Burkholder’s submartingales from a stochastic calculus perspective
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Abstract

We provide a simple proof, as well as several generalizations, of a recent result by Davis and Suh, characterizing a class of continuous submartingales and supermartingales that can be expressed in terms of a squared Brownian motion and of some appropriate powers of its maximum. Our techniques involve elementary stochastic calculus, as well as the Doob-Meyer decomposition of continuous submartingales. These results can be used to obtain an explicit expression of the constants appearing in the Burkholder-Davis-Gundy inequalities. A connection with some balayage formulae is also established.

Key Words: Balayage; Burkholder-Davis-Gundy inequalities; Continuous Submartingales; Doob-Meyer decomposition

AMS 2000 classification: 60G15, 60G44

1 Introduction

Let \( W = \{ W_t : t \geq 0 \} \) be a standard Brownian motion initialized at zero, set \( W^*_t = \max_{s \leq t} |W_s| \) and write \( \mathcal{F}^W_t = \sigma \{ W_u : u \leq t \}, t \geq 0 \). In [3], Davis and Suh proved the following result.

Theorem 1 ([3, Th. 1.1]) For every \( p > 0 \) and every \( c \in \mathbb{R} \), set

\[
Y_t = Y_t(c,p) = (W^*_t)^{p-2} [W^2_t - t] + c (W^*_t)^p, \quad t > 0,
\]
\[
Y_0(c,p) = Y_0 = 0.
\]

1. For every \( p \in (0, 2] \), the process \( Y_t \) is a \( \mathcal{F}^W_t \)-submartingale if, and only if, \( c \geq \frac{2-p}{p} \).

2. For every \( p \in [2, +\infty) \), the process \( Y_t \) is a \( \mathcal{F}^W_t \)-supermartingale if, and only if, \( c \leq \frac{2-p}{p} \).

As pointed out in [3, p. 314] and in Section 4 below, part 1 of Theorem 1 can be used to derive explicit expressions of the constants appearing in the Burkholder-Davis-Gundy (BDG) inequalities (see [8], or [3, Ch. IV, §4]). The proof of Theorem 1 given in [3] uses several delicate estimates related to a class of Brownian hitting times: such an approach can be seen as a ramification of the discrete-time techniques developed in [2]. In particular, in [3] it is observed that the submartingale (or supermartingale) characterization of \( Y_t(c,p) \) basically relies on the properties of the random subset of \([0, +\infty)\) composed of the instants \( t \) where \( |W_t| = W^*_t \). The aim of this note is to bring this last connection into further light, by providing an elementary proof of Theorem 1, based on a direct application of Itô formula and on an appropriate version of the Doob-Meyer decomposition of submartingales. We will see that our techniques lead naturally to some substantial generalizations (see Theorem 4 below).
Theorem 2

Throughout this section, \( \mathcal{F} = \{ \mathcal{F}_t : t \geq 0 \} \) stands for a filtration satisfying the usual conditions. We will write \( X = \{ X_t : t \geq 0 \} \) to indicate a continuous \( \mathcal{F}_t \)-submartingale issued from zero and such that \( \mathbb{P} \{ X_t \geq 0, \forall t \} = 1 \). We will suppose that the Doob-Meyer decomposition of \( X \) (see for instance [4, Th. 1.4.14]) is of the type \( X_t = M_t + A_t, t \geq 0 \), where \( M \) is a square-integrable continuous \( \mathcal{F}_t \)-martingale issued from zero, and \( A \) is an increasing (integrable) natural process. We assume that \( A_0 = M_0 = 0 \); the symbol \( (M) \) \( = \{ (M)_t : t \geq 0 \} \) stands for the quadratic variation of \( M \). We note \( X^*_t = \max_{s \leq t} X_s \), and we also suppose that \( \mathbb{P} \{ X^*_t > 0 \} = 1 \) for every \( t > 0 \). The following result is a an extension of Theorem 1.

**Theorem 2.** Fix \( \varepsilon > 0 \).

1. Suppose that the function \( \phi : (0, +\infty) \mapsto \mathbb{R} \) is of class \( C^1 \), non-increasing, and such that
   \[
   \mathbb{E} \left[ \int_{\varepsilon}^T \phi(X^*_s)^2 d(M)_s \right] < +\infty, \tag{2}
   \]
   for every \( T > \varepsilon \). For every \( x \geq z > 0 \), we set
   \[
   \Phi(x, z) = -\int_z^x y \phi'(y) dy; \tag{3}
   \]
   then, for every \( \alpha \geq 1 \) the process
   \[
   Z_\varepsilon(\phi, \alpha; t) = \phi(X^*_t) (X_t - A_t) + \alpha \Phi(X^*_t, X^*_s), \quad t \geq \varepsilon, \tag{4}
   \]
   is a \( \mathcal{F}_t \)-submartingale on \( [\varepsilon, +\infty) \).

2. Suppose that the function \( \phi : (0, +\infty) \mapsto \mathbb{R} \) is of class \( C^1 \), non-decreasing and such that (3) holds for every \( T > \varepsilon \). Define \( \Phi(\cdot, \cdot) \) according to (3), and \( Z_\varepsilon(\phi, \alpha; t) \) according to (4). Then, for every \( \alpha \geq 1 \) the process \( Z_\varepsilon(\phi, \alpha; t) \) is a \( \mathcal{F}_t \)-supermartingale on \( [\varepsilon, +\infty) \).

**Remarks.** (i) Note that the function \( \phi(y) \) (and \( \phi'(y) \)) need not be defined at \( y = 0 \).

(ii) In Section 4, where we will focus on the Brownian setting, we will exhibit specific examples where the condition \( \alpha \geq 1 \) is necessary and sufficient to have that the process \( Z_\varepsilon(\alpha, \phi; t) \) is a submartingale (when \( \phi \) is non-increasing) or a supermartingale (when \( \phi \) is non-decreasing).

**Proof of Theorem 2.** (Proof of Point 1.) Observe first that, since \( M_t = X_t - A_t \) is a continuous martingale, \( X^* \) is non-decreasing and \( \phi \) is differentiable, then a standard application of Itô formula gives that
   \[
   \phi(X^*_t) (X_t - A_t) - \phi(X^*_s) (X_s - A_s) = \phi(X^*_t) M_t - \phi(X^*_s) M_s
   \]
   \[
   = \int_s^t \phi(X^*_s) dM_s + \int_s^t (X_s - A_s) \phi'(X^*_s) dX^*_s. \tag{5}
   \]
The assumptions in the statement imply that the application \( \tilde{M}_{\varepsilon,t} := \int_{\varepsilon}^{t} \phi(X^*_s) dM_s \) is a continuous square integrable \( \mathcal{F}_t \)-martingale on \([\varepsilon, +\infty)\). Moreover, the continuity of \( X \) implies that the support of the random measure \( dX^*_t \) (on \([0, +\infty)\)) is contained in the (random) set \( \{ t \geq 0 : X_t = X^*_t \} \), thus yielding that

\[
\int_{\varepsilon}^{t} (X_s - A_s) \phi^\prime (X^*_s) dX^*_s = \int_{\varepsilon}^{t} (X^*_s - A_s) \phi^\prime (X^*_s) dX^*_s
\]

\[
\quad = - \int_{\varepsilon}^{t} A_s \phi^\prime (X^*_s) dX^*_s - \Phi (X^*_t, X^*_s),
\]

where \( \Phi \) is defined in (3). As a consequence,

\[
Z^*_\varepsilon (\phi, \alpha; t) = \tilde{M}_{\varepsilon,t} + \int_{\varepsilon}^{t} (-A_s \phi^\prime (X^*_s)) dX^*_s + (\alpha - 1) \Phi (X^*_t, X^*_s). \tag{6}
\]

Now observe that the application \( t \mapsto \Phi (X^*_t, X^*_s) \) is non-decreasing (a.s.-\( P \)), and also that, by assumption, \(-A_s \phi^\prime (X^*_s) \geq 0\) for every \( s > 0 \). This entails immediately that \( Z^*_\varepsilon (\phi, \alpha; t) \) is a \( \mathcal{F}_t \)-submartingale for every \( \alpha \geq 1 \).

(Proof of Point 2.) By using exactly the same line of reasoning as in the proof of Point 1., we obtain that

\[
Z^*_\varepsilon (\phi, \alpha; t) = \int_{\varepsilon}^{t} \phi (X^*_s) dM_s + \int_{\varepsilon}^{t} (-A_s \phi^\prime (X^*_s)) dX^*_s + (\alpha - 1) \Phi (X^*_t, X^*_s). \tag{7}
\]

Since (3) is in order, we deduce that \( t \mapsto \int_{\varepsilon}^{t} \phi (X^*_s) dM_s \) is a continuous (square-integrable) \( \mathcal{F}_t \)-martingale on \([\varepsilon, +\infty)\). Moreover, \(-\alpha \phi^\prime (X^*_s) \leq 0\) for every \( s > 0 \), and we also have that \( t \mapsto \Phi (X^*_t, X^*_s) \) is a.s. decreasing. This implies that \( Z^*_\varepsilon (\phi, \alpha; t) \) is a \( \mathcal{F}_t \)-supermartingale for every \( \alpha \geq 1 \). \( \square \)

The next result allows to characterize the nature of the process \( Z \) appearing in (3) on the whole positive axis. Its proof can be immediately deduced from formulae (3) (for Part 1) and (7) (for Part 2).

**Proposition 3** Let the assumptions and notation of this section prevail.

1. Consider a decreasing function \( \phi : (0, +\infty) \to \mathbb{R} \) verifying the assumptions of Part 1 of Theorem 3 and such that

\[
\Phi (x, 0) := - \int_{0}^{x} y \phi^\prime (y) dy \text{ is finite } \forall x > 0. \tag{8}
\]

Assume moreover that

\[
\mathbb{E} \left[ \int_{0}^{T} \phi (X^*_s)^2 \, d\langle M \rangle_s \right] < +\infty, \tag{9}
\]

and also

\[
\phi (X^*_s) M_t = \phi (X^*_s) (X_\varepsilon - A_\varepsilon) \text{ converges to zero in } L^1 (\mathbb{P}), \text{ as } \varepsilon \downarrow 0, \tag{10}
\]

\[
\Phi (X^*_t, 0) \in L^1 (\mathbb{P}). \tag{11}
\]

Then, for every \( \alpha \geq 1 \) the process

\[
Z (\phi, \alpha; t) = \begin{cases} 0 & \text{for } t = 0, \\ \phi (X^*_t) (X_t - A_t) + \alpha \Phi (X^*_t, 0) & \text{for } t > 0, \end{cases}
\]

is a \( \mathcal{F}_t \)-submartingale.

2. Consider an increasing function \( \phi : (0, +\infty) \to \mathbb{R} \) as in Part 2 of Theorem 3 and such that assumptions (3)–(7) are satisfied. Then, for every \( \alpha \geq 1 \) the process \( Z (\phi, \alpha; t) \) appearing in (7) is a \( \mathcal{F}_t \)-supermartingale.
for every $T > \varepsilon$ such that to have (10) is the following:

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ \phi (X^\ast_t)^2 \right] = \lim_{\varepsilon \to 0} \mathbb{E} [M_t^2] = \mathbb{E} \left[ \phi (X^\ast_t)^2 \right] \times \mathbb{E} [M_t] = 0 \tag{13}$$

(observe that $\lim_{\varepsilon \to 0} \mathbb{E} [M_t^2] = 0$, since $M_t = 0$ by assumption). In other words, when (13) is verified the quantity $\mathbb{E} [M_t^2]$ ‘takes care’ of the possible explosion of $\varepsilon \mapsto \mathbb{E} \left[ \phi (X^\ast_t)^2 \right]$ near zero.

(ii) Let $\phi$ be non-increasing or non-decreasing on $(0, +\infty)$, and suppose that $\phi$ satisfies the assumptions of Theorem 3 and Proposition 3. Then, the process $t \mapsto \int_0^t \phi(X^\ast_s) dM_s$ is a continuous square-integrable $\mathcal{F}_t^W$-martingale. Moreover, for any choice of $\alpha \in \mathbb{R}$, the process $Z(\phi, \alpha; t) = \int_0^t \phi(X^\ast_s) dM_s + \int_0^t (\alpha - 1)X^\ast_s - A_s \phi' (X^\ast_s) dX^\ast_s$.

3 A generalization of Theorem 1

The forthcoming Theorem 4 is a generalization of Theorem 1. Recall the notation: $W$ is a standard Brownian motion issued from zero, $W^*_t = \max_{s \leq t} |W_s|$ and $\mathcal{F}_t^W = \sigma \{ W_u : u \leq t \}$. We also set for every $m \geq 1$, every $p > 0$ and every $c \in \mathbb{R}$:

$$J_t = J_t (m, c, p) = (W^*_t)^{p-m} [ |W_t|^m - A_{m,t} ] + c (W^*_t)^p, \quad t > 0, \tag{14}$$
$$J_0 (m, c, p) = J_0 = 0,$$

where $t \mapsto A_{m,t}$ is the increasing natural process in the Doob-Meyer decomposition of the $\mathcal{F}_t^W$-submartingale $t \mapsto |W_t|^m$. Of course, $J_t (2, c, p) = Y_t (c, p)$, as defined in (1).

Theorem 4 Under the above notation:

1. For every $p \in (0, m]$, the process $J_t$ is a $\mathcal{F}_t^W$-submartingale if, and only if, $c \geq \frac{m-p}{p}$.

2. For every $p \in [m, +\infty)$, the process $J_t$ is a $\mathcal{F}_t^W$-supermartingale if, and only if, $c \leq \frac{m-p}{p}$.

Proof. Recall first the following two facts: (i) $W^*_t \overset{law}{=} \sqrt{t} W^*_1$ (by scaling), and (ii) there exists $\eta > 0$ such that $\mathbb{E} \left[ \exp (\eta (W^*_1)^{-2}) \right] < +\infty$ (this can be deduced e.g. from [2, Ch. II, Exercice 3.10]), so that the random variable $(W^*_1)^{-1}$ has finite moments of all orders. Note also that the conclusions of both Point 1 and Point 2 are trivial in the case where $p = m$. In the rest of the proof we will therefore assume that $p \neq m$.

To prove Point 1, we shall apply Theorem 3 and Proposition 3 in the following framework: $X_t = |W_t|^m$ and $\phi (x) = x^{\frac{m-p}{m}} = x^{\frac{m}{m}}$. In this case, the martingale $M_t = |W_t|^m - A_{m,t}$ is such that $\langle M \rangle_t = m^2 \int_0^t W_s^{2m-2} ds$, $t \geq 0$, and $\Phi (x, z) = - \int_z^x y \phi' (y) dy = - (\frac{z}{m} - 1) \int_z^x y^{\frac{m}{m} - 1} dy = \frac{m-p}{p} (x^{\frac{m}{m}} - z^{\frac{m}{m}})$. Also, for every $T > \varepsilon > 0$

$$\mathbb{E} [ \int_\varepsilon^T \phi (X^\ast_s)^2 d \langle M \rangle_s ] = m^2 \mathbb{E} [ \int_\varepsilon^T (W^*_s)^{2p-2m} W_s^{2m-2} ds ] \leq m^2 \mathbb{E} [ \int_\varepsilon^T (W^*_s)^{2p-2} ds ] = m^2 \mathbb{E} [ (W^*_1)^{2p-2} ] \int_\varepsilon^T s^{\frac{m}{m}-1} ds, \tag{15}$$
so that \( \phi \) verifies (2) and (3). Relations (8) and (11) are trivially satisfied. To see that (10) holds, use the relations

\[
\mathbb{E} \left\{ \phi (X^*_t) (X_t - A_t) \right\} = \mathbb{E} \left\{ (W^*_t)^{p-m} \left[ |W_t|^m - A_{m,t} \right] \right\} = \mathbb{E} \left\{ (W^*_t)^{p-m} |W_t|^m \right\} \leq \mathbb{E} \left\{ (W^*_t)^{2p-2m} \right\} \mathbb{E} \left\{ (M)_t \right\}^{1/2} = \frac{m}{\alpha} \mathbb{E} \left\{ (W^*_t)^{2p-2m} \right\} \left[ \varepsilon \left( \int_0^t |W_s|^{-1} ds \right) \right]^{1/2} \to 0, \quad \text{as } \varepsilon \downarrow 0.
\]

From Point 1 of Proposition 3, we therefore deduce that the process \( Z(t) \) defined as \( Z(0) = 0 \) and, for \( t > 0 \),

\[
Z(t) = \phi (W^*_t)^{m} \left[ |W_t|^m - A_{m,t} \right] + \alpha \Phi \left( (W^*_t)^m, 0 \right) = (W^*_t)^{p-m} \left[ |W_t|^m - A_{m,t} \right] + \alpha \frac{m-p}{p} (W^*_t)^p,
\]

is a \( F_t^W \)-submartingale for every \( \alpha \geq 1 \). By writing \( c = \alpha \frac{m-p}{p} \) in the previous expression, and by using the fact that \( \frac{m-p}{p} \geq 0 \) by assumption, we deduce immediately that \( J_t (m,c,p) \) is a submartingale for every \( c \geq \frac{m-p}{p} \). Now suppose \( c < \frac{m-p}{p} \). One can use formulae (2), (3) and (4) to prove that

\[
J_t (m,c,p) = \int_0^t \phi (W_s^*)^p dM_s + \int_0^t [-A_{m,s} \phi' \left( (W_s^*)^m \right)] d(W_s^*)^m + (\alpha - 1) \Phi \left( (W_s^*)^m, 0 \right)
\]

where \( 1 - \alpha = 1 - pc/(m-p) > 0 \). Note that \( \int_0^t (W_s^*)^{p-m} dM_s \) is a square-integrable martingale, due to (13). To conclude that, in this case, \( J_t (m,c,p) \) cannot be a submartingale (nor a supermartingale), it is sufficient to observe that (for every \( m \geq 1 \) and every \( \alpha < 1 \)) the paths of the finite variation process

\[
t \mapsto \int_0^t \left[ (1 - \alpha) (W_s^*)^m - A_{m,s} (W_s^*)^{p-2m} \right] d(W_s^*)^m
\]

are neither non-decreasing nor non-increasing, with \( P \)-probability one.

To prove Point 2, one can argue in exactly the same way, and use Point 2 of Proposition 3 to obtain that the process \( Z(t) \) defined as \( Z(0) = 0 \) and, for \( t > 0 \),

\[
Z(t) = (W^*_t)^{p-m} \left[ |W_t|^m - A_{m,t} \right] + \alpha \frac{m-p}{p} (W^*_t)^p
\]

is a \( F_t^W \)-supermartingale for every \( \alpha \geq 1 \). By writing once again \( c = \alpha \frac{m-p}{p} \) in the previous expression, and since \( \frac{m-p}{p} \leq 0 \), we immediately deduce that \( J_t (m,c,p) \) is a supermartingale for every \( c \leq \frac{m-p}{p} \). One can show that \( J_t (m,c,p) \) cannot be a supermartingale, whenever \( c > \frac{m-p}{p} \), by using arguments analogous to those displayed in the last part of the proof of Point 1.

The following result is obtained by specializing Theorem 4 to the case \( m = 1 \) (via Tanaka’s formula).

**Corollary 5** Denote by \( \ell_t : t \geq 0 \) the local time at zero of the Brownian motion \( W \). Then, the process

\[
J_t(p) = (W^*_t)^{p-1} \left[ |W_t| - \ell_t \right] + c (W^*_t)^p, \quad t > 0,
\]

\[
J_0(p) = 0,
\]

is such that: (i) for \( p \in (0,1) \), \( J_t(p) \) is a \( F_t^W \)-submartingale if, and only if, \( c \geq 1/p - 1 \), and (ii) for \( p \in [1, +\infty) \), \( J_t(p) \) is a \( F_t^W \)-supermartingale if, and only if, \( c \leq 1/p - 1 \).
4 Burkholder-Davis-Gundy (BDG) inequalities

We reproduce an argument taken from [3, p. 314], showing that the first part of Theorem 4 can be used to obtain a strong version of the BDG inequalities (see e.g. [3, Ch. IV, §4]).

Fix $p \in (0, 2)$ and define $c = (2 - p)/p = 2/p - 1$. Since, according to the first part of Theorem 4, $Y_t = Y_t(c, p)$ is a $\mathcal{F}^W_t$-submartingale starting from zero, we deduce that, for every bounded and strictly positive $\mathcal{F}^W_t$-stopping time $\tau$, one has $\mathbb{E}(Y_\tau) \geq 0$. In particular, this yields

$$\mathbb{E} \left( \tau \left( W^n_\tau \right)^p \right) \leq \frac{2}{p} \mathbb{E}((W^*_\tau)^p).$$

(18)

Formula (18), combined with an appropriate use of Hörmander’s inequality, entails finally that, for $0 < p < 2$,

$$\mathbb{E} \left( \tau^{\frac{p}{2}} \right) \leq \left[ \frac{2}{p} \mathbb{E}((W^*_\tau)^p) \right]^{\frac{p}{2}} \left[ \mathbb{E}((W^*_\tau)^p) \right]^{\frac{2-p}{2}} = \left[ \frac{2}{p} \right]^{\frac{p}{2}} \mathbb{E}((W^*_\tau)^p).$$

(19)

Of course, relation (19) extends to general stopping times $\tau$ (not necessarily bounded) by monotone convergence (via the increasing sequence $\{\tau \land n : n \geq 1\}$).

Remark. Let $\{\mathcal{A}_n : n \geq 0\}$ be a discrete filtration of the reference $\sigma$-field $\mathcal{A}$, and consider a $\mathcal{A}_n$-adapted sequence of measurable random elements $\{f_n : n \geq 0\}$ with values in a Banach space $\mathcal{B}$. We assume that $f_n$ is a martingale, i.e. that, for every $n$, $\mathbb{E}[f_n - f_{n-1} | \mathcal{A}_{n-1}] = \mathbb{E}[d_n | \mathcal{A}_{n-1}] = 0$, where $d_n := f_n - f_{n-1}$. We note

$$S_n(f) = \sqrt{\sum_{k=0}^{n} |d_k|^2} \quad \text{and} \quad f^*_n = \sup_{0 \leq m \leq n} |f_m|,$$

and write $S(f)$ and $f^*$, respectively, to indicate the pointwise limits of $S_n(f)$ and $f^*_n$, as $n \to +\infty$. In [2], D.L. Burkholder proved that

$$\mathbb{E}(S(f)) \leq \sqrt{3} \mathbb{E}(f^*),$$

(20)

where $\sqrt{3}$ is the best possible constant, in the sense that for every $\eta \in (0, \sqrt{3})$ there exists a Banach space-valued martingale $f(\eta)$ such that $\mathbb{E}(S(f(\eta))) > \eta \mathbb{E}(f^*(\eta))$. As observed in [3], Burkholder’s inequality (20) should be compared with (19) for $p = 1$, which yields the relation $\mathbb{E}(\tau^{1/2}) \leq \sqrt{2} \mathbb{E}(W^*_\tau)$ for every stopping time $\tau$. This shows that in such a framework, involving uniquely continuous martingales, the constant $\sqrt{3}$ is no longer optimal.

5 Balayage

Keep the assumptions and notation of Section 4 and Theorem 4, fix $\varepsilon > 0$ and consider a finite variation function $\psi : (0, +\infty) \to \mathbb{R}$. In this section we focus on the formula

$$\psi(X^*_\tau)(X_\tau - A_\tau) - \psi(X^*_\varepsilon)(X_\varepsilon - A_\varepsilon) = \int_\varepsilon^\tau \psi(X^*_s)d(X_s - A_s) + \int_\varepsilon^\tau (X^*_s - A_s) d\psi(X^*_s),$$

(21)

where $\varepsilon > 0$. Note that by choosing $\psi = \phi$ in (21), where $\phi \in C^1$ is monotone, one recovers formula (3), which was crucial in the proof Theorem 2. We shall now show that (21) can be obtained by means of the balayage formulae proved in [4].

To see this, let $U = \{U_t : t \geq 0\}$ be a continuous $\mathcal{F}_t$-semimartingale issued from zero. For every $t > 0$ we define the random time

$$\sigma(t) = \sup\{s < t : U_s = 0\}.$$

(22)

The following result is a particular case of [4, Th. 1].
Proposition 6 (Balayage Formula) Consider a stochastic process \( \{K_t : t > 0\} \) such that the restriction \( \{K_t : t \geq \varepsilon\} \) is locally bounded and \( \mathcal{F}_t \)-predictable on \( [\varepsilon, +\infty) \) for every \( \varepsilon > 0 \). Then, for every fixed \( \varepsilon > 0 \), the process \( K_{\sigma(t)}, t \geq \varepsilon \), is locally bounded and \( \mathcal{F}_t \)-predictable, and moreover

\[
U_t K_{\sigma(t)} = U_\varepsilon K_{\sigma(\varepsilon)} + \int_\varepsilon^t K_{\sigma(s)} dU_s. \tag{23}
\]

To see how (21) can be recovered from (23), set \( U_t = X_t - X_\sigma^* \) and \( K_t = \psi(X_\sigma^*) \) by construction, where \( \sigma(t) \) is defined as in (22). As a consequence, (23) gives

\[
\psi(X_\sigma^*) (X_t - X_\sigma^*) = \psi(X_\sigma^*) (X_\varepsilon - X_\sigma^*) + \int_\varepsilon^t \psi(X_s^*) d(X_s - X_s^*).
\]

Finally, a standard integration by parts applied to \( \psi(X_\sigma^*) (X_t - A_t) \) yields

\[
\psi(X_\sigma^*) (X_t - A_t) = \psi(X_\sigma^*) (X_t - X_\sigma^*) + \psi(X_\sigma^*) (X_t^* - A_t)
= \psi(X_\sigma^*) (X_\varepsilon - X_\sigma^*) + \int_\varepsilon^t \psi(X_s^*) d(X_s - X_s^*)
+ \psi(X_\sigma^*) (X_\varepsilon^* - A_\varepsilon) + \int_\varepsilon^t \psi(X_s^*) d(X_s^* - A_s)
+ \int_\varepsilon^t (X_s^* - A_s) d\psi(X_s^*),
\]

which is equivalent to (21).

References


