$

That means that, when acting on $W^{1,2}$ with \mathbb{L}^2 -type pairing, the operator $\mathbb{V}(u) = u G(\sigma^2(u))$ belongs to $W^{-1,2}(\mathcal{E})$ and

$$\|\mathbb{V}(u)\|_{W^{-1,2}(\mathcal{E})} \leq C \|u\|_{W^{1,2}}$$

with $C = (\|\mathcal{J}\|_\infty + \max(c_F, B))$.

Let us now turn to Lipschitz estimate. Suppose $v \neq u$ and g are still in $W^{1,2}(\mathcal{E})$. From the first part of the proof and $\mathbb{V}(0) = 0$, one may assume that $u \neq 0$ and $v \neq 0$. Let us set $u_\alpha \equiv \alpha u + (1 - \alpha)v$, $\alpha \in [0, 1]$, and let $w \equiv u - v$. Assume first that $u_\alpha \neq 0$ for any α . Then we have

$$\left| \mu \left(g \left[uG(\sigma^2(u)) - vG(\sigma^2(v)) \right] \right) \right| \leq \int_0^1 d\alpha \left| \mu \left(g \frac{d}{d\alpha} \left[u_\alpha G(\sigma^2(u_\alpha)) \right] \right) \right|,$$

with $\frac{d}{d\alpha} u_\alpha G(\sigma^2(u_\alpha))$ explicitly given by

$$wG(\sigma^2(u_\alpha)) + 2\sigma^2(u_\alpha)G'(\sigma^2(u_\alpha))w - 2\sigma^3(u_\alpha)G'(\sigma^2(u_\alpha))\mu(\sigma(u_\alpha)w).$$

Since by our assumption $\sigma^2|G'(\sigma^2)| \leq \tilde{B}$, we get

$$\begin{aligned} \left| \mu \left(g \left[uG(\sigma^2(u)) - vG(\sigma^2(v)) \right] \right) \right| &\leq \int_0^1 d\alpha \left| \mu \left(g w G(\sigma^2(u_\alpha)) \right) \right| \\ &\quad + 2\tilde{B}\mu(|g||w|) + 2\tilde{B} \int_0^1 d\alpha \mu(|g||\sigma(u_\alpha)|) \mu(|\sigma(u_\alpha)||w|). \end{aligned}$$

Now, by similar arguments as above, one has

$$\left| \mu \left(g w G(\sigma^2(u_\alpha)) \right) \right| \leq (\|\mathcal{J}\|_\infty + \max(c_F, B)) \|g\|_{W^{1,2}} \|w\|_{W^{1,2}}. \quad (6)$$

On the other hand, Cauchy-Schwarz inequality applied twice gives

$$\mu(|g||\sigma(u_\alpha)|) \mu(|\sigma(u_\alpha)||w|) \leq \|w\|_{\mathbb{L}^2} \|g\|_{\mathbb{L}^2},$$

so that finally

$$\left| \mu \left(g \left[uG(\sigma^2(u)) - vG(\sigma^2(v)) \right] \right) \right| \leq C \|g\|_{W^{1,2}} \|w\|_{W^{1,2}}$$

with a constant $C = \|\mathcal{J}\|_\infty + \max(c_F, B) + 4\tilde{B}$. We conclude by noting that, in the case when $u_\alpha = 0$ for some $\alpha \in (0, 1)$, one has $\sigma^2(u) = \sigma^2(v)$ so that (6) with u instead of u_α provides the corresponding estimate. \square

3.1.1 Existence

Given $f \in \mathbb{L}_2(\mu)$ and a fixed parameter $\lambda \in \mathbb{R}$, we define a sequence $u_n : \mathbb{R}^+ \times \mathbb{M} \rightarrow \mathbb{R}$, $n \in \mathbb{Z}^+$, such that u_0 is a unique solution of

$$\begin{cases} \partial_t u_0 &= L u_0 \\ u_0|_{t=0} &= f \end{cases}$$

and

$$\begin{cases} \partial_t u_{n+1} &= L u_{n+1} + \lambda \mathcal{V}(u_n) \\ u_{n+1}|_{t=0} &= f, \end{cases} \quad (\mathbb{A})$$

We would like to argue that, for any $T \in (0, \infty)$, $u_n \in \mathbb{H}_{T,+}$, and a function given by

$$u_{n+1}(t) \equiv P_t f + \lambda \int_0^t ds P_{t-s} \mathbb{V}(u_n(s)) \quad (7)$$

is a weak solution of (A) on $[0, T]$ and $u_{n+1} \in \mathbb{H}_{T,+}(\mathcal{E})$ provided (FS) with a constant c_F is satisfied, if we take $\lambda \in [0, 1/c_F)$ and $T > 0$ sufficiently small. Later we will show that such sequence of solutions converges strongly to a weak solution of our problem in a corresponding small time interval.

In fact, equation (7) has only a formal meaning as in general $\|P_{t-s}\|_{W^{-1,2} \rightarrow W^{1,2}} \leq C/(t-s)$ and not better. For that reason one has to introduce an additional smoothing by replacing \mathbb{V} by $\mathbb{V}^{(\varepsilon)}(u) = P_\varepsilon \mathbb{V}(u)$. Now, $\mathbb{V}^{(\varepsilon)}$ is a Lipschitz continuous operator from $W^{1,2}$ to itself. Thus, if we define $\mathcal{V}^{(\varepsilon)}(u)(s) \equiv \mathbb{V}^{(\varepsilon)}(u(s))$, then, for any $\varepsilon > 0$, we have $\mathcal{V}^{(\varepsilon)}(u) \in \mathbb{H}_{T,+}(\mathcal{E})$ when $u \in \mathbb{H}_{T,+}(\mathcal{E})$. It allows us to consider an ε -mollified approximate solution $\bar{u}_n = u_n^{(\varepsilon)}$ on the whole half-line $[0, +\infty)$.

Mollified problem. We introduce the following sequence of functions $(\bar{u}_n)_{n \in \mathbb{N}}$. For $f \in \mathbb{L}_2$, we set $\bar{u}_0(t) \equiv u_0(t) \equiv P_t f$, and for $n \in \mathbb{N}$, we define $\bar{u}_n \equiv \bar{u}_n^{(\varepsilon)}$ inductively as follows

$$\bar{u}_n(t) \equiv P_t f + \lambda \int_0^t ds P_{t-s} \mathcal{V}^{(\varepsilon)}(\bar{u}_{n-1})(s)$$

Then, for any $T > 0$, $\bar{u}_n \in \mathbb{L}^2([0, T], W^{1,2}(\mathcal{E})) \cap C([0, T], \mathbb{L}^2(\mu))$. Moreover, for any $t > 0$, $\bar{u}_n(t)$ (which is well defined) belongs to the domain $\mathcal{D}(L)$ of L and is differentiable with respect to t . More particularly, for any $T > 0$, $\bar{u}_n \in W^{1,2}((0, T), W^{1,2}(\mathcal{E}))$.

And we have in a strong sense (in $\mathbb{L}^2(\mu)$)

$$\begin{cases} \partial_t \bar{u}_n &= L \bar{u}_n(t) + \lambda \mathbb{V}^{(\varepsilon)}(\bar{u}_{n-1}(t)) \\ \bar{u}_n(0) &= f \end{cases} \quad (\mathbb{A}_\varepsilon)$$

Using that we show the following uniform boundedness property.

Proposition 5. Uniform bound in $C([0, T], \mathbb{L}^2) \cap \mathbb{H}_{T,+}(\mathcal{E})$.

Suppose $\mu \in \mathbf{FS}(c_F)$ with a constant $c_F \in (0, \infty)$ and let $\lambda \in [0, c_F^{-1})$. Then for any $T \in (0, \infty)$ such that

$$\eta_T \equiv \left\{ \frac{\lambda c_F}{2 - \lambda c_F} + \lambda \|\mathcal{J}\|_\infty T \right\} e^{\lambda D T} < 1$$

where $D \equiv B + \|\mathcal{J}\|_\infty$, we have, for any $n \in \mathbb{N}$,

$$\sup_{0 \leq t \leq T} \left(\mu \bar{u}_n^2(t) + (2 - \lambda c_F) \int_0^t \mu |\nabla \bar{u}_n|^2(s) ds \right) \leq \frac{2e^{\lambda D T}}{1 - \eta_T} \mu(f^2). \quad (8)$$

Hence we have in particular

$$\|\bar{u}_n\|_{C([0,T],\mathbb{L}^2)}^2 + \|\bar{u}_n\|_{\mathbb{H}_{T,+}}^2 \leq \frac{2(T+2)e^{\lambda DT}}{1-\eta_T} \mu(f^2)$$

with the right hand sides independent of $\varepsilon > 0$.

Remark 6. Note here that time $T > 0$ for which $\eta_T < 1$, and so uniform bound (8) holds, only depends on λ, c_F and $\|\mathcal{J}\|_\infty$ but not on the initial condition f . This will be a crucial step for global existence for weak solutions of (MCP) .

Proof. We note first that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu(\bar{u}_n^2) &= \mu(\bar{u}_n \partial_t \bar{u}_n) = -\mu|\nabla \bar{u}_n|^2 + \lambda \mu \left(\bar{u}_n \nabla^{(\varepsilon)}(\bar{u}_{n-1}) \right) \\ &= -\mu|\nabla \bar{u}_n|^2 + \lambda \mu (P_\varepsilon(\bar{u}_n) \nabla(\bar{u}_{n-1})) \end{aligned}$$

where in the last line we used the symmetry of P_ε in $\mathbb{L}_2(\mu)$. Simplifying the notation by setting $\tilde{u}_n \equiv P_\varepsilon(\bar{u}_n)$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu(\bar{u}_n^2) &= \int \bar{u}_n L \bar{u}_n d\mu + \lambda \int \tilde{u}_n \bar{u}_{n-1} G(\sigma^2(\bar{u}_{n-1})) d\mu \\ &\leq - \int |\nabla \bar{u}_n|^2 d\mu + \lambda \|\mathcal{J}\|_\infty \int |\tilde{u}_n \bar{u}_{n-1}| d\mu \\ &\quad + \lambda \int \tilde{u}_n \bar{u}_{n-1} F(\sigma^2(\bar{u}_{n-1})) d\mu. \end{aligned} \tag{9}$$

To estimate the second term on the right hand side of (9), we use the simple bound

$$\int |\tilde{u}_n \bar{u}_{n-1}| d\mu \leq \frac{1}{2} \mu(\tilde{u}_n^2) + \frac{1}{2} \mu(\bar{u}_{n-1}^2).$$

Next, because F is nonnegative, we can estimate the third term on the right hand side of (9) as follows

$$\begin{aligned} \left| \int \tilde{u}_n \bar{u}_{n-1} F(\sigma^2(\bar{u}_{n-1})) d\mu \right| &\leq \int |\tilde{u}_n \bar{u}_{n-1}| F(\sigma^2(\bar{u}_{n-1})) d\mu \\ &\leq \frac{1}{2} \int \tilde{u}_n^2 F(\sigma^2(\bar{u}_{n-1})) d\mu + \frac{1}{2} \int \bar{u}_{n-1}^2 F(\sigma^2(\bar{u}_{n-1})) d\mu \\ &\leq \frac{1}{2} \int \tilde{u}_n^2 F(\sigma^2(\tilde{u}_n)) d\mu + \frac{B}{2} \mu(\tilde{u}_n^2) + \frac{1}{2} \int \bar{u}_{n-1}^2 F(\sigma^2(\bar{u}_{n-1})) d\mu \\ &\leq \frac{c_F}{2} \mu|\nabla \tilde{u}_n|^2 + \frac{B}{2} \mu(\tilde{u}_n^2) + \frac{c_F}{2} \mu|\nabla \bar{u}_{n-1}|^2 \\ &\leq \frac{c_F}{2} \mu|\nabla \bar{u}_n|^2 + \frac{B}{2} \mu(\bar{u}_n^2) + \frac{c_F}{2} \mu|\nabla \bar{u}_{n-1}|^2 \end{aligned}$$

thanks to **(GREI)** and the **(FS)** inequality. Combining our bounds, after simple rearrangement we arrive at the following differential inequality

$$\frac{d}{dt}\mu(\bar{u}_n^2) + (2 - \lambda c_F)\mu|\nabla \bar{u}_n|^2 \leq \lambda\|\mathcal{J}\|_\infty\mu(\bar{u}_{n-1}^2) + \lambda c_F\mu(|\nabla \bar{u}_{n-1}|^2) + \lambda D\mu(\bar{u}_n^2)$$

with the constant $D \equiv \|\mathcal{J}\|_\infty + B$. After integration with respect to time, using $2 - \lambda c_F \geq 0$, for any $t \in \mathbb{R}^+$, we obtain

$$\begin{aligned} & \mu(\bar{u}_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla \bar{u}_n(s)|^2) ds \\ & \leq \mu(f^2) + \lambda\|\mathcal{J}\|_\infty \int_0^t \mu(\bar{u}_{n-1}^2(s)) ds + \lambda c_F \int_0^t \mu(|\nabla \bar{u}_{n-1}(s)|^2) ds \\ & \quad + \lambda D \int_0^t \mu(\bar{u}_n^2(s)) ds \\ & \leq \mu(f^2) + \lambda\|\mathcal{J}\|_\infty \int_0^t \mu(\bar{u}_{n-1}^2(s)) ds + \lambda c_F \int_0^t \mu(|\nabla \bar{u}_{n-1}(s)|^2) ds \\ & \quad + \lambda D \int_0^t \left(\mu(\bar{u}_n^2(s)) + (2 - \lambda c_F) \int_0^s \mu(|\nabla \bar{u}_n(r)|^2) dr \right) ds. \end{aligned}$$

Hence by Gronwall type arguments, we get

$$\begin{aligned} & \mu(\bar{u}_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla \bar{u}_n(s)|^2) ds \\ & \leq e^{\lambda Dt} \cdot \left\{ \mu(f^2) + \lambda\|\mathcal{J}\|_\infty \int_0^t \mu(\bar{u}_{n-1}^2(s)) ds + \lambda c_F \int_0^t \mu(|\nabla \bar{u}_{n-1}(s)|^2) ds \right\} \end{aligned}$$

Setting

$$\mathcal{Z}_n(t) \equiv \mu(\bar{u}_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla \bar{u}_n(s)|^2) ds$$

we can see that the following inductive inequality is true

$$\mathcal{Z}_n(t) \leq \mu(f^2)e^{\lambda Dt} + \frac{\lambda c_F}{2 - \lambda c_F} e^{\lambda Dt} \mathcal{Z}_{n-1}(t) + \lambda\|\mathcal{J}\|_\infty e^{\lambda Dt} \int_0^t \mathcal{Z}_{n-1}(s) ds$$

Using this for all $t \in [0, T]$, with $Z_n \equiv \mathcal{Z}_n(T) \equiv \sup_{t \in [0, T]} \mathcal{Z}_n(t)$, we obtain

$$Z_n \leq \mu(f^2)e^{\lambda DT} + \eta_T Z_{n-1} \quad (10)$$

with

$$\eta_T \equiv \left\{ \frac{\lambda c_F}{2 - \lambda c_F} + \lambda\|\mathcal{J}\|_\infty T \right\} e^{\lambda DT}$$

Assuming that $0 < \lambda c_F < 1$, $\eta_T \in (0, 1)$ for all $T \in (0, \infty)$ small enough. In this case (10) can be iterated to obtain the following bound uniform in $n \in \mathbb{N}$ as well as $\varepsilon > 0$

$$Z_n \leq \frac{2e^{\lambda DT}}{1 - \eta_T} \mu(f^2) \quad (11)$$

as $Z_0 \equiv Z_0(u_0) \leq 2\mu(f^2)$. That is we have shown that, for $0 < \lambda c_F < 1$ and $T \in (0, \infty)$ sufficiently small, we have

$$\sup_{0 \leq t \leq T} \left(\mu \bar{u}_n^2(t) + (2 - \lambda c_F) \int_0^t \mu |\nabla \bar{u}_n|^2(s) ds \right) \leq \frac{2e^{\lambda DT}}{1 - \eta_T} \mu(f^2)$$

This naturally implies that, under the same conditions for $t \in [0, T]$, we have

$$\|\bar{u}_n\|_{\mathbb{H}_{T,+}}^2 \leq \frac{2(1+T)e^{\lambda DT}}{1 - \eta_T} \mu(f^2) \quad \text{and} \quad \|\bar{u}_n\|_{C([0,T], \mathbb{L}^2(\mu))}^2 \leq \frac{2e^{\lambda DT}}{1 - \eta_T} \mu(f^2)$$

uniformly in $n \in \mathbb{N}$ and $\varepsilon > 0$. \square

Proposition 7. Convergence scheme in $\mathbf{C}([0, \mathbf{T}], \mathbb{L}^2) \cap \mathbb{H}_{\mathbf{T},+}(\mathcal{E})$.

Suppose $\mu \in \mathbf{FS}(c_F)$ with a constant $c_F \in (0, \infty)$ and let $\lambda \in [0, c_F^{-1})$. For $T \in (0, \infty)$, let

$$\eta_*(T) \equiv \left[\lambda a T + \frac{\lambda c_F}{2 - \lambda c_F} \right] e^{\lambda a T},$$

where $a = \left(\|\mathcal{J}\|_\infty + B + 4\tilde{B} \right)$. Let $T_0 \in (0, \infty)$ be small enough so that $\eta_*(T_0) < 1$. Then, for any $0 < T \leq T_0$, the function $w_n \equiv \bar{u}_{n+1} - \bar{u}_n$, satisfies the following bound

$$\sup_{t \in [0, T]} \left(\mu(w_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla w_n(s)|^2) ds \right) \leq C(\eta_*(T))^{n-1} \mu f^2,$$

with a constant $C \in (0, \infty)$ independent of $\varepsilon > 0$ and T . As a consequence,

$$\|\bar{u}_{n+1} - \bar{u}_n\|_{C([0, T], \mathbb{L}^2)}^2 + \|\bar{u}_{n+1} - \bar{u}_n\|_{\mathbb{H}_{T,+}}^2 \leq C(T+2) \mu(f^2) (\eta_*(T))^{n-1}$$

uniformly in $\varepsilon > 0$.

Proof. Since \bar{u}_n is a strong solution of (\mathbb{A}_ε) , the difference $w_n \equiv \bar{u}_{n+1} - \bar{u}_n$ is a strong solution of

$$\begin{cases} \partial_t w_n &= Lw_n + \lambda P_\varepsilon [\bar{u}_n G(\sigma^2(\bar{u}_n)) - \bar{u}_{n-1} G(\sigma^2(\bar{u}_{n-1}))] \\ w_n(0) &= 0. \end{cases}$$

Hence, with $\tilde{w}_n \equiv P_\varepsilon w_n$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mu(w_n^2) &= \int w_n Lw_n d\mu \\ &\quad + \lambda \int \tilde{w}_n [\bar{u}_n G(\sigma^2(\bar{u}_n)) - \bar{u}_{n-1} G(\sigma^2(\bar{u}_{n-1}))] d\mu \\ &= - \int |\nabla w_n|^2 d\mu + \lambda \int \int_0^1 \tilde{w}_n \frac{d}{d\alpha} v_\alpha G(\sigma^2(v_\alpha)) d\alpha d\mu \end{aligned}$$

where $v_\alpha = \alpha \bar{u}_n + (1 - \alpha) \bar{u}_{n-1}$. As already seen in the proof of lemma 4

$$\begin{aligned} \frac{d}{d\alpha} v_\alpha G(\sigma^2(v_\alpha)) &= w_{n-1} G(\sigma^2(v_\alpha)) + 2 \sigma^2(v_\alpha) G'(\sigma^2(v_\alpha)) w_{n-1} \\ &\quad - 2 \sigma^3(v_\alpha) G'(\sigma^2(v_\alpha)) \mu(\sigma(v_\alpha) w_{n-1}). \end{aligned} \quad (12)$$

We will deal with the corresponding three different terms as follows. For the first term, using our assumption $G = \mathcal{J} + F$, we have

$$\begin{aligned} \left| \int \tilde{w}_n w_{n-1} G(\sigma^2(v_\alpha)) d\mu \right| &\leq \int |\tilde{w}_n w_{n-1}| \cdot |\mathcal{J}(\sigma^2(v_\alpha))| d\mu \\ &\quad + \frac{1}{2} \int \tilde{w}_n^2 F(\sigma^2(v_\alpha)) d\mu + \frac{1}{2} \int w_{n-1}^2 F(\sigma^2(v_\alpha)) d\mu \\ &\leq \frac{1}{2} (\|\mathcal{J}\|_\infty + B) (\mu(\tilde{w}_n^2) + \mu(w_{n-1}^2)) \\ &\quad + \frac{1}{2} \int \tilde{w}_n^2 F(\sigma^2(\tilde{w}_n)) d\mu + \frac{1}{2} \int w_{n-1}^2 F(\sigma^2(w_{n-1})) d\mu \end{aligned}$$

where we used $|xy| \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$, boundedness of \mathcal{J} and (GREI). Hence, using **(FS)** and the fact that $\mu(\tilde{w}_n^2) \leq \mu(w_n^2)$ and $\mu(|\nabla \tilde{w}_n|^2) \leq \mu(|\nabla w_n|^2)$, we arrive at

$$\begin{aligned} \left| \int \tilde{w}_n w_{n-1} G(\sigma^2(v_\alpha)) d\mu \right| &\leq \frac{1}{2} (\|\mathcal{J}\|_\infty + B) (\mu(w_n^2) + \mu(w_{n-1}^2)) \\ &\quad + \frac{c_F}{2} (\mu(|\nabla w_n|^2) + \mu(|\nabla w_{n-1}|^2)). \end{aligned}$$

For the second term, since by condition **(C2)** we have $|xG'(x)| \leq \tilde{B}$, we get

$$\left| \int \tilde{w}_n \sigma^2(v_\alpha) G'(\sigma^2(v_\alpha)) w_{n-1} d\mu \right| \leq \frac{1}{2} \tilde{B} (\mu(w_n^2) + \mu(w_{n-1}^2)).$$

For analysis of the third term, recall that $\mu(\sigma^2(v_\alpha)) = 1$, so that

$$\begin{aligned} \left| \int \tilde{w}_n \sigma^3(v_\alpha) G'(\sigma^2(v_\alpha)) \mu(\sigma(v_\alpha) w_{n-1}) d\mu \right| &\leq \tilde{B} \mu(|\tilde{w}_n \sigma(v_\alpha)|) \mu(|\sigma(v_\alpha) w_{n-1}|) \\ &\leq \tilde{B} \mu(\tilde{w}_n^2)^{\frac{1}{2}} \mu(w_{n-1}^2)^{\frac{1}{2}} \leq \frac{1}{2} \tilde{B} (\mu(w_n^2) + \mu(w_{n-1}^2)). \end{aligned}$$

Combining all bounds we obtain a differential inequality which after integration with respect to time and taking into the account the time zero condition

$w_n(0) = 0$, leads to

$$\begin{aligned}
& \mu(w_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla w_n(s)|^2) ds \\
& \leq \lambda a \int_0^t \mu(w_{n-1}^2(s)) ds + \lambda c_F \int_0^t \mu(|\nabla w_{n-1}(s)|^2) ds \\
& \quad + \lambda a \int_0^t \mu(w_n^2(s)) ds \\
& \leq \lambda a \int_0^t \mu(w_{n-1}^2(s)) ds + \lambda c_F \int_0^t \mu(|\nabla w_{n-1}(s)|^2) ds \\
& \quad + \lambda a \int_0^t \left(\mu(w_n^2(s)) + (2 - \lambda c_F) \int_0^s \mu(|\nabla w_n(r)|^2) dr \right) ds
\end{aligned}$$

with a constant $a = (\|\mathcal{J}\|_\infty + B + 4\tilde{B})$ for every $t \in [0, T]$. Setting

$$\Xi_n(t) \equiv \mu(w_n^2(t)) + (2 - \lambda c_F) \int_0^t \mu(|\nabla w_n(s)|^2) ds$$

the last bound implies the following relation

$$\Xi_n(t) \leq \frac{\lambda c_F}{2 - \lambda c_F} \Xi_{n-1}(t) + \lambda a \int_0^t \Xi_{n-1}(s) ds + \lambda a \int_0^t \Xi_n(s) ds \quad (13)$$

Applying Gronwall type arguments, we get for any $t \in [0, T]$,

$$\Xi_n(t) \leq \frac{\lambda c_F}{2 - \lambda c_F} e^{\lambda a t} \Xi_{n-1}(t) + \lambda a e^{\lambda a t} \int_0^t \Xi_{n-1}(s) ds. \quad (14)$$

This leads to the following inductive bound

$$\sup_{t \in [0, T]} \Xi_n(t) \leq \eta_*(T) \sup_{t \in [0, T]} \Xi_{n-1}(t) \quad (15)$$

with $\eta_*(T) \equiv \left[\lambda a T + \frac{\lambda c_F}{2 - \lambda c_F} \right] e^{\lambda a T}$. If $0 < \lambda c_F < 1$, then there exists $T_0 > 0$ such that $\eta_*(T) \in (0, 1)$ for any $0 < T < T_0$. In this situation using the uniform bound of Proposition 5, we arrive at

$$\sup_{t \in [0, T]} \Xi_n(t) \leq C \mu(f^2) (\eta_*(T))^{n-1}$$

with a constant $C \in (0, \infty)$ independent of $\varepsilon > 0$. As a consequence we conclude that there exists $T \in (0, \infty)$, independent of $\varepsilon > 0$ and of the initial value $f \in \mathbb{L}_2(\mu)$, such that the sequence $(\bar{u}_n(t))_{n \in \mathbb{N}}$, $t \in [0, T]$, converges in $\mathbb{H}_{T,+}(\mathcal{E}) \cap C([0, T], \mathbb{L}^2(\mu))$ uniformly in $\varepsilon > 0$. \square

Proposition 8. Uniqueness for Mollified Problem.

Assume $\mu \in \mathbf{FS}(c_F)$ with a constant $c_F \in (0, \infty)$, let $\lambda \in [0, c_F^{-1})$. Then, for any $T > 0$, there exists at most one weak solution on $[0, T]$ of the mollified Cauchy problem

$$\begin{cases} \partial_t u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda \mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) \\ u^{(\varepsilon)}|_{t=0} &= f \end{cases} \quad (\mathbb{C}_\varepsilon)$$

Proof. Assume there are two distinct weak solutions $u^{(\varepsilon)}$ and $v^{(\varepsilon)}$ on $[0, T]$ with the same initial value f . Let $w = u^{(\varepsilon)} - v^{(\varepsilon)}$. Then w belongs to the space $\mathbb{L}^2([0, T], W^{1,2}(\mathcal{E})) \cap C([0, T], \mathbb{L}^2(\mu))$, admits a weak derivative $\partial_t w$ in $\mathbb{L}^1([0, T], W^{-1,2}(\mathcal{E}))$ and one has $\partial_t w = \nu$ with

$$\nu \equiv Lw + \lambda \left[\mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) - \mathcal{V}^{(\varepsilon)}(v^{(\varepsilon)}) \right] \in \mathbb{L}^2([0, T], W^{-1,2}(\mathcal{E}))$$

and $w(0) = 0$. Extend w to $\mathbb{L}^2(\mathbb{R}, W^{1,2}(\mathcal{E}))$ by setting $w \equiv 0$ outside $[0, T]$. Making use of regularization via convolution with mollifiers $\rho_\delta \in C^\infty(\mathbb{R})$ with support in $[-\delta, \delta]$, $\delta > 0$, one has $\rho_\delta * w \in C^\infty(\mathbb{R}, W^{1,2}(\mathcal{E}))$ and, for any $t \in \mathbb{R}$, $\frac{d}{dt}(\rho_\delta * w)(t) = (\frac{d}{dt}\rho_\delta) * w(t)$ in $W^{1,2}(\mathcal{E})$. For any $0 < \delta < \delta_0$, and any $t \in (\delta_0, T - \delta_0)$, one has furthermore $\frac{d}{dt}(\rho_\delta * w)(t) = \rho_\delta * \partial_t w(t)$ as $\rho_\delta(t - \cdot) \in C_c^\infty((0, T))$. As a consequence, for any $t \in (\delta_0, T - \delta_0)$, one gets

$$\frac{d}{dt} \mu((\rho_\delta * w)^2)(t) = 2 \langle (\rho_\delta * \nu)(t), (\rho_\delta * w)(t) \rangle_{W^{-1,2}, W^{1,2}}$$

with $\nu \in \mathbb{H}_{T,-}(\mu)$ as defined above. Integrating between δ_0 and t , and then letting δ go to 0, we arrive at

$$\begin{aligned} \mu((w(t))^2) &= \mu((w(\delta_0))^2) - 2 \int_{\delta_0}^t \mu(|\nabla w(s)|^2) ds \\ &\quad + 2\lambda \int_{\delta_0}^t \mu \left(P_\varepsilon w(s) \left[\mathbb{V}(u^{(\varepsilon)}(s)) - \mathbb{V}(v^{(\varepsilon)}(s)) \right] \right) ds. \end{aligned}$$

One then use linear interpolation $v_\alpha = \alpha u^{(\varepsilon)} + (1 - \alpha)v^{(\varepsilon)}$ to obtain the bound

$$\left| \mathbb{V}(u^{(\varepsilon)}(s)) - \mathbb{V}(v^{(\varepsilon)}(s)) \right| \leq \int_0^1 d\alpha \left| \frac{d}{d\alpha} (v_\alpha(s) G(\sigma^2(v_\alpha(s)))) \right|.$$

This derivative was computed in (12), and by similar arguments as in the proof of proposition 7, one obtains the following inequality

$$\begin{aligned} \mu((w(t))^2) + 2(1 - \lambda c_F) \int_{\delta_0}^t \mu(|\nabla w(s)|^2) ds \\ \leq \mu((w(\delta_0))^2) + 2\lambda a \int_{\delta_0}^t \mu((w(s))^2) ds \end{aligned}$$

with $a = \left(\|\mathcal{J}\|_\infty + B + 4\tilde{B} \right)$. After letting $\delta_0 \rightarrow 0$, Gronwall lemma leads to $\mu \left((w(t))^2 \right) = 0$, for any $t \in (0, T)$. This contradicts our assumption that two distinct weak solutions exist. \square

Theorem 9. Solution of the Mollified Problem.

Suppose $\mu \in \mathbf{FS}(c_F)$, with a constant $c_F \in (0, \infty)$, and let $\lambda \in [0, c_F^{-1})$. For $T \in (0, \infty)$, let

$$\eta_*(T) \equiv \left[\lambda a T + \frac{\lambda c_F}{2 - \lambda c_F} \right] e^{\lambda a T},$$

where $a = \left(\|\mathcal{J}\|_\infty + B + 4\tilde{B} \right)$ and choose $T_0 \in (0, \infty)$ such that $\eta^*(T_0) < 1$. Then

1. The function $u^{(\varepsilon)} \equiv \lim_{n \rightarrow \infty} \bar{u}_n^{(\varepsilon)}$, with the limit in space $\mathbb{H}_{T_0, +}(\mu) \cap C([0, T_0], \mathbb{L}^2(\mu))$, is a unique weak solution on $[0, T_0]$ of the Mollified Cauchy problem

$$\begin{cases} \partial_t u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda \mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) \\ u^{(\varepsilon)}|_{t=0} &= f \end{cases} \quad (\mathbb{C}_\varepsilon)$$

2. The later solution can be extended to a unique global weak solution of problem (\mathbb{C}_ε) .
3. Moreover if $\mathcal{J} \equiv 0$, then the following contractivity property is true for any $t \geq 0$

$$\mu \left(u^{(\varepsilon)}(t) \right)^2 \leq \mu f^2.$$

Remark 10. A posteriori, by lemma 4, it follows that

$$u^{(\varepsilon)} = P_t f + \lambda \int_0^t ds P_{t-s} \mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)})(s)$$

and so $u^{(\varepsilon)}$ is a strong solution, as the left hand side belongs to the domain of L and is differentiable with respect to t .

As follows from proposition 5 (uniform bound), $\|u^{(\varepsilon)}\|_{\mathbb{H}_{T, +}}$ is uniformly bounded in $\varepsilon > 0$.

Proof of theorem 9. By definition of $u^{(\varepsilon)}$ and completeness of $C([0, T], \mathbb{L}^2(\mu))$, we have $u^{(\varepsilon)} \in C([0, T], \mathbb{L}^2(\mu))$. Fix a test function $v \in C^\infty([0, T], W^{1,2}(\mathcal{E})) \subset \mathbb{H}_{T, +}(\mathcal{E})$. First, for any $t \in (0, T]$,

$$\int_0^t ds \mu \left(v(s) \mathcal{V}^{(\varepsilon)}(\bar{u}_n^{(\varepsilon)})(s) \right) \longrightarrow \int_0^t ds \mu \left(v(s) \mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)})(s) \right) \quad (16)$$

as n goes to ∞ . Namely, from Lemma 4, it follows that $\mathcal{V}^{(\varepsilon)} : \mathbb{H}_{T,+} \rightarrow \mathbb{H}_{T,+}$ is Lipschitz continuous. In particular, $\mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) = \mathbb{H}_{T,+} - \lim_{n \rightarrow \infty} \mathcal{V}^{(\varepsilon)}(\bar{u}_n^{(\varepsilon)})$ (with short hand notation for limit in space $\mathbb{H}_{T,+}(\mathcal{E})$). And so $\mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) = \mathbb{H}_{T,-} - \lim_{n \rightarrow \infty} \mathcal{V}^{(\varepsilon)}(\bar{u}_n^{(\varepsilon)})$ when acting on $\mathbb{H}_{T,+}(\mathcal{E})$ with $\mathbb{L}^2(\mu)$ -type pairing. Thus (16) follows.

Recall that by classical arguments $L : \mathbb{H}_{T,+}(\mathcal{E}) \rightarrow \mathbb{H}_{T,-}(\mu)$ acting by $\langle Lu, v \rangle_{\mathbb{H}_{T,-}, \mathbb{H}_{T,+}} \equiv - \int_0^T \mu(\nabla u(s) \cdot \nabla v(s)) ds$ is continuous so that

$$\int_0^t \mu(\nabla \bar{u}_n^{(\varepsilon)}(s) \cdot \nabla v(s)) ds \longrightarrow \int_0^t \mu(\nabla u^{(\varepsilon)}(s) \cdot \nabla v(s)) ds.$$

Convergence of $\bar{u}_n^{(\varepsilon)}$ to $u^{(\varepsilon)}$ in $\mathbb{L}^2([0, T], \mathbb{L}^2(\mu))$ leads to

$$\int_0^t \mu(\bar{u}_n^{(\varepsilon)}(s) \partial_s v(s)) ds \longrightarrow \int_0^t \mu(u^{(\varepsilon)}(s) \partial_s v(s)) ds$$

whereas convergence in $C([0, T], \mathbb{L}^2(\mu))$ ensures that $\mu(\bar{u}_n^{(\varepsilon)}(t) v(t))$ goes to $\mu(u^{(\varepsilon)}(t) v(t))$ and $u^{(\varepsilon)}(0) = f$. This completes the proof that $u^{(\varepsilon)}$ is a weak solution of (\mathbb{C}_ε) . Uniqueness of the solution was proved in proposition 8.

As for existence of a global weak solution, it follows from the fact that the time T_0 in the foregoing does not depend on initial condition f . Hence one can glue at T_0 the solution u_f with initial value f and the solution starting from $u_f(T_0) \in \mathbb{L}^2(\mu)$ to get a solution on $[0, 2T_0]$, and so on...

By our construction $u^{(\varepsilon)}$ is in fact a strong solution (cf. Remark 10). Using this a simple calculation shows that we have

$$\frac{d}{dt} \mu \left(\left[u^{(\varepsilon)}(t) \right]^2 \right) = -2 \mu \left(\left| \nabla u^{(\varepsilon)}(t) \right|^2 \right) + 2 \lambda \mu \left(\left[u^{(\varepsilon)}(t) \right]^2 G(\sigma^2(u^{(\varepsilon)}(t))) \right)$$

If **(FS)** holds and $\lambda \in [0, \frac{1}{c_F})$ this leads to the following differential inequality

$$\frac{d}{dt} \mu \left(\left[u^{(\varepsilon)}(t) \right]^2 \right) \leq 2 \lambda \|\mathcal{J}\|_\infty \mu \left(\left[u^{(\varepsilon)}(t) \right]^2 \right)$$

whence one obtains

$$\mu \left(\left[u^{(\varepsilon)}(t) \right]^2 \right) \leq e^{\lambda \|\mathcal{J}\|_\infty t} \mu(f^2).$$

In particular, if $\mathcal{J} \equiv 0$, we arrive at the following contractivity property

$$\mu \left(u^{(\varepsilon)}(t) \right)^2 \leq \mu f^2.$$

□

Φ -bounds. In this section, we investigate regularity for mollified solutions in the Orlicz space $\mathbb{L}^{\Phi_q}(\mu)$ provided the initial value also belongs to this space.

Theorem 11. *Suppose conditions (C1 – C3) are satisfied. Let $\Phi_q(x) \equiv x^2 e^{qF(x^2)}$ for some fixed $q \in (0, \infty)$. Suppose that $\mu\Phi_q(f) < \infty$ and $\lambda \in (0, (1+q\bar{B})^{-2} c_F^{-1})$. Fix $\varepsilon > 0$. Then the weak solution $u^{(\varepsilon)}(t)$ of the mollified Cauchy problem*

$$\begin{cases} \frac{\partial u^{(\varepsilon)}(t)}{\partial t} &= Lu^{(\varepsilon)}(t) + \lambda P_\varepsilon [u^{(\varepsilon)}(t) G(\sigma^2(u^{(\varepsilon)}(t)))] \\ u^{(\varepsilon)}(0) &= f \end{cases},$$

satisfies the following bound

$$\mu \left(\Phi_q(u^{(\varepsilon)}(t)) \right) + 2C(q, \lambda, c_F) \int_0^t ds \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 \leq e^{\tilde{a}\lambda t} \mu(\Phi_q(f))$$

with some constants $C(q, \lambda, c_F) = (1 - \lambda c_F (1 + q\bar{B})^2) / (1 + q\bar{B}) > 0$ and $\tilde{a} = 2(1 + q\bar{B})(B + \|\mathcal{J}\|_\infty)$.

Remark 12. In the proof, we make computation on $\mu(\Phi_q(u^{(\varepsilon)}(t)))$ without knowing a priori that it is finite. In fact, we can truncate $u^{(\varepsilon)}$ to a bounded function $u^{(\varepsilon, N)} \equiv (u^{(\varepsilon)} \wedge N) \vee (-N)$ still in the domain, then perform the same computations with $u^{(\varepsilon, N)}$ and at last remove the truncation thanks to Fatou lemma (or limit results for Dirichlet forms and truncation). Which provides a proof that $\sqrt{\Phi_q(u^{(\varepsilon)})}$ belongs to $\mathbb{H}_{T,+}(\mathcal{E})$ provided $f \in \mathbb{L}^{\Phi_q}(\mu)$. Note also that when \mathcal{E} is a local regular Dirichlet form, property $\mathcal{E}(\Phi'_q(u), u) \geq k\mathcal{E}(\sqrt{\Phi_q(u)})$, for $u \in W^{1,2}(\mathcal{E})$ bounded and $k = 2/(1 + q\bar{B})$, still holds true thanks to Lejan chain rule formula for energy measures (see [FOT]).

Proof of theorem 11. Let $u^\delta(t) \equiv \frac{1}{\delta} \int_0^\delta u^{(\varepsilon)}(t+s) ds$. Then we have

$$\begin{aligned} & \frac{\partial}{\partial t} \mu \left(\Phi_q(u^\delta(t)) \right) \\ &= \mu \left(\Phi'_q(u^\delta(t)) \frac{1}{\delta} \int_0^\delta (Lu^{(\varepsilon)}(t+s) + \lambda P_\varepsilon u^{(\varepsilon)}(t+s) G(\sigma^2(u^{(\varepsilon)}(t+s)))) ds \right) \end{aligned}$$

Integrating with respect to t and passing to the limit with $\delta \rightarrow 0$, after simple rearrangements one arrives at the following inequality

$$\begin{aligned} \mu \left(\Phi_q(u^{(\varepsilon)}(t)) \right) &\leq \mu(\Phi_q(f)) + \int_0^t ds \mu \left(-\Phi''_q(u^{(\varepsilon)}(s)) |\nabla u^{(\varepsilon)}(s)|^2 \right) \\ &\quad + \lambda \int_0^t ds \mu \left(\left| P_\varepsilon(\Phi'_q(u^{(\varepsilon)}(s))) u^{(\varepsilon)}(s) \right| F(\sigma^2(u^{(\varepsilon)}(s))) \right) \\ &\quad + \lambda \|\mathcal{J}\|_\infty \int_0^t ds \mu \left(\left| P_\varepsilon(\Phi'_q(u^{(\varepsilon)}(s))) u^{(\varepsilon)}(s) \right| \right). \end{aligned}$$

First note that, with $k = 2/(1 + q\bar{B}) \leq 2$, one has

$$\Phi_q''(x) \geq k \frac{(\Phi_q'(x))^2}{4\Phi_q(x)} = k \left(\frac{d}{dx} \left(\sqrt{\Phi_q(x)} \right) \right)^2.$$

To show this, we recall that $\Phi_q(x) = \Psi_q(x^2)$, with Ψ_q convex, and so

$$\Phi_q''(x) = 2\Psi_q'(x^2) + 4x^2\Psi_q''(x^2) \geq 2\Psi_q'(x^2) = \Phi_q'(x)/x.$$

Thus the relation

$$x \Phi_q'(x) = 2\Phi_q(x)(1 + qx^2F'(x^2)) \leq 2(1 + q\bar{B})\Phi_q(x) \quad (17)$$

leads to the announced differential inequality on Φ_q . And consequently,

$$\mu \left(-\Phi_q''(u^{(\varepsilon)}(s)) |\nabla u^{(\varepsilon)}(s)|^2 \right) \leq -k \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2.$$

Next, we note that by Young inequality and Jensen inequality for the semigroup, we have

$$\begin{aligned} |P_\varepsilon(\Phi_q'(u^{(\varepsilon)}(s))) \cdot u^{(\varepsilon)}(s)| &\leq \Phi_q^*(P_\varepsilon(\Phi_q'(u^{(\varepsilon)}(s)))) + \Phi_q(u^{(\varepsilon)}(s)) \\ &\leq P_\varepsilon \Phi_q^*(\Phi_q'(u^{(\varepsilon)}(s))) + \Phi_q(u^{(\varepsilon)}(s)) \end{aligned}$$

with $\Phi_q^*(y) = \sup_{x \in \mathbb{R}} |xy| - \Phi_q(x)$, the conjugate of Φ_q . Since $\Phi_q^*(\Phi_q'(x)) = x\Phi_q'(x) - \Phi_q(x)$, thanks to (17), we have $\Phi_q^*(\Phi_q'(x)) \leq (1 + 2q\bar{B})\Phi_q(x)$. Hence,

$$|P_\varepsilon(\Phi_q'(u^{(\varepsilon)}(s))) \cdot u^{(\varepsilon)}(s)| \leq (1 + 2q\bar{B})P_\varepsilon \Phi_q(u^{(\varepsilon)}(s)) + \Phi_q(u^{(\varepsilon)}(s)).$$

Using this, the definition of the constant k , (GREI) twice and then invariance property for P_ε w.r.t. μ , we have on the one hand,

$$\mu \left(\left| P_\varepsilon \left(\Phi_q'(u^{(\varepsilon)}(s)) \right) u^{(\varepsilon)}(s) \right| \right) \leq 2(1 + q\bar{B}) \mu \left(\Phi_q(u^{(\varepsilon)}(s)) \right)$$

and

$$\begin{aligned} &\mu \left(\left| P_\varepsilon \left(\Phi_q'(u^{(\varepsilon)}(s)) \right) u^{(\varepsilon)}(s) \right| F(\sigma^2(u^{(\varepsilon)}(s))) \right) \\ &\leq \mu \left(\Phi_q(u^{(\varepsilon)}(s)) F(\sigma^2(u^{(\varepsilon)}(s))) \right) \\ &\quad + (1 + 2q\bar{B}) \mu \left(P_\varepsilon \left(\Phi_q(u^{(\varepsilon)}(s)) \right) F(\sigma^2(u^{(\varepsilon)}(s))) \right) \\ &\leq \mu \left(\Phi_q(u^{(\varepsilon)}(s)) F(\sigma^2(\sqrt{\Phi_q(u^{(\varepsilon)}(s))})) \right) + 2B(1 + q\bar{B}) \mu \left(\Phi_q(u^{(\varepsilon)}(s)) \right) \\ &\quad + (1 + 2q\bar{B}) \mu \left(P_\varepsilon \left(\Phi_q(u^{(\varepsilon)}(s)) \right) F(\sigma^2(\sqrt{P_\varepsilon(\Phi_q(u^{(\varepsilon)}(s))})) \right) \end{aligned}$$

on the other hand. Since $xF(x)$ is convex by condition **(C3)**, one has

$$\begin{aligned} \mu \left(P_\varepsilon \left(\Phi_q(u^{(\varepsilon)}(s)) \right) F(\sigma^2(\sqrt{P_\varepsilon(\Phi_q(u^{(\varepsilon)}(s)))})) \right) \\ \leq \mu \left(\Phi_q(u^{(\varepsilon)}(s)) F \left(\sigma^2 \left(\sqrt{\Phi_q(u^{(\varepsilon)}(s))} \right) \right) \right). \end{aligned}$$

Hence

$$\begin{aligned} \mu \left(-\Phi_q''(u^{(\varepsilon)}(s)) |\nabla u^{(\varepsilon)}(s)|^2 \right) + \lambda \|\mathcal{J}\|_\infty \mu \left(\left| \Phi_q'(u^{(\varepsilon)}(s)) P_\varepsilon u^{(\varepsilon)}(s) \right| \right) \\ + \lambda \mu \left(\left| \Phi_q'(u^{(\varepsilon)}(s)) P_\varepsilon u^{(\varepsilon)}(s) \right| F(\sigma^2(u^{(\varepsilon)}(s))) \right) \\ \leq -k \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 + 2\lambda(1+q\bar{B}) \mu \left(\Phi_q(u^{(\varepsilon)}(s)) F(\sigma^2(\sqrt{\Phi_q(u^{(\varepsilon)}(s))})) \right) \\ + 2\lambda(1+q\bar{B}) (B + \|\mathcal{J}\|_\infty) \mu \left(\Phi_q(u^{(\varepsilon)}(s)) \right) \end{aligned}$$

With the use of (FS) inequality, the last can be bounded by

$$-2\left(\frac{k}{2} - \tilde{\lambda} c_F\right) \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 + 2\tilde{\lambda} (B + \|\mathcal{J}\|_\infty) \mu \left(\Phi_q(u^{(\varepsilon)}(s)) \right)$$

where $\tilde{\lambda} \equiv \lambda(1+q\bar{B})$. Combining all the above we arrive at the following inequality

$$\begin{aligned} \mu \Phi_q(u^{(\varepsilon)}(t)) + 2\left(\frac{k}{2} - \tilde{\lambda} c_F\right) \int_0^t ds \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 \\ \leq \mu \Phi_q(f) + 2\tilde{\lambda} (B + \|\mathcal{J}\|_\infty) \int_0^t ds \mu \Phi_q(u^{(\varepsilon)}(s)) \end{aligned}$$

Increasing the integrand on the right hand side so it resembles the expression on the left hand side, we obtain a Gronwall type inequality which leads to the following bound

$$\mu \Phi_q(u^{(\varepsilon)}(t)) + 2\left(\frac{k}{2} - \tilde{\lambda} c_F\right) \int_0^t ds \mu |\nabla \sqrt{\Phi_q(u^{(\varepsilon)}(s))}|^2 \leq e^{2\tilde{\lambda}(B+\|\mathcal{J}\|_\infty)t} \mu \Phi_q(f)$$

provided $\tilde{\lambda} c_F < \frac{k}{2}$. □

Removing the smoothing.

Theorem 13 (Convergence in $\mathbb{H}_{T,+} \cap C([0, T], \mathbb{L}^2(\mu))$ when $\varepsilon \rightarrow 0$). *Let $F(x) = g_\delta * F_\alpha$ be like in example 3, with $\delta > 0$ and $\alpha \in (1, 2]$. Recall that function F satisfies Conditions **(C1)**, **(C3)** and **(C4)**. For a fixed*

$\lambda \in [0, c_F^{-1})$, let $u^{(\varepsilon)}(t)$ denote the solution on $[0, T_0]$ of the approximated Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u^{(\varepsilon)} &= Lu^{(\varepsilon)} + \lambda \mathcal{V}^{(\varepsilon)}(u^{(\varepsilon)}) \\ u^{(\varepsilon)}(0) &= f \end{cases} \quad (\mathbb{C}_\varepsilon)$$

with $\varepsilon > 0$ and initial value $f \in L^2(\mu)$. Here $T_0 > 0$ is like chosen in Theorem 9 or Proposition 7. Let $q > 0$ small enough such that one has $\lambda < (1 + q\bar{B})^{-2} c_F^{-1}$ and assume furthermore that $\mu(\Phi_q(f)) < \infty$. Then, when $\varepsilon \rightarrow 0$, the solutions $u^{(\varepsilon)}$ converge in the Banach space $\mathbb{H}_{T_0,+}(\mu) \cap C([0, T_0], \mathbb{L}^2(\mu))$.

Remark 14. Recall from Proposition 7 that T_0 is independent of the initial value f and $\varepsilon > 0$.

Proof. For $\varepsilon > \varepsilon' > 0$ define $w \equiv w_{\varepsilon, \varepsilon'} \equiv u^{(\varepsilon)} - u^{(\varepsilon')}$. Note that we have $w(0) = 0$. Recall that $u^{(\varepsilon)}$ and $u^{(\varepsilon')}$ is a strong solutions of (\mathbb{C}_ε) and $(\mathbb{C}_{\varepsilon'})$, respectively. Computing the time derivative of $\frac{1}{2}\mu w^2(t)$ and integrating between time 0 and t , we arrive at

$$\begin{aligned} \frac{1}{2}\mu w^2(t) &= - \int_0^t ds \mu |\nabla w(s)|^2 \\ &\quad + \lambda \int_0^t \mu \left[P_\varepsilon w(s) \cdot \mathbb{V}(u^{(\varepsilon)}(s)) - P_{\varepsilon'} w(s) \cdot \mathbb{V}(u^{(\varepsilon')}(s)) \right] ds. \end{aligned} \quad (18)$$

First we note that

$$\begin{aligned} &\mu \left[P_\varepsilon w(s) \cdot \mathbb{V}(u^{(\varepsilon)}(s)) - P_{\varepsilon'} w(s) \cdot \mathbb{V}(u^{(\varepsilon')}(s)) \right] \\ &= \mu \left[P_\varepsilon w(s) \cdot \left\{ \mathbb{V}(u^{(\varepsilon)}(s)) - \mathbb{V}(u^{(\varepsilon')}(s)) \right\} \right] + \mu \left[(P_\varepsilon - P_{\varepsilon'}) w(s) \cdot \mathbb{V}(u^{(\varepsilon')}(s)) \right]. \end{aligned}$$

Using interpolation $u(\alpha, s) \equiv \alpha u^{(\varepsilon)}(s) + (1 - \alpha) u^{(\varepsilon')}(s)$ and suitable bounds already detailed, we have

$$\begin{aligned} &\mu \left[P_\varepsilon w(s) \cdot \left\{ \mathbb{V}(u^{(\varepsilon)}(s)) - \mathbb{V}(u^{(\varepsilon')}(s)) \right\} \right] \\ &\leq \int_0^1 d\alpha \left\{ \mu \left(P_\varepsilon w(s) w(s) G(\sigma^2(u(\alpha, s))) \right) \right. \\ &\quad \left. + 2 \mu \left(P_\varepsilon w(s) w(s) \left[\sigma^2(u(\alpha, s)) G'(\sigma^2(u(\alpha, s))) \right] \right) \right. \\ &\quad \left. - 2 \mu \left(P_\varepsilon w(s) \sigma^3(u(\alpha, s)) G'(\sigma^2(u(\alpha, s))) \right) \mu \left(w(s) \sigma(u(\alpha, s)) \right) \right\} \\ &\leq \frac{1}{2} \left[\mu w^2(s) F(\sigma^2(w(s))) + \mu (P_\varepsilon w)^2(s) F(\sigma^2(P_\varepsilon w(s))) \right] + a \mu w^2(s) \\ &\leq c_F \mu |\nabla w|^2 + a \mu w^2(s) \end{aligned}$$

with $a = \|J\|_\infty + B + 4\tilde{B}$. Combining this with (18), we arrive at

$$\begin{aligned} \frac{1}{2}\mu w^2(t) + (1 - \lambda_{c_F}) \int_0^t ds \mu |\nabla w(s)|^2 &\leq \lambda a \int_0^t ds \mu w^2(s) \\ &+ \int_0^t ds \mu \left[(P_\varepsilon - P_{\varepsilon'}) w(s) \cdot \mathbb{V}(u^{(\varepsilon')}(s)) \right] \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{1}{2}\mu w^2(t) + (1 - \lambda_{c_F}) \int_0^t ds \mu |\nabla w(s)|^2 &\leq e^{\lambda a t} \int_0^t ds \mu \left[(P_\varepsilon - P_{\varepsilon'}) w(s) \cdot \mathbb{V}(u^{(\varepsilon')}(s)) \right] \\ &\leq e^{\lambda a t} \left(\int_0^t ds \mu [(P_\varepsilon - P_{\varepsilon'}) w(s)]^2 \right)^{\frac{1}{2}} \left(\int_0^t ds \mu \mathbb{V}^2(u^{(\varepsilon')}(s)) \right)^{\frac{1}{2}} \end{aligned} \quad (19)$$

First we note that, since $w(s)$ belongs to the domain of L ,

$$\begin{aligned} \int_0^t ds \mu [(P_\varepsilon - P_{\varepsilon'}) w(s)]^2 &= \int_0^t ds \mu \left[(-L)^{\frac{1}{4}} w(s) (P_\varepsilon - P_{\varepsilon'})^2 (-L)^{-\frac{1}{4}} w(s) \right] \\ &\leq \left(\int_0^t ds \mu \left((-L)^{\frac{1}{4}} w(s) \right)^2 \right)^{\frac{1}{2}} \cdot \left(\int_0^t ds \mu \left((P_\varepsilon - P_{\varepsilon'})^2 (-L)^{-\frac{1}{4}} w(s) \right)^2 \right)^{\frac{1}{2}} \end{aligned}$$

Next we observe that, (using symmetry of L and Schwartz inequality),

$$\left(\int_0^t ds \mu \left((-L)^{\frac{1}{4}} w(s) \right)^2 \right)^{\frac{1}{2}} \leq \left(\int_0^t ds [\mu(w^2(s)) + \mu|\nabla w(s)|^2] \right)^{\frac{1}{2}} \leq \|w\|_{\mathbb{H}_{T_0,+}}$$

But uniform L_2 bound ensures that $\|w\|_{\mathbb{H}_{T_0,+}} \leq 2C(T_0)^{\frac{1}{2}} (\mu f^2)^{\frac{1}{2}}$ with some constant $C(T_0) \in (0, \infty)$ specified in Proposition 5. Moreover by spectral theory we have

$$\mu \left((P_\varepsilon - P_{\varepsilon'})^2 (-L)^{-\frac{1}{4}} w(s) \right)^2 = \int_0^\infty e^{-4\varepsilon'\eta} (e^{-(\varepsilon-\varepsilon')\eta} - 1)^4 \eta^{-\frac{1}{2}} dE_{w(s)}(\eta)$$

which we bound by

$$\begin{aligned} \sup_{\eta>0} \left(e^{-4\varepsilon'\eta} (e^{-(\varepsilon-\varepsilon')\eta} - 1)^4 \cdot \eta^{-1} \right) \int_0^\infty \eta^{\frac{1}{2}} dE_{w(s)}(\eta) &\leq (\varepsilon - \varepsilon') \int_0^\infty \eta^{\frac{1}{2}} dE_{w(s)}(\eta) \\ &\leq (\varepsilon - \varepsilon') (\mu w^2(s) + \mu |\nabla w(s)|^2) \end{aligned}$$

To bound the supremum we notice that in the case when $(\varepsilon - \varepsilon')\eta \leq 1$, we have $|e^{-(\varepsilon-\varepsilon')\eta} - 1|^4 \cdot \eta^{-1} \leq |(\varepsilon - \varepsilon')\eta|^4 / \eta \leq (\varepsilon - \varepsilon')$, while for $(\varepsilon - \varepsilon')\eta \geq 1$,

we have $\left(e^{-4\varepsilon'\eta}(e^{-(\varepsilon-\varepsilon')\eta} - 1)^4 \cdot \eta^{-1}\right) \leq \eta^{-1} \leq (\varepsilon - \varepsilon')$. Hence we obtain the following bound

$$\left(\int_0^t ds \mu \left((P_\varepsilon - P_{\varepsilon'})^2 (-L)^{-\frac{1}{4}} w(s) \right)^2 \right)^{\frac{1}{2}} \leq 2C(T_0)^{\frac{1}{2}} (\mu f^2)^{\frac{1}{2}} (\varepsilon - \varepsilon')^{\frac{1}{2}}$$

Combining the above estimates we arrive at the following bound

$$\left(\int_0^t ds \mu [(P_\varepsilon - P_{\varepsilon'}) w(s)]^2 \right)^{\frac{1}{2}} \leq 2C(T_0)^{\frac{1}{2}} (\mu f^2)^{\frac{1}{2}} (\varepsilon - \varepsilon')^{\frac{1}{4}}. \quad (20)$$

Now, provided the initial value f satisfies $\mu(\Phi_q(f)) < +\infty$ for some $q > 0$ such that $\lambda < (1 + q\bar{B})^{-2}c_F^{-1}$, the function $\mathcal{V}(u^{(\varepsilon')})$ is more regular than in $\mathbb{H}_{T_0,-}(\mu)$. It belongs to $\mathbb{L}^2([0, T_0], \mathbb{L}^2(\mu))$ with the corresponding norm uniformly bounded in $\varepsilon' > 0$ in terms of $\mu(\Phi_q(f))$. This is the aim of lemma 15 below which provides an estimate of the term

$$\left(\int_0^t ds \mu \left(\mathbb{V}^2(u^{(\varepsilon')}(s)) \right)\right)^{\frac{1}{2}}$$

in (19). Together with the bound (20), it proves that $(u^{(\varepsilon)})_{\varepsilon>0}$ is Cauchy in the space $\mathbb{H}_{T_0,+}(\mu) \cap C([0, T_0], \mathbb{L}^2(\mu))$ as ε goes to 0. \square

Lemma 15. *Let $F(x) = g_\delta * F_\alpha$ be like in example 3, with $\delta > 0$ and $\alpha \in (1, 2]$. Let $u^{(\varepsilon)}$ be as in theorem 13 for an initial condition $f \in \mathbb{L}^{\Phi_q}(\mu)$ for some $q > 0$ such that $\lambda < (1 + q\bar{B})^{-2}c_F^{-1}$. Then, for any $\varepsilon > 0$,*

$$\left(\int_0^t ds \mu \mathbb{V}^2(u^{(\varepsilon)}(s))\right)^{\frac{1}{2}} \leq A e^{(1+q\bar{B})\lambda Bt} (\mu \Phi_q(f) + 1)^{\frac{1}{2}}$$

with some constant $A = A(q) \in (1, \infty)$ which is independent of ε .

Proof. Using Young's inequality we have

$$\begin{aligned} \mu \left(\mathbb{V}^2(u^{(\varepsilon)}(s)) \right) &= \mu \left[(u^{(\varepsilon)}(s))^2 F^2(\sigma^2(u^{(\varepsilon)}(s))) \right] \\ &\leq \mu \Psi_q((u^{(\varepsilon)}(s))^2) + \mu \Psi_q^*(F^2(\sigma^2(u^{(\varepsilon)}(s)))). \end{aligned} \quad (21)$$

So, let's now bound $\Psi_q^*(F^2(z))$ for $z \geq 0$. Let $y \geq 0$. One has

$$\Psi_q^*(y) = \sup_{x \in \mathbb{R}} [yx - \Psi_q(x)] = yx_* - \Psi_q(x_*)$$

for some point $x_* \geq 0$ satisfying either $x_* = 0$ or $\Psi_q'(x_*) = y$. So assume y satisfies $\Psi_q^*(y) > 0$ (in particular, $y > 1$). Computing $\Psi_q'(x_*) = (1 + qx_*F'(x_*))e^{qF(x_*)}$, one gets

$$e^{qF(x_*)} \leq y \leq (1 + q\bar{B})e^{qF(x_*)}.$$

Thus

$$x_* \leq F_{|\bar{\theta}}^{-1}\left(\frac{1}{q} \log y\right) \quad (22)$$

where we denote by $F_{|\bar{\theta}}$ the restriction of F to $[\bar{\theta}, \infty)$. Recall that F is invertible from $[\bar{\theta}, \infty)$ to $[0, \infty)$ (with $\bar{\theta} = \theta - \delta$) whereas $F_\alpha(x) = ((\log x)^\beta - (\log \theta)^\beta) \mathbb{1}_{x \geq \theta}$ is invertible from $[\theta, \infty)$ to $[0, \infty)$. From (22) follows the bound

$$\Psi_q^*(y) \leq |y| F_{|\bar{\theta}}^{-1}\left(\frac{1}{q} \log(1 + |y|)\right) \quad (23)$$

for any $y \in \mathbb{R}$. Recall approximation estimate

$$F_\alpha(x) \leq F(x) \leq F_\alpha(x) + \kappa$$

from example 3 (for some constant $\kappa = \kappa_{\theta, \delta}$) from which it follows that, for any $\xi \geq 0$, $F_{|\bar{\theta}}^{-1}(\xi) \leq (F_\alpha)_{|\theta}^{-1}(\xi)$. From this and (23) we get the following bound

$$\Psi^*(F^2(z)) \leq (F_\alpha(z) + \kappa)^2 \cdot (F_\alpha)_{|\theta}^{-1}\left(\frac{1}{q} \log(1 + (F_\alpha(z) + \kappa)^2)\right).$$

Next we observe that, for any $z \geq 0$

$$(F_\alpha)_{|\theta}^{-1}\left(\frac{1}{q} \log(1 + (F_\alpha(z) + \kappa)^2)\right) \leq C(z^{\frac{1}{2}} + 1)$$

with some constant $C \equiv C_{q, \beta, \delta} \in (1, \infty)$ as well as $(F_\alpha(z) + \kappa)^2 \leq \tilde{C}(z^{\frac{1}{2}} + 1)$, with some constant $\tilde{C} \equiv \tilde{C}_{\beta, \delta} \in (1, \infty)$. Coming back to (21) we conclude that

$$\begin{aligned} \mu\left(\mathbb{V}^2(u^{(\varepsilon)}(s))\right) &\leq \mu\Phi_q(u^{(\varepsilon)}(s)) + C\tilde{C}\mu\left((\sigma(u^{(\varepsilon)}(s)) + 1)^2\right) \\ &\leq \mu\Phi_q(u^{(\varepsilon)}(s)) + 4C\tilde{C} \end{aligned}$$

Hence, using the Φ -bound from theorem 11 we arrive at

$$\begin{aligned} \left(\int_0^t ds \mu \mathbb{V}^2(u^{(\varepsilon)}(s))\right)^{\frac{1}{2}} &\leq \left(\int_0^t ds \left[e^{2(1+q\bar{B})\lambda Bs} \mu\Phi_q(f) + 4C\tilde{C}\right]\right)^{\frac{1}{2}} \\ &\leq Ae^{(1+q\bar{B})B\lambda t} (\mu\Phi_q(f) + 1)^{\frac{1}{2}} \end{aligned}$$

with some constant $A \in (1, \infty)$ which is independent of ε . \square

Global existence and uniqueness for (MCP).

Theorem 16 (Existence completed). *Let $F = g_\delta * F_\alpha$ be like in example 3, with $\delta > 0$ and $\alpha \in (1, 2]$. Assume that L satisfies F -Sobolev inequality with*

constant c_F , that is $\mu \in \mathbf{FS}(c_F)$. Let $\lambda \in [0, c_F^{-1})$. Then, for any $T \in (0, \infty)$, the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u &= Lu + \lambda u F\left(\frac{u^2}{\mu(u^2)}\right) \\ u(0) &= f \end{cases}$$

admits a unique weak solution on $[0, T]$, and this for any $f \in \mathbb{L}^2(\mu)$.

Proof. First, we mimic arguments given in the proof of Proposition 8 to get uniqueness for weak solutions on any interval $[0, T]$.

Let us turn to the proof of the local existence. Choose $q > 0$ small enough so that $\lambda c_F < (1 + q\bar{B})^{-2}$. Then, provided the initial value $f \in \mathbb{L}^{\Phi_q}(\mu)$, we can use theorem 13 to exhibit a function $u \in \mathbb{H}_{T_0,+}(\mu) \cap C([0, T_0], \mathbb{L}^2(\mu))$ such that

$$\|u^{(\varepsilon)} - u\|_{\mathbb{H}_{T_0,+}(\mu)} + \sup_{t \in [0, T_0]} \|u^{(\varepsilon)}(t) - u(t)\|_{\mathbb{L}^2(\mu)} \longrightarrow 0$$

when ε goes to 0. Thus, by lemma 15, one has

$$\|\mathcal{V}(u^{(\varepsilon)}) - \mathcal{V}(u)\|_{\mathbb{H}_{T_0,-}(\mu)} \longrightarrow 0.$$

Hence, for any $v \in \mathbb{H}_{T_0,+}(\mu)$ and $t \in [0, T_0]$,

$$\begin{aligned} \int_0^t ds \left\langle \mathcal{V}(u^{(\varepsilon)})(s), v(s) \right\rangle_{W^{-1,2}, W^{1,2}} &= \int_0^t ds \left\langle \mathcal{V}(u^{(\varepsilon)})(s), P_\varepsilon v(s) \right\rangle_{W^{-1,2}, W^{1,2}} \\ &\longrightarrow \int_0^t ds \left\langle \mathcal{V}(u)(s), v(s) \right\rangle_{W^{-1,2}, W^{1,2}} \end{aligned}$$

as additionally $P_\varepsilon v$ goes to v in $\mathbb{H}_{T_0,+}(\mu)$. This together with other arguments developed in the proof of theorem 9 shows that u is a weak solution on $[0, T_0]$ of problem (\mathbb{MCP}) . Hence, we are done provided the initial value $f \in \mathbb{L}^{\Phi_q}(\mu)$.

Let's now consider a general initial value $f \in \mathbb{L}^2(\mu)$. By density of $\mathbb{L}^{\Phi_q}(\mu)$, there exists $(f_n)_{n \in \mathbb{N}} \subset \mathbb{L}^{\Phi_q}(\mu)$ with $(f_n)_n \rightarrow f$ in $\mathbb{L}^2(\mu)$. Let's denote by u_n the unique weak solution (on $[0, T_0]$) of (\mathbb{MCP}) with initial value f_n . Define $w \equiv w_{n,m} \equiv u_n - u_m$. One has $w(0) = f_n - f_m$. Let $w^\delta(t) \equiv \frac{1}{\delta} \int_0^\delta w(t+s) ds$, for $\delta > 0$. Then we have

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \mu(w^\delta(t))^2 &= \frac{1}{\delta} \int_0^\delta ds \mu(w^\delta(t) L w(t+s)) \\ &\quad + \lambda \frac{1}{\delta} \int_0^\delta ds \mu \left[w^\delta(t) \cdot (\mathbb{V}(u_n(t+s)) - \mathbb{V}(u_m(t+s))) \right] \\ &= -\frac{1}{\delta} \int_0^\delta ds \mu \left(\nabla w^\delta(t) \cdot \nabla w(t+s) \right) \\ &\quad + \lambda \frac{1}{\delta} \int_0^\delta ds \mu \left[w^\delta(t) \cdot (\mathbb{V}(u_n(t+s)) - \mathbb{V}(u_m(t+s))) \right] \end{aligned}$$

Integrating with respect to time and passing to zero with δ , we arrive at

$$\begin{aligned} \mu(w^2(t)) &= \mu((f_n - f_m)^2) - 2 \int_0^t ds \mu |\nabla w(s)|^2 \\ &\quad + 2\lambda \int_0^t \mu(w(s) \cdot [\mathbb{V}(u_n(s)) - \mathbb{V}(u_m(s))]) ds. \end{aligned}$$

Using linear interpolation between $u_n(s)$ and $u_m(s)$, and now routine bounds (see proof of theorem 13 for instance), this leads to

$$\mu(w^2(t)) + 2(1 - \lambda c_F) \int_0^t ds \mu |\nabla w(s)|^2 \leq \mu((f_n - f_m)^2) + 2\lambda a \int_0^t ds \mu(w^2(s))$$

with $a = \|J\|_\infty + B + 4\tilde{B}$. So that, for any $t \in [0, T_0]$,

$$\mu(w^2(t)) + 2(1 - \lambda c_F) \int_0^t ds \mu |\nabla w(s)|^2 \leq e^{2\lambda at} \mu((f_n - f_m)^2).$$

Then $(u_n)_n$ is Cauchy in $\mathbb{H}_{T_0,+}(\mu) \cap C([0, T_0], \mathbb{L}^2(\mu))$. Let u be the corresponding limit. As in the beginning of this proof, one can easily check that u is a weak solution on $[0, T_0]$ of (MCP) with $u(0) = f$. Arguments to extend u to any interval $[0, T]$ have been given in the proof of theorem 9. The proof is complete. \square

4 Properties of Solutions of (MCP)

4.1 Markovian behaviour

Proposition 17 (Positivity). *Assume that $\mu \in \mathbf{FS}(c_F)$ with a constant $c_F \in (0, \infty)$. Then, for any $\lambda \in [0, c_F^{-1})$, any solution $u(t)$ of (MCP) with initial value $f \geq 0$ satisfies $u(t) \geq 0$ for any $t \geq 0$.*

Proof. Let $u(t)$ be a weak solution of (MCP) with initial value $f \geq 0$. We will prove that, μ -a.s., $u^-(t) = \min(0, u(t)) = 0$. Then $w_h \equiv \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$, for $h > 0$, is differentiable with respect to t ; thus it is a test function and so is $w_h^- \equiv \min(0, w_h)$. Hence, using the definition of the weak solution, we get

$$\begin{aligned} \frac{1}{2} \mu(w_h^-(t))^2 &= \frac{1}{2} \mu(w_h^-(0))^2 + \frac{1}{2} \int_0^t ds \partial_s \mu(w_h^-(s))^2 \\ &= \frac{1}{2} \mu(w_h^-(0))^2 + \int_0^t ds \mu \left(w_h^-(s) \frac{1}{h} (u(s+h) - u(s)) \right) \\ &= \frac{1}{2} \mu(w_h^-(0))^2 + \int_0^t ds \frac{1}{h} \int_s^{s+h} d\tau [-\mu(\nabla w_h^-(s) \nabla u(\tau))] \\ &\quad + \lambda \int_0^t ds \frac{1}{h} \int_s^{s+h} d\tau \mu(w_h^-(s) u(\tau) G(\sigma^2(u(\tau)))) \end{aligned}$$

In the final integral we can pass to the limit with $h \rightarrow 0$ which yields (as $\mu((f^-)^2) = 0$)

$$\begin{aligned} \frac{1}{2}\mu(u^-(t))^2 &= -\int_0^t ds \mu |\nabla u^-(s)|^2 + \lambda \int_0^t ds \mu \left((u^-(s))^2 G \left(\frac{(u^-(s))^2}{\|u(s)\|_2^2} \right) \right) \\ &\leq -\int_0^t ds \mu |\nabla u^-(s)|^2 + \lambda \int_0^t ds \mu \left((u^-(s))^2 F \left(\frac{(u^-(s))^2}{\|u(s)\|_2^2} \right) \right) \\ &\leq 0 \end{aligned}$$

provided $\lambda \leq c_F^{-1}$ thanks to the F -Sobolev inequality. Hence,

$$0 \leq \mu((u^-(t))^2) \leq 0.$$

The proof is complete. \square

Proposition 18 (Boundedness). *Assume that μ satisfies the F -Sobolev inequality (1) with constant c_F . Then, for any $\lambda \in [0, c_F^{-1})$, any solution $u(t)$ of (MCP) satisfies for any $t \geq 0$,*

$$\|u(t)\|_\infty \leq \|f\|_\infty.$$

Proof. Let $f \in \mathbb{L}^\infty(\mu)$. We write χ_+ for the indicatrice function $\chi_{u(t) \geq \|f\|_\infty}$ of the set $\{u(t) \geq \|f\|_\infty\}$. As in the proof of proposition 17, we start from analysing the properties of mollification of the function $(u(t) - \|f\|_\infty)_+ \equiv (u(t) - \|f\|_\infty) \cdot \chi_+$ using the definition of the weak solution and locality of the generator L . This procedure leads to the following integral relation

$$\begin{aligned} \frac{1}{2}\mu((u(t) - \|f\|_\infty)_+^2) &= -\int_0^t ds \mu |\nabla (u(s) - \|f\|_\infty)_+|^2 \\ &\quad + \lambda \int_0^t ds \int (u(s) - \|f\|_\infty) u(s) \chi_+ G \left(\frac{u(s)^2}{\|u(s)\|_2^2} \right) d\mu. \end{aligned}$$

Next we note that $\mu(|\nabla (u(s) - \|f\|_\infty)_+|^2) = \mu(|\nabla (u(s)\chi_+)|^2)$ and that, thanks to $(u(s) - \|f\|_\infty) u(s) \chi_+ \geq 0$, we can use $G \leq F$ to get

$$\begin{aligned} \frac{1}{2}\mu((u(t) - \|f\|_\infty)_+^2) &\leq -\int_0^t ds \mu (|\nabla (u(s)\chi_+)|^2) \\ &\quad + \lambda \int_0^t ds \int (u(s)\chi_+)^2 F \left(\frac{(u(s)\chi_+)^2}{\|u(s)\|_2^2} \right) d\mu \\ &\quad - \lambda \|f\|_\infty \int_0^t ds \int u(s)\chi_+ F \left(\frac{u(s)^2}{\|u(s)\|_2^2} \right) d\mu. \end{aligned}$$

Since $u(s)\chi_+ \geq 0$ and $F \geq 0$, the last term is non positive. Hence, using $\mu((u(t)\chi_+)^2) \leq \|u(t)\chi_+\|_2^2 \leq \|u(t)\|_2^2$, and the F -Sobolev inequality, one gets

$$\begin{aligned} & \frac{1}{2}\mu\left((u(t) - \|f\|_\infty)_+^2\right) \leq \\ & \leq \int_0^t ds \left\{ -\mu|\nabla(u(s)\chi_+)|^2 + \lambda\mu\left((u(s)\chi_+)^2 F(\sigma^2(u(s)\chi_+))\right) \right\} \leq 0. \end{aligned}$$

And so $\mu\left((u(t) - \|f\|_\infty)_+^2\right) \leq 0$. Therefore $u(t) \leq \|f\|_\infty$, μ -a.s.. By homogeneity of the equation, $-u(t)$ is a solution with initial value $-f$. And the above argument also leads to $u(t) \geq -\|f\|_\infty$. This ends the proof. \square

4.2 Further properties

For simplicity we set $\hat{u}^2(t) \equiv \sigma^2(u(t)) \equiv u^2(t)/\mu(u^2(t))$.

Theorem 19 (Exponential decay in \mathbb{L}^2). *Assume that $\mu \in \mathbf{FS}(c_F)$. Suppose also that μ satisfies the following spectral gap inequality*

$$m\mu(g - \mu g)^2 \leq \mu|\nabla g|^2$$

with $m \in (0, \infty)$ independent of $g \in W^{1,2}(\mathcal{E})$. Choose $\lambda \in (0, (c_F + (a/m))^{-1})$ where $a \equiv (\|\mathcal{J}\|_\infty + 4\tilde{B} + B)$ with $B > 0$ as in the generalized relative entropy inequality, and \tilde{B} and \mathcal{J} as in condition **(C2)**. Then, any solution $u(t)$ of the problem (\mathbb{MCP}) with initial data $f \in \mathbb{L}^2(\mu)$ satisfies for any $t \geq 0$,

$$\mu((u(t) - \mu(u(t)))^2) \leq e^{-2Mt} \mu((f - \mu(f))^2) \quad (24)$$

with $M \equiv m(1 - \lambda c_F) - \lambda a$.

Proof. Set $w(t) = u(t) - \mu(u(t))$ and recall $w_h(t) \equiv \frac{1}{h} \int_t^{t+h} w(\tau) d\tau$ (and similarly for $u(t)$) so that $w_h(t) = u_h(t) - \mu(u_h(t))$. Since $w_h(t)$ is differentiable and $\mu(w_h(t)) = 0$ for any t , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (e^{2Mt} \mu((w_h(t))^2)) - M e^{2Mt} \mu((w_h(t))^2) \\ & = e^{2Mt} \mu(w_h(t)) \frac{1}{h} (u(t+h) - u(t)) \\ & = e^{2Mt} \frac{1}{h} \int_t^{t+h} d\tau \left\{ -\mu(\nabla w_h(t) \nabla u(\tau)) + \lambda \int w_h(t) u(\tau) G(\hat{u}^2(\tau)) d\mu \right\} \end{aligned}$$

Integrating from 0 to t , and passing to the limit with $h \rightarrow 0$, we arrive at

$$\begin{aligned} & \frac{e^{2Mt}}{2} \mu(w(t))^2 = \frac{1}{2} \mu(f - \mu f)^2 \\ & + \int_0^t ds e^{2Ms} \left\{ -\mu|\nabla u(s)|^2 + \lambda\mu[w(s)u(s)G(\hat{u}^2(s))] + M\mu(w^2(s)) \right\} \quad (25) \end{aligned}$$

Now, as G vanishes at one, we have $u(s)G(\hat{u}^2(s)) = \int_0^1 d\alpha \frac{d}{d\alpha} [u_{[\alpha]}(s)G(\hat{u}_{[\alpha]}^2(s))]$ with $u_{[\alpha]}(s) \equiv \alpha u(s) + (1 - \alpha)\mu(u(s))$ and $\hat{u}_{[\alpha]}^2(s) \equiv (u_{[\alpha]}(s))^2 / \mu(u_{[\alpha]}(s))^2$. Evaluating this derivative as in (12), one gets

$$\begin{aligned} \mu[w(s)u(s)G(\hat{u}^2(s))] &= \int_0^1 d\alpha \mu(w^2(s) [G(\hat{u}_{[\alpha]}^2(s)) + 2\hat{u}_{[\alpha]}^2(s)G'(\hat{u}_{[\alpha]}^2(s))]) \\ &\quad - 2 \int_0^1 d\alpha \mu[\hat{u}_{[\alpha]}^3(s)G'(\hat{u}_{[\alpha]}^2(s))w(s)] \mu(\hat{u}_{[\alpha]}(s)w(s)) \\ &\leq \int_0^1 d\alpha \mu(w^2(s)F(\hat{u}_{[\alpha]}^2(s))) + (\|\mathcal{J}\|_\infty + 2\tilde{B})\mu(w^2(s)) + 2\tilde{B}[\mu(\hat{u}_{[\alpha]}(s)w(s))]^2 \\ &\leq \mu(w^2(s)F(w^2(s))) + a\mu(w^2(s)), \end{aligned}$$

where we used (GREI) and Cauchy-Schwarz inequality in the last bound, and we set $a \equiv (\|\mathcal{J}\|_\infty + 4\tilde{B} + B)$. Coming back to (25) and applying F-Sobolev inequality, we obtain (using the fact that $\nabla w(s) = \nabla u(s)$)

$$\begin{aligned} \mu(w(t))^2 &= e^{-2Mt} \mu(f - \mu f)^2 \\ &\quad + 2 \int_0^t ds e^{-2M(t-s)} \{ -(1 - \lambda c_F) \mu |\nabla u(s)|^2 + (\lambda a + M) \mu(w^2(s)) \} \end{aligned}$$

If $m \in (0, \infty)$ is the best constant in the following Poincaré inequality

$$m \cdot \mu(g - \mu g)^2 \leq \mu |\nabla g|^2,$$

for any g in the domain of the form, then we get

$$\begin{aligned} \mu(w(t))^2 &\leq e^{-2Mt} \mu(f - \mu f)^2 \\ &\quad - 2 \int_0^t ds e^{-2M(t-s)} \{ [m(1 - \lambda c_F) - M - \lambda a] \mu w^2(s) \} \end{aligned}$$

Thus, if $\lambda \in [0, \{c_F + (a/m)\}^{-1})$, then $M = m(1 - \lambda c_F) - \lambda a > 0$ and we obtain

$$\mu(w(t))^2 \leq e^{-2Mt} \mu(f - \mu f)^2$$

□

We define nonlinear parabolic transfer operator by

$$\mathbb{S}_\infty(f) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mu(u(s)) ds.$$

Proposition 20.

Under the assumptions of the previous theorem, there exists a constant $K' \in (0, \infty)$ such that for any $t \geq 0$,

$$|\mu(u(t)) - \mathbb{S}_\infty(f)| \leq K' e^{-Mt} \|f - \mu(f)\|_2.$$

Consequently, there exists a constant $K'' \in (0, \infty)$ such that

$$\|u(t) - \mathbb{S}_\infty(f)\|_2 \leq K'' e^{-Mt} \|f - \mu(f)\|_2$$

Proof. We first prove the convergence of $\mu(u(t))$. As in the previous proof, let $u_{[\alpha]}(t) = \alpha u(t) + (1 - \alpha)\mu(u(t))$, $w(t) = u(t) - \mu(u(t))$ and $u_h(t) = \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$. We have

$$\begin{aligned} \partial_t \mu(u_h(t)) &= \frac{1}{h} \mu(u(t+h) - u(t)) = \lambda \frac{1}{h} \int_t^{t+h} ds \mu(u(s) G(\hat{u}(s)^2)) \\ &= \frac{1}{h} \int_t^{t+h} ds \mu \left(\lambda \int_0^1 \frac{d}{d\alpha} (u_{[\alpha]}(s) G(\hat{u}_{[\alpha]}^2(s))) d\alpha \right). \end{aligned}$$

with

$$\begin{aligned} &\mu \left(\frac{d}{d\alpha} (u_{[\alpha]}(s) G(\hat{u}_{[\alpha]}^2(s))) \right) \\ &= \int w(s) G(\hat{u}_{[\alpha]}^2(s)) d\mu + 2 \int w(s) \hat{u}_{[\alpha]}^2(s) G'(\hat{u}_{[\alpha]}^2(s)) d\mu \\ &\quad - 2 \int \hat{u}_{[\alpha]}^2(s) G'(\hat{u}_{[\alpha]}^2(s)) \frac{u_{[\alpha]}(s) \mu(u_{[\alpha]}(s) w(s))}{\mu(u_{[\alpha]}^2(s))} d\mu. \end{aligned}$$

It follows from condition **(C2)** that $\tilde{B} \equiv \sup |x G'(x)| < \infty$ and that $|G(x)| \leq C + |x|^{\frac{1}{2}}$ with some constant $C \in (0, \infty)$. Hence, using Hölder inequality, we get

$$\left| \int \frac{d}{d\alpha} (u_{[\alpha]}(s) G(\hat{u}_{[\alpha]}^2(s))) d\mu \right| \leq (C + 1 + 4\tilde{B}) (\mu w^2(s))^{\frac{1}{2}}.$$

Combining our considerations, we obtain

$$|\partial_t \mu(u_h(t))| \leq \lambda(C + 1 + 4\tilde{B}) \frac{1}{h} \int_t^{t+h} ds (\mu w(s)^2)^{\frac{1}{2}}.$$

Now using the bound of Theorem 19 gives (uniformly in $h > 0$)

$$|\partial_t \mu(u_h(t))| \leq \lambda(C + 1 + 4\tilde{B}) e^{-Mt} (\mu(f - \mu f)^2)^{\frac{1}{2}}.$$

Thus, if $T \geq t$, one gets

$$|\mu(u_h(T)) - \mu(u_h(t))| \leq \lambda(C + 1 + 4\tilde{B}) \|f - \mu(f)\|_2 \int_t^\infty e^{-Ms} ds$$

so that, after passing to the limit $h \rightarrow 0$,

$$|\mu(u(T)) - \mu(u(t))| \leq \frac{e^{-Mt}}{M} \lambda(C + 1 + 4\tilde{B}) \|f - \mu(f)\|_2.$$

Hence, $(\mu(u(t)))_{t \geq 0}$ is Cauchy. Set $K' = \frac{\lambda}{M}(C + 1 + 4\tilde{B})$. For a fix $T > t$, we have

$$\begin{aligned}
& \left| \mu(u(t)) - \frac{1}{T} \int_0^T \mu(u(s)) ds \right| \\
& \leq \frac{1}{T} \int_0^t |\mu(u(t)) - \mu(u(s))| ds + \frac{1}{T} \int_t^T |\mu(u(t)) - \mu(u(s))| ds \\
& \leq \frac{1}{T} \left(\int_0^t K' e^{-Ms} ds + \int_t^T K' e^{-Mt} ds \right) \|f - \mu(f)\|_2 \\
& \leq K' \left(\frac{T-t}{T} e^{-Mt} + \frac{1 - e^{-Mt}}{TM} \right) \|f - \mu(f)\|_2.
\end{aligned}$$

Letting T go to infinity proves the first part of the proposition.

The second part follows from the following inequality

$$\mu((u(t) - \mathbb{S}_\infty(f))^2) \leq 2\mu((u(t) - \mu(u(t)))^2) + 2|\mu(u(t)) - \mathbb{S}_\infty(f)|^2,$$

the previous bound and Theorem 19. \square

5 Uniform Hypercontractivity

In the linear case the hypercontractivity in the norm sense is equivalent to the one with respect to the metric. For nonlinear semigroups this may be no longer true. The following result below shows that under certain conditions this stronger property holds.

Define for any $r \geq 0$, $\tau_r(x) := x^2 e^{rF(x^2)}$ and assume that there exists a constant $k > 0$ such that for all $r \geq 0$: $\tau_r'' \tau_r \geq \frac{k}{4} \tau_r'^2$.

In particular if we consider the function F defined in Example 3, thanks to Lemma 37 of [BCR04], we have for any $r \geq 0$,

$$(\tau_r^{(\alpha)})'' \tau_r^{(\alpha)} \geq \frac{3 - 2(2 - \alpha)/(\alpha \log(\theta))}{4} (\tau_r^{(\alpha)})'^2. \quad (26)$$

Suppose $\lambda \in [0, \min(c_F^{-1}, k(2c_F)^{-1})]$. Let $q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a C^1 non-decreasing function satisfying $-k + c_F [\lambda 2(1 + q(t)\bar{B}) + q'(t)] \leq 0$ and $q(0) = 0$. In particular, one may choose $q(t) \equiv \frac{\eta}{2\lambda(\bar{B})}(1 - e^{-2\lambda \bar{B}t})$ with $0 \leq \eta \leq \frac{k}{c_F} - 2\lambda$. We set $\Phi_t := \tau_{q(t)}$, $t \geq 0$. We have $x^2 \equiv \Phi_0(x) \leq \Phi_t(x^2)$ and consequently $\|f\|_2 \leq \|f\|_{\Phi_t}$.

Theorem 21. *Assume the hypothesis described in section 2 are satisfied. Assume that μ satisfies the F -Sobolev inequality (1) with constant c_F . Suppose $\lambda \in [0, \min(c_F^{-1}, k(2c_F)^{-1})]$. Then, any solutions $u(t)$ and $v(t)$ of the (MCP) with initial data $f \in \mathbb{L}^2(\mu)$ and $g \in \mathbb{L}^2(\mu)$, respectively, satisfy, for all $t \geq 0$,*

$$\|u(t)\|_{\Phi_t} \leq \exp \left\{ \lambda(B + \|\mathcal{J}\|_\infty) \int_0^t ds (1 + q(s)\bar{B}) \right\} \|f\|_2 \quad (27)$$

and

$$\|u(t) - v(t)\|_{\Phi_t} \leq C_{u,v}(t) \|f - g\|_2$$

where $C_{u,v}(t)$ is given by

$$\begin{aligned} C_{u,v}(t) &= \\ &= \exp \left(\lambda \int_0^t \left[2\bar{B} \left(1 + 2 \int_0^1 \frac{\|u_{[\alpha]}(s)\|_{\Phi_s}}{\|u_{[\alpha]}(s)\|_2} d\alpha \right) + B + \|\mathcal{J}\|_\infty \right] (1 + q(s)\bar{B}) ds \right) \end{aligned}$$

with $u_{[\alpha]}(s) \equiv \alpha u(s) + (1 - \alpha)v(s)$.

Proof. Let $u(t)$ and $v(t)$ be a solution of the Cauchy problem with smooth initial data f and g , respectively. The desired hypercontractivity once proven for the case of bounded smooth initial data, can later be extended to the general case. Let $w(t) \equiv u(t) - v(t)$. Let $q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a general non-decreasing function with $q(0) = 0$ and consider first $N_h(t) = \|w_h(t)\|_{\tau_{q(t)}}$,

where $w_h(t) \equiv \frac{1}{h} \int_t^{t+h} ds w(s)$. For simplicity, we set $T(x, q) = \tau_q(x)$. Then by definition of the Luxemburg norm, we have

$$\int T(\sigma_t(w_h(t)), q(t)) d\mu = 1, \quad \forall t \geq 0,$$

where $\sigma_t(w_h(t)) \equiv \frac{w_h(t)}{N_h(t)}$. Thus, by differentiation, we get

$$\begin{aligned} \frac{N'_h(t)}{N_h^2(t)} \int w_h(t) \partial_1 T(\sigma_t(w_h(t)), q(t)) d\mu \\ = \int \frac{\partial_t w_h(t)}{N_h(t)} \partial_1 T(\sigma_t(w_h(t)), q(t)) d\mu \\ + q'(t) \int \partial_2 T(\sigma_t(w_h(t)), q(t)) d\mu, \end{aligned} \quad (28)$$

where ∂_1 and ∂_2 are short hand notations for the partial derivative with respect to the first and second variable, respectively.

If $N'_h(t) \leq 0$, there is nothing to prove. On the other hand in case when $N'_h(t) \geq 0$, using convexity of T , we get

$$\frac{N'_h}{N_h} \int \frac{w_h(t)}{N_h} \partial_1 T(\sigma_t(w_h(t)), q) d\mu \geq \frac{2N'_h}{N_h} \int T(\sigma_t(w_h(t)), q) d\mu = \frac{2N'_h}{N_h}. \quad (29)$$

Next, because $\partial_t w_h(t) = \frac{1}{h}(w(t+h) - w(t))$, using the definition of the weak solution, we obtain

$$\begin{aligned} \int \frac{\partial_t w_h(t)}{N_h} \partial_1 T(\sigma_t(w_h(t)), q(t)) d\mu = \\ = \frac{1}{h} \int_t^{t+h} ds \left\{ - \int \frac{\nabla w(s) \cdot \nabla w_h(t)}{N_h^2} \partial_{11} T(\sigma_t(w_h(t)), q(t)) d\mu \right. \\ \left. + \lambda \int \frac{1}{N_h} [u(s)G(\sigma_0^2(u(s))) - v(s)G(\sigma_0^2(v(s)))] \partial_1 T(\sigma_t(w_h(t)), q(t)) d\mu \right\} \end{aligned} \quad (30)$$

where $\sigma_0^2(u(s)) \equiv u(s)^2/||u(s)||_2^2$ and similarly for $\sigma_0^2(v(s))$. Combining (28)-(30), after passing to the limit $h \rightarrow 0$, we arrive at the following inequality with $N \equiv N(t) \equiv N_{\Phi_t}(w(t))$

$$\begin{aligned} \frac{2N'}{N} \leq - \int \frac{|\nabla w(t)|^2}{N^2} \partial_{11} T(\sigma_t(w(t)), q(t)) d\mu \\ + \lambda \int \frac{1}{N} [u(t)G(\sigma_0^2(u(t))) - v(t)G(\sigma_0^2(v(t)))] \partial_1 T(\sigma_t(w(t)), q(t)) d\mu \\ + q'(t) \int \partial_2 T(\sigma_t(w(t)), q(t)) d\mu \end{aligned} \quad (31)$$

For $\alpha \in [0, 1]$, set $u_{[\alpha]} \equiv u_{[\alpha]}(t) \equiv \alpha u(t) + (1 - \alpha)v(t)$. Using this interpolation we can estimate the second term as follows

$$\begin{aligned} & \int \frac{1}{N} [u(t)G(\sigma_0^2(u(t))) - v(t)G(\sigma_0^2(v(t)))] \partial_1 T d\mu \\ & \leq \int_0^1 d\alpha \int d\mu \left\{ [F(\sigma_0^2(u_{[\alpha]}(t))) + \|\mathcal{J}\|_\infty] |\sigma_t(w(t))| \partial_1 T \right. \\ & \quad \left. + \frac{2}{N} [\sigma_0^2(u_{[\alpha]}(t))G'(\sigma_0^2(u_{[\alpha]}(t))) \left(w(t) - \frac{u_{[\alpha]}(t)}{\|u_{[\alpha]}(t)\|_2} \frac{d}{d\alpha} \|u_{[\alpha]}(t)\|_2 \right) \partial_1 T] \right\} \\ & \leq \text{(I)} + \text{(II)} \quad (32) \end{aligned}$$

where we wrote $\partial_1 T$ for $\partial_1 T(\sigma_t(w(t)), q(t))$. Because by our assumption $\sup |xG'(x)| \equiv \tilde{B} < \infty$ and one has $|\frac{d}{d\alpha} \|u_{[\alpha]}(t)\|_2| \leq \|w(t)\|_2$ (by Minkowski inequality), the second term (II) on the right hand side of (32) can be bounded as follows

$$\begin{aligned} \text{(II)} & \leq \int_0^1 d\alpha \int d\mu \frac{2}{N} [\tilde{B} (|w(t)| + |\sigma_0(u_{[\alpha]}(t))| \|w(t)\|_2) \partial_1 T(\sigma_t(w(t)), q(t))] \\ & \leq 2\tilde{B} \int_0^1 d\alpha \int d\mu (|\sigma_t(w(t))| + |\sigma_0(u_{[\alpha]}(t))|) |\partial_1 T(\sigma_t(w(t)), q(t))| \end{aligned}$$

Since $|\partial_1 T(x, q)| = 2|x|(1 + qx^2 F'(x^2))e^{qF(x^2)} \leq 2(1 + q\bar{B})|x|e^{qF(x^2)}$, one has

$$\text{(II)} \leq 4\tilde{B}(1 + q(t)\bar{B}) + 4\tilde{B}(1 + q(t)\bar{B}) \int_0^1 d\alpha \int d\mu \frac{|\sigma_0(u_{[\alpha]}(t))|}{|\sigma_t(w(t))|} T(\sigma_t(w(t)), q(t))$$

But

$$\begin{aligned} & \int d\mu \frac{|\sigma_0(u_{[\alpha]}(t))|}{|\sigma_t(w(t))|} T(\sigma_t(w(t)), q(t)) \\ & = \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2} \int d\mu \frac{|\sigma_t(u_{[\alpha]}(t))|}{|\sigma_t(w(t))|} T(\sigma_t(w(t)), q(t)) \leq 2 \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2} \end{aligned}$$

where to get the last bound we split the integral with respect to μ into two parts $\{|\sigma_t(u_{[\alpha]}(t))| \leq |\sigma_t(w(t))|\}$ and the complement of this set, applying the fact that the function $y \mapsto ye^{q(t)F(y^2)}$ is monotone, and finally applying the identity $\mu T(\sigma_t(z), q(t)) = 1$. Consequently, we conclude that

$$\text{(II)} \leq 4\tilde{B}(1 + q(t)\bar{B}) \left(1 + 2 \int_0^1 \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2} d\alpha \right). \quad (33)$$

We'll write $\zeta_{u,v}(t) = 1 + 2 \int_0^1 \frac{\|u_{[\alpha]}(t)\|_{\Phi_t}}{\|u_{[\alpha]}(t)\|_2} d\alpha$.

The first term (I) on the right hand side of (32) can be represented as $(I) = \int_0^1 d\alpha (I_\alpha)$ with

$$(I_\alpha) \equiv \int d\mu \left\{ [F(\sigma_0^2(u_{[\alpha]}(t))) + \|\mathcal{J}\|_\infty] \sigma_t(w(t)) \partial_1 T(\sigma_t(w(t)), q(t)) \right\}.$$

Using $x\partial_1 T(x, q) = 2T(x, q) + 2qx^2 F'(x^2)T(x, q) \leq 2(1 + q\bar{B})T(x, q)$, the quantity (I_α) may be bounded as follows

$$\begin{aligned} (I_\alpha) &\leq \int d\mu \{ [F(\sigma_0^2(u_{[\alpha]}(t))) + \|\mathcal{J}\|_\infty] 2(1 + q(t)\bar{B})T(\sigma_t(w(t)), q(t)) \} \\ &\leq 2(1 + q(t)\bar{B}) \int d\mu \left[T(\sigma_t(w(t)), q(t)) F(\sigma_0^2(T^{\frac{1}{2}}(\sigma_t(w(t)), q(t)))) \right] \\ &\quad + 2(B + \|\mathcal{J}\|_\infty)(1 + q(t)\bar{B}) \int T(\sigma_t(w(t)), q(t)) d\mu \end{aligned}$$

where in the last line we have applied the generalized entropy inequality. If $\mu \in \mathbf{FS}(c_F)$, we conclude that

$$\begin{aligned} (I) &\leq 2(1 + q(t)\bar{B})c_F \int d\mu \left| \nabla \sqrt{T(\sigma_t(w(t)), q(t))} \right|^2 \\ &\quad + 2(B + \|\mathcal{J}\|_\infty)(1 + q(t)\bar{B}) \quad (34) \end{aligned}$$

Winding up (31-34), we get

$$\begin{aligned} \frac{2N'}{N} &\leq - \int \frac{|\nabla w(t)|^2}{N^2} \partial_{11} T(\sigma_t(w(t)), q(t)) d\mu \\ &\quad + \lambda 2(1 + q(t)\bar{B})c_F \int d\mu \left| \nabla \sqrt{T(\sigma_t(w(t)), q(t))} \right|^2 \\ &\quad + 2\lambda(2\tilde{B}\zeta_{u,v}(t) + B + \|\mathcal{J}\|_\infty)(1 + q(t)\bar{B}) + q'(t) \int \partial_2 T(\sigma_t(w(t)), q(t)) d\mu. \end{aligned} \quad (35)$$

Next, it follows from our assumption on τ_q that

$$\begin{aligned} - \int |\nabla g|^2 \partial_{11} T(g, q) d\mu &\leq -k \int |\nabla g|^2 \frac{\partial_1 T(g, q)^2}{4T(g, q)} d\mu \\ &= -k \int |\nabla \sqrt{T(g, q)}|^2 d\mu. \end{aligned} \quad (36)$$

On the other hand $\partial_2 T(x, q) = T(x, q)F(x^2) \leq T(x, q)F(T(x, q))$ since F is non-decreasing and $x^2 \leq T(x, q)$. Hence, using **(FS)**, we have

$$\int \partial_2 T(g, q(t)) d\mu \leq \int T(g, q(t)) F(T(g, q(t))) d\mu \leq c_F \int |\nabla \sqrt{T(g, q(t))}|^2 d\mu. \quad (37)$$

Thus (35-37) lead to

$$\begin{aligned} \frac{2N'}{N} &\leq (-k + c_F [\lambda 2(1 + q(t)\bar{B}) + q'(t)]) \int |\nabla \sqrt{T(\sigma_t(w(t)), q(t))}|^2 d\mu \\ &\quad + 2\lambda(2\tilde{B}\zeta_{u,v}(t) + B + \|\mathcal{J}\|_\infty)(1 + q(t)\bar{B}) \end{aligned} \quad (38)$$

Now, we choose $q(t)$ such that $-k + c_F [\lambda 2(1 + q(t)\bar{B}) + q'(t)] \leq 0$. Hence

$$\frac{2N'}{N} \leq 2\lambda\tilde{a}(t)(1 + q(t)\bar{B})$$

with $\tilde{a}(t) \equiv (2\tilde{B}\zeta_{u,v}(t) + B + \|\mathcal{J}\|_\infty)$. And this for any t such that $N'(t) \geq 0$. Thus by integration we arrive at the following bound

$$\|u(t) - v(t)\|_{\Phi_t} \leq \exp \left\{ \lambda \int_0^t ds \tilde{a}(s) (1 + q(s)\bar{B}) \right\} \|f - g\|_2$$

which ends the proof of the metric type hypercontractivity. As for the hypercontractivity for the norm (27), the proof is simpler and may be developed by similar arguments from (31) by taking $w = u$ and $v = 0$. Interpolation is no longer necessary. \square

6 Local problems

In [FZ04] (and references therein), abstract semilinear problems of the following type are considered

$$\begin{cases} \partial_t u = \mathbf{L}(u) + \mathbf{V}(u) \\ u(0) = f \end{cases} \quad (39)$$

where u is a continuous curve $t \in [0, T] \mapsto u(t) \in X$ in some Banach space X . The operators \mathbf{L} and \mathbf{V} satisfy dual properties which may be sketch out by saying that, the strongest \mathbf{L} is, the worse \mathbf{V} can be. Let us now summarize the setting and results from [FZ04].

We assume that there exists another Banach space $Y \supset X$ of "worse regularity" such that

(SH) **Smoothing properties:** the operator \mathbf{L} generates a \mathcal{C}^0 semigroup on X which extends to the \mathcal{C}^0 semigroup $(\mathbf{P}_t)_{t \geq 0}$ on Y . The semigroup maps continuously Y to X for any $t > 0$ (gain of regularity). Moreover, for any $T \in (0, \infty)$, there is a positive function $h_T \in \mathbb{L}_{1+\varepsilon}[0, T]$ such that

$$\forall t \in [0, T], \quad \|\mathbf{P}_t\|_{Y \rightarrow X} \leq h_T(t).$$

(BLH) **Lipschitz continuity on bounded sets:** the operator $\mathbf{V} : X \rightarrow Y$ is locally Lipschitz on bounded sets that is, for any $M \in (0, \infty)$, there exists a constant $\kappa_M \in (0, \infty)$ such that, for any $u, v \in B_X(0, M)$,

$$\|\mathbf{V}u - \mathbf{V}v\|_Y \leq \kappa_M \|u - v\|_X.$$

Here, $B_X(0, M) = \{u \in X : \|u\|_X \leq M\}$.

Hypothesis (SH) specifies how strong \mathbf{L} has to be to allow the loss of regularity due to \mathbf{V} described in (BLH). These two hypothesis are already sufficient to imply existence and uniqueness of the solution in a "mild" or "integral" sense on a maximal interval $[0, T(f))$. And this, for any initial condition $f \in X$. As we are interested in nonlinear semigroups associated to this problem we need to assume a further entanglement between \mathbf{L} and \mathbf{V} to ensure non explosion for any initial value. This is given by the following additional hypothesis:

(DH) **Duality between loss of regularity and smoothing:** There exists a non decreasing positive continuous function W on $[0, \infty)$ such that

1. for any $u \in X$,

$$\|\mathbf{V}(u)\|_Y \leq W(\|u\|_X) \quad (40)$$

2. one may find a pair $(p_0, q_0) \in (1, \infty)^2$, $1/p_0 + 1/q_0 = 1$, satisfying jointly

- (a) $\int^\infty (s/W(s))^{p_0} \frac{ds}{s} = \infty$.
- (b) For any $T \in (0, \infty)$, $h_T \in \mathbb{L}_{q_0}[0, T]$.

Theorem 22 (Global existence). *Assume \mathbf{L} and \mathbf{V} satisfy hypothesis (SH), (BLH) and (DH). Then, for any $f \in X$, the Cauchy problem (39) admits a unique integral solution $u(t)$ on $[0, \infty)$. Consequently, there exists a nonlinear \mathcal{C}^0 semigroup $(S_t)_{t \geq 0}$ on X such that for any $f \in X$, $u(t) = S_t f$.*

When $X, Y \subset \mathbb{L}_0(\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mu)$, the space of measurable functions on a measurable space \mathbb{M} and the action of \mathbf{V} is a composition with a (nice) Young function $V : \mathbb{R} \rightarrow [0, \infty)$ satisfying the Δ_2 condition, then hypothesis (BLH) is actually a consequence of (40), see [FZ04, Lemma 4.2].

Consider that \mathbf{L} is a self-adjoint (non positive) Markovian generator on $\mathbb{L}_2(\mu)$ for a probability measure μ . It is shown in [FZ04] that, provided \mathbf{L} satisfies a logarithmic Sobolev inequality, for any f smooth enough,

$$\int f^2 \log \left(\frac{f^2}{\mu(f^2)} \right) d\mu \leq C_{LS} \int f \cdot (-\mathbf{L})f d\mu,$$

hypothesis (SH) is satisfied with $h_T(t) = c_T t^{-1/2} \mathbb{1}_{[0, T]}(t)$ when $Y = \mathbb{L}_2(\mu)$ and X is an Orlicz space $X = \mathbb{L}_{N_2}(\mu)$ for some Young function N_2 .

Finally we may exhibit new Young functions V for which Theorem 22 applies.

We start with hypothesis (SH). Here the role played by the logarithmic Sobolev inequality is fulfilled by an Orlicz-Sobolev inequality. It means demanding a bit more regularity on the initial value by replacing \mathbb{L}_{N_2} by a more restrictive Orlicz space.

Theorem 23 (Smoothing via Orlicz-Sobolev). *Let Φ be a Young function. Assume that for any f smooth enough,*

$$\|(f - \mu(f))^2\|_\Phi \leq C_\Phi \int |\nabla f|^2 d\mu,$$

for some constant C_Φ independent on f . Set $\Phi_2(x) := \Phi(x^2)$. Then, for any $t > 0$, \mathbf{P}_t maps $\mathbb{L}^2(\mu)$ to $\mathbb{L}^{\Phi_2}(\mu)$ and, for any $T \in (0, \infty)$, any $t \in (0, T)$,

$$\|\mathbf{P}_t f\|_{\Phi_2} \leq \frac{C_T}{\sqrt{t}} \|f\|_2$$

with $C_T^2 = \frac{C_\Phi}{e} + 2T \|\mathbb{I}\|_\Phi$.

Proof. We follow [FZ04, Theorem 4.3]. Using the Orlicz-Sobolev inequality, we have (recall that $\|g\|_{\Phi_2}^2 = \|g^2\|_\Phi$)

$$\begin{aligned} \|\mathbf{P}_t f\|_{\Phi_2}^2 &\leq 2\|\mathbf{P}_t(f - \mu(f))\|_{\Phi_2}^2 + 2\|\mu(f)\|_{\Phi_2}^2 \\ &= 2\|(\mathbf{P}_t f - \mu(f))^2\|_\Phi + 2\mu(f)^2 \|\mathbb{I}\|_\Phi \\ &\leq 2C_\Phi \int |\nabla \mathbf{P}_t f|^2 d\mu + 2\mu(f)^2 \|\mathbb{I}\|_\Phi. \end{aligned}$$

By spectral theory we have

$$\int |\nabla \mathbf{P}_t f|^2 d\mu = \int_0^\infty \lambda e^{-2\lambda t} dE_\lambda(f) \leq \sup_{\lambda \geq 0} \lambda e^{-2\lambda t} \int f^2 d\mu \leq \frac{1}{2et} \int f^2 d\mu.$$

Hence, by Cauchy-Schwarz inequality, we get

$$\|\mathbf{P}_t f\|_{\Phi_2}^2 \leq \left(\frac{C_\Phi}{et} + 2\|\mathbf{I}\|_\Phi \right) \mu(f^2).$$

The result follows. \square

As said before, hypothesis (BLH) is a consequence of (40). We now prove this inequality on a class of Young functions.

Proposition 24. *Let $\Phi_\beta = |x| \log(1 + |x|)^\beta$, $\beta \in (0, 1)$ and $\Phi_{\beta,2} = \Phi_\beta(x^2)$. Set $V_\beta = \sqrt{\Phi_{\beta,2}}$. Then, for any $f \in \mathbb{L}_{\Phi_{\beta,2}}$, any $\beta \in (0, 1)$,*

$$\|V_\beta(f)\|_2 \leq W_\beta \left(\|f\|_{\Phi_{\beta,2}} \right)$$

for $W_\beta(x) = x + V_\beta(x) = x + x \log(1 + x^2)^{\frac{\beta}{2}}$.

Proof. Fix $\beta \in (0, 1)$. Let $f \in \mathbb{L}_{\Phi_{\beta,2}}$ and define $\tilde{f} = f/\|f\|_{\Phi_{\beta,2}}$. Then, by definition, $\int \Phi_{\beta,2}(\tilde{f}) d\mu = 1$. Note that $(1 + (ab)^2) \leq (1 + a^2)(1 + b^2)$. Thus, $\Phi_{\beta,2}(ab) \leq a^2 \Phi_{\beta,2}(b) + b^2 \Phi_{\beta,2}(a)$. It follows that

$$\begin{aligned} \|V_\beta(f)\|_2 &= \left(\int \Phi_{\beta,2}(f) d\mu \right)^{\frac{1}{2}} = \left(\int \Phi_{\beta,2}(\|f\|_{\Phi_{\beta,2}} \tilde{f}) d\mu \right)^{\frac{1}{2}} \\ &\leq \left(\Phi_{\beta,2}(\|f\|_{\Phi_{\beta,2}}) \int \tilde{f}^2 d\mu + \|f\|_{\Phi_{\beta,2}}^2 \int \Phi_{\beta,2}(\tilde{f}) d\mu \right)^{\frac{1}{2}} \\ &\leq [\Phi_{\beta,2}(\|f\|_{\Phi_{\beta,2}})]^{\frac{1}{2}} + \|f\|_{\Phi_{\beta,2}}. \end{aligned}$$

the result follows. \square

We are now in position to apply Theorem 22.

Theorem 25. *Let $\Phi_\beta = |x| \log(1 + |x|)^\beta$, $\beta \in (0, 1)$ and $\Phi_{\beta,2} = \Phi_\beta(x^2)$. Set $V_\beta = \sqrt{\Phi_{\beta,2}}$. Let μ_α be the Gibbs measure defined in Theorem 28 on $\mathbb{R}^{\mathbb{Z}^d}$. Then, for any $\beta \in (0, 1)$, for any $f \in \mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$, the Cauchy problem (39), with $\mathbf{V}(u) = V_\beta \circ u$ acting by composition with V_β , admits a unique integral solution $u(t)$ on $[0, \infty)$. Consequently, there exists a nonlinear \mathcal{C}^0 semigroup $(S_t)_{t \geq 0}$ on $\mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$ such that for any $f \in \mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$, $u(t) = S_t f$.*

Proof. We will apply Theorem 22 with $X = \mathbb{L}_{\Phi_{\beta,2}}(\mu_\alpha)$ and $Y = \mathbb{L}^2(\mu_\alpha)$. Thus we have to verify that assumptions (SH), (BLH) and (DH) are satisfied.

Since by Theorem 29 μ_α satisfies an Orlicz-Sobolev inequality, Theorem 23 applies and assumption (SH) is satisfied with $h_T(t) = C_T/\sqrt{t}$.

From Proposition 24, inequality (40) is satisfied, and thus (BLH), with $W_\beta(x) = x + x \log(1 + x^2)^{\frac{\beta}{2}}$.

Let $p_0 = 2/\beta$. Note that $p_0 > 2$ and thus that $q_0 < 2$, where $1/p_0 + 1/q_0 = 1$. It follows that for any $T \in (0, \infty)$, $h_T \in \mathbb{L}^{q_0}([0, T])$. On the other hand, $\int^\infty (s/W_\beta(s))^{p_0} \frac{ds}{s}$ has the same behaviour as

$$\int^\infty \frac{ds}{s \log(1 + s)}.$$

Consequently it diverges. This ends the proof. □

7 Functional inequalities for Gibbs measures

In this section we describe briefly the functional inequalities like F -Sobolev inequality and Orlicz-Sobolev inequality for a class of non-product Gibbs measures in infinite dimensions; for more details we refer to [BCR04, RZ05].

We start by recalling some useful results on these functional inequalities in \mathbb{R}^n [BCR04, RZ05].

The strategy to prove such inequalities relies on the notion of capacity of a set introduced in [BR03]. Let μ and ν be two absolutely continuous measures on \mathbb{R}^n . Then, for any Borel set $A \subset \Omega$, we set

$$\text{Cap}_\nu(A, \Omega) := \inf \left\{ \int |\nabla f|^2 d\nu; f \geq \mathbb{1}_A \text{ and } f|_{\Omega^c} = 0 \right\}.$$

If μ is a probability measure on \mathbb{R}^n , then, for $A \subset \mathbb{R}^n$ such that $\mu(A) < \frac{1}{2}$, the capacity of A with respect to μ and ν is

$$\begin{aligned} \text{Cap}_\nu(A, \mu) &:= \inf \left\{ \int |\nabla f|^2 d\nu; \mathbb{1} \geq f \geq \mathbb{1}_A \text{ and } \mu(f = 0) \geq \frac{1}{2} \right\} \\ &= \inf \left\{ \text{Cap}_\nu(A, \Omega); \Omega \subset \mathbb{R}^n \text{ s.t. } \Omega \supset A \text{ and } \mu(\Omega) = \frac{1}{2} \right\}. \end{aligned}$$

We will write $\text{Cap}_\mu(A)$ for $\text{Cap}_\mu(A, \mu)$. We refer the reader to [BCR04, section 5.2] for a general introduction and discussion on the notion of capacity. The second equality in the definition comes from the fact that $\text{Cap}_\nu(A, \Omega)$ is non-increasing in Ω and an easy truncation argument (see [BCR04]).

Then we have the following two theorems. The first is of Orlicz-Sobolev type (used in local problems) while the second give us **FS** inequality.

Theorem 26. *Let μ an absolutely continuous probability measures on \mathbb{R}^n . Let $T : [0, 1] \rightarrow \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Denote by C_T the optimal constant such that for every smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ one has*

$$\sup_{p \in (1, 2)} \frac{\int f^2 d\mu - (\int |f|^p d\mu)^{\frac{2}{p}}}{T(2-p)} \leq C_T \int |\nabla f|^2 d\mu.$$

Finally, assume that there exists a positive constant c_1 such that

$$c_1 x T\left(\frac{1}{\log(1+x)}\right) \leq \Phi^{-1}(x), \quad \forall x > 2.$$

Let Φ be a Young function and fix a constant $k \in (0, +\infty)$ such that for any function f with $f^2 \in \mathbb{L}_\Phi(\mu)$, $\|\mu(f)^2\|_\Phi \leq k \|f^2\|_\Phi$.

Then, every smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$\|(f - \mu(f))^2\|_\Phi \leq \frac{48(1+k)C_T}{c_1} \int |\nabla f|^2 d\mu.$$

The proof can be found in [RZ05].

Theorem 27. *Let $T : [0, 1] \rightarrow \mathbb{R}^+$ be non-decreasing and such that $x \mapsto T(x)/x$ is non-increasing. Let μ and ν be two absolutely continuous measures on \mathbb{R}^n with $\mu(\mathbb{R}^n) = 1$. Let C_T be the optimal constant such that for every smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ one has*

$$\sup_{p \in (1, 2)} \frac{\int f^2 d\mu - (\int |f|^p d\mu)^{\frac{2}{p}}}{T(2-p)} \leq C_T \int |\nabla f|^2 d\nu. \quad (41)$$

Let $F : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing function. Assume that $F(x) = 0$ if $x \leq \theta$ for some $\theta > 2$ and that there exists a constant c such that $F(\theta y/2) \leq c/T(1/\log(1+y))$ for any $y \geq \theta$.

Then, for every smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ one has

$$\int f^2 F\left(\frac{f^2}{\mu(f^2)}\right) d\mu \leq 3cC_T \left(\frac{\theta}{\sqrt{\theta} - \sqrt{2}}\right)^2 \int |\nabla f|^2 d\nu.$$

Proof. Combining Theorem 9 and Lemma 8 of [BCR04] we get that any Borel set $A \subset \mathbb{R}^n$ with $\mu(A) < \frac{1}{2}$ satisfies

$$\frac{\mu(A)}{T(1/\log(1 + \frac{1}{\mu(A)}))} \leq 6C_T \text{Cap}_\nu(A, \mu).$$

This leads to

$$\mu(A) F\left(\frac{\theta}{2\mu(A)}\right) \leq 6cC_T \text{Cap}_\nu(A, \mu)$$

for any A with $\mu(A) < \frac{1}{\theta}$. The result follows by Theorem 20 of [BCR04]. \square

In the rest of this section we consider the following infinite dimensional models on a space $\Omega \equiv \mathbb{R}^{\mathbb{Z}^d} \equiv \{\omega = (\omega_i \in \mathbb{R})_{i \in \mathbb{Z}^d}\}$.

Let $\mathcal{U}_i \equiv \mathcal{U}_i(\omega_i)$, $i \in \mathbb{Z}$, be smooth convex functions such that

$$0 < \inf_{i \in \mathbb{Z}^d} \int e^{-\mathcal{U}_i(x)} dx \leq \sup_{i \in \mathbb{Z}^d} \int e^{-\mathcal{U}_i(x)} dx < \infty.$$

Let $\mathcal{I} \equiv \{\mathcal{I}_X\}$, $X \Subset \mathbb{Z}^d$, $|X| \geq 1$, be a collection of smooth bounded cylinder functions, (dependent only on $\omega_X \equiv (\omega_i : i \in X)$, respectively), and such that

$$\|\mathcal{I}\|_{u,2} \equiv \sup_{i \in \mathbb{Z}^d} \left(\sum_{\substack{X \Subset \mathbb{Z}^d \\ X \ni i}} \left\{ \|\mathcal{I}_X\|_u + \sum_{j \in \mathbb{Z}^d} [\|\nabla_j \mathcal{I}_X\|_u + \|\nabla_j \nabla_i \Phi_X\|_u] \right\} \right) < \infty \quad (42)$$

where $\|\cdot\|_u$ denotes the uniform norm. For $\Lambda \Subset \mathbb{Z}^d$, setting

$$U_\Lambda \equiv \sum_{i \in \Lambda} \mathcal{U}_i(\omega_i) + \sum_{X \cap \Lambda \neq \emptyset} \mathcal{I}_X(\omega_X)$$

we define

$$E_\Lambda^\omega(f) \equiv \frac{\int e^{-U_\Lambda(\tilde{\omega} \circ_\Lambda \omega)} f(\tilde{\omega} \circ_\Lambda \omega) d\tilde{\omega}_\Lambda}{\int e^{-U_\Lambda(\tilde{\omega} \circ_\Lambda \omega)} d\tilde{\omega}_\Lambda}$$

where

$$(\tilde{\omega} \circ_\Lambda \omega)_i \equiv \begin{cases} \tilde{\omega}_i & \text{if } i \in \Lambda \\ \omega_i & \text{if } i \in \Lambda^c \end{cases}$$

A measure μ is called a Gibbs measure on Ω for local specification $\{E_\Lambda\}_{\Lambda \Subset \mathbb{Z}^d}$ iff for any integrable function f one has

$$\mu(E_\Lambda f) = \mu(f)$$

for all $\Lambda \Subset \mathbb{Z}^d$. For any $\Lambda \subset \mathbb{Z}^d$ and $i \in \Lambda$ we have

$$E_\Lambda(f L_i g) \equiv -E_\Lambda \nabla_i f \cdot \nabla_i g$$

for any functions f and g for which both sides make sense. Thus we have

$$L_i f = e^{U_i \operatorname{div}_i} (e^{-U_i} \nabla_i f) = \Delta_i f - \nabla_i U_i \cdot \nabla_i f$$

where div_i and ∇_i are with respect to ω_i and $U_i \equiv U_{\{i\}}$.

We introduce the following Markov generator

$$L \equiv \sum_{i \in \mathbb{Z}^d} L_i$$

which is well defined on a domain including all smooth cylinder functions. Consequently we have

$$-\mu(f L g) = \sum_{i \in \mathbb{Z}^d} \mu(\nabla_i f \cdot \nabla_i g)$$

and if $P_t f \equiv e^{tL} f \equiv f_t$ is the corresponding Markov semigroup, we also have

$$\mu(f P_t g) = \mu(g P_t f).$$

For a construction of the semigroup $(P_t)_{t \geq 0}$ in the space of bounded continuous functions we refer to [GZ03], (see also [Zeg96], [Yos01], [Hel02], [DPZ96], and references therein).

We note that in the present setup one has

$$|\nabla z|_2^2 \equiv \frac{1}{2}(Lz^2 - 2zLz) = \sum_i |\nabla_i z|^2$$

and the generator L has the following diffusion property (or chain rule): for any (localized) smooth vector functions $f = (f_1, \dots, f_\nu)$ on Ω ($\nu \in \mathbb{N}$) and any smooth function Ψ on \mathbb{R}^ν ,

$$L\Psi(f_1, \dots, f_\nu) = \sum_{k=1}^{\nu} \partial_k \Psi(f) Lf_k + \sum_{k,l=1}^{\nu} \partial_{k,l}^2 \Psi(f) L\nabla f_k \cdot \nabla f_l.$$

In the above described setup we have the following results.

Theorem 28 ([RZ05],[GZ03]). *Fix $\alpha \in (1, 2)$. Assume that for all $i \in \mathbb{N}$, $\mathcal{U}_i(x) = \mathcal{U}_\alpha(x)$ where \mathcal{U}_α is the following \mathcal{C}^2 function*

$$\mathcal{U}_\alpha(x) = \begin{cases} |x|^\alpha & \text{for } |x| > 1 \\ \frac{\alpha(\alpha-2)}{8}x^4 + \frac{\alpha(4-\alpha)}{4}x^2 + (1 - \frac{3}{4}\alpha + \frac{1}{8}\alpha^2) & \text{for } |x| \leq 1. \end{cases}$$

Define the Gibbs measure μ_α in this case. Then, there exists a constant C_α such that for any function f in the domain of the Dirichlet form

$$\sup_{p \in (1,2)} \frac{\int f^2 d\mu_\alpha - (\int |f|^p d\mu_\alpha)^{\frac{2}{p}}}{(2-p)^{2(1-\frac{1}{\alpha})}} \leq C_\alpha \int |\nabla f|^2 d\mu_\alpha.$$

Moreover if $\|\mathcal{I}\|_{u,2}$ is sufficiently small, then the same results (with appropriate constants) remain true for the corresponding Gibbs measures.

We are now in position to give example of Gibbs measures satisfying a F -Sobolev inequality and an Orlicz-Sobolev inequality.

Theorem 29 ([RZ05]). *Fix $\alpha \in (1, 2)$ and set $\beta = 2(1 - \frac{1}{\alpha})$. Consider the function F defined in Example 3 and $\Phi_\beta(x) = |x| \log(1 + |x|)^\beta$. Under the assumption and notations of Theorem 28, there exists a constant $D_\alpha = D_\alpha(C_\alpha, \beta, \bar{\theta}, \theta, \varepsilon_0)$ such that any function f in the domain of the Dirichlet form satisfies*

$$\|(f - \mu_\alpha(f))^2\|_{\Phi_\beta} \leq 179C_\alpha \int |\nabla f|^2 d\mu_\alpha$$

and

$$\int f^2 F\left(\frac{f^2}{\mu_\alpha(f^2)}\right) d\mu_\alpha \leq D_\alpha \int |\nabla f|^2 d\mu_\alpha.$$

Moreover if $\|\mathcal{I}\|_{u,2}$ is sufficiently small, then the same results (with appropriate constants) remain true for the corresponding Gibbs measures.

Proof. Set $T(x) = |x|^\beta$. It is not difficult (see [RZ05]) to show that for any $x > 2$,

$$xT\left(\frac{1}{\log(1+x)}\right) \leq \Phi_\beta^{-1}(x).$$

On the other hand, thanks to Remark 32, $\|\mu_\alpha(f)^2\|_{\Phi_\beta} \leq e\|f^2\|_{\Phi_\beta}$. Consider a smoothed cylinder function f . We can apply Theorems 26 and 28 to get the result for the Orlicz-Sobolev inequality (note that $48(1+e) \leq 179$). A density argument ends the proof.

Note that $\bar{\theta} > 2$, thus $\bar{\theta}y/2 \leq (1+y)^{\bar{\theta}/2}$. It follows that

$$\log\left(\frac{\bar{\theta}y}{2}\right)^\beta \leq \left(\frac{\bar{\theta}}{2}\right)^\beta \log(1+y)^\beta.$$

Set $c = 2 \max\left(\left(\frac{\bar{\theta}}{2}\right)^\beta; \left[-(\log \theta)^\beta + \log\left(\frac{\max(\theta, e) + \varepsilon_0}{\theta}\right)\right] / \log(1 + \bar{\theta})^\beta\right)$. It follows from (2) that for any $y \geq \bar{\theta}$,

$$F(y) \leq F_\alpha(x) + \log\left(\frac{\max(\theta, e) + \varepsilon_0}{\theta}\right) \leq c \log(1+y)^\beta = \frac{c}{T\left(\frac{1}{\log(1+y)}\right)}.$$

Hence we can apply Theorem 27 for smooth cylinder functions. The result follows by density. \square

8 Appendix: Young functions and Orlicz spaces

In this section we collect some results on Orlicz spaces. We refer the reader to [RR91] for demonstrations and complements.

Definition 30 (Young function). A function $\Phi : \mathbb{R} \rightarrow [0, \infty]$ is a *Young function* if it is convex, even, such that $\Phi(0) = 0$, and $\lim_{x \rightarrow +\infty} \Phi(x) = +\infty$.

The Legendre transform Φ^* of Φ defined by

$$\Phi^*(y) = \sup_{x \geq 0} \{x|y| - \Phi(x)\}$$

is a lower semicontinuous Young function. It is called the *complementary function* or *conjugate* of Φ .

Among the Young functions, we will consider those continuous with finite values such that $\Phi(x)/x \rightarrow \infty$ as $x \rightarrow \infty$ (for stability reasons w.r.t. duality). When additionally $\Phi(x) = 0 \Leftrightarrow x = 0$ and $\Phi'(0_+) = 0$, Φ is called a *N-function*.

For any lower semicontinuous Young function Φ (in particular if Φ has finite values), the conjugate of Φ^* is Φ . The pair (Φ, Ψ) is said to be a *complementary pair* if $\Psi = \Phi^*$ (or equivalently $\Phi = \Psi^*$). When $\Phi(1) + \Phi^*(1) = 1$, the pair (Φ, Φ^*) is said to be *normalized*. The conjugate of an *N-function* is an *N-function*.

Let Φ be an *N-function*. Then, for any $a > 0$,

$$a < \Phi^{-1}(a)(\Phi^*)^{-1}(a) \leq 2a.$$

The simplest example of *N-function* is $\Phi(x) = \frac{|x|^p}{p}$, $p > 1$, in which case, $\Phi^*(x) = \frac{|x|^q}{q}$, with $1/p + 1/q = 1$. The function $\Phi(x) = |x|^\alpha \ln(1 + |x|)^\beta$ is also a Young function for $\alpha \geq 1$ and $\beta \geq 0$ and an *N-function* when $\alpha > 1$ or $\beta > 0$.

Now let (\mathcal{X}, μ) be a measurable space, and Φ a Young function. The space

$$\mathbb{L}_\Phi(\mu) = \{f : \mathcal{X} \rightarrow \mathbb{R} \text{ measurable; } \exists \alpha > 0, \int_{\mathcal{X}} \Phi(\alpha f) < +\infty\}$$

is called the *Orlicz space* associated to Φ . When $\Phi(x) = |x|^p$, then $\mathbb{L}_\Phi(\mu)$ is the standard Lebesgue space $\mathbb{L}_p(\mu)$.

There exist two equivalent norms which give to $\mathbb{L}_\Phi(\mu)$ a structure of Banach space. Namely

$$\|f\|_\Phi = \inf \left\{ \lambda > 0; \int_{\mathcal{X}} \Phi \left(\frac{f}{\lambda} \right) d\mu \leq 1 \right\}$$

and

$$N_\Phi(f) = \sup \left\{ \int_{\mathcal{X}} |fg| d\mu; \int_{\mathcal{X}} \Phi^*(g) d\mu \leq 1 \right\}.$$

Note that we invert the notation with respect to [RR91]. We will use the notation \mathcal{G}_Φ , or more simply \mathcal{G} when no confusion, the set $\mathcal{G}_\Phi = \{|g| : \int_{\mathcal{X}} \Phi^*(g) d\mu \leq 1\}$. Note in particular that \mathcal{G}_Φ is a space of non negative functions. Moreover

$$\|f\|_\Phi \leq N_\Phi(f) \leq 2\|f\|_\Phi. \quad (43)$$

By definition of the norm and the previous result, it is easy to see that for any measurable subset A of \mathcal{X} ,

$$\|\mathbb{1}_A\|_\Phi = \frac{1}{\Phi^{-1}\left(\frac{1}{\mu(A)}\right)}. \quad (44)$$

Then, the following result generalizes Hölder inequality. Let Φ_1, Φ_2 and Φ_3 be three Young functions satisfying for all $x \geq 0$, $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \Phi_3^{-1}(x)$. Then, for any $(f, g) \in \mathbb{L}_{\Phi_1}(\mu) \times \mathbb{L}_{\Phi_2}(\mu)$,

$$\|fg\|_{\Phi_3} \leq 2\|f\|_{\Phi_1}\|g\|_{\Phi_2}. \quad (45)$$

In particular, when $\Phi_3(x) = |x|$, we get $\int_{\mathcal{X}} |fg| d\mu \leq 2\|f\|_{\Phi_1}\|g\|_{\Phi_2}$. In the case of complementary pairs of Young functions, we have the following more precise result, see [RR91, Proposition 1 in section 3]:

$$\int_{\mathcal{X}} |fg| d\mu \leq 2\|f\|_\Phi\|g\|_{\Phi^*}. \quad (46)$$

Finally, for any constant $c > 0$, it is easy to see that for any function f ,

$$c\|f\|_{\Phi(\cdot/c)} = \|f\|_\Phi. \quad (47)$$

Comparison of norms

In what follows, we will often have to compare Orlicz norms associated to different Young functions. Let us notice that any Young function Φ satisfies $|x| = O(\Phi(x))$ as x goes to ∞ . It leads to the following lemma.

Lemma 31. *Any Orlicz space may be continuously embedded in \mathbb{L}_1 . More precisely, let M and τ in $(0, \infty)$ such that $|x| \leq \tau\Phi(x)$ for any $|x| \geq M$. Then, for any $f \in \mathbb{L}_\Phi$,*

$$\|f\|_1 \leq (M + \tau)\|f\|_\Phi. \quad (48)$$

Consequently, if Φ and Ψ are two Young functions satisfying, for some constants $A, B \geq 0$, $\Phi(x) \leq A|x| + B\Psi(x)$, then

$$\|f\|_\Phi \leq \max\left(1, A\|\text{Id}\|_{\mathbb{L}_\Psi \rightarrow \mathbb{L}_1} + B\right)\|f\|_\Psi. \quad (49)$$

Remark 32. When $\Phi(x)/x \rightarrow \infty$ as $x \rightarrow \infty$, we may choose $\tau = 1$ or any other positive constant. We get in particular the estimate

$$\|\mu(f)^2\|_{\Phi} \leq (M + 1) \|\mathbb{I}\|_{\Phi} \|f^2\|_{\Phi}, \quad (50)$$

where M is such that $|x| \leq \Phi(x)$ for any $|x| \geq M$.

Proof of lemma 31. Let $f \in \mathbb{L}_{\Phi}(\mu)$. We may assume by homogeneity that $\|f\|_{\Phi} = 1$. Then $\int \Phi(f) d\mu = 1$ and so

$$\begin{aligned} \int |f| d\mu &= \int_{\{|f| \leq M\}} |f| d\mu + \int_{\{|f| \geq M\}} |f| d\mu \\ &\leq M\mu(|f| \leq M) + \tau \int_{\{|f| \geq M\}} \Phi(f) d\mu \leq M + \tau. \end{aligned}$$

As for bound (48), assume now that $\|f\|_{\Psi} = 1$ and hence $\int \Psi(f) d\mu = 1$ as well. For any $\lambda \geq 1$,

$$\begin{aligned} \int \Phi(f/\lambda) d\mu &\leq \frac{A}{\lambda} \|f\|_1 + B \int \Psi(f/\lambda) d\mu \\ &\leq \frac{A}{\lambda} \|\text{Id}\|_{\mathbb{L}_{\Psi} \rightarrow \mathbb{L}_1} \|f\|_{\Psi} + \frac{B}{\lambda} \int \Psi(f) d\mu \leq 1 \end{aligned}$$

provided $\lambda \geq A \|\text{Id}\|_{\mathbb{L}_{\Psi} \rightarrow \mathbb{L}_1} + B$. Note that for the second inequality we used convexity of Ψ . \square

References

- [BCR04] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and gaussian, orlicz hypercontractivity and application to isoperimetry. to appear in *Revista Matematica Iberoamericana*.
- [BR03] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. *Studia Math.* 159 (2003), no3, 481-497.
- [DPZ96] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.
- [Dav95] E. B. Davies. *Spectral theory and differential operators*, volume 42 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [FZ04] P. Fougères and B. Zegarlinski. Semi-linear problems in infinite dimensions. *J. of Funct. Analysis* 228 (2005) 39-88.
- [FOT] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. Walter de Gruyter 1995.
- [Gross] L. Gross. Logarithmic Sobolev inequalities *Amer. J. Math.* 97 (1975), Issue 4, 1061-1083.
- [GZ03] A. Guionnet and B. Zegarlinski. Lectures on logarithmic Sobolev inequalities. In *Séminaire de Probabilités, XXXVI*, volume 1801 of *Lecture Notes in Math.*, pages 1–134. Springer, Berlin, 2003.
- [Hel02] B. Helffer. *Semiclassical analysis, Witten Laplacians, and statistical mechanics*, volume 1 of *Series on Partial Differential Equations and Applications*. World Sci. Publishing Co. Inc., River Edge, NJ, 2002.
- [LSU] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Moscow, 1967
- [LZ] P. Lugiewicz, B. Zegarlinski, Coercive Inequalities for Hörmander Type Generators in Infinite Dimensions, to appear in *J. Funct. Anal.* (2007)
- [MZR] Z.-M. Ma, M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag 1992.

- [RR91] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*. Marcel Dekker Inc., 1991.
- [RZ05] C. Roberto and B. Zegarliński. Orlicz-sobolev inequalities for sub-gaussian measure and ergodicity of Markov semigroups. *J. of Funct. Analysis* 243 (2007) 28-66
- [Wlo87] J. Wloka. *Partial differential equations*, Cambridge University Press, Cambridge, 1987.
- [Yos01] N. Yoshida. The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on the lattice. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(2):223–243, 2001.
- [Zeg96] B. Zegarliński. The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice. *Comm. Math. Phys.*, 175(2):401–432, 1996.

Pierre Fougères,
Équipe Modal'X, UFR Segmi,
Université Paris X - Nanterre,
200 avenue de la République,
92001 Nanterre cedex, FRANCE
pierre.fougeres@u-paris10.fr

and

Cyril Roberto,
Laboratoire d'Analyse et de Mathématiques Appliquées UMR 8050,
Université de Marne-La-Vallée,
Cité Descartes - 5 bd Descartes,
Champs-Sur-Marne,
77454 MARNE-LA-VALLEE Cedex 2, FRANCE
cyril.roberto@univ-mlv.fr

and

Boguslaw Zegarliński,
Department of Mathematics,
Imperial College London,
180 Queen's Gate, LONDON SW7 2AZ,
UNITED KINGDOM
b.zegarlinski@imperial.ac.uk