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# CONFIGURATIONS OF SADDLE CONNECTIONS OF QUADRATIC DIFFERENTIALS ON $\mathbb{CP}^1$ AND ON HYPERELLIPTIC RIEMANN SURFACES

CORENTIN BOISSY

ABSTRACT. Configurations of rigid collections of saddle connections are invariants of connected components of strata of the moduli space of quadratic differentials. They have been classified for strata of Abelian differentials in [EMZ]. Similar work for strata of quadratic differentials has been done in [MZ], although in that case the connected components were not distinguished.

We classify the configurations for quadratic differentials on  $\mathbb{CP}^1$  and on hyperelliptic connected components of the moduli space of quadratic differentials. We show, in particular, that, when the genus is greater than 5, any configuration that appears in the hyperelliptic connected component of a stratum also appears in the nonhyperelliptic one.

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## 1. INTRODUCTION

We study flat surfaces with conical singularities of angle multiple of  $\pi$ . The quotient of this set by isometries is isomorphic to the moduli space of quadratic differentials on Riemann surfaces and is naturally stratified. Flat surfaces which correspond to squares of Abelian differentials are often called *translation surfaces*. Flat surfaces appear in the study of billiards in rational polygons to the extent that there is an “unfolding” procedure of a rational billiard that gives a translation surface.

A sequence of quadratic differentials or Abelian differentials leaves any compact set of a stratum when the length of a saddle connection tends to zero. This might force some other saddle connections to shrink. The corresponding collections of saddle connections on a flat surface are said to be  $\hat{\text{homologous}}$  (in the case of an Abelian differential this correspond to homologous saddle connections). According to Masur and Smillie [MS] (see also [EMZ, MZ]), a “typical degeneration” corresponds to the case when all the “short” saddle connections are  $\hat{\text{homologous}}$  together (*i.e.* all short saddle connections are pairwise  $\hat{\text{homologous}}$ ). Therefore the study of configurations of  $\hat{\text{homologous}}$  saddle connections (or homologous saddle connection in the case of Abelian differential) is related to the study of compactification of a given stratum. A configuration of  $\hat{\text{homologous}}$  saddle connections on a generic surface is also a natural invariant of a connected component of the ambient stratum.

In a recent article, Eskin, Masur and Zorich [EMZ] study collections of homologous saddle connections for Abelian differentials. They describe configurations for each connected component of the strata of Abelian differentials. Collections of  $\hat{\text{homologous}}$  saddle connections are studied for quadratic differentials by Masur and Zorich [MZ]: they describe all the configurations that can arise in a stratum of quadratic differentials, but they do not distinguish connected components of such stratum.

According to Lanneau [L2], the nonconnected strata of quadratic differentials admit two connected components. They are of one of the following two types:

- “hyperelliptic” stratum: the stratum admits a connected component that consists of hyperelliptic quadratic differentials (note that some of these strata are connected).
- exceptional stratum: there exist four nonconnected strata that do not belong to the previous case.

In this article, we give the classification of the configurations that appear in the hyperelliptic connected components. This gives therefore a sufficient condition for a surface in a stratum that admits a hyperelliptic connected component to be in the nonhyperelliptic component. Unfortunately, this is not a necessary condition since, as we show, any configuration that appears in a hyperelliptic connected component also appears in the other component of the stratum when the genus is greater than 5. We postpone to a future work the description of configurations for exceptional strata.

We deduce configurations for hyperelliptic components from configurations for strata of quadratic differentials on  $\mathbb{CP}^1$ . Configurations for  $\mathbb{CP}^1$  are deduced from general results on configurations that appear in [MZ]. Note that these configurations are needed in the study of asymptotics in billiards in polygons with “right” angles [AEZ]. For such a polygon, there is a simple unfolding procedure that consists of gluing along their boundaries two copies of the polygon. This gives a flat surface of genus zero with conical singularities, whose angles are multiples of  $\pi$  (*i.e.* a quadratic differential on  $\mathbb{CP}^1$ ). Then a generalised diagonal or a periodic trajectory in the polygon gives a saddle connection on the corresponding flat surface.

We also give in annex an explicit formula that gives a relation between the genus of a surface and the ribbon graph of connected components associated to a collection of  $\hat{\text{homologous}}$  saddle connections.

Some particular splittings are sometime used to compute the closure  $SL(2, \mathbb{R})$ -orbits of surfaces (see [Mc, HLM]). These splittings of surfaces can be reformulated as configurations of homologous or  $\hat{\text{homologous}}$  saddle connections on these surfaces. It would be interesting to find some configurations that appears in *any* surface of a connected component of a stratum, as it was done in [Mc].

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**1.1. Basic definitions.** Let  $S$  be a compact Riemann surface of genus  $g$ . A quadratic differential  $q$  on  $S$  is given locally by  $q(z) = \phi(z)dz^2$ , for  $(U, z)$  a local chart with  $\phi$  a meromorphic function with at most simple poles. We define the poles and zeroes of  $q$  in a local chart to be the poles and zeroes of the corresponding meromorphic function  $\phi$ . It is easy to check that they do not depend on the choice of the local chart. Slightly abusing notations, a pole will be referred to as a zero of order  $-1$ . An Abelian differential on  $S$  is a holomorphic 1-form.

Outside its poles and zeros,  $q$  is locally the square of an Abelian differential. Integrating this 1-form gives a natural atlas such that the transition functions are of the kind  $z \mapsto \pm z + c$ . Thus  $S$  inherits a flat metric with singularities, where a zero of order  $k \geq -1$  becomes a conical singularity of angle  $(k + 2)\pi$ . The flat metric has trivial holonomy if and only if  $q$  is globally the square of an Abelian differential. If not, then the holonomy is  $\mathbb{Z}/2\mathbb{Z}$  and  $(S, q)$  is sometimes called a *half-translation* surface. In order to simplify the notation, we will usually denote by  $S$  a surface with a flat structure.

We associate to a quadratic differential the set  $\{k_1, \dots, k_r\}$  of orders of its poles and zeros. The Gauss-Bonnet formula asserts that  $\sum_i k_i = 4g - 4$ . Conversely, if we fix a collection  $\{k_1, \dots, k_r\}$  of integers greater than or equal to  $-1$  satisfying the previous equality, we denote by  $\mathcal{Q}(k_1, \dots, k_r)$  the (possibly empty) moduli space of quadratic differentials which are not globally the square of Abelian differential, having  $\{k_1, \dots, k_r\}$  as orders of poles and zeros. It is well known that  $\mathcal{Q}(k_1, \dots, k_r)$  is a complex analytic orbifold, which is usually called a *stratum* of the moduli space of quadratic differentials. We mostly restrict ourselves to the subspace  $\mathcal{Q}_1(k_1, \dots, k_r)$  of area one surfaces, where the area is given by the flat metric. In a similar way, we denote by  $\mathcal{H}_1(n_1, \dots, n_s)$  the moduli space of Abelian differentials of area 1 having zeroes of degree  $\{n_1, \dots, n_s\}$ , where  $n_i \geq 0$  and  $\sum_{i=1}^s n_i = 2g - 2$ .

A saddle connection is a geodesic segment (or geodesic loop) joining two singularities (or a singularity to itself) with no singularities in its interior. Even if  $q$  is not globally a square of an Abelian differential we can find a square root of it along the saddle connection. Integrating it along the saddle connection we get a complex number (defined up to multiplication by  $-1$ ). Considered as a planar vector, this complex number represents the affine holonomy vector along the saddle connection. In particular, its euclidean length is the modulus of its holonomy vector. Note that a saddle connection lingers under small deformation of the surface.

Local coordinates on a stratum of Abelian differentials are obtained by integrating the holomorphic 1-form along a basis of the relative homology  $H_1(S, \text{sing}, \mathbb{Z})$ , where *sing* is the set of conical singularities. Equivalently, this means that local coordinates are defined by the relative cohomology  $H^1(S, \text{sing}, \mathbb{C})$ .

Local coordinates in a stratum of quadratic differentials are obtained in the following way: one can naturally associate to a quadratic differential  $(S, q) \in \mathcal{Q}(k_1, \dots, k_r)$  a double cover  $p: \hat{S} \rightarrow S$  such that  $p^*q$  is the square of an Abelian differential  $\omega$ . The surface  $\hat{S}$  admits a natural involution  $\tau$ , that induces on the relative cohomology  $H^1(S, \text{sing}, \mathbb{C})$  an involution  $\tau^*$ . It decomposes  $H^1(S, \text{sing}, \mathbb{C})$  into an invariant subspace  $H_+^1(S, \text{sing}, \mathbb{C})$  and an antiinvariant subspace  $H_-^1(S, \text{sing}, \mathbb{C})$ . One can show that the antiinvariant subspace  $H_-^1(S, \text{sing}, \mathbb{C})$  gives local coordinates to the stratum  $\mathcal{Q}(k_1, \dots, k_r)$ . It is well known that Lebesgue measure on these coordinates defines a finite measure  $\mu$  on the stratum  $\mathcal{Q}_1(k_1, \dots, k_r)$ .

A hyperelliptic quadratic differential is a quadratic differential such that there exists an orientation preserving involution  $\tau$  with  $\tau^*q = q$

and such that  $S/\tau$  is a sphere. We can construct families of hyperelliptic quadratic differentials in the following way: to all quadratic differentials on  $\mathbb{CP}^1$ , we associate a double covering ramified over some singularities satisfying some fixed combinatorial conditions. The resulting Riemann surfaces naturally carry hyperelliptic quadratic differentials.

Some strata admit an entire connected component that is made of hyperelliptic quadratic differentials. These components arise from the previous construction and have been classified by M. Kontsevich and A. Zorich for Abelian differentials [KZ] and by E. Lanneau for quadratic differentials [L1].

**Theorem** (M. Kontsevich, A. Zorich). *The strata of Abelian differentials that have a hyperelliptic connected component are the following ones.*

- (1)  $\mathcal{H}(2g - 2)$ , where  $g \geq 2$ . It arises from  $\mathcal{Q}(2g - 3, -1^{2g+1})$ . The ramifications points are located over all the singularities.
- (2)  $\mathcal{H}(g - 1, g - 1)$ , where  $g \geq 2$ . It arises from  $\mathcal{Q}(2g - 2, -1^{2g+2})$ . The ramifications points are located over all the poles.

*In the above presented list, the strata  $\mathcal{H}(1, 1)$  and  $\mathcal{H}(2)$  are the ones that are connected.*

**Theorem** (E. Lanneau). *The strata of quadratic differentials that have a hyperelliptic connected component are the following ones.*

- (1)  $\mathcal{Q}(2(g - k) - 3, 2(g - k) - 3, 2k + 1, 2k + 1)$  where  $k \geq -1$ ,  $g \geq 1$  and  $g - k \geq 2$ . It arises from  $\mathcal{Q}(2(g - k) - 3, 2k + 1, -1^{2g+2})$ . The ramifications points are located over  $2g + 2$  poles.
- (2)  $\mathcal{Q}(2(g - k) - 3, 2(g - k) - 3, 4k + 2)$  where  $k \geq 0$ ,  $g \geq 1$  and  $g - k \geq 1$ . It arises from  $\mathcal{Q}(2(g - k) - 3, 2k, -1^{2g+1})$ . The ramifications points are located over  $2g + 1$  poles and over the zero of order  $2k$ .
- (3)  $\mathcal{Q}(4(g - k) - 6, 4k + 2)$  where  $k \geq 0$ ,  $g \geq 2$  and  $g - k \geq 2$ . It arises from  $\mathcal{Q}(2(g - k) - 4, 2k, -1^{2g})$ . The ramifications points are located over all the singularities

*In the above presented list, the strata  $\mathcal{Q}(-1, -1, 1, 1)$ ,  $\mathcal{Q}(-1, -1, 2)$ ,  $\mathcal{Q}(1, 1, 1, 1)$ ,  $\mathcal{Q}(1, 1, 2)$  and  $\mathcal{Q}(2, 2)$  are the ones that are connected.*

**1.2.  $\hat{\mathbf{H}}$ omologous saddle connections.** Let  $S \in \mathcal{Q}(k_1, \dots, k_r)$  and denote by  $p : \hat{S} \rightarrow S$  its canonical double cover and by  $\tau$  the corresponding involution. Let  $P$  denote the set of singularities of  $S$  and let  $\hat{P} = p^{-1}(P)$ .

To an oriented saddle connection  $\gamma$  on  $S$ , we can associate  $\gamma_1$  and  $\gamma_2$  its preimages by  $p$ . If the relative cycle  $[\gamma_1]$  satisfies  $[\gamma_1] = -[\gamma_2] \in$

$H_1(\hat{S}, \hat{P}, \mathbb{Z})$ , then we define  $[\tilde{\gamma}] = [\gamma_1]$ . Otherwise, we define  $[\tilde{\gamma}] = [\gamma_1] - [\gamma_2]$ . Note that in all cases, the cycle  $[\tilde{\gamma}]$  is antiinvariant with respect to the involution  $\tau$ .

**Definition 1.1.** Two saddle connections  $\gamma$  and  $\gamma'$  are  $\hat{\text{homologous}}$  if  $[\tilde{\gamma}] = \pm[\tilde{\gamma}']$ .

**Theorem** (H. Masur; A. Zorich). *Consider two distinct saddle connections  $\gamma, \gamma'$  on a half-translation surface. The following assertion are equivalent:*

- *The two saddle connections  $\gamma$  and  $\gamma'$  are  $\hat{\text{homologous}}$ .*
- *The ratio of their lengths is constant under any small deformation of the surface inside the ambient stratum.*
- *They have no interior intersection and one of the connected components of  $S \setminus \{\gamma \cup \gamma'\}$  has trivial linear holonomy.*

*Furthermore, if  $\gamma$  and  $\gamma'$  are  $\hat{\text{homologous}}$ , then the ratio of their lengths belongs to  $\{\frac{1}{2}, 1, 2\}$ .*

Consider a set of  $\hat{\text{homologous}}$  saddle connections  $\gamma = \{\gamma_1, \dots, \gamma_s\}$  on a half-translation surface  $(S, q)$ . Slightly abusing notation, we will denote by  $S \setminus \gamma$  the subset  $S \setminus (\cup_{i=1}^s \gamma_i)$ . This subset is a finite union of connected half-translation surfaces with boundary. We define a graph  $\Gamma(S, \gamma)$  called the graph of connected components in the following way (see [MZ]): the vertices are the connected components of  $S \setminus \gamma$ , labelled as “o” if the corresponding surface is a cylinder, as “+” if it has trivial holonomy (but is not a cylinder), and as “−” if it has non-trivial holonomy. The edges are given by the saddle connections in  $\gamma$ . Each  $\gamma_i$  is on the boundary of one or two connected components of  $S \setminus \gamma$ . In the first case it becomes an edge joining the corresponding vertex to itself. In the second case, it becomes an edge joining the two corresponding vertices.

In [MZ], Masur and Zorich describe the set of all possible graphs of connected components for a quadratic differential.

**Theorem** (H. Masur, A. Zorich). *Let  $(S, q)$  be quadratic differential ; let  $\gamma$  be a collection of  $\hat{\text{homologous}}$  saddle connections  $\{\gamma_1, \dots, \gamma_n\}$ , and let  $\Gamma(S, \gamma)$  be the graph of connected components encoding the decomposition  $S \setminus (\gamma_1 \cup \dots \cup \gamma_n)$ .*

*The graph  $\Gamma(S, \gamma)$  either has one of the basic types listed below or can be obtained from one of these graphs by placing additional “o”-vertices of valence two at any subcollection of edges subject to the following restrictions. At most one “o”-vertex may be placed at the same edge;*

a “o”-vertex cannot be placed at an edge adjacent to a “o”-vertex of valence 3 if this is the edge separating the graph.

The graphs of basic types, presented in Figure 1, are given by the following list:

- a) An arbitrary (possibly empty) chain of “+”-vertices of valence two bounded by a pair of “-”-vertices of valence one;
- b) A single loop of vertices of valence two having exactly one “-”-vertex and arbitrary number of “+”-vertices (possibly no “+”-vertices at all);
- c) A single chain and a single loop joined at a vertex of valence three. The graph has exactly one “-”-vertex of valence one; it is located at the end of the chain. The vertex of valence three is either a “+”-vertex, or a “o”-vertex (vertex of the cylinder type). Both the chain, and the cycle may have in addition an arbitrary number of “+”-vertices of valence two (possibly no “+”-vertices at all);
- d) Two nonintersecting cycles joined by a chain. The graph has no “-”-vertices. Each of the two cycles has a single vertex of valence three (the one where the chain is attached to the cycle); this vertex is either a “+”-vertex or a “o”-vertex. If both vertices of valence three are “o”-vertices, the chain joining two cycles is nonempty: it has at least one “+”-vertex. Otherwise, each of the cycles and the chain may have arbitrary number of “+”-vertices of valence two (possibly no “+”-vertices of valence two at all);
- e) “Figure-eight” graph: two cycles joined at a vertex of valence four, which is either a “+”-vertex or a “o”-vertex. All the other vertices (if any) are the “+”-vertices of valence two. Each of the two cycles may have arbitrary number of such “+”-vertices of valence two (possibly no “+”-vertices of valence two at all).

Every graph listed above corresponds to some flat surface  $S$  and to some collection of saddle connections  $\gamma$ .

Each connected component of  $S \setminus \gamma$  is a non-compact surface which can be naturally compactified (for example considering the distance induced by the flat metric on a connected component of  $S \setminus \gamma$ , and the corresponding completion). We denote this compactification by  $S_j$ . We warn the reader that  $S_j$  might differ from the closure of the component in the surface  $S$ : for example, if  $\gamma_i$  is on the boundary of just one connected component  $S_j$  of  $S \setminus \gamma$ , then the compactification of  $S_j$  carries two copies of  $\gamma_i$  in its boundary, while in the closure of the corresponding connected component of  $S \setminus \gamma$ , these two copies are identified. The



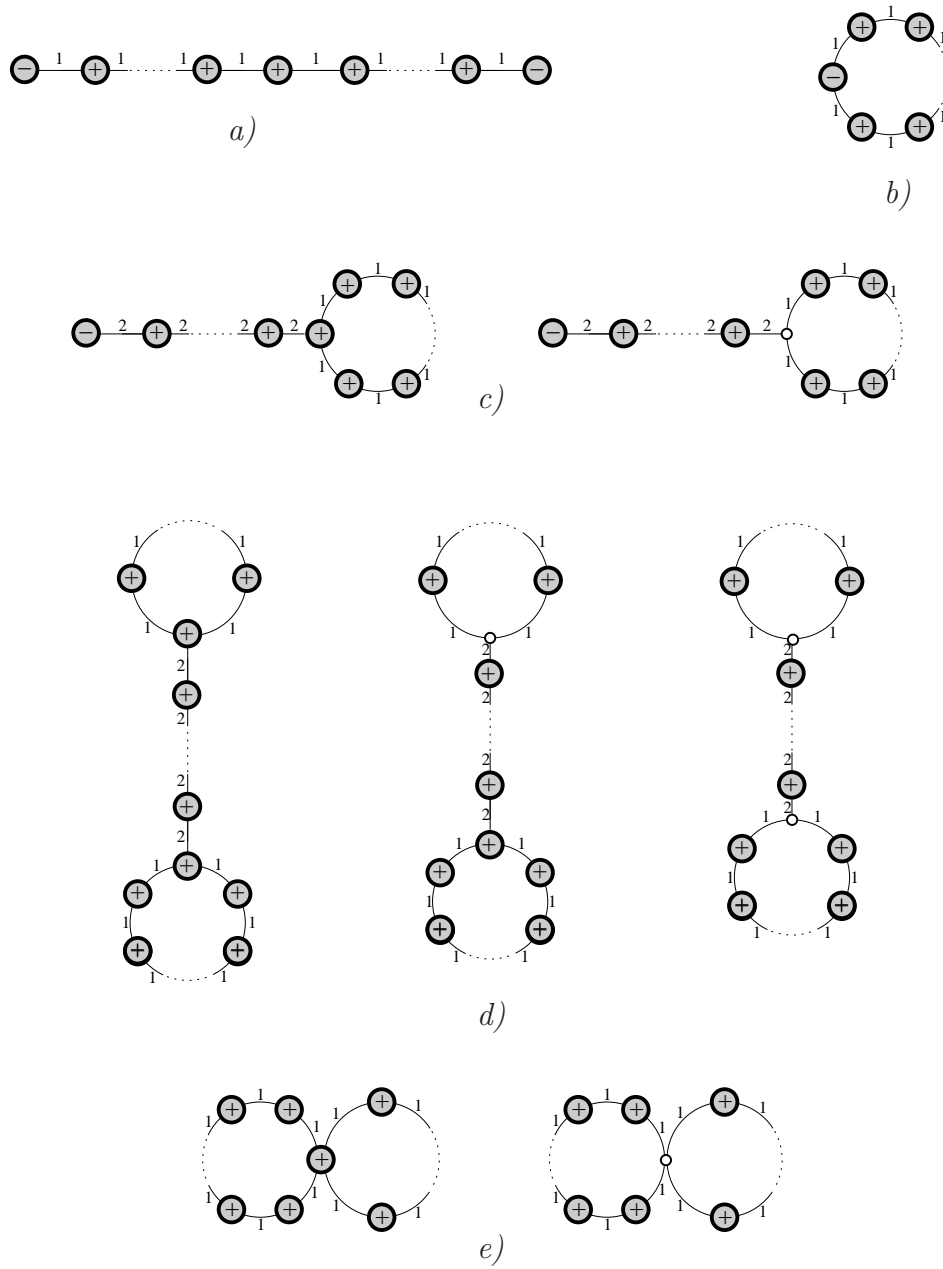


FIGURE 1. Classification of admissible graphs.

boundary of each  $S_i$  is a union of saddle connections; it has one or several connected components. Each of them is homeomorphic to  $\mathbb{S}^1$

and therefore defines a cyclic order in the set of boundary saddle connections. Each consecutive pair of saddle connections for that cyclic order defines a *boundary singularity* with an associated angle which is an integer multiple of  $\pi$  (since the boundary saddle connections are parallel). The surface with boundary  $S_i$  might have singularities in its interior. We call them *interior singularities*.

**Definition 1.2.** Let  $\gamma = \{\gamma_1, \dots, \gamma_r\}$  be a maximal collection of homologous saddle connections. Then a *configuration* is the following combinatorial data:

- The graph  $\Gamma(S, \gamma)$
- For each vertex of this graph, a permutation of the edges adjacent to the vertex (encoding the cyclic order of the saddle connections on each connected component of the boundary of the  $S_i$ ).
- For each pair of consecutive elements in that cyclic order, an integer  $k \geq 0$  such that the angle between the two corresponding saddle connections is  $(k + 1)\pi$ . This integer will be referred as the *order of the boundary singularity*.
- For each  $S_i$ , a collection of integers that are the orders of the interior singularities of  $S_i$ .

Following [MZ], we will encode the permutation of the edges adjacent to each vertex by a *ribbon graph*.

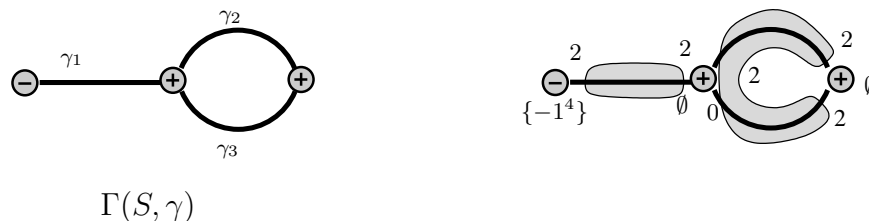


FIGURE 2. An example of configuration

*Example 1.3.* Figure 2 represents a configuration on a flat surface. The corresponding collection  $\{\gamma_1, \gamma_2, \gamma_3\}$  of homologous saddle connections decomposes the surface into three connected components. The first connected component  $\ominus \text{---} \oplus$  has four interior singularities of order  $-1$ , and its boundary consists of a single saddle connection with the corresponding boundary singularity of angle  $(2 + 1)\pi = 3\pi$ . The second connected component  $\oplus \text{---} \oplus$  has no interior singularities. It has two boundary components, one consists of a single saddle connection with

corresponding singularity of angle  $(2 + 1)\pi$ , and one that is a union of two saddle connections with corresponding boundary singularities of angle  $(0+1)\pi$  and  $(2+1)\pi$ . The last connected component  $\ominus\oplus$  has no interior singularities, and admits two boundary components that consists each of a single saddle connection (with corresponding boundary singularities of angles  $(2 + 1)\pi$ ).

Figure 3 represents a flat surface with a collection of three  $\hat{\text{homologous}}$  saddle connections realizing this configuration.

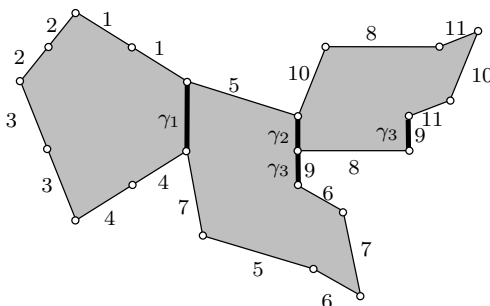


FIGURE 3. Unfolded flat surface realizing configuration as in figure 2

*Remark 1.4.* In description of a configuration of a collection of  $\hat{\text{homologous}}$  saddle connections  $\gamma = \{\gamma_1, \dots, \gamma_r\}$ , we will always assume that the quadratic differential is *generic*, and therefore, each saddle connection parallel to the  $\gamma_i$  is  $\hat{\text{homologous}}$  to the  $\gamma_i$  (see [MZ]).

*Remark 1.5.* A maximal collection of  $\hat{\text{homologous}}$  saddle connections and the associated configuration linger under any small deformation of the flat surface inside the ambient stratum. They also linger under the well know  $SL(2, \mathbb{R})$  action on the stratum which is ergodic with respect to the Lebesgue measure  $\mu$  (see [M, V1, V2]). Hence, all admissible configurations that exists in a connected component are realised in a generic surface of that component. Furthermore, the number of collections realizing a given configuration in a generic surface has quadratic asymptotics (see [EM]).

## 2. CONFIGURATIONS FOR THE RIEMANN SPHERE

In this section we describe all admissible configurations of  $\hat{\text{homologous}}$  saddle connections that arise on  $\mathbb{CP}^1$ . We first start with several preliminary lemmas which are applicable to flat surfaces of arbitrary genus. Let  $S$  be a generic flat surface of genus  $g \geq 0$  in some stratum of

quadratic differentials, and let  $\gamma$  be a maximal collection of  $\hat{\text{homologous}}$  saddle connections on it. Taking the natural compactification of each connected component of  $S \setminus \gamma$ , we get a collection  $\{S_i\}_{i \in I}$  of compact surfaces with boundary. The boundary of each  $S_i$  is topologically a union of disjoint circles. We can glue a disc to each connected component of the boundary of  $S_i$  and get a closed surface  $\overline{S}_i$ ; we denote by  $g_i$  the genus of  $\overline{S}_i$ .

**Lemma 2.1.** *Let  $g$  be the genus of  $S$ , then  $g \geq \sum g_i$ .*

*Proof.* For each  $S_i$ , we consider a collection of paths  $(c_{i,1}, \dots, c_{i,2g_i})$  of  $S_i$  that represent a symplectic basis of  $H_1(\overline{S}_i, \mathbb{R})$  and that avoid the boundary of  $S_i$ . When we glue the  $S_i$  together, the  $c_{i,j}$  provides a collection of cycles of  $H_1(S, \mathbb{R})$ . It is a symplectic family because two paths arising from two different surfaces do not intersect. Therefore we get a free family of  $H_1(S, \mathbb{R})$ , so:

$$2g = \dim(H_1(S, \mathbb{R})) \geq \sum \dim(H_1(\overline{S}_i, \mathbb{R})) = \sum 2g_i$$

□

**Lemma 2.2.** *If  $S_{i_0}$  is not a cylinder and has trivial holonomy, then  $g_{i_0} > 0$ .*

*Proof.* We recall that the initial collection of  $\hat{\text{homologous}}$  saddle connections is assumed to be maximal, therefore there is no interior saddle connections  $\hat{\text{homologous}}$  to any boundary saddle connection. Let  $(k_1, \dots, k_s)$  be the order of the interior conical singularities of  $S_{i_0}$  and  $(l_1, \dots, l_{s'})$  be the orders of the boundary singularities. Let  $X$  be the closed flat surface obtained by gluing  $S_{i_0}$  and a copy of itself taken with opposite orientation along their boundary. It's easy to see that  $g_X = 2g_{i_0} + r - 1$ , where  $r$  the number of connected components of the boundary of  $S_{i_0}$  and  $g_X$  is the genus of  $X$ . The singularities of  $X$  are of order  $(k_1, \dots, k_s, k_1, \dots, k_s, 2l_1, \dots, 2l_{s'})$ . Furthermore,  $k_i, l_j$  are nonnegative integers since  $X$  has trivial holonomy. Using the Gauss-Bonnet formula for quadratic differentials, we have:

$$g_X = 1 + \sum_{j=1}^s \frac{k_j}{2} + \sum_{i=1}^{s'} \frac{l_i}{2} = 2g_{i_0} + r - 1$$

And therefore

$$2g_{i_0} \geq 2 - r + \sum_{i=1}^{s'} \frac{l_i}{2}$$

To conclude, we need few elementary remarks (which are already written in [MZ]) about the order of the conical singularities of the boundary:

- a) If a connected component of the boundary is just a single saddle connection, then the corresponding angle cannot be  $\pi$  because the saddle connection would then be a boundary of a cylinder. Then the other boundary of the cylinder would be a saddle connection  $\hat{h}$  homologous to the previous one. So  $S_i$  would be that cylinder contradicting the hypothesis. Furthermore, the holonomy of a path homotopic to the saddle connection is trivial if and only if the conical angle of the boundary singularity is an odd multiple of  $\pi$ . Therefore that angle is greater or equal to  $3\pi$ , and hence, the order  $l_1$  of the boundary singularity has order at least 2.
- b) If a connected component of the boundary is given by two saddle connections, then as before, the two corresponding conical angles cannot be both equal to  $\pi$  (because  $S_i$  is not a cylinder) and are of same parity (because  $S_i$  has trivial holonomy).

The case  $r = 1$  is trivial. If  $r = 2$  then there is a connected component of the boundary of  $S_{i_0}$  with one or two saddle connections (recall that  $S_{i_0}$  is of valence at most four). In both cases, the previous remarks imply that  $S_{i_0}$  admits a boundary singularity of order  $l_1 > 0$ , and therefore  $2g_{i_0} \geq l_1 > 0$ .

If  $r \in \{3, 4\}$ , then there are at least two boundary components that consists of a single saddle connection. From remark a), this implies that  $S_{i_0}$  admits two boundary singularities  $l_1$  and  $l_2$  of order greater than or equal to two. Applying remarks a) and b) on the other boundary components, we show that  $S_{i_0}$  admits at least an other boundary singularity of order  $l_3 > 0$ . Therefore  $2g_{i_0} > 2 - r + l_1/2 + l_2/2 \geq 4 - r$ . So  $g_{i_0} > 0$ . □

Now, we find all the configurations when the genus  $g$  of the surface is zero. To avoid confusion of notation, we specify the following convention: in the next theorem, we denote by  $\{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}\}$  the set with multiplicity  $\{k_1, k_1, \dots, k_r\}$ , where  $\alpha_i$  is the multiplicity of  $k_i$ . We assume that  $k_i \neq k_j$  for  $i \neq j$ . For example the notation  $\mathcal{Q}(1^2, -1^6)$  means  $\mathcal{Q}(1, 1, -1, -1, -1, -1, -1, -1)$ .

**Theorem 2.3.** *Let  $\mathcal{Q}(k_1^{\alpha_1}, \dots, k_r^{\alpha_r}, -1^s)$  be a stratum of quadratic differentials on  $\mathbb{CP}^1$  different from  $\mathcal{Q}(-1^4)$ , and let  $\gamma$  be a maximal collection of  $\hat{h}$  homologous saddle connections on a surface in that stratum.*

Then the possible configurations for  $\gamma$  are given in the list below (see also table 1).

- a) Let  $\{k, k'\} \subset \{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}, -1^s\}$  be an unordered pair of integers with  $(k, k') \neq (-1, -1)$ . The set  $\gamma$  consists of a single saddle connection joining a singularity of order  $k$  to a distinct singularity of order  $k'$ .
- b) Consider  $(a_1, a_2)$  a pair of strictly positive integers such that  $a_1 + a_2 = k \in \{k_1, \dots, k_r\}$  (with  $k \neq 1$ ), and a partition  $A_1 \sqcup A_2$  of  $\{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}\} \setminus \{k\}$ . The set  $\gamma$  consists of a simple saddle connection that decomposes the sphere into two one-holed spheres  $S_1$  and  $S_2$ , such that each  $S_i$  has interior singularities of positive order given by  $A_i$  and  $s_i = (\sum_{a \in A_i} a) + a_i + 2$  poles, and has a single boundary singularity of order  $a_i$ .
- c) Consider  $\{a_1, a_2\} \subset \{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}\}$  a pair of integers. Let  $A_1 \sqcup A_2$  be a partition of  $\{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}\} \setminus \{a_1, a_2\}$ . The set  $\gamma$  consists of two closed saddle connections that decompose the sphere into two one-holed spheres  $S_1$  and  $S_2$  and a cylinder, and such that each  $S_i$  has interior singularities of positive orders given by  $A_i$  and  $s_i = (\sum_{a \in A_i} a) + a_i + 2$  poles and has a boundary singularity of order  $a_i$ .
- d) Let  $k \in \{k_1, \dots, k_r\}$ . The set  $\gamma$  is a pair of saddle connections of different lengths, and such that the largest one starts and ends from a singularity of order  $k$  and decompose the surface into a one-holed sphere and a half-pillowcase, while the shortest one joins a pair of poles and lies on the other end of the half-pillowcase.

*Proof.* The existence of each of these configurations is a direct consequence of the Main Theorem of [MZ].

It follows from the previous lemmas that  $\Gamma(S, \gamma)$  has no “+” components. Furthermore, a loop of the graph  $\Gamma(S, \gamma)$  cannot have any cylinders (this would add a handle to the surface). Now using the description from [MZ] of admissible graphs (see figure 1), we can list all possible graphs.

a) A single “−” vertex of valence two and an edge joining it to itself. This can represent two possible cases: either the boundary of the closure of  $S \setminus \gamma$  has two connected components, or it has only one. In the first case each connected component of the boundary is a single saddle connection. But gluing these two boundary components together adds a handle to the surface, so, this case doesn’t appear.

In the other case, the single boundary component consists of two saddle connections, so  $\gamma$  consists of a single saddle connection joining a

	Tological picture	Configuration
a)		
b)		
c)		
d)		

On $\mathcal{Q}(-1^4)$		
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TABLE 1. Configurations in genus zero

singularity of order  $k$  to a distinct singularity of order  $k'$ . Note that  $k$  and  $k'$  cannot be both equal to  $-1$  otherwise there would be another saddle connection in the collection  $\gamma$ .

b) Two “-” vertices of valence one joined by a single edge. That means that  $\gamma$  consists of a single closed saddle connection  $\gamma_1$  which separates the surface in two part. We get a pair  $(S_1, S_2)$  of one-holed

spheres with boundary singularities of angles  $(a_1 + 1)\pi$  and  $(a_2 + 1)\pi$  correspondingly. The saddle connection of the initial surface is adjacent to a singularity of order  $a_1 + a_2 = k$ . None of the  $a_i$  can be equal to 0 otherwise the saddle connection would bound a cylinder, and there would exist a saddle connection  $\hat{\gamma}_1$  homologous to  $\gamma_1$  on the other end of that cylinder (see remark 1.4).

Now considering the interior singularities of positive order of  $S_1$  and  $S_2$  respectively, this defines a partition  $A_1 \sqcup A_2$  of  $\{k_1^{\alpha_1}, \dots, k_r^{\alpha_r}\} \setminus \{k\}$ . Each  $S_i$  also has  $s_i$  poles, with  $s_1 + s_2 = s$ . If we decompose the boundary saddle connection of  $S_i$  in two segments starting from the boundary singularity, and glue together these two segments, then we get a closed flat surface with  $A_i \sqcup \{a_i - 1, -1\} \sqcup \{-1^{s_i}\}$  as order of singularities. The Gauss-Bonnet theorem implies that:

$$\left(\sum_{a \in A_i} a\right) + a_i - 2 - s_i = -4$$

c) Two “-” vertex of valence one and a “o” vertex of valence 2. This case is analogous to the previous one.

d) A “-” vertex of valence one, joined by an edge to a valence three “o” vertex and an edge joining the “o” vertex to itself.

The “-” vertex represents a one-holed sphere. It has a single boundary component which is a closed saddle connection. The cylinder has two boundary components of equal lengths. One has two saddle connections of length 1 (after normalization) the other component has a single saddle connection of length 2. So, the only possible configuration is obtained by gluing the two saddle connections of length 1 together (creating a “half-pillowcase”) and gluing the other one with the boundary of the “-” component. The boundary singularity of the “-” component has an angle of  $(k + 2 - 1)\pi$  (equivalently, has order  $k$ ) for some  $k \in \{k_1, \dots, k_r\}$ .

e) A valence four “o” vertex with two edges joining the vertex to itself. The cylinder has two boundary components, each of them is composed of two saddle connections. All the saddle connections have the same length. If we glue a saddle connection with one of the other connected component of the boundary, we get a flat torus, which has trivial holonomy. So, we have to glue each saddle connection with the other saddle connection of its boundary component. That means that we get a (twisted) “pillowcase” and the surface lies in  $\mathcal{Q}(-1, -1, -1, -1)$

In each of the first four cases, the surface necessarily has a singularity of order at least 1. So, they cannot appear in  $\mathcal{Q}(-1, -1, -1, -1)$ , which means that the fifth case is the only possibility in that stratum.  $\square$



### 3. CONFIGURATIONS FOR HYPERELLIPTIC CONNECTED COMPONENTS

In this section, we describe the configurations of  $\hat{\text{homologous}}$  saddle connections in a hyperelliptic connected component. We first reformulate Lanneau's description of such components, see [L1].

**Theorem** (E. Lanneau). *The hyperelliptic connected components are given by the following list:*

- (1) *The subset of surfaces in  $\mathcal{Q}(k_1, k_1, k_2, k_2)$ , that are a double covering of a surface in  $\mathcal{Q}(k_1, k_2, -1^s)$  ramified over  $s$  poles. Here  $k_1$  and  $k_2$  are odd,  $k_1 \geq -1$  and  $k_2 \geq 1$ , and  $k_1 + k_2 - s = -4$ .*
- (2) *The subset of surfaces in  $\mathcal{Q}(k_1, k_1, 2k_2 + 2)$ , that are a double covering of a surface in  $\mathcal{Q}(k_1, k_2, -1^s)$  ramified over  $s$  poles and over the singularity of order  $k_2$ . Here  $k_1$  is odd and  $k_2$  is even,  $k_1 \geq -1$  and  $k_2 \geq 0$ , and  $k_1 + k_2 - s = -4$ .*
- (3) *The subset of surfaces in  $\mathcal{Q}(2k_1 + 2, 2k_2 + 2)$ , that are a double covering of a surface in  $\mathcal{Q}(k_1, k_2, -1^s)$  ramified over all the singularities. Here  $k_1$  and  $k_2$  are even,  $k_1 \geq 0$  and  $k_2 \geq 0$ , and  $k_1 + k_2 - s = -4$ .*

We deduce configurations for hyperelliptic components from the configurations for  $\mathbb{CP}^1$ . This leads to the following theorem:

**Theorem 3.1.** *In notations of the classification theorem above, the admissible configurations of  $\hat{\text{homologous}}$  saddle connections for hyperelliptic connected components are given by tables 2, 3, 4 and 5. No other configuration can appear.*

*Remark 3.2.* Integer parameters  $k_1, k_2 \geq -1$  in tables 2, 3, 4 are allowed to take values  $-1$  and  $0$  as soon as this does not contradict explicit restrictions. In table 5, we list several additional configurations which appear only when at least one of  $k_1, k_2$  is equal to zero.

*Remark 3.3.* In the description of configurations for  $\mathcal{Q}_{hyp}(k_1, k_1, k_2, k_2)$  with  $k_1 = k_2$ , the notation  $k_i, k_i$  (*resp.*  $k_j, k_j$ ) still represents the orders of a pair of singularities that are *interchanged* by the hyperelliptic involution. For example in a generic surface in the hyperelliptic component  $\mathcal{Q}_{hyp}(k, k, k, k)$ , the second line of table 3 means that, between any pair of singularities that are interchanged by the hyperelliptic involution, there exists a saddle connection with no other saddle connections  $\hat{\text{homologous}}$  to it. But if  $\gamma$  is a saddle connection between two singularities that are not interchanged by the involution  $\tau$ , then  $\tau(\gamma)$  is an other saddle connection  $\hat{\text{homologous}}$  to  $\gamma$  (see below).

$\mathcal{Q}(k_1, k_2, -1^s)$ ( $\mathbb{CP}^1$ )	$\mathcal{Q}_{hyp}(k_1, k_1, 2k_2 + 2)$	$k_1$ odd $k_2$ even
$k_1 = a_1 + a_2$ $a_1, a_2 \geq 1$ 	$a_1$ odd, $a_2$ even 	$a_1$ even, $a_2$ odd 
$k_2 = a_1 + a_2$ $a_1, a_2 \geq 1$ 	$a_1, a_2$ odd 	$a_1, a_2$ even 
$k_1, k_2 \geq 1$ 		

TABLE 2. Configurations for  $\mathcal{Q}_{hyp}(k_1, k_1, 2k_2 + 2)$

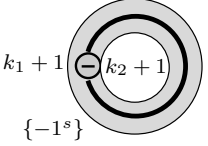
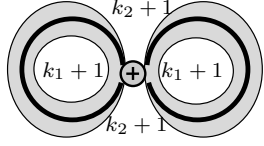
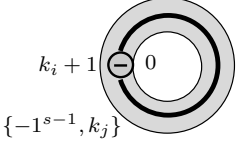
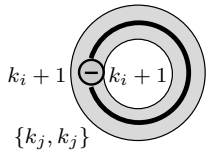
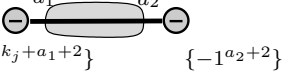
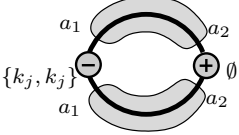
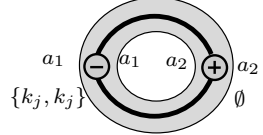
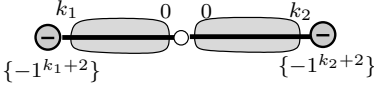
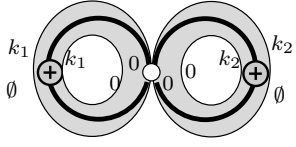
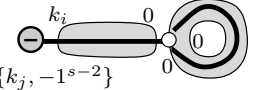
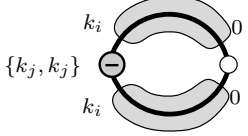
$\mathcal{Q}(k_1, k_2, -1^s)$ ( $\mathbb{CP}^1$ )	$\mathcal{Q}_{hyp}(k_1, k_1, k_2, k_2)$	$k_1, k_2$ odd $(k_1, k_2) \neq (-1, -1)$	
			
		$k_i \neq -1$	
			$k_i = a_1 + a_2$ $a_1, a_2 \geq 1$
		$k_1, k_2 \geq 1$	
		$k_i \geq 1$	

TABLE 3. Configurations for  $\mathcal{Q}_{hyp}(k_1, k_1, k_2, k_2)$ 

*Proof.* Let  $\mathcal{Q}_{hyp}$  be a hyperelliptic connected component from the list above and  $\mathcal{Q} = \mathcal{Q}(k_1, k_2, -1^s)$  the corresponding stratum on  $\mathbb{CP}^1$ . The projection  $p : \tilde{S} \rightarrow \tilde{S}/\tau = S$ , where  $\tilde{S} \in \mathcal{Q}_{hyp}$  and  $\tau$  is the corresponding hyperelliptic involution, induces a covering from  $\mathcal{Q}_{hyp}$  to  $\mathcal{Q}$ . This is not necessarily a one-to-one map because there might be a choice of the ramification points on  $\mathbb{CP}^1$ . But if we fix the ramification points, there is a locally one-to-one correspondence, and the ratio of lengths of a pair of saddle connections in  $\tilde{S} \in \mathcal{Q}_{hyp}$  is constant under any small deformation of  $\tilde{S}$  in the stratum  $\mathcal{Q}_{hyp}$  if and only if the ratio of the

$Q(k_1, k_2, -1^s)$ ( $\mathbb{C}P^1$ )	$Q_{hyp}(2k_1 + 2, 2k_2 + 2)$	$k_1$ even $k_2$ even
$k_i = a_1 + a_2$ $a_1, a_2 \geq 1$	$a_1, a_2$ odd	$a_1, a_2$ even
$k_1, k_2 \geq 1$		
$k_i \geq 1$		

TABLE 4. Configurations for  $Q_{hyp}(2k_1 + 2, 2k_2 + 2)$

corresponding pair in  $S$  is constant under any small deformation in  $Q$ . Hence by theorem of Masur and Zorich cited after definition 1.1, the image by  $p$  of a maximal collection  $\tilde{\gamma}$  of homologous saddle connections on  $\hat{S}$  is a collection  $\gamma$  of homologous saddle connections on  $S$ . Note that  $\gamma$  is not necessary maximal since the preimage of a pole by  $p$  is a marked point on  $\tilde{S}$  and we do not consider saddle connections starting

from a marked point. However, we can deduce *all* configurations on  $Q_{hyp}$  from the list of configurations of  $Q$ .

We give details for a few configurations, the other ones are similar and the proofs are left to the reader.

-*First line of table 2:* the configuration for  $Q = \mathcal{Q}(k_1, k_2, -1^s)$  corresponds to a single saddle connection  $\gamma$  on a surface  $S$  that joins a singularity  $P_1$  of degree  $k_1$  to the distinct singularity  $P_2$  of degree  $k_2$ . The double covering is ramified over  $P_2$  but not over  $P_1$ . Therefore, the preimage of  $\gamma$  in  $\tilde{S}$  is a pair  $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$  of saddle connections of the same length that join each preimage of  $P_1$  to the preimage of  $P_2$ . The boundary of compactification of  $\tilde{S} \setminus \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$  admits only one connected component that consists of four saddle connections. The angles of the boundary singularities corresponding to the preimages of  $P_1$  are both  $(k_1+2)\pi$ , and the angles of the other boundary singularities are  $(k_2+2)\pi$  since  $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$  are interchanged by the hyperelliptic involution.

-*Fourth line of table 2:* the configuration for  $Q = \mathcal{Q}(k_1, k_2, -1^s)$  corresponds to a single closed saddle connection  $\gamma$  on a flat surface  $S$  that separates the surface into two parts  $S_1$  and  $S_2$ . Each  $S_i$  contains some ramification points, so the preimage of  $\gamma$  separates  $\tilde{S}$  into two parts  $\tilde{S}_1$  and  $\tilde{S}_2$  that are double covers of  $S_1$  and  $S_2$ . One of the  $\tilde{S}_i$  has an interior singularity of order  $2k_2 + 2$ , while the other one does not have interior singularities. The description from Masur and Zorich of possible graphs of connected components (see figure 1) implies that  $\tilde{S}_1$  and  $\tilde{S}_2$  cannot have the same holonomy. Let  $\tilde{S}_2$  be the component with trivial holonomy, and choose  $\omega$  to be a square root of the quadratic differential that defines its flat structure. If  $\tilde{S}_2$  has two boundary components, each consisting of a single saddle connection, then the corresponding boundary singularity must be of even order  $a_2$ . If  $\tilde{S}_2$  has a single boundary component, then integrating  $\omega$  along that boundary must give zero ( $\omega$  is closed), which is only possible if the order  $a_2$  of the boundary singularities are odd. Applying lemma 3.4 below we see that  $\tilde{S}_2$  does not have interior singularity. Hence,  $\tilde{S}_1$  has an interior singularity of order  $2k_2 + 2$ . The order of the boundary singularities of  $\tilde{S}_1$  are  $a_1 = k_1 - a_2$ , which is of parity opposite to the one of  $a_2$ . Applying again lemma 3.4, we get the number of boundary components of  $\tilde{S}_1$ .

-*Last line of table 2:* the configuration for  $Q = \mathcal{Q}(k_1, k_2, -1^s)$  corresponds to a pair of saddle connections on a surface  $S \in Q$  that separate the surface into a cylinder  $C$  and a one-holed sphere  $S_1$ . The double cover  $\tilde{S}_1$  of  $S_1$  is connected, and applying lemma 3.4 we see that it has two boundary components. The double cover  $\tilde{C}$  of the cylinder  $C$

admits no ramification points. So *a priori*, there are two possibilities:  $\tilde{C}$  is either a cylinder with the same length and a width twice bigger as the width of  $C$ , or is a pair of copies of  $C$ . Here, the first possibility is not realisable otherwise the double covering  $\tilde{S} \rightarrow S$  would be necessary ramified over  $k_1$ . Finally we get  $\tilde{S}$  by gluing a boundary component of each cylinder to each boundary component of  $\tilde{S}_1$ , and glue together the remaining boundary components of the cylinders.

Note that in that case, the preimage of the saddle connection joining a pair of poles on  $S$  is a *regular* closed geodesic in  $\tilde{S}$ , and hence, we do not consider it in the collection of  $\hat{\text{homologous}}$  saddle connections on  $\tilde{S}$ .

When at least one of  $k_1$  or  $k_2$  equals zero, there is a marked point on  $\mathbb{CP}^1$  that is a ramification point of the double covering. Hence we have to start from a configuration of saddle connections on  $\mathbb{CP}^1$  that might have marked points as end points:

- If a maximal collection of  $\hat{\text{homologous}}$  saddle connection does not intersect a marked point, then the collection has already been described in theorem 2.3, and hence, the corresponding configuration in  $Q_{hyp}$  is already presented in tables 2 and 4.
- If a nonclosed saddle connection admits a marked point as end point, then this saddle connection is simple since we can move freely that saddle connection. Hence the corresponding configuration in  $Q_{hyp}$  is already written in tables 2 and 4.
- If a closed saddle connection admits a marked point as end point, then it is a closed geodesic. This corresponds to a new configuration on  $\mathbb{CP}^1$  and the corresponding configuration in  $Q_{hyp}$  is described in table 5. The proof is analogous to the other cases.

This concludes the proof of theorem 3.1. □

**Lemma 3.4.** *Let  $S_i$  be a flat surface whose boundary consists of a single closed saddle connection and let  $a > 0$  be the order of the corresponding boundary singularity. Let  $\tilde{S}_i$  be a connected ramified double cover of the interior of  $S_i$  and let  $(\tilde{k}_1, \dots, \tilde{k}_l)$  be interior singularities. The sum  $\sum_i \tilde{k}_i$  is even and:*

- *If  $\frac{\sum_i \tilde{k}_i}{2} + a$  is even, then the compactification of  $\tilde{S}_i$  has two boundary components, each of them consists of a single saddle connection, with corresponding boundary singularity of order  $a$ .*

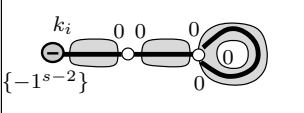
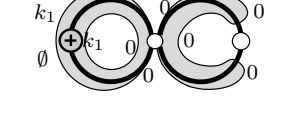
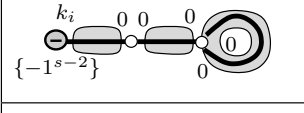
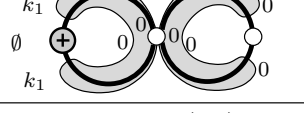
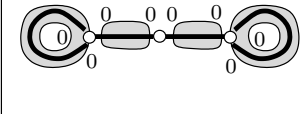
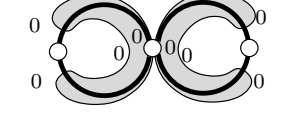
$\mathcal{Q}(k_i, 0, -1^s)$	$\mathcal{Q}_{hyp}$
$k_i \geq 1, k_i \text{ odd}$ 	$\mathcal{Q}_{hyp} = \mathcal{Q}(k_i, k_i, 2)$ 
$k_i \geq 1, k_i \text{ even}$ 	$\mathcal{Q}_{hyp} = \mathcal{Q}(2k_i + 2, 2), k_i > 0$ 
$k_1 = k_2 = 0$ 	$\mathcal{Q}_{hyp} = \mathcal{Q}(2, 2)$ 

TABLE 5. Additionnal configurations which appears when at least one of  $k_1$  or  $k_2$  equals 0.

- If  $\frac{\sum_i \tilde{k}_i}{2} + a$  is odd, then the compactification of  $\tilde{S}_i$  has a single boundary component which consists of a pair of saddle connections of equal lengths, with corresponding boundary singularities of order  $a$ .

*Proof.* By construction, the boundary of the compactification of  $\tilde{S}_i$  necessary consists of two saddle connections of equal length. It has one or two connected components.

Now we claim that  $\sum_i \tilde{k}_i + 2a \equiv 2r \pmod{4}$ , with  $r$  the number of connected components of the boundary of  $\tilde{S}_i$ . This equality (that already appears in [MZ]) clearly implies our lemma. The proof is the following: consider as in lemma 2.2 the surface  $\tilde{X}$  of genus  $g_{\tilde{X}}$  obtained by gluing  $\tilde{S}_i$  and a copy of itself with opposite orientation along their boundary. The order of the singularities of  $\tilde{X}$  are  $\{\tilde{k}_1, \dots, \tilde{k}_l, \tilde{k}_1, \dots, \tilde{k}_l, 2a, 2a\}$ , so we get  $4g_{\tilde{X}} - 4 = 2\sum_i \tilde{k}_i + 4a = 4(2\tilde{g}_i + r - 1) - 4$ , and therefore  $\sum_i \tilde{k}_i + 2a = 4g_i - 4 + 2r \equiv 2r \pmod{4}$ .  $\square$

Given a concrete flat surface, we do not necessary see at once whether it belongs or not to a hyperelliptic connected component. Indeed, there exists hyperelliptic flat surfaces that are not in a hyperelliptic connected component. As a direct corrolary of theorem 3.1, we have the following quick test.

**Corollary 3.5.** *Let  $S$  be a flat surface with non-trivial holonomy and  $\gamma$  be a collection of  $\hat{h}$ omologous saddle connections on  $S$ . The surface  $S$  does not belong to a hyperelliptic connected component if one of the following holds:*

- $S \setminus \gamma$  admits three connected components and neither of them is a cylinder.
- $S \setminus \gamma$  admits four connected components or more.

#### 4. CONFIGURATIONS FOR NON-HYPERELLIPTIC CONNECTED COMPONENTS

Using [MZ], one can get a list of all configurations of  $\hat{h}$ omologous saddle connections realizable for a stratum, but it is not clear which configuration realizes in which component. In the previous section we have described configurations for hyperelliptic components.

Now we show that *any* configuration realizable for a stratum is also realizable in its non-hyperelliptic connected component, provided the genus  $g$  is sufficiently large.

We remind the following theorem.

**Theorem** (Kontsevich-Zorich / Lanneau). *The following strata consists entirely of hyperelliptic surfaces and are connected:*

- $\mathcal{H}(1, 1)$  and  $\mathcal{H}(2)$  in the moduli spaces of Abelian differentials.
- $\mathcal{Q}(-1, -1, 1, 1)$ ,  $\mathcal{Q}(-1, -1, 2)$ ,  $\mathcal{Q}(1, 1, 1, 1)$ ,  $\mathcal{Q}(1, 1, 2)$  and  $\mathcal{Q}(2, 2)$  in the moduli spaces of quadratic differentials.

*Any other stratum that contains a hyperelliptic connected component admit at least one other connected component that contains a subset of full measure of flat surfaces that do not admit any isometric involution.*

**Lemma 4.1.** *Let  $\mathcal{Q}$  be a nonconnected stratum that contains a hyperelliptic connected component. If the set of order of singularities defining  $\mathcal{Q}$  contains  $\{k, k\}$ , for some  $k \geq 1$ , then there exists a nonhyperelliptic flat surface in  $\mathcal{Q}$  that has a simple saddle connection joining two different singularities of the same order  $k$ .*

Here we call a saddle connection “simple” when they are no other saddle connections  $\hat{h}$ omologous to it.

*Proof.* According to Masur and Smillie [MS], any stratum is nonempty except the ones that are in the following list:  $\mathcal{Q}(\emptyset)$ ,  $\mathcal{Q}(1, -1)$ ,  $\mathcal{Q}(3, 1)$  and  $\mathcal{Q}(4)$ .

According to Masur and Zorich [MZ], if  $S \in \mathcal{Q}(k_1 + k_2, k_3, \dots, k_r)$ , then there is a path  $(S_t)_{t \in [0, 1]}$ , continuous in the moduli space of quadratic differentials, such that  $S_0 = S$  and  $S_t \in \mathcal{Q}(k_1, k_2, k_3, \dots, k_r)$  for



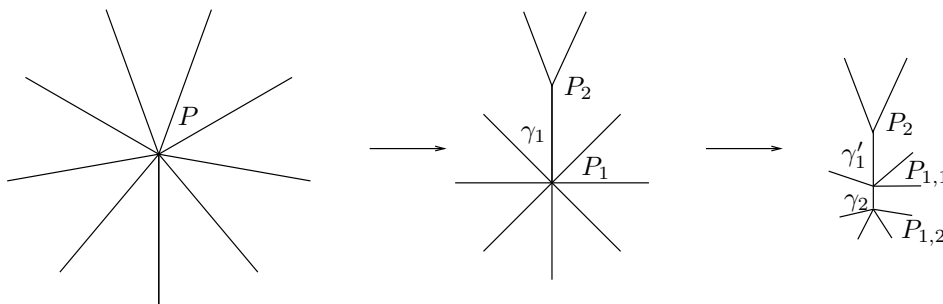


FIGURE 4. Construction of a simple saddle connection in a nonhyperelliptic surface

$t > 0$ , and such that the smallest saddle connection in  $S_t$ , for  $t > 0$  is simple and joins a singularity of order  $k_1$  to a singularity of order  $k_2$ .

We first consider the stratum  $\mathcal{Q} = \mathcal{Q}(k_1, k_1, k_2, k_2)$ . By assumption,  $\mathcal{Q}$  is nonconnected, so, either the genus is greater than 3, or  $k_1 = 3$  and  $k_2 = -1$ . Hence the stratum  $\mathcal{Q}(2k_1 + k_2, k_2)$  is nonempty. Now, we start from a surface  $S_0$  in that stratum, and break the singularity  $P$  of order  $2k_1 + k_2$  into two singularities  $P_1$  and  $P_2$  of orders  $2k_1$  and  $k_2$  respectively. We get a surface  $S_1$  with a short vertical saddle connection  $\gamma_1$  between  $P_1$  and  $P_2$ . Since the singularity breaking procedure is continuous, there are no other short saddle connections on  $S_1$ . Then, we break the singularity  $P_1$  of order  $2k_1$  into a pair of singularities  $P_{1,1}$  and  $P_{1,2}$  of orders  $k_1$ . We get by construction a surface  $S_2$  in the stratum  $\mathcal{Q}$  with a simple saddle connection  $\gamma_2$  between  $P_{1,1}$  and  $P_{1,2}$ , and of length very small compared to the length of  $\gamma_1$ . The fact that the singularity breaking procedure is continuous implies that there lingers a saddle connection  $\gamma'_1$  between  $P_2$  and one of the  $P_{1,i}$  (see figure 4). By construction, we can assume there is no other saddle connection of length  $l(\gamma'_1)$ ,  $2l(\gamma'_1)$  or  $l(\gamma'_1)/2$ , where  $l(\gamma'_1)$  denotes the length of  $\gamma'_1$ . Hence,  $\gamma'_1$  is simple. According to theorem 3.1, this cannot exist in the hyperelliptic connected component since the corresponding configuration is not present in table 3. Thus  $S_2$  belongs to the nonhyperelliptic connected component and we can assume, after a slight perturbation, that  $S_2$  is not hyperelliptic. Since by construction, the saddle connection  $\gamma_2$  is simple and joins two singularities of order  $k = k_1 \geq 1$ , the lemma is proven for the stratum  $\mathcal{Q}(k_1, k_1, k_2, k_2)$ .

The proof for  $\mathcal{Q}(k_1, k_1, 2k_2+2)$  and for  $\mathcal{Q}(2k_1+2, 2k_2+2)$  is analogous: note that this case does not occur for the genera 1 or 2, because all corresponding strata are connected. Therefore the genus is greater than or equal to 3 and the stratum  $\mathcal{Q}(2k_1 + 2k_2 + 2)$  is nonempty.  $\square$

**Proposition 4.2.** *If  $g \geq 5$ , any configuration that appears in a hyperelliptic connected component of a stratum of meromorphic quadratic differentials with at most simple poles also appears in the nonhyperelliptic component of the same stratum.*

*Proof.* Let  $S$  be a flat surface in a hyperelliptic component and  $\gamma$  a maximal collection of  $\hat{\Gamma}$  homologous saddle connections. The hyperelliptic involution  $\tau$  maps  $S \setminus \gamma$  to itself and hence, induces an involution  $\tau^*$  on the set of connected components of  $S \setminus \gamma$ . We claim that  $\tau^*$  does not interchange two connected components  $S_i, S_j$  of  $S \setminus \gamma$ , for otherwise we can continuously deform  $S_i$  outside a neighborhood of its boundary and reconstruct a new flat surface  $S'$  in the same stratum that is not any more hyperelliptic. Therefore, if  $S$  is in a hyperelliptic component, then  $\tau$  must induce an isometric and orientation preserving involution on each connected component of  $S \setminus \gamma$ .

Using the formula for the genus of a compound surface proved in the appendix and the list of configurations for hyperelliptic connected components given in the previous section, we derive the following fact: if  $S$  has genus  $g \geq 5$  and  $\gamma$  is a maximal collection of  $\hat{\Gamma}$  homologous saddle connections, then at least one of the following propositions is true.

- a)  $S \setminus \gamma$  admits a connected component  $S_0$  of genus  $g_0 \geq 3$ , that has a single boundary component and whose corresponding vertex in the graph  $\Gamma(S, \gamma)$  is of valence 2.
- b)  $S \setminus \gamma$  admits a connected component  $S_0$  of genus  $g_0 \geq 2$ , that has exactly two boundary components and whose corresponding vertex in the graph  $\Gamma(S, \gamma)$  is of valence 2.
- c)  $S \setminus \gamma$  is connected and the corresponding vertex in the graph  $\Gamma(S, \gamma)$  is of valence 4.

We conclude the proof by applying lemmas 4.3, 4.4, 4.5 to situations a), b), c) correspondingly.  $\square$

**Lemma 4.3.** *Let  $S$  be a flat surface in a hyperelliptic connected component and  $\gamma$  a maximal collection of  $\hat{\Gamma}$  homologous saddle connections. We assume that  $S \setminus \gamma$  admits a connected component  $S_0$  of genus  $g_0 \geq 3$ , whose corresponding vertex in the graph  $\Gamma(S, \gamma)$  is of valence 2, and such that  $S_0$  has a single boundary component.*

*Then there exists  $(S', \gamma')$  that has the same configuration as  $(S, \gamma)$ , with  $S'$  in the complementary component of the same stratum.*

*Proof.* The boundary components of  $S_0$  consists of two saddle connections of the same length and the corresponding boundary singularities have the same orders  $k \geq 1$ . Identifying together these two boundary saddle connections, we get a hyperelliptic surface  $\overline{S}_0$ . If we continuously

deform this surface, it keeps being hyperelliptic otherwise  $S$  would not be in a hyperelliptic connected component. Hence,  $\overline{S}_0$  belongs to a hyperelliptic component, and the hyperelliptic involution interchange two singularities of order  $k - 1$ .

The genus of  $\overline{S}_0$  is greater than 3, so the corresponding stratum admits an other connected component. Now we start from a closed flat surface  $X$  in that other connected component. According to lemma 4.1, we can choose  $X$  such that it admits a simple saddle connection between the two singularities of order  $k - 1$ . Now we cut  $X$  along that saddle connection and get a surface  $S_1$  that have, after rescaling, the same boundary as  $S_0$ . By construction,  $S_1$  admits no interior saddle connection  $\hat{\text{homologous}}$  to one of its boundary saddle connections. So, we can reconstruct a pair  $(S', \gamma')$  such that  $\gamma'$  has the same configuration as  $\gamma$  in  $S$ .

The surface  $S_1$  admits a nontrivial isometric involution if and only if  $X$  has one. So, we can choose  $X$  such as it admits no nontrivial isometric involution, and therefore the surface  $S'$  is nonhyperelliptic.

This argument also works when  $\overline{S}_0$  is in the stratum  $\mathcal{Q}(3, 3, -1, -1)$  (here  $g_0 = 2$  and  $k = 4$ ). In any other case for  $g_0 \leq 2$ , it is not possible to replace  $S_0$  by a surface  $S_1$  with no involution.  $\square$

**Lemma 4.4.** *Let  $S$  be a flat surface in a hyperelliptic connected component and  $\gamma$  a maximal collection of  $\hat{\text{homologous}}$  saddle connections. We assume that  $S \setminus \gamma$  admits a connected component  $S_0$  of genus  $g_0 \geq 2$ , that has two boundary components, and whose corresponding vertex in the graph  $\Gamma(S, \gamma)$  is of valence 2.*

*Then there exists  $(S', \gamma')$  that has the same configuration as  $(S, \gamma)$ , with  $S'$  in the complementary component of the same stratum.*

*Proof.* Each boundary components of  $S_0$  consists of one saddle connection and the corresponding boundary singularities have the same orders  $k \geq 1$ . Now we start from a closed flat surface  $X$  with the same holonomy as  $S_0$  and whose singularities consists of the interior singularities of  $S_0$  and two singularities  $P_1$  and  $P_2$  of order  $k - 2$ . We can always choose  $X$  such that it admits a saddle connection  $\eta$  between  $P_1$  and  $P_2$ .

Now we construct a pair of holes by removing a parallelogram as in figure 5 and gluing together the two long sides. Note that the holes can be chosen arbitrary small, and therefore, the resulting surface with boundary does not have any interior saddle connection  $\hat{\text{homologous}}$  to one of its boundary components. We denote by  $S_1$  this surface, and up to rescaling, we can assume that  $S_0$  and  $S_1$  have isometric boundaries.

Hence replacing  $S_0$  by  $S_1$  in the decomposition of  $S$ , we get a new pair  $(S', \gamma')$  that have the same configuration as  $(S, \gamma)$ .

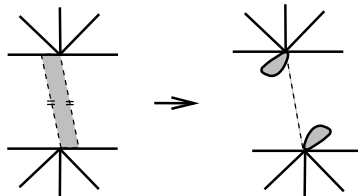


FIGURE 5. Construction of a pair of holes

Now assume  $S_1$  admits a nontrivial isometric involution  $\tau$  that is orientation preserving. Then this involution interchanges the two boundary components of the surface. It is easy to check that we can perform the reverse surgery as the one described previously and we get a closed surface that admits a nontrivial involution. Hence if  $X$  belongs to a stratum that does not consist entirely of hyperelliptic flat surfaces, then we can choose  $X$  such that  $S'$  is not in a hyperelliptic connected component.

Hypothesis on the genus and the theorem from Kontsevich-Zorich and Lanneau implies that this arguments works except when  $X$  belongs to  $\mathcal{H}(1, 1)$ ,  $\mathcal{Q}(2, 1, 1)$ ,  $\mathcal{Q}(1, 1, 1, 1)$ , or  $\mathcal{Q}(2, 2)$ .

We remark that if  $X \in \mathcal{Q}(2, 2)$ , then  $S_0$  has nontrivial linear holonomy and have no interior singularities. But according to the list of configurations for hyperelliptic connected components given in section 3, this cannot happen.

We exhibit in figure 6 three explicit surfaces with boundary that corresponds to the three cases left. We represent these three surfaces as having a one-cylinder decomposition and by describing the identifications on the boundary of that cylinder. The length parameters can be chosen freely under the obvious condition that the sum of the lengths corresponding to the top of the cylinder must be equal to the sum of the lengths corresponding to the bottom of the cylinder. Bold lines represents the boundary of the flat surface. Now we remark that a nontrivial isometric involution must preserve the interior of the cylinder, and must exchange the boundary components. This induces some supplementary relations on the length parameters. Therefore, we can choose them such that there is no nontrivial isometric involution.  $\square$

**Lemma 4.5.** *Let  $S$  be a flat surface of genus  $g \geq 3$  with nontrivial linear holonomy that belongs to a hyperelliptic connected component*

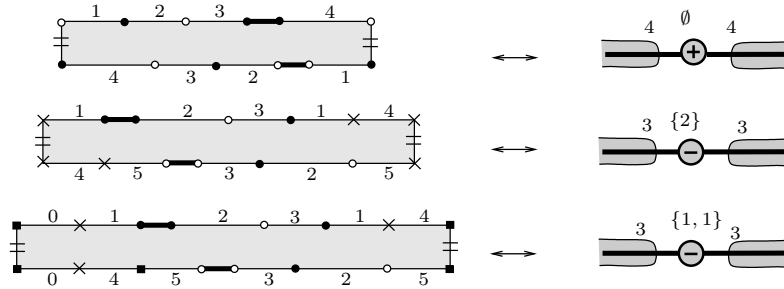


FIGURE 6. Surfaces with two boundary components and no involutions in low genus

and let  $\gamma = \{\gamma_1, \gamma_2\}$  be a maximal collection of  $\hat{h}$  homologous saddle connections on  $S$ . If  $S \setminus \gamma$  is connected, then there exists  $(S', \gamma')$  that has the same configuration as  $(S, \gamma)$ , with  $S'$  in the complementary component of the same stratum.

*Proof.* Since  $S \setminus \gamma$  connected, then the graph  $\Gamma(S, \gamma)$  contains a single vertex, and it has valence four. According to theorem 3.1, two different cases appear:

a) The surface  $\overline{S \setminus \gamma}$  has one boundary component. In this case we start from a surface in  $\mathcal{H}(k_1 + k_2 + 1)$  and perform a local surgery in a neighborhood of the singularity, as described in figure 7 (see also [MZ], section 5). We get a surface and a pair of small saddle connections of length  $\delta$  that have the same configuration as  $\gamma$ . The stratum  $\mathcal{H}(k_1 + k_2 + 1)$  admits nonhyperelliptic components and the same argument as before works: if we start from a generic surface in a nonhyperelliptic component, then the resulting surface after surgery does not have any nontrivial involution.

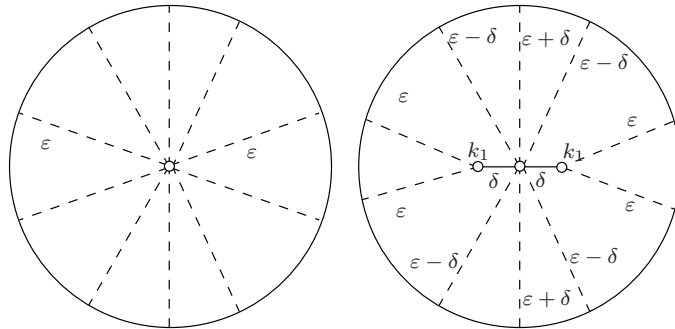


FIGURE 7. Breaking up a zero in three ones

b) The surface  $\overline{S \setminus \gamma}$  has two boundary components, each of them consists of a pair of saddle connections with boundary singularities of order  $k_1 + 1$  and  $k_2 + 1$ . We construct explicit surfaces with the same configuration as  $\gamma$ , but that have no nontrivial involution. Let  $2n = k_1 + k_2 + 2$  and we start from a surface  $S_0$  of genus  $n$  in  $\mathcal{H}(n-1, n-1)$ , that have a one-cylinder decomposition and such as identification on the boundary of that cylinder is given by the permutation

$$\begin{pmatrix} 1 & 2 & \dots & 2n \\ 2n & 2n-1 & \dots & 1 \end{pmatrix}$$

when  $n$  is even, and otherwise by the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n-1 & 2n \\ n-1 & n-2 & \dots & 1 & n & 2n-1 & 2n-2 & \dots & n+1 & 2n \end{pmatrix}$$

We assume that  $k_1$  and  $k_2$  are odd and perform a surgery on  $S_0$  to get a surface  $S_1$  with boundary as pictured on figure 8. The surface  $S_1$  admits two boundary components that consist of two saddle connections each and which are represented by the bold segments. Each symbol  $\circ$ ,  $\bullet$ ,  $\square$ ,  $\blacksquare$  represents a different boundary singularity. It is easy to check that the boundary angles corresponding to  $\blacksquare$  and  $\square$  are both  $(k_1 + 2)\pi$  and that the angles corresponding to  $\bullet$  and  $\circ$  are  $(k_2 + 2)\pi$ . Hence after suitable identifications of the boundary of  $S_1$ , we get a surface  $S'$  and a pair of  $\hat{\text{homologous}}$  saddle connections  $\gamma'$  that have the same configuration as  $(S, \gamma)$ . However,  $S'$  does not admit any nontrivial involution if the length parameters are chosen generically. Note that this construction does not work when  $n = 2$ , but according to section 3, and since  $k_1$  and  $k_2$  are odd, we have  $g = n \geq 3$ .

The case  $k_1$  and  $k_2$  even is analogous and left to the reader (note that in this case,  $g = n + 1$ , and the construction works also for  $n = 2$ ).  $\square$

#### APPENDIX A. COMPUTATION OF THE GENUS IN TERMS OF CONFIGURATION

Here we improve lemma 2.1 and give the relation between the genus of a surface and the genera of the connected components of  $S \setminus \gamma$ , where  $\gamma$  is a collection of  $\hat{\text{homologous}}$  saddle connections.

We first remark that this relation depends not only on the graph of connected components, but also on the permutation on each of its vertices (*i.e.* on the ribbon graph). Indeed, consider a pair of  $\hat{\text{homologous}}$  saddle connections that decompose the surface into two connected components  $S_1$  and  $S_2$ . Then either both  $S_1$  and  $S_2$  has only one boundary component, or at least one of them has two boundary components. In

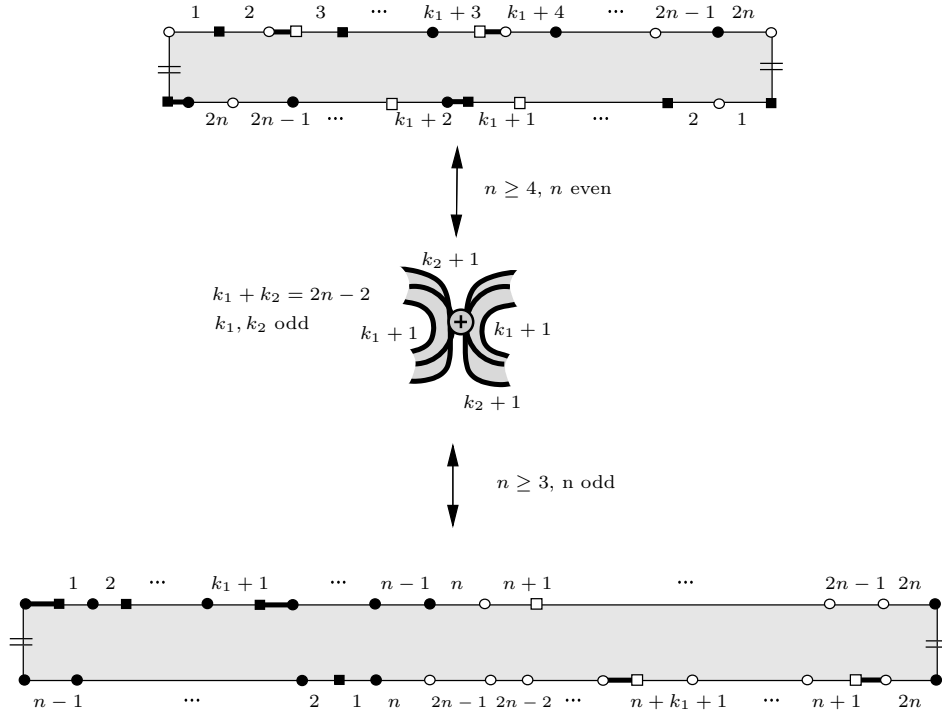


FIGURE 8. Valence four component with no involution

the first case,  $S$  is the connected sum of  $\tilde{S}_1$  and  $\tilde{S}_2$ , so  $g = g_1 + g_2$ , while in the second case, we we have  $g = g_1 + g_2 + 1$ .

**Definition A.1.** Let  $(S, \gamma)$  be a flat surface with a collection of homologous saddle connections. Then the *pure* ribbon graph associated to  $(S, \gamma)$  is the 2-dimensional topological manifold obtained from the ribbon graph by forgetting the graph  $\Gamma(S, \gamma)$ , as in figure 9.

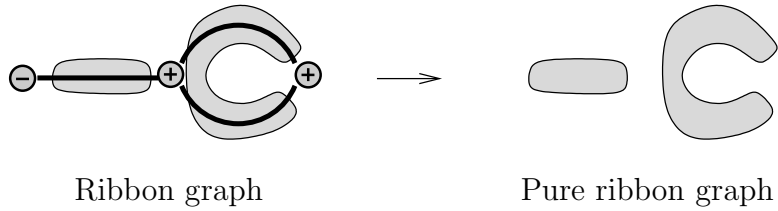


FIGURE 9. Pure ribbon graph

**Proposition A.2.** Let  $\chi_1$  be the Euler characteristic of  $\Gamma(S, \gamma)$  and let  $\chi_2$  be the Euler characteristic of the pure ribbon graph associated to the configuration.

- If the pure ribbon graph has only one connected component and does not embedde into the plane (see figure 10), then

$$g = \left( \sum_i g_i \right) + 1$$

- In any other case,

$$g = \left( \sum_i g_i \right) + (\chi_2 - n) - (\chi_1 - 1)$$

*Remark A.3.* Simply connected components of the pure ribbon graph do not contribute to the term  $(n - \chi_2)$ , since the Euler characteristic of a disc is 1.

Note also that in the first case, we have  $\chi_1 = -1$  and  $\chi_2 = -1$ , and therefore  $(\sum_i g_i) + 1 \neq (\sum_i g_i) + (\chi_2 - n) - (\chi_1 - 1)$ .

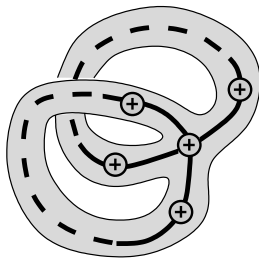


FIGURE 10. Example of a ribbon graph that does not embedde into  $\mathbb{R}^2$ .

*Proof.* Here we do not assume that the collection  $\gamma$  is necessary maximal. When  $\Gamma(S, \gamma)$  has a single vertex, then we prove the proposition using direct computation and the description of the boundary components corresponding to each possible ribbon graph. We refer to [MZ] for this description. Then our goal is to reduce ourselves to that case by removing successively from the collection  $\gamma = \{\gamma_1, \dots, \gamma_k\}$  some  $\gamma_i$  whose corresponding edges joins a vertex to a distinct one.

We define a new graph  $G(S, \gamma)$ , which is a deformation retract of the pure ribbon graph: the vertices of  $G(S, \gamma)$  are the boundary components of each  $S_i$ , while the edges correspond to the saddle connections in  $\gamma$  (see figure 11). For each vertex, there is a cyclic order on the set of edges adjacent to the vertex consistent with the orientation of the plane. If the initial pure ribbon graph does not embedde into the plane, then it is also the case for  $G(S, \gamma)$ . By construction, the Euler characteristic of  $G(S, \gamma)$  is the same as the pure ribbon graph associated to  $(S, \gamma)$ , and is easier to compute.



Let  $\Gamma(S, \gamma)$  contain at least two vertices. Choose a saddle connection representing an edge joining two distinct vertices of  $\Gamma(S, \gamma)$ , and up to renumeration, we can assume that this saddle connection is  $\gamma_1$ . Let us study the resulting configuration of  $\gamma' = \gamma \setminus \{\gamma_1\}$ . The saddle connection  $\gamma_1$  is on the boundary of two surfaces  $S_1$  and  $S_2$ . Then the connected components of  $S \setminus \gamma'$  are the same as the connected component of  $S \setminus \gamma$  except that the surfaces  $S_1$  and  $S_2$  are now glued along  $\gamma_1$ , and hence define a single surface  $S_{1,2}$ . The genus of  $S_{1,2}$  (after gluing disks on its boundary) is  $g_1 + g_2$ .

The graph  $G(S, \gamma')$  is obtained from  $G(S, \gamma)$  by shrinking an edge that joins two different vertices, so these two graphs have the same Euler characteristic  $\chi_1$ .

Furthermore, if  $\gamma_1$  was in a boundary component of  $S_1$  (resp.  $S_2$ ) defined by the ordered collection  $(\gamma_1, \gamma_{i_1}, \dots, \gamma_{i_s})$  (resp.  $(\gamma_1, \gamma_{j_1}, \dots, \gamma_{j_t})$ ). Then the cyclic order in the corresponding boundary component of  $S_{1,2}$  is defined by  $(\gamma_{i_1}, \dots, \gamma_{i_s}, \gamma_{j_1}, \dots, \gamma_{j_t})$ . Therefore  $G(S, \gamma')$  is obtained from  $G(S, \gamma)$  by shrinking the edge corresponding to  $\gamma_1$  and removing a isolated vertex that might appear (see figure 11). It is clear that the difference  $(\chi_2 - n)$  between the Euler characteristic of  $G(S, \gamma)$  and its number of connected component is constant under this procedure. One can also remark that if  $G(S, \gamma)$  is connected and does not embedde into the plane (case 1 of the proposition), then this is also true for  $G(S, \gamma')$ .

Forgetting successively these  $\gamma_i$  will lead to the case when  $\Gamma(S, \gamma)$  has a single vertex. At each steps of the removing procedure, the numbers  $\chi_1$  and  $\chi_2 - n$  do not change, and the sum of the genera associated to the vertices does not change either. This concludes the proof.  $\square$

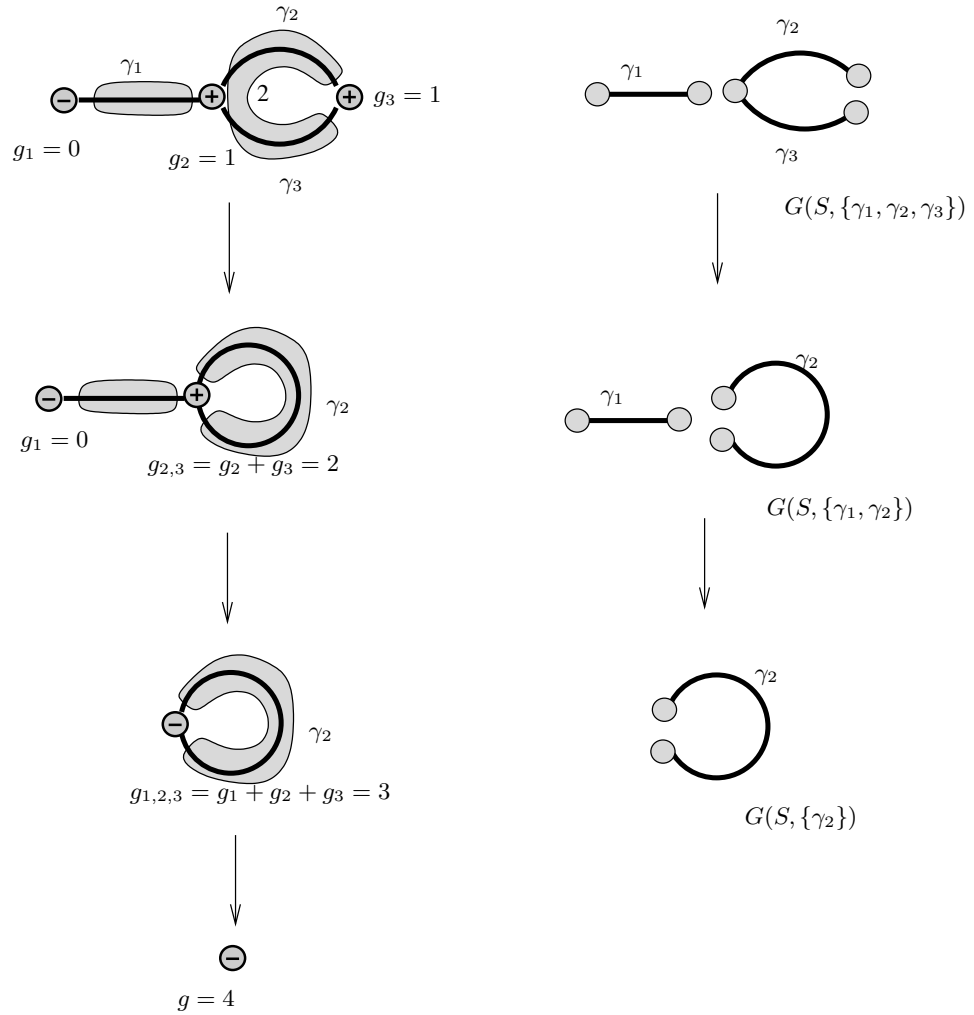


FIGURE 11. Removing successively some elements of a collection  $(\gamma_1, \gamma_2, \gamma_3)$ .

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