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Jean-Yves Charbonnel, Anne Moreau. Nilpotent bicone and characteristic submodule of a reductive Lie algebra. 2008. hal-00147409v2

HAL Id: hal-00147409

<https://hal.science/hal-00147409v2>

Preprint submitted on 19 Jan 2008 (v2), last revised 2 Dec 2014 (v5)

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NILPOTENT BICONE AND CHARACTERISTIC SUBMODULE OF A REDUCTIVE LIE ALGEBRA.

by

Jean-Yves Charbonnel & Anne Moreau

Abstract. — Let \mathfrak{g} be a finite dimensional complex reductive Lie algebra and $S(\mathfrak{g})$ its symmetric algebra. The *nilpotent bicone* of \mathfrak{g} is the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that the subspace generated by x and y is contained in the nilpotent cone. The nilpotent bicone is naturally endowed with a scheme structure, as nullvariety of the augmentation ideal of the subalgebra of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ generated by the 2-order polarizations of invariants of $S(\mathfrak{g})$. The main result of this note is that the nilpotent bicone is a complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$, where $\mathfrak{b}_{\mathfrak{g}}$ and $\text{rk } \mathfrak{g}$ are the dimension of Borel subalgebras and the rank of \mathfrak{g} respectively. This answers a conjecture of Kraft-Wallach [21] concerning the nullcone affirmatively. In addition, we introduce and study in this note the *characteristic submodule* of \mathfrak{g} . The properties of the nilpotent bicone and the characteristic submodule are known to be very important for the understanding of the commuting variety and its ideal of definition. The main difficulty encountered for this work is that the nilpotent bicone is not reduced. To deal with this problem, we introduce an auxiliary reduced variety, the *principal bicone*. The nilpotent bicone, as well as the principal bicone, are linked to jet schemes. We study their dimensions using arguments from motivic integration. Namely, we follow methods developed by M. Mustața in [25]. At last, we give applications of our results to invariant theory.

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2000 Mathematics Subject Classification. — 17B20, 14B05, 14E15.

Key words and phrases. — nilpotent cone, nilpotent bicone, polarizations, nullcone, rational singularities, jet scheme, motivic integration.

1. Introduction

1.1. — Let \mathfrak{g} be a finite dimensional complex reductive Lie algebra, let G be its adjoint group and let $\mathfrak{N}_{\mathfrak{g}}$ be its nilpotent cone. For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, we denote by $P_{x,y}$ the subspace generated by x and y . The *nilpotent bicone* $\mathcal{N}_{\mathfrak{g}}$ of \mathfrak{g} is the subset

$$\mathcal{N}_{\mathfrak{g}} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \subset \mathfrak{N}_{\mathfrak{g}}\} .$$

A subset of $\mathfrak{g} \times \mathfrak{g}$ is called *bicone* if it is stable under the maps $(x, y) \mapsto (sx, ty)$ where s and t are in $\mathbb{C} \setminus \{0\}$. Since $\mathfrak{N}_{\mathfrak{g}}$ is a G -invariant closed cone of \mathfrak{g} , the subset $\mathcal{N}_{\mathfrak{g}}$ is a closed bicone of $\mathfrak{g} \times \mathfrak{g}$, invariant under the diagonal action of G on $\mathfrak{g} \times \mathfrak{g}$.

Let $S(\mathfrak{g})$ be the symmetric algebra of \mathfrak{g} and let $S(\mathfrak{g})^{\mathfrak{g}}$ be the subalgebra of G -invariant elements of $S(\mathfrak{g})$. According to a classical result, the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in $\mathrm{rk} \mathfrak{g}$ variables, where $\mathrm{rk} \mathfrak{g}$ is the rank of \mathfrak{g} . Let $p_1, \dots, p_{\mathrm{rk} \mathfrak{g}}$ be homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \dots, d_{\mathrm{rk} \mathfrak{g}}$ respectively. We can suppose that the sequence $d_1, \dots, d_{\mathrm{rk} \mathfrak{g}}$ is nondecreasing. For $i = 1, \dots, \mathrm{rk} \mathfrak{g}$, the *2-order polarizations* $p_{i,m,n}$ of p_i are the unique elements of $(S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}))^{\mathfrak{g}}$ satisfying the following relation:

$$(1) \quad p_i(ax + by) = \sum_{m+n=d_i} a^m b^n p_{i,m,n}(x, y) ,$$

for all (a, b) in \mathbb{C}^2 and (x, y) in $\mathfrak{g} \times \mathfrak{g}$.

The nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety in \mathfrak{g} of the ideal generated by the polynomials $p_1, \dots, p_{\mathrm{rk} \mathfrak{g}}$. Therefore, according to relation (1) and the definition of the nilpotent bicone, $\mathcal{N}_{\mathfrak{g}}$ is the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of the ideal of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ generated by the polynomials $p_{i,m,n}$, for $i = 1, \dots, \mathrm{rk} \mathfrak{g}$ and $m + n = d_i$. Thus, $\mathcal{N}_{\mathfrak{g}}$ is naturally endowed with a scheme structure. From now on, we study $\mathcal{N}_{\mathfrak{g}}$ as the subscheme of $\mathfrak{g} \times \mathfrak{g}$ corresponding to the ideal generated by the polynomials $p_{i,m,n}$, for $i = 1, \dots, \mathrm{rk} \mathfrak{g}$ and $m + n = d_i$.

Let $b_{\mathfrak{g}}$ be the dimension of a Borel subalgebra of \mathfrak{g} . By a classical result [2], we have $d_1 + \dots + d_{\mathrm{rk} \mathfrak{g}} = b_{\mathfrak{g}}$. So $\mathcal{N}_{\mathfrak{g}}$ is the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of $b_{\mathfrak{g}} + \mathrm{rk} \mathfrak{g}$ polynomial functions. As a result, the dimension of any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ is at least $3(b_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$, since $\mathfrak{g} \times \mathfrak{g}$ has dimension $2(2b_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$. The main result of this note is the following theorem (see Theorem 4.1, (iv) and Corollary 4.13, (iv)):

Theorem 1.1. — *The nilpotent bicone is a non-reduced complete intersection in $\mathfrak{g} \times \mathfrak{g}$ of dimension $3(b_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$. Moreover, the images of any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ by the first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} are equal to $\mathfrak{N}_{\mathfrak{g}}$.*

This result answers a conjecture of H. Kraft and N. Wallach [21] concerning the *nullcone* affirmatively (see Theorem 5.3). Clearly, the images of $\mathcal{N}_{\mathfrak{g}}$ by the first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} are equal to $\mathfrak{N}_{\mathfrak{g}}$. Theorem 1.1 specifies that this is true for any irreducible component of $\mathcal{N}_{\mathfrak{g}}$. This statement is a much stronger result.

We introduce in addition in this note the *characteristic submodule of \mathfrak{g}* . It is a sub- $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ -module of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$. The properties of the nilpotent bicone and the characteristic submodule are known to be very important for the understanding of the commuting variety. Recall that the *commuting variety* $\mathcal{C}_{\mathfrak{g}}$ of \mathfrak{g} is the set of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. The commuting variety has been studied for many years. According to a result of R.W. Richardson [30], $\mathcal{C}_{\mathfrak{g}}$ is irreducible. In addition, $\mathcal{C}_{\mathfrak{g}}$ is the nullvariety of the ideal generated by the elements $(x, y) \mapsto \langle v, [x, y] \rangle$, where v runs through \mathfrak{g} . An old unsolved question is to know whether this ideal is prime [22]. In other words, we want to know if this ideal is the ideal of definition of $\mathcal{C}_{\mathfrak{g}}$, since $\mathcal{C}_{\mathfrak{g}}$ is irreducible. The study of the commuting variety and of its ideal of definition is a main motivation for our work.

1.2. — We denote by $\mathfrak{g}_{\text{reg}}$ the subset of regular elements of \mathfrak{g} and we denote by $\Omega_{\mathfrak{g}}$ the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y}$ has dimension 2 and such that $\mathfrak{g}_{\text{reg}}$ contains $P_{x,y} \setminus \{0\}$. The properties of the subset $\Omega_{\mathfrak{g}}$ were studied by A. V. Bolsinov in [1] and recently by D. I. Panyushev and O. Yakimova in [26]. It is an open subset of $\mathfrak{g} \times \mathfrak{g}$ and its intersection with the nilpotent bicone turns out to be the set of smooth points of $\mathcal{N}_{\mathfrak{g}}$ (see Proposition 2.12 and Remark 2.13). The description of this subset is therefore very important for us. We give various properties of the subset $\Omega_{\mathfrak{g}}$ in Section 2. In addition, we introduce in Section 2 the *characteristic submodule $B_{\mathfrak{g}}$ of \mathfrak{g}* (see Definition 2.9). We study it using properties of the open subset $\Omega_{\mathfrak{g}}$. The characteristic submodule was introduced by the first author a few years ago in work on the commuting variety [4]. We prove that $B_{\mathfrak{g}}$ is a free $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ -module of rank $b_{\mathfrak{g}}$, and we provide an explicit basis for $B_{\mathfrak{g}}$ (see Theorem 2.10). This result is not directly useful for the nilpotent bicone but it is of independent interest.

The results obtained in Section 2 concerning $\Omega_{\mathfrak{g}}$ do not provide “enough” smooth points for $\mathcal{N}_{\mathfrak{g}}$ to apply the criterion of Kostant [19] (see Remark 2.13). Actually, the intersection of $\mathcal{N}_{\mathfrak{g}}$ and $\Omega_{\mathfrak{g}}$ turns out to be an empty set in many cases. This observation makes the study of the scheme $\mathcal{N}_{\mathfrak{g}}$ very difficult. In order to deal with this problem, we introduce an auxiliary reduced variety, the *principal bicone* (see Definition 3.6). By definition, the *principal cone* $\mathfrak{X}_{\mathfrak{g}}$ of \mathfrak{g} is the closure of the set of principal semisimple elements (see Definition 3.1). Then, the *principal bicone* $\mathcal{X}_{\mathfrak{g}}$ of \mathfrak{g} is the subset

$$\mathcal{X}_{\mathfrak{g}} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \subset \mathfrak{X}_{\mathfrak{g}}\} .$$

The study of the varieties $\mathfrak{X}_{\mathfrak{g}}$ and $\mathcal{X}_{\mathfrak{g}}$ is the main topic of Section 3.

We observe that $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{X}_{\mathfrak{g}}$ can be identified to subsets of jet schemes of $\mathfrak{N}_{\mathfrak{g}}$ and $\mathfrak{X}_{\mathfrak{g}}$ respectively. In [25], M. Mustața uses the theory of motivic integration, as developed by M. Kontsevich [17], J. Denef and F. Loeser [6], and V. Batyrev, to prove a result concerning the jet schemes of locally complete intersections. In the appendix of [25], D. Eisenbud and E. Frenkel apply this result to the nilpotent cone of a reductive Lie algebra. We study Section 4 the principal bicone and the nilpotent bicone using arguments from motivic integration. Namely, we follow methods developed by M. Mustața in [25]. We prove in this section the following theorem (see Theorem 4.1, (i) and Corollary 4.13, (i)):

Theorem 1.2. — *The principal bicone of \mathfrak{g} is a reduced complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$. Moreover, the images of any irreducible component of $\mathcal{X}_{\mathfrak{g}}$ by the first and second projection from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} are equal to the principal cone $\mathfrak{X}_{\mathfrak{g}}$.*

Then we deduce Theorem 1.1 from Theorem 1.2. We give Section 5 applications of our results to invariant theory, mainly in relation with the *nullcone*. In Section 6, we obtain additional properties about the irreducible components of the nilpotent bicone. We prove that $\mathcal{N}_{\mathfrak{g}}$ is not reduced (see Theorem 6.1) and we give a lower bound for the number of irreducible components of $\mathcal{N}_{\mathfrak{g}}$ (see Proposition 6.8).

1.3. Additional notations and conventions. — In this note, the ground field is \mathbb{C} . All topological terms refer to the Zariski topology. If X is an algebraic variety, an open subset of X whose complement in X has codimension at least 2 is called a *big open subset* of X .

If E is a finite set, its cardinality is denoted by $|E|$.

For x in \mathfrak{g} , we denote by $\mathfrak{g}(x)$ the centralizer of x in \mathfrak{g} . Thus x belongs to $\mathfrak{g}_{\text{reg}}$ if and only if $\mathfrak{g}(x)$ has dimension $\text{rk } \mathfrak{g}$.

The G -action of G in $\mathfrak{g} \times \mathfrak{g}$ is the diagonal action.

If \mathfrak{g} is commutative, then $\mathcal{N}_{\mathfrak{g}}$ is reduced to $\{(0,0)\}$. From now on, we suppose that \mathfrak{g} is not commutative.

Let $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ be the simple factors of \mathfrak{g} . Since the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ is the product of the nilpotent cones $\mathfrak{N}_{\mathfrak{g}_1}, \dots, \mathfrak{N}_{\mathfrak{g}_m}$ of $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ respectively, $\mathcal{N}_{\mathfrak{g}}$ is the product of the nilpotent bicones $\mathcal{N}_{\mathfrak{g}_1}, \dots, \mathcal{N}_{\mathfrak{g}_m}$. Furthermore, as the

equality

$$3(\mathfrak{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) = 3(\mathfrak{b}_{\mathfrak{g}_1} - \mathrm{rk} \mathfrak{g}_1) + \cdots + 3(\mathfrak{b}_{\mathfrak{g}_m} - \mathrm{rk} \mathfrak{g}_m)$$

holds, it is enough to prove Theorem 1.1 in the case where \mathfrak{g} is simple. It will be sometimes useful to make the assumption that \mathfrak{g} is simple (see e.g. Sections 3, 4 and 5).

For Y a subset of $\mathfrak{g} \times \mathfrak{g}$ and x in \mathfrak{g} , Y_x is the subset of elements y of \mathfrak{g} such that (x, y) belongs to Y .

The first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} are denoted by ϖ_1 and ϖ_2 respectively. The following lemma will be very useful throughout the note.

Lemma 1.3. — *Let Y be a G -invariant closed bicone in $\mathfrak{g} \times \mathfrak{g}$. Then the subsets $\varpi_1(Y)$ and $\varpi_2(Y)$ are G -invariant closed cones of \mathfrak{g} .*

Proof. — As Y is a G -invariant bicone, $\varpi_1(Y)$ and $\varpi_2(Y)$ are G -invariant cones of \mathfrak{g} . Moreover, $\varpi_1(Y) \times \{0\}$ and $\{0\} \times \varpi_2(Y)$ are the intersections of Y with $\mathfrak{g} \times \{0\}$ and $\{0\} \times \mathfrak{g}$ respectively, since Y is a closed bicone. So $\varpi_1(Y)$ and $\varpi_2(Y)$ are closed subsets of \mathfrak{g} . \square

We fix a principal \mathfrak{sl}_2 -triple (e, h, f) of \mathfrak{g} . In particular, the following relations

$$[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f,$$

are satisfied, e and f are regular nilpotent elements and h is a regular semisimple element. If ad denotes the adjoint representation of \mathfrak{g} , then $\mathrm{ad} h$ induces a \mathbb{Z} -grading on \mathfrak{g} , and we have $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$, where \mathfrak{g}_i is the eigenspace of $\mathrm{ad} h$ of eigenvalue i . Moreover, all the eigenvalues of $\mathrm{ad} h$ are even integers, the centralizer $\mathfrak{g}(x) = \mathfrak{g}_0$ of x in \mathfrak{g} is a Cartan subalgebra, that we will denote also by \mathfrak{h} , the unique Borel subalgebra \mathfrak{b} containing e is

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{i>0} \mathfrak{g}_i,$$

and the subset \mathfrak{u} of nilpotent elements of \mathfrak{b} is $\mathfrak{u} = \sum_{i>0} \mathfrak{g}_i$. We also set:

$$\mathfrak{b}_- = \sum_{i \leq 0} \mathfrak{g}_i, \quad \mathfrak{u}_- = \sum_{i < 0} \mathfrak{g}_i.$$

Let \mathbf{B} and \mathbf{B}_- be the normalizers of \mathfrak{b} and \mathfrak{b}_- in G and let \mathbf{H} and $N_G(\mathfrak{h})$ be the centralizer and the normalizer of \mathfrak{h} in G . Then the quotient $W_G(\mathfrak{h})$ of $N_G(\mathfrak{h})$ by \mathbf{H} is the Weyl group of \mathfrak{g} with respect to \mathfrak{h} . The element of biggest length of $W_G(\mathfrak{h})$ is denoted by w_0 . So for any representative g_0 of w_0 in $N_G(\mathfrak{h})$, $g_0(\mathfrak{b})$ is equal to \mathfrak{b}_- .

Let \mathcal{R} be the root system of \mathfrak{h} in \mathfrak{g} , let \mathcal{R}_+ be the positive root system of \mathcal{R} defined by \mathfrak{b} , and let Π be the basis of \mathcal{R}_+ . For any α in \mathcal{R} , we denote by \mathfrak{g}^α the corresponding root subspace. In particular, $w_0(\Pi)$ is equal to $-\Pi$.

Let $\langle \cdot, \cdot \rangle$ be a non-degenerate G -invariant bilinear form on $\mathfrak{g} \times \mathfrak{g}$ which extends the Killing form on the semisimple part of \mathfrak{g} . In the remainder of this note, the orthogonality will refer to $\langle \cdot, \cdot \rangle$.

Acknowledgment. — We are grateful to Michel Duflo for bringing [25] to our attention. We would like to thank Oksana Yakimova for her explanations, François Loeser for his e-mails concerning motivic integration questions, and Karin Baur for her comments. In addition, we thank Vladimir Popov for his support.

2. Characteristic submodule

In this section, we introduce the characteristic submodule of \mathfrak{g} and we describe some of its properties.

2.1. — Let us recall that $p_1, \dots, p_{\text{rk } \mathfrak{g}}$ are homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ of degrees $d_1, \dots, d_{\text{rk } \mathfrak{g}}$ respectively such that the sequence $d_1, \dots, d_{\text{rk } \mathfrak{g}}$ is non-decreasing. For $i = 1, \dots, \text{rk } \mathfrak{g}$, let ε_i be the element of $S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ defined by the following relation:

$$\langle \varepsilon_i(x), v \rangle = p'_i(x)(v) ,$$

for x, v in \mathfrak{g} , where $p'_i(x)$ is the differential of p_i at x .

Lemma 2.1. — *Let x be in \mathfrak{g} .*

- i) *For $i = 1, \dots, \text{rk } \mathfrak{g}$, $\varepsilon_i(x)$ belongs to the center of $\mathfrak{g}(x)$.*
- ii) *The elements $\varepsilon_1(x), \dots, \varepsilon_{\text{rk } \mathfrak{g}}(x)$ are linearly independent in \mathfrak{g} if and only if x is regular. Moreover, if so, $\varepsilon_1(x), \dots, \varepsilon_{\text{rk } \mathfrak{g}}(x)$ is a basis of $\mathfrak{g}(x)$.*

Proof. — i) For any v in \mathfrak{g} ,

$$\langle v, [x, \varepsilon_i(x)] \rangle = \langle [v, x], \varepsilon_i(x) \rangle = -p'_i(x)([x, v]) .$$

The right hand side of these equalities is equal to 0 since p_i is invariant under G . Hence $\varepsilon_i(x)$ belongs to $\mathfrak{g}(x)$. Let g be in G . Then for $i = 1, \dots, \text{rk } \mathfrak{g}$, $g \circ \varepsilon_i(x)$ is equal to $\varepsilon_i(g(x))$. In particular, when $g(x)$ is equal to x , $g \circ \varepsilon_i(x)$ is equal to $\varepsilon_i(x)$. So for any v in $\mathfrak{g}(x)$, $[v, \varepsilon_i(x)]$ is equal to 0.

Statement (ii) comes from [19](Theorem 9). □

Let i be in $\{1, \dots, \text{rk } \mathfrak{g}\}$. The 2-order polarizations p_i of p_i are defined by the following relation:

$$p_i(ax + by) = \sum_{m+n=d_i} a^m b^n p_{i,m,n}(x, y) ,$$

for any (a, b) in \mathbb{C}^2 and any (x, y) in $\mathfrak{g} \times \mathfrak{g}$. For (m, n) in \mathbb{N}^2 such that $m + n = d_i$, let $\varepsilon_{i,m,n}$ be the element of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ such that the linear functional $v \mapsto \langle \varepsilon_{i,m,n}(x, y), v \rangle$ on \mathfrak{g} is the differential at x of the function $x \mapsto p_{i,m,n}(x, y)$ for (x, y) in $\mathfrak{g} \times \mathfrak{g}$. In particular, $\varepsilon_{i,0,d_i} = 0$ for $i = 1, \dots, \text{rk } \mathfrak{g}$. In addition, since $d_1 + \dots + d_{\text{rk } \mathfrak{g}}$ is equal to $b_{\mathfrak{g}}$, the cardinality of the family $\{\varepsilon_{i,1,d_i-1}, \dots, \varepsilon_{i,d_i,0}, i = 1, \dots, \text{rk } \mathfrak{g}\}$ is equal to $b_{\mathfrak{g}}$.

Lemma 2.2. — For $i = 1, \dots, \text{rk } \mathfrak{g}$, we have

$$\varepsilon_i(ax + by) = \sum_{m=1}^{d_i} a^{m-1} b^{d_i-m} \varepsilon_{i,m,d_i-m}(x, y) ,$$

for any (a, b) in \mathbb{C}^2 . In particular, $\varepsilon_{i,1,d_i-1}(x, y)$ and $\varepsilon_{i,d_i,0}(x, y)$ are equal to $\varepsilon_i(y)$ and $\varepsilon_i(x)$ respectively, for any (x, y) in $\mathfrak{g} \times \mathfrak{g}$.

Proof. — Let v be in \mathfrak{g} . For any (a, b) in \mathbb{C}^2 such that a is not equal to 0, we have

$$\begin{aligned} \langle \varepsilon_i(ax + by), v \rangle &= \frac{d}{dt} p_i(ax + by + tv) \Big|_{t=0} \\ &= \sum_{m+n=d_i} a^m b^n \frac{d}{dt} p_{i,m,n}(x + ta^{-1}v, y) \Big|_{t=0} \\ &= \sum_{m+n=d_i} a^m b^n \langle \varepsilon_{i,m,n}(x, y), a^{-1}v \rangle , \end{aligned}$$

whence the lemma, since $\langle \cdot, \cdot \rangle$ is a non degenerate bilinear form on $\mathfrak{g} \times \mathfrak{g}$. \square

Let us recall that we have introduced the subset

$$\Omega_{\mathfrak{g}} := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \setminus \{0\} \subset \mathfrak{g}_{\text{reg}}, \dim P_{x,y} = 2\} .$$

The subset $\Omega_{\mathfrak{g}}$ is clearly invariant under the action of G in $\mathfrak{g} \times \mathfrak{g}$. We denote by \mathfrak{h}' , \mathfrak{g}'_2 and \mathfrak{g}'_{-2} the intersections of $\mathfrak{g}_{\text{reg}}$ with \mathfrak{h} , \mathfrak{g}_2 and \mathfrak{g}_{-2} respectively. We set:

$$\mathfrak{u}_+ = \sum_{i \geq 4} \mathfrak{g}_i .$$

In the following lemma, we explicitly provide elements of $\Omega_{\mathfrak{g}}$.

Lemma 2.3. — Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$.

- i) If x and y are in $\mathfrak{h}' \oplus \mathfrak{u}_+$ and $\mathfrak{g}'_2 \oplus \mathfrak{u}_+$ respectively, then (x, y) is in $\Omega_{\mathfrak{g}}$.
- ii) If x and y are in \mathfrak{g}'_{-2} and \mathfrak{g}'_2 respectively, then (x, y) is in $\Omega_{\mathfrak{g}}$.

Proof. — i) Let us suppose that x and y are in $\mathfrak{h}' \oplus \mathfrak{u}$ and $\mathfrak{g}'_2 \oplus \mathfrak{u}_+$ respectively. The elements x and y are clearly linearly independent. In addition, for any t in \mathbb{C} , $x + ty$ is conjugate to x under G . So $x + ty$ is regular for any t in \mathbb{C} . Since y is regular, (x, y) belongs to $\Omega_{\mathfrak{g}}$.

ii) Let us suppose that x and y are in \mathfrak{g}'_{-2} and \mathfrak{g}'_2 respectively. The elements x and y are clearly linearly independent. As x is regular, $x + sy$ is regular for any s in an open subset of \mathbb{C} containing 0. Let $t \mapsto g(t)$ be the one-parameter subgroup of \mathbf{H} generated by $\text{ad}h$. As $\mathfrak{g}_{\text{reg}}$ is G -invariant, for any t in $\mathbb{C} \setminus \{0\}$, $g(t)(x + sy)$ is regular for any s in an open subset of \mathbb{C} , containing 0. So, from the relations $[h, x] = -2x$ and $[h, y] = 2y$, we deduce that for any t in $\mathbb{C} \setminus \{0\}$, $t^{-2}x + st^2y$ is regular for any s in an open subset of \mathbb{C} containing 0. As $\mathfrak{g}_{\text{reg}}$ is an open cone, $x + st^4y$ is consequently regular for any s in an open subset of \mathbb{C} containing $\{0\}$. So $x + ty$ is regular for any t in \mathbb{C} . As y is regular, we deduce that (x, y) belongs to $\Omega_{\mathfrak{g}}$. \square

Remark 2.4. — The subset $\Omega_{\mathfrak{g}}$ is stable under the involution $(x, y) \mapsto (y, x)$. So, by Lemma 2.3, the images of the subsets $(\mathfrak{h}' \oplus \mathfrak{u}) \times (\mathfrak{g}'_2 \oplus \mathfrak{u}_+)$ and $\mathfrak{g}'_{-2} \times \mathfrak{g}'_2$ by this involution are contained in $\Omega_{\mathfrak{g}}$. For example, the elements (e, h) , (f, h) and (e, f) are in $\Omega_{\mathfrak{g}}$.

The following lemma is well-known:

Lemma 2.5. — *Let x be in $\mathfrak{g}_{\text{reg}}$.*

- i) *If x belongs to \mathfrak{b} , then $\mathfrak{g}(x)$ is contained in \mathfrak{b} .*
- ii) *If x belongs to \mathfrak{u} , then $\mathfrak{g}(x)$ is contained in \mathfrak{u} .*

For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, we denote by $\mathfrak{V}(x, y)$ the subspace generated by the elements

$$\varepsilon_{i,1,d_i-1}(x, y), \dots, \varepsilon_{i,d_i,0}(x, y), \quad i = 1, \dots, \text{rk } \mathfrak{g},$$

and we set:

$$\mathfrak{V}'(x, y) := \sum_{(a,b) \in \mathbb{C}^2 \setminus (0,0)} \mathfrak{g}(ax + by).$$

We collect in the following lemma some results concerning the subset $\Omega_{\mathfrak{g}}$ in part obtained by A. V. Bolsinov in [1] and recently by D. I. Panyushev and O. Yakimova in [26]:

Lemma 2.6 (Bolsinov). — *Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$.*

- i) *The subspace $\mathfrak{V}(x, y)$ has dimension at most $\mathfrak{b}_{\mathfrak{g}}$. Moreover, it has dimension $\mathfrak{b}_{\mathfrak{g}}$ if and only if (x, y) belongs to $\Omega_{\mathfrak{g}}$.*
- ii) *The subspace $\mathfrak{V}(x, y)$ is contained in $\mathfrak{V}'(x, y)$. Moreover, the equality occurs when (x, y) belongs to $\Omega_{\mathfrak{g}}$.*
- iii) *If (x, y) belongs to $\mathfrak{b} \times \mathfrak{b}$, then $\mathfrak{V}(x, y)$ is contained in \mathfrak{b} .*

Proof. — i) As the sum of $d_1, \dots, d_{b_{\mathfrak{g}}}$ is equal to $b_{\mathfrak{g}}$, $\mathfrak{V}(x, y)$ has dimension at most $b_{\mathfrak{g}}$. Let us suppose that (x, y) belongs to $\Omega_{\mathfrak{g}}$. Then by [26] (Theorem 2.4), the subspace $\mathfrak{V}(x, y)$ has dimension $b_{\mathfrak{g}}$. Conversely, let us suppose that $\mathfrak{V}(x, y)$ has dimension $b_{\mathfrak{g}}$. In particular, $\varepsilon_1(x), \dots, \varepsilon_{b_{\mathfrak{g}}}(x)$ are linearly independent. Hence by Lemma 2.1, (ii), x is regular. By Lemma 2.2, for any (a, b) in $\mathbb{C}^2 \setminus \{(0, 0)\}$, $\mathfrak{V}(x, y)$ is equal to $\mathfrak{V}(ax + by, y)$. Hence (x, y) belongs to $\Omega_{\mathfrak{g}}$.

ii) By Lemma 2.1, $\mathfrak{V}'(x, y)$ contains $\varepsilon_i(ax + by)$ for $i = 1, \dots, \text{rk } \mathfrak{g}$ and any (a, b) in \mathbb{C}^2 . Hence by Lemma 2.2, $\mathfrak{V}'(x, y)$ contains $\mathfrak{V}(x, y)$. Moreover, by (i) and Lemma 2.1, $\mathfrak{V}(x, y)$ is equal to $\mathfrak{V}'(x, y)$ when (x, y) belongs to $\Omega_{\mathfrak{g}}$.

iii) Let us suppose that (x, y) belongs to $\mathfrak{b} \times \mathfrak{b}$. By Lemma 2.3 and Remark 2.4, the intersection of $\Omega_{\mathfrak{g}}$ and $\mathfrak{b} \times \mathfrak{b}$ is a nonempty open subset. Moreover, when (x, y) belongs to this intersection, for any (a, b) in $\mathbb{C}^2 \setminus \{(0, 0)\}$, $\mathfrak{g}(ax + by)$ is contained in \mathfrak{b} , by Lemmas 2.1, (ii) and 2.5, (i). Hence for $i = 1, \dots, \text{rk } \mathfrak{g}$ and $m = 1, \dots, d_i$, $\varepsilon_{i,m,d_i-m}(x, y)$ belongs to \mathfrak{b} , by (ii). So $\mathfrak{V}(x, y)$ is contained in \mathfrak{b} . \square

2.2. — The following automorphisms give an action of $\text{GL}_2(\mathbb{C})$ in $\mathfrak{g} \times \mathfrak{g}$:

$$(x, y) \longmapsto (ax + by, cx + dy), \text{ where } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let \mathfrak{s} be the subspace of \mathfrak{g} generated by e, h, f and let \mathbf{S} be the closed connected subgroup of G whose Lie algebra is the image of \mathfrak{s} by the adjoint representation of \mathfrak{g} . Then \mathfrak{s} is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The nilpotent cone of \mathfrak{s} is denoted by $\mathfrak{N}_{\mathfrak{s}}$. Let T_5 be the subset of elements (x, y) of $\mathfrak{s} \times \mathfrak{s}$ such that x and y belong to the same Borel subalgebra of \mathfrak{s} .

Lemma 2.7. — *Let T_3 and T_4 be the images of $\mathfrak{N}_{\mathfrak{s}} \times \mathbb{C}$ and $\mathfrak{s} \times \mathbb{C}$ by the map $(x, t) \mapsto (x, tx)$ from $\mathfrak{s} \times \mathbb{C}$ to $\mathfrak{s} \times \mathfrak{s}$ respectively.*

i) *The subsets T_3, T_4, T_5 are closed, irreducible subsets of $\mathfrak{s} \times \mathfrak{s}$, invariant under the actions of \mathbf{S} and $\text{GL}_2(\mathbb{C})$. Moreover, they have dimension 3, 4, 5 respectively.*

ii) *Let T be a nonempty, irreducible closed subset of $\mathfrak{s} \times \mathfrak{s}$, invariant under the actions of \mathbf{S} and $\text{GL}_2(\mathbb{C})$. Then T is one of the following subsets $\{0\}, T_3, T_4, T_5, \mathfrak{s} \times \mathfrak{s}$.*

Proof. — i) As the condition defining T_5 is a closed condition, the subset T_5 is closed in $\mathfrak{s} \times \mathfrak{s}$. In addition, T_5 is the image of the map

$$\mathbf{S} \times \mathfrak{b}_{\mathfrak{s}} \times \mathfrak{b}_{\mathfrak{s}} \rightarrow \mathfrak{s} \times \mathfrak{s}, (g, x, y) \mapsto (g(x), g(y)),$$

where $\mathfrak{b}_{\mathfrak{s}}$ is generated by h and e . Hence T_5 is irreducible and has dimension 5. Moreover, it is invariant under the actions of \mathbf{S} and $\text{GL}_2(\mathbb{C})$. By definition, T_3 is the subset of elements (x, y) of $\mathfrak{N}_{\mathfrak{s}} \times \mathfrak{N}_{\mathfrak{s}}$ such that x and y are colinear. Hence T_3 is closed in $\mathfrak{s} \times \mathfrak{s}$ since $\mathfrak{N}_{\mathfrak{s}}$ is closed in \mathfrak{s} . Moreover, T_3 has dimension

3 and it is invariant under the action of \mathbf{S} since $\mathfrak{N}_{\mathfrak{s}}$ has dimension 2 and it is \mathbf{S} -invariant. As $\mathfrak{N}_{\mathfrak{s}}$ is conic, T_3 is invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. By the same arguments, T_4 is a closed subset of $\mathfrak{s} \times \mathfrak{s}$, invariant under the actions of \mathbf{S} and $\mathrm{GL}_2(\mathbb{C})$. Moreover, its dimension is equal to 4. As images of irreducible varieties, T_3 and T_4 are irreducible.

ii) By Lemma 1.3, $\varpi_1(T)$ is a \mathbf{S} -invariant closed cone. If T has dimension 0, $\varpi_1(T)$ is equal to $\{0\}$. We suppose that the dimension of T is positive and at most 3. Then $\varpi_1(T)$ has dimension smaller than 2. As the \mathbf{S} -orbits in $\mathfrak{s} \setminus \{0\}$ have dimension 2, $\varpi_1(T)$ is equal to $\mathfrak{N}_{\mathfrak{s}}$, since the \mathbf{S} -orbit of a non zero semisimple element is not conic. Then T contains T_3 . As T is irreducible of dimension smaller than 3, T is equal to T_3 by (i).

We suppose that T has dimension bigger than 4 and that $\varpi_1(T)$ is contained in $\mathfrak{N}_{\mathfrak{s}}$. We expect a contradiction. In this case $\varpi_1(T)$ is equal to $\mathfrak{N}_{\mathfrak{s}}$ since $\varpi_1(T)$ is \mathbf{S} -invariant and contains non zero elements. Moreover, as T has dimension bigger than 4, there exist elements a, b, c in \mathbb{C} with the following properties: T contains $(e, ae + bf + ch)$ and b or c is not equal to 0. Hence T contains $(e, bf + ch)$. As $bf + ch$ is semisimple as soon as c is not equal to 0, c is equal to 0, and T contains (e, f) . So $\varpi_1(T)$ contains $e + f$ which is semisimple, whence the expected contradiction. As $\varpi_1(T)$ is not contained in $\mathfrak{N}_{\mathfrak{s}}$, $\varpi_1(T)$ is equal to \mathfrak{s} since $\varpi_1(T)$ is a \mathbf{S} -invariant, closed cone of \mathfrak{s} . Then T contains T_4 . Moreover, T is equal to T_4 whenever its dimension is 4.

We suppose that T has dimension bigger than 5 and that T is not equal to T_5 . Then $\varpi_1(T)$ is equal to \mathfrak{s} and, for any x in a non empty open subset of \mathfrak{s} , there exists an element y of \mathfrak{s} with the following two properties: T contains (x, y) and there is no Borel subalgebra which contains x and y . As T is invariant under \mathbf{S} and $\mathrm{GL}_2(\mathbb{C})$, this is still true for any x in a \mathbf{S} -invariant open cone. Hence there exist a and b in $\mathbb{C} \setminus \{0\}$, such that T contains $(h, ae + bf)$ since T is $\mathrm{GL}_2(\mathbb{C})$ -invariant. As T is invariant under the action of the one-parameter subgroup generated by $\mathrm{ad}h$, T contains $(h, at^2e + bt^{-2}f)$ for any t in $\mathbb{C} \setminus \{0\}$. Hence T contains $(h, st^2ae + bst^{-2}f)$ for any s and t in $\mathbb{C} \setminus \{0\}$, since T is $\mathrm{GL}_2(\mathbb{C})$ -invariant. Then T contains $(h, ue + vf)$ for any (u, v) in \mathbb{C}^2 , since T is closed. So T contains $\{x\} \times \mathfrak{s}$ for any x in the cone generated by the \mathbf{S} -orbit of h . Hence T is equal to $\mathfrak{s} \times \mathfrak{s}$. \square

Corollary 2.8. — *The subset $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$.*

Proof. — Let us suppose that $\Omega_{\mathfrak{g}}$ is not a big open subset of $\mathfrak{g} \times \mathfrak{g}$. We expect a contradiction. In this case, the complement of $\Omega_{\mathfrak{g}}$ in $\mathfrak{g} \times \mathfrak{g}$ has an irreducible component Σ of codimension 1 in $\mathfrak{g} \times \mathfrak{g}$. As $\Omega_{\mathfrak{g}}$ is invariant under the action of G and $\mathrm{GL}_2(\mathbb{C})$, Σ is invariant under these actions. Let T be the intersection of Σ and $\mathfrak{s} \times \mathfrak{s}$. Then T is a nonempty closed cone of $\mathfrak{s} \times \mathfrak{s}$ since it contains $(0, 0)$. As Σ is an hypersurface of $\mathfrak{g} \times \mathfrak{g}$, the codimension of T in $\mathfrak{s} \times \mathfrak{s}$ is smaller than 1. Moreover, T is invariant under the actions of \mathbf{S} and $\mathrm{GL}_2(\mathbb{C})$. Hence

by Lemma 2.11, (ii), (h, e) belongs to T . But by Lemma 2.3, (i), (h, e) belongs to $\Omega_{\mathfrak{g}}$, whence the expected contradiction. \square

2.3. — By a result of J. Dixmier [9](§2), the sub- $S(\mathfrak{g})$ -module of elements φ in $S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ such that $\varphi(x)$ belongs to $\mathfrak{g}(x)$ for all x in \mathfrak{g} , is a free module of basis $\varepsilon_1, \dots, \varepsilon_{\text{rk } \mathfrak{g}}$.

Definition 2.9. — The *characteristic submodule* of \mathfrak{g} , denoted by $B_{\mathfrak{g}}$, is the sub- $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ -module of elements φ in $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ such that $\varphi(x, y)$ belongs to $\mathfrak{V}'(x, y)$, for any (x, y) in a nonempty open subset of $\mathfrak{g} \times \mathfrak{g}$.

The following result can be viewed as a generalization of the previous result of J. Dixmier.

Theorem 2.10. — *The submodule $B_{\mathfrak{g}}$ of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ is a free $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ -module of rank $\text{rk } \mathfrak{g}$. Moreover, the family $\{\varepsilon_{i,1,d_i}, \dots, \varepsilon_{i,d_i,0}, i = 1, \dots, \text{rk } \mathfrak{g}\}$ is a basis of $B_{\mathfrak{g}}$.*

Proof. — By Lemma 2.6, (ii), for $i = 1, \dots, \text{rk } \mathfrak{g}$ and $m = 1, \dots, d_i$, ε_{i,m,d_i-m} belongs to $B_{\mathfrak{g}}$. By Lemma 2.6, (i), these elements are linearly independent over $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$. It remains to prove that they generate $B_{\mathfrak{g}}$ as $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ -module. Let φ be in $B_{\mathfrak{g}}$. By Lemma 2.6, (i) and (ii), $\varphi(x, y)$ belongs to $\mathfrak{V}(x, y)$ for any (x, y) in $\Omega_{\mathfrak{g}}$. So there exist regular functions ψ_{i,m,d_i-m} on $\Omega_{\mathfrak{g}}$ for $i = 1, \dots, \text{rk } \mathfrak{g}$ and $m = 1, \dots, d_i$, such that:

$$\varphi(x, y) = \sum_{\substack{i=1, \dots, \text{rk } \mathfrak{g} \\ m=1, \dots, d_i}} \psi_{i,m,d_i-m}(x, y) \varepsilon_{i,m,d_i-m}(x, y) ,$$

for any (x, y) in $\Omega_{\mathfrak{g}}$. By Corollary 2.8, $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$. Hence the regular functions ψ_{i,m,d_i-m} have regular extensions to $\mathfrak{g} \times \mathfrak{g}$ since $\mathfrak{g} \times \mathfrak{g}$ is normal. As a result, the family

$$\{\varepsilon_{i,1,d_i}, \dots, \varepsilon_{i,d_i,0}, i = 1, \dots, \text{rk } \mathfrak{g}\}$$

is a basis of $B_{\mathfrak{g}}$. \square

2.4. — In this subsection, we establish a link between the open subset $\Omega_{\mathfrak{g}}$ and the nilpotent bicone. For $i = 1, \dots, \text{rk } \mathfrak{g}$, we denote by σ_i the map

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}^{d_i+1}, (x, y) \mapsto (p_{i,0,d_i}(x, y), \dots, p_{i,d_i,0}(x, y)) .$$

Lemma 2.11. — *For (x, y) in $\mathfrak{g} \times \mathfrak{g}$ and $i = 1, \dots, \text{rk } \mathfrak{g}$, the differential of σ_i at (x, y) is the linear map*

$$(v, w) \longmapsto (\langle \varepsilon_{i,1,d_i-1}(x, y), w \rangle, \langle \varepsilon_{i,1,d_i-1}(x, y), v \rangle + \langle \varepsilon_{i,2,d_i-2}(x, y), w \rangle, \\ \dots, \langle \varepsilon_{i,d_i-1,1}(x, y), v \rangle + \langle \varepsilon_{i,d_i,0}(x, y), w \rangle, \langle \varepsilon_i(x), v \rangle) .$$

Proof. — For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, we denote by $p'_{i,m,d_i-m}(x, y)$ the differential of p_{i,m,d_i-m} at (x, y) . From Lemma 2.2 and the equality

$$p_i(tx + y) = \sum_{m=0}^{d_i} t^m p_{i,m,d_i-m}(x, y)$$

for (x, y) in $\mathfrak{g} \times \mathfrak{g}$, we deduce the equality

$$\sum_{m=1}^{d_i} t^m \langle \varepsilon_{i,m,d_i-m}(x, y), v + t^{-1}w \rangle = \sum_{m=0}^{d_i} t^m p'_{i,m,d_i-m}(x, y)(v, w) ,$$

for x, y, v, w in \mathfrak{g} and t in $\mathbb{C} \setminus \{0\}$, since $p_{i,0,d_i}$ is the map $(x, y) \mapsto p_i(y)$. Hence we get, for $m = 0, \dots, d_i - 1$,

$$p'_{i,m,d_i-m}(x, y)(v, w) = \langle \varepsilon_{i,m,d_i-m}(x, y), v \rangle + \langle \varepsilon_{i,m+1,d_i-m-1}(x, y), w \rangle .$$

In addition $p'_{i,d_i,0}(x, y)$ is the linear functional $v \mapsto \langle \varepsilon_i(x), v \rangle$. \square

Let σ be the map:

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}^{\mathfrak{b}_{\mathfrak{g}} + \text{rk } \mathfrak{g}} , (x, y) \mapsto (\sigma_1(x, y), \dots, \sigma_{\text{rk } \mathfrak{g}}(x, y)) .$$

Proposition 2.12. — *Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$. Then σ is smooth at (x, y) if and only if (x, y) belongs to $\Omega_{\mathfrak{g}}$.*

Proof. — We denote by $\sigma'(x, y)$ the tangent linear map to σ at (x, y) and we denote by $\ker \sigma'(x, y)$ its kernel in $\mathfrak{g} \times \mathfrak{g}$. Let us suppose that (x, y) belongs to $\Omega_{\mathfrak{g}}$. For v in \mathfrak{g} , we denote by X_v the subset of elements w in \mathfrak{g} such that (v, w) is in $\ker \sigma'(x, y)$. By Lemmas 2.11 and 2.1, (ii), v belongs to the orthogonal of $\mathfrak{g}(x)$ in \mathfrak{g} . In addition, X_v is an affine subspace whose tangent space is equal to the orthogonal of $\mathfrak{V}(x, y)$ in \mathfrak{g} . By Lemma 2.6, (i), $\mathfrak{V}(x, y)$ has dimension $\mathfrak{b}_{\mathfrak{g}}$ since (x, y) belongs to $\Omega_{\mathfrak{g}}$. Consequently, $\ker \sigma'(x, y)$ has dimension $\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 2(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) = 3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$, since x is regular. Hence $\sigma'(x, y)$ is surjective. So σ is smooth at (x, y) .

Conversely, suppose that σ is smooth at (x, y) . For (a, b) in \mathbb{C}^2 , we denote by $\pi_{a,b}$ the linear map

$$(z_{i,0,d_i}, \dots, z_{i,d_i,0}, i = 1, \dots, \text{rk } \mathfrak{g}) \longmapsto \left(\sum_{m+n=d_i} a^m b^n z_{i,m,n}, i = 1, \dots, \text{rk } \mathfrak{g} \right) ,$$

from $\mathbb{C}^{\mathfrak{b}_{\mathfrak{g}} + \text{rk } \mathfrak{g}}$ to $\mathbb{C}^{\text{rk } \mathfrak{g}}$. The linear map $\pi_{a,b}$ is surjective as soon as (a, b) is different from $(0, 0)$. Since σ is smooth at (x, y) , we deduce that the compound map $\sigma_{a,b} := \pi_{a,b} \circ \sigma$ is smooth at (x, y) , for any (a, b) in $\mathbb{C}^2 \setminus \{(0, 0)\}$. As

$$\sum_{m+n=d_i} a^m b^n p_{i,m,n}(x', y') = p_i(ax' + by') ,$$

for $i = 1, \dots, \text{rk } \mathfrak{g}$ and (x', y') in $\mathfrak{g} \times \mathfrak{g}$, $\sigma_{a,b}$ maps (x', y') to $(p_1(ax' + by'), \dots, p_{\text{rk } \mathfrak{g}}(ax' + by'))$. Moreover, it is the compound map of the two following maps:

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g}, (x', y') \mapsto (ax' + by') \\ \mathfrak{g} &\longrightarrow \mathbb{C}^{\text{rk } \mathfrak{g}}, z \mapsto (p_1(z), \dots, p_{\text{rk } \mathfrak{g}}(z)). \end{aligned}$$

Therefore, the second map is smooth at $(ax + by)$, for any (a, b) in $\mathbb{C}^2 \setminus \{(0, 0)\}$. Therefore, by Lemma 2.1, (ii), (x, y) belongs to $\Omega_{\mathfrak{g}}$. \square

Remark 2.13. — Let us recall that $\mathcal{N}_{\mathfrak{g}}$ is the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal generated by the polynomials $p_{i,m,n}$. Therefore, by Proposition 2.12, the intersection of $\mathcal{N}_{\mathfrak{g}}$ and $\Omega_{\mathfrak{g}}$ is the set of smooth points of $\mathcal{N}_{\mathfrak{g}}$. The elements of $\Omega_{\mathfrak{g}}$ provided by Lemma 2.3, (i) are not in $\mathcal{N}_{\mathfrak{g}}$. Consider now the elements described in Lemma 2.3, (ii). If \mathfrak{g} is equal to \mathfrak{sl}_3 , the subset $\mathfrak{g}'_{-2} \times \mathfrak{g}'_2$ has a non empty intersection with $\mathcal{N}_{\mathfrak{g}}$. Indeed, if we set

$$x := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad y := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then we can check that (x, y) belongs to the intersection of $\mathcal{N}_{\mathfrak{g}}$ and $\mathfrak{g}'_{-2} \times \mathfrak{g}'_2$. In general this intersection can be empty. There are many cases when \mathfrak{g} is equal to \mathfrak{sl}_n , with $n > 3$. Whatever the case, we will see a posteriori that there exists at least one irreducible component of $\mathcal{N}_{\mathfrak{g}}$ which has an empty intersection with $\Omega_{\mathfrak{g}}$ (see Theorem 6.1). So we cannot hope to apply the criterion of B. Kostant [19]. This observation explains the introduction of another subvariety in the following section.

3. Principal cone and principal bicone

In this section, we suppose that \mathfrak{g} is simple. Then we can suppose that p_1 is the Casimir element of $S(\mathfrak{g})^{\mathfrak{g}}$. Namely, $p_1(x)$ is equal to $\langle x, x \rangle$ for any x in \mathfrak{g} . We recall that (e, h, f) is a principal \mathfrak{sl}_2 -triple of \mathfrak{g} and we use the notations of Subsection 1.3.

3.1. — Since e is a regular nilpotent element of \mathfrak{g} , the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ is the G -invariant closed cone generated by e . According to Kostant's results [19], the nilpotent cone is a complete intersection of codimension $\text{rk } \mathfrak{g}$. Moreover, it is proved in [15] that it has rational singularities. In this subsection, we intend to prove analogous properties for *the principal cone* introduced in the following definition:

Definition 3.1. — The principal cone of \mathfrak{g} , denoted by $\mathfrak{X}_{\mathfrak{g}}$, is the G -invariant closed cone generated by h .

We recall that w_0 is the element of biggest length of the Weyl group $W_G(\mathfrak{h})$ of \mathfrak{g} with respect to the centralizer \mathfrak{h} of h . The following lemma turns out to be very useful.

Lemma 3.2. — *The element $w_0(h)$ is equal to $-h$. Moreover, there exists a representative g_0 of w_0 in $N_G(\mathfrak{h})$ such that $g_0(e)$ is equal to f .*

Proof. — As w_0 is the element of $W_G(\mathfrak{h})$ of biggest length, $w_0(\Pi)$ is equal to $-\Pi$. But, by the choices of \mathfrak{b} , \mathfrak{h} and Π , $\alpha(h)$ is equal to 2 for any α in Π , since (e, h, f) is a principal \mathfrak{sl}_2 -triple. As a consequence $w_0(h)$ is equal to $-h$. Moreover, if g_0 is a representative of w_0 in $N_G(\mathfrak{h})$, then $g_0(e)$ belongs to \mathfrak{g}'_{-2} . As \mathfrak{g}'_{-2} is the orbit of f under \mathbf{H} , we can choose g_0 such that $g_0(e)$ is equal to f . \square

For $i = 2, \dots, \text{rk } \mathfrak{g}$, we define the element q_i of $S(\mathfrak{g})^{\mathfrak{g}}$ as follows:

$$q_i = \begin{cases} p_i, & \text{if } d_i \text{ is odd} \\ p_1(h)^{d_i/2} p_i - p_i(h) p_1^{d_i/2}, & \text{otherwise} \end{cases}.$$

The polynomial q_i is homogenous of degree d_i . As the eigenvalues of h are integers not all equal to 0, $p_1(h)$ is not equal to 0. In addition, as

$$p_i(h) = p_i(w_0(h)) = p_i(-h) = (-1)^{d_i} p_i(h),$$

$p_i(h)$ is equal to 0 when d_i is odd. As a result, $q_i(h)$ is equal to 0 for $i = 2, \dots, \text{rk } \mathfrak{g}$. So $\mathfrak{X}_{\mathfrak{g}}$ is contained in the nullvariety of the functions $q_2, \dots, q_{\text{rk } \mathfrak{g}}$.

Lemma 3.3. — i) *The nullvariety of p_1 in $\mathfrak{X}_{\mathfrak{g}}$ is equal to $\mathfrak{N}_{\mathfrak{g}}$.*

ii) *The principal cone $\mathfrak{X}_{\mathfrak{g}}$ is the nullvariety of the functions $q_2, \dots, q_{\text{rk } \mathfrak{g}}$.*

iii) *The codimension of $(\mathfrak{X}_{\mathfrak{g}} \setminus \mathfrak{g}_{\text{reg}})$ in $\mathfrak{X}_{\mathfrak{g}}$ is equal to 3.*

Proof. — i) Let us prove first that $\mathfrak{N}_{\mathfrak{g}}$ is contained in $\mathfrak{X}_{\mathfrak{g}}$. Since $\mathfrak{X}_{\mathfrak{g}}$ is a G -invariant, closed, cone, e belongs to $\mathfrak{X}_{\mathfrak{g}}$ since $\exp(-tade)(h)$ is equal to $h + 2te$ for any t in \mathbb{C} . So $\mathfrak{X}_{\mathfrak{g}}$ contains $\mathfrak{N}_{\mathfrak{g}}$ since $\mathfrak{N}_{\mathfrak{g}}$ is the closure of $G.e$. Let X be the nullvariety of the functions $q_2, \dots, q_{\text{rk } \mathfrak{g}}$. As $p_1(h)$ is not equal to 0, the nullvariety of p_1 in X is the nullvariety of $p_1, \dots, p_{\text{rk } \mathfrak{g}}$. So $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of p_1 in X . Moreover, $\mathfrak{N}_{\mathfrak{g}}$ is also the nullvariety of p_1 in $\mathfrak{X}_{\mathfrak{g}}$ since $\mathfrak{X}_{\mathfrak{g}}$ is a subset of X which contains $\mathfrak{N}_{\mathfrak{g}}$.

ii) We have to prove that X is contained in $\mathfrak{X}_{\mathfrak{g}}$. Let x be an element of X which is not nilpotent. Then by (i), $p_1(x)$ is not equal to 0. So there exists t in $\mathbb{C} \setminus \{0\}$ such that $p_1(tx)$ is equal to $p_1(h)$. Then $p_i(tx)$ is equal to $p_i(h)$ for $i = 1, \dots, \text{rk } \mathfrak{g}$ since x is in X . As h is regular and semisimple, tx and h are G -conjugate. Hence x belongs to $\mathfrak{X}_{\mathfrak{g}}$.

iii) By (i) and (ii), the subset of elements of $\mathfrak{X}_{\mathfrak{g}}$ which are not regular in \mathfrak{g} , is equal to $\mathfrak{N}_{\mathfrak{g}} \setminus G.e$. As this subset has codimension 2 in $\mathfrak{N}_{\mathfrak{g}}$, it has codimension 3 in $\mathfrak{X}_{\mathfrak{g}}$ by (i). \square

We recall that \mathfrak{b}_- is the Borel subalgebra containing \mathfrak{h} and opposite to \mathfrak{b} . The subset of nilpotent elements of \mathfrak{b}_- is denoted by \mathfrak{u}_- . Let \mathfrak{b}_0 be the subspace of \mathfrak{b} generated by h and \mathfrak{u} and let $\mathfrak{b}_{0,-}$ be the subspace of \mathfrak{b}_- generated by h and \mathfrak{u}_- . The following result is partially proved in [32](Proposition 10.3) (see also Remarque 3.5 below). We give here an independent proof.

Corollary 3.4. — *The principal cone of \mathfrak{g} is normal and it is a complete intersection of codimension $\mathrm{rk} \mathfrak{g} - 1$ in \mathfrak{g} . Moreover, the regular elements of \mathfrak{g} which belong to $\mathfrak{X}_{\mathfrak{g}}$ are smooth points of $\mathfrak{X}_{\mathfrak{g}}$ and $\mathfrak{X}_{\mathfrak{g}}$ has rational singularities.*

Proof. — By Lemma 3.3, (i) and (ii), $\mathfrak{X}_{\mathfrak{g}}$ is a complete intersection of codimension $\mathrm{rk} \mathfrak{g} - 1$. By Lemma 2.1, (ii), the differentials at x of $p_1, \dots, p_{\mathrm{rk} \mathfrak{g}}$ are linearly independent as soon as x is a regular element of \mathfrak{g} . Hence the same goes for the differentials at x of $q_2, \dots, q_{\mathrm{rk} \mathfrak{g}}$. Then, any x in the union of $G.e$ and $(\mathbb{C} \setminus \{0\})G.h$ is a smooth point of $\mathfrak{X}_{\mathfrak{g}}$. So by Lemma 3.3, (ii), $\mathfrak{X}_{\mathfrak{g}}$ is regular in codimension 1 and by Serre's normality criterion [24](Ch. 8, Theorem 23.8), $\mathfrak{X}_{\mathfrak{g}}$ is a normal subvariety.

The subalgebra \mathfrak{b}_0 is an ideal of \mathfrak{b} . The contracted product $G \times_{\mathbf{B}} \mathfrak{b}_0$ is defined as the quotient of $G \times \mathfrak{b}_0$ under the right action of \mathbf{B} given by $(g, x).b = (gb, b^{-1}(x))$. The map $(g, x) \mapsto g(x)$ from $G \times \mathfrak{b}_0$ to \mathfrak{g} factorizes through the canonical map from $G \times \mathfrak{b}_0$ to $G \times_{\mathbf{B}} \mathfrak{b}_0$. Since the G -orbit of an element of $\mathfrak{X}_{\mathfrak{g}}$ has a non empty intersection with \mathfrak{b}_0 , we get a morphism π from $G \times_{\mathbf{B}} \mathfrak{b}_0$ onto $\mathfrak{X}_{\mathfrak{g}}$. As G/\mathbf{B} is a projective variety, π is a proper morphism. Let K be the field of rational functions on $G \times_{\mathbf{B}} \mathfrak{b}_0$ and let K_0 be the field of rational functions on $\mathfrak{X}_{\mathfrak{g}}$. Let g_0 be as in Lemma 3.2. By Lemma 3.2, $g_0(h)$ is equal to $-h$. In particular, the fiber of π at h has at least two elements. Let (g, x) be an element of $G \times \mathfrak{b}_0$ such that $g(x)$ is equal to h . In particular, x is a regular semisimple element of \mathfrak{g} which belongs to \mathfrak{b}_0 . Hence there exists b in \mathbf{B} such that $b(x)$ belongs to the line generated by h . As $\mathrm{ad} h$ and $\mathrm{ad} b(x)$ have the same eigenvalues, $b(x)$ is equal to h or $-h$, since the eigenvalues of h are integers not all equal to 0. If $b(x)$ is equal to h , then (g, x) and $(\mathbf{1}_{\mathfrak{g}}, h)$ are equal in $G \times_{\mathbf{B}} \mathfrak{b}_0$. Otherwise, (g, x) and $(g_0, -h)$ are equal in $G \times_{\mathbf{B}} \mathfrak{b}_0$. Hence the fiber of π at h has two elements. So the same applies for the fiber of π at any element of the G -invariant cone generated by h since π is G -equivariant. Hence K is an algebraic extension of degree 2 of K_0 . In particular, it is a Galois extension. Let A be the integral closure of $\mathbb{C}[\mathfrak{X}_{\mathfrak{g}}]$ in K and let $\mathfrak{X}'_{\mathfrak{g}}$ be an affine algebraic variety such that $\mathbb{C}[\mathfrak{X}'_{\mathfrak{g}}]$ is equal to A . Then A is stable under the Galois group of the extension K of K_0 and $\mathbb{C}[\mathfrak{X}_{\mathfrak{g}}]$ is the subalgebra of invariant elements of this action since $\mathfrak{X}_{\mathfrak{g}}$ is normal. Hence by [10](Lemma 1), it is enough to prove that $\mathfrak{X}'_{\mathfrak{g}}$ has rational singularities.

The variety $G \times_{\mathbf{B}} \mathfrak{b}_0$ is smooth as a vector bundle over G/\mathbf{B} . Hence the morphism π factors through the canonical morphism from $\mathfrak{X}'_{\mathfrak{g}}$ to $\mathfrak{X}_{\mathfrak{g}}$. Let π'

be the morphism from $G \times_{\mathbf{B}} \mathfrak{b}_0$ to $\mathfrak{X}'_{\mathfrak{g}}$ such that π is the compound map of π' with the canonical morphism from $\mathfrak{X}'_{\mathfrak{g}}$ to $\mathfrak{X}_{\mathfrak{g}}$. Then $(G \times_{\mathbf{B}} \mathfrak{b}_0, \pi')$ is a desingularisation of $\mathfrak{X}'_{\mathfrak{g}}$ since π' is proper and birational. By the corollary of [15](Theorem B), for i positive integer, the i -th cohomology group $H^i(G \times_{\mathbf{B}} \mathfrak{b}_0, \mathcal{O}_{G \times_{\mathbf{B}} \mathfrak{b}_0})$ is equal to zero. Hence by [14](Ch. III, Proposition 8.5), for any positive integer i , $R^i \pi'_*(\mathcal{O}_{G \times_{\mathbf{B}} \mathfrak{b}_0})$ is equal to zero and $\mathfrak{X}'_{\mathfrak{g}}$ has rational singularities since $\mathfrak{X}'_{\mathfrak{g}}$ is normal. \square

Remark 3.5. — The G -invariant closed cone generated by a semisimple element of \mathfrak{g} is not a normal variety in general (see [32](Proposition 10.1)). Nevertheless, whenever x is the central element of an \mathfrak{sl}_2 -triple of \mathfrak{g} , the closed cone generated by the regular orbit whose closure contains x is normal and Cohen-Macaulay [32](Proposition 10.3).

3.2. — We study in this subsection various properties of the *principal bicone*.

Definition 3.6. — The *principal bicone* $\mathcal{X}_{\mathfrak{g}}$ of \mathfrak{g} is the subset

$$\begin{aligned} \mathcal{X}_{\mathfrak{g}} &:= \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \subset \mathfrak{X}_{\mathfrak{g}}\} \\ &= \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid ax + by \in \mathfrak{X}_{\mathfrak{g}}, \forall (a, b) \in \mathbb{C}^2\}. \end{aligned}$$

As $\mathcal{N}_{\mathfrak{g}}$, the subset $\mathcal{X}_{\mathfrak{g}}$ is a closed bicone of $\mathfrak{g} \times \mathfrak{g}$, and it is invariant under the actions of G and $\mathrm{GL}_2(\mathbb{C})$. For $i = 2, \dots, \mathrm{rk} \mathfrak{g}$, we define the elements $q_{i,m,n}$ of $(S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}))^{\mathfrak{g}}$ by the following relation:

$$(2) \quad q_i(ax + by) = \sum_{m+n=d_i} a^m b^n q_{i,m,n}(x, y),$$

for (a, b) in \mathbb{C}^2 and (x, y) in $\mathfrak{g} \times \mathfrak{g}$. For w in $W_G(\mathfrak{h})$, we denote by $\mathfrak{b}_{0,w}$ the subspace generated by $w(h)$ and \mathfrak{u} . The following result is well known and will be useful for the understanding of the principal bicone.

Lemma 3.7. — *If x is in \mathfrak{h} , then the intersection of $G \cdot x$ and \mathfrak{h} is the orbit of x under $W_G(\mathfrak{h})$.*

We wish to prove that $\mathcal{X}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$. We will prove this result next section.

Lemma 3.8. — i) *The subset $\mathcal{X}_{\mathfrak{g}}$ is the nullvariety of the polynomial functions $q_{i,m,n}$, for $i = 2, \dots, \mathrm{rk} \mathfrak{g}$ and $m + n = d_i$. In particular, any irreducible component of $\mathcal{X}_{\mathfrak{g}}$ has dimension at least $3(\mathfrak{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$.*

ii) *The subset $\mathfrak{s} \times \mathfrak{s}$ is contained in $\mathcal{X}_{\mathfrak{g}}$.*

iii) *For x in \mathfrak{b} , $(w(h), x)$ belongs to $\mathcal{X}_{\mathfrak{g}}$ if and only if x is in $\mathfrak{b}_{0,w}$.*

Proof. — i) As $\mathfrak{X}_{\mathfrak{g}}$ is the nullvariety of the q_i by Lemma 3.3, (ii), $\mathcal{X}_{\mathfrak{g}}$ is the nullvariety of the polynomial functions $q_{i,m,n}$, for $i = 2, \dots, \text{rk } \mathfrak{g}$ and $m+n = d_i$. As the degree d_1 of p_1 is 2, $\mathcal{X}_{\mathfrak{g}}$ is the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of $\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} - 3$ regular functions, whence the second statement.

ii) Let T be the intersection of $\mathcal{X}_{\mathfrak{g}}$ and $\mathfrak{s} \times \mathfrak{s}$. Then T is a closed, subset of $\mathfrak{s} \times \mathfrak{s}$ invariant under $\text{GL}_2(\mathbb{C})$ and \mathbf{S} . For any t in \mathbb{C} , $h + te$ belongs to $G.h$. So T contains (h, e) . Hence by Lemma 2.11, the dimension of T is bigger than 5 and T is equal to T_5 if and only if its dimension is equal to 5. As th and $e + t^2f$ are in the same G -orbit for any t in $\mathbb{C} \setminus \{0\}$, (e, f) is in $\mathcal{X}_{\mathfrak{g}}$. So T is equal to $\mathfrak{s} \times \mathfrak{s}$ since (e, f) is not in T_5 .

iii) For any t in $\mathbb{C} \setminus \{0\}$ and any x in \mathfrak{u} , $tw(h) + x$ belongs to the G -orbit of $tw(h)$. Hence $(w(h), tw(h) + x)$ belongs to $\mathcal{X}_{\mathfrak{g}}$. So $\{w(h)\} \times \mathfrak{b}_{0,w}$ is contained in $\mathcal{X}_{\mathfrak{g}}$. Let x_1 be in \mathfrak{h} and let x_2 be in \mathfrak{u} . We suppose that $(w(h), x_1 + x_2)$ is in $\mathcal{X}_{\mathfrak{g}}$. Then for any t in \mathbb{C} , $w(h) + tx_1 + tx_2$ is in $\mathfrak{X}_{\mathfrak{g}}$. In particular, $x_1 + x_2$ is tangent at $w(h)$ to the cone $\mathfrak{X}_{\mathfrak{g}}$. As the tangent space at $w(h)$ of the cone $\mathfrak{X}_{\mathfrak{g}}$ is generated by $w(h)$ and $[w(h), \mathfrak{g}]$, x_1 and $w(h)$ are colinear. \square

3.3. — As the nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ is contained in the principal cone $\mathfrak{X}_{\mathfrak{g}}$, the nilpotent bicone $\mathcal{N}_{\mathfrak{g}}$ is contained in the principal bicone $\mathcal{X}_{\mathfrak{g}}$. We introduce two other varieties “squeezed in between” $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{X}_{\mathfrak{g}}$. Let us set

$$\begin{aligned} \mathcal{Y}_{\mathfrak{g}} &:= \{ (x, y) \in \mathcal{X}_{\mathfrak{g}} \mid \langle x, y \rangle = 0 \}, \\ \mathcal{Z}_{\mathfrak{g}} &:= \{ (x, y) \in \mathcal{Y}_{\mathfrak{g}} \mid y \in \mathfrak{N}_{\mathfrak{g}} \}. \end{aligned}$$

Lemma 3.9. — i) The nilpotent bicone is the nullvariety in $\mathcal{X}_{\mathfrak{g}}$ of $p_{1,2,0}, p_{1,1,1}, p_{1,0,2}$.

ii) The subset $\mathcal{Y}_{\mathfrak{g}}$ is the nullvariety in $\mathcal{X}_{\mathfrak{g}}$ of $p_{1,1,1}$. Moreover, $\mathcal{Y}_{\mathfrak{g}}$ is a G -invariant closed bicone and any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ has dimension at least $3(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 2$.

iii) The subset $\mathcal{Z}_{\mathfrak{g}}$ is the nullvariety in $\mathcal{Y}_{\mathfrak{g}}$ of $p_{1,0,2}$. Moreover, $\mathcal{Z}_{\mathfrak{g}}$ is a G -invariant closed bicone and any irreducible component of $\mathcal{Z}_{\mathfrak{g}}$ has dimension at least $3(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$.

Proof. — i) As $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of p_1 in $\mathfrak{X}_{\mathfrak{g}}$, by Lemma 3.3 (i), $\mathcal{N}_{\mathfrak{g}}$ is the nullvariety in $\mathcal{X}_{\mathfrak{g}}$ of the 2-order polarizations of p_1 , that is to say $p_{1,2,0}, p_{1,1,1}, p_{1,0,2}$.

ii) As $p_{1,1,1}$ is the Killing form, $\mathcal{Y}_{\mathfrak{g}}$ is the nullvariety of $p_{1,1,1}$ in $\mathcal{X}_{\mathfrak{g}}$. Moreover, $\mathcal{Y}_{\mathfrak{g}}$ is a G -invariant closed bicone since $p_{1,1,1}$ is G -invariant and bihomogeneous. By Lemma 3.8, (i), any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ has dimension at least $3(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 2$.

iii) By Lemma 3.3, (i), $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of p_1 in $\mathfrak{X}_{\mathfrak{g}}$. Hence $\mathcal{Z}_{\mathfrak{g}}$ is the nullvariety of $p_{1,0,2}$ in $\mathcal{Y}_{\mathfrak{g}}$. Moreover, $\mathcal{Z}_{\mathfrak{g}}$ is a G -invariant closed bicone since

$p_{1,0,2}$ is G -invariant and bihomogeneous. By (ii), any irreducible component of $\mathcal{Z}_{\mathfrak{g}}$ has dimension at least $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$. \square

Remark 3.10. — By Lemma 3.9, the subsets $\mathcal{Y}_{\mathfrak{g}}$ and $\mathcal{Z}_{\mathfrak{g}}$ inherit a natural structure of scheme.

Let P be a subset of the set of polynomials

$$\{p_{1,2,0}, p_{1,1,1}, p_{1,0,2}, q_{i,m,d_i-m}, i = 2, \dots, \text{rk } \mathfrak{g}, m = 0, \dots, d_i\}.$$

We denote by $\mathbb{C}[P]$ the subalgebra of $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$ generated by P , σ_P the morphism of affine varieties whose comorphism is the canonical injection from $\mathbb{C}[P]$ to $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g})$, \mathcal{X}_P the nullvariety of P in $\mathfrak{g} \times \mathfrak{g}$.

Lemma 3.11. — *Let (x, y) be in $\Omega_{\mathfrak{g}}$.*

- i) *The morphism σ_P is smooth at (x, y) .*
- ii) *If (x, y) is in \mathcal{X}_P , then (x, y) is a smooth point of \mathcal{X}_P . Moreover, the unique irreducible component of \mathcal{X}_P which contains (x, y) has dimension $2\dim \mathfrak{g} - |P|$.*

Proof. — i) Let ρ be the morphism whose comorphism is the canonical injection from $\mathbb{C}[P]$ to the subalgebra generated by the polynomials p_{i,m,d_i-m} where $i = 1, \dots, \text{rk } \mathfrak{g}$ and $m = 0, \dots, d_i$. Then σ_P is equal to $\rho \circ \sigma$, where σ is the morphism introduced in Subsection 2.4. As q_{i,m,d_i-m} is a linear combination of p_{i,m,d_i-m} and homogeneous elements in the subalgebra generated by $p_{1,2,0}, p_{1,1,1}, p_{1,0,2}$, for $i = 2, \dots, \text{rk } \mathfrak{g}$ and $m = 0, \dots, d_i$, the morphism ρ is smooth. Then by Proposition 2.12, σ_P is a smooth morphism at (x, y) .

ii) We suppose that (x, y) is in \mathcal{X}_P . By definition, \mathcal{X}_P is the fiber at 0 of the morphism σ_P . So by (i), (x, y) is a regular point of \mathcal{X}_P and the codimension in $\mathfrak{g} \times \mathfrak{g}$ of the irreducible component of \mathcal{X}_P which contains (x, y) is equal to the dimension of $\mathbb{C}[P]$. By Proposition 2.12, the polynomials p_{i,m,d_i-m} are algebraically independent for $i = 1, \dots, \text{rk } \mathfrak{g}$ and $m = 0, \dots, d_i$. So the elements of P are algebraically independent since q_{i,m,d_i-m} is a linear combination of p_{i,m,d_i-m} and homogeneous elements in the subalgebra generated by $p_{1,2,0}, p_{1,1,1}, p_{1,0,2}$, for $i = 2, \dots, \text{rk } \mathfrak{g}$ and $m = 0, \dots, d_i$. Hence the dimension of the irreducible component of \mathcal{X}_P containing (x, y) is equal to $2\dim \mathfrak{g} - |P|$. \square

The following proposition assures that $\mathcal{X}_{\mathfrak{g}}$ has irreducible components of the expected dimension. Actually this proposition will be not useful in the following.

Proposition 3.12. — *Let w be in $W_G(\mathfrak{h})$. There is an unique irreducible component $\chi(w)$ of $\mathcal{X}_{\mathfrak{g}}$ containing $(w(h), e)$. The irreducible component $\chi(w)$ satisfies the following properties:*

- 1) $\chi(w)$ has dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$,
- 2) $\chi(w w_0)$ is equal to $\chi(w)$,

3) $\chi(w)$ contains $(w_0w(h), f)$, $(e, w(h))$, $(f, w_0w(h))$.

Moreover, $\chi(w)$ contains $\mathfrak{s} \times \mathfrak{s}$ if $w = 1$ or $w = w_0$.

Proof. — Let P be the subset $\{q_{i,m,d_i-m}, i = 2, \dots, \text{rk } \mathfrak{g}, m = 0, \dots, d_i\}$. Then \mathcal{X}_P is equal to $\mathcal{X}_{\mathfrak{g}}$ by Lemma 3.8, (i). By Lemma 3.8, (iii), $\mathcal{X}_{\mathfrak{g}}$ contains $(w(h), e)$ and by Lemma 2.3, (i), $(w(h), e)$ belongs to $\Omega_{\mathfrak{g}}$. Hence by Lemma 3.11, (ii), $(w(h), e)$ is a regular element of $\mathcal{X}_{\mathfrak{g}}$. So there is an unique irreducible component of $\mathcal{X}_{\mathfrak{g}}$ which contains $(w(h), e)$. Moreover, this component has dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$. Let us denote it by $\chi(w)$. Then $\chi(w)$ satisfies condition (1). As $ww_0(h)$ is equal to $-w(h)$, by Lemma 3.2, $\chi(w)$ contains $(ww_0(h), e)$ since $\chi(w)$ is a bicone. Moreover, by Lemma 3.2, $\chi(w)$ contains $(w_0w(h), f)$ since $\chi(w)$ is G -invariant and $g_0.(w(h), e)$ is equal to $(w_0w(h), f)$. As $\chi(w)$ is also $\text{GL}_2(\mathbb{C})$ -invariant, it is invariant under the involution $(x, y) \mapsto (y, x)$. So $\chi(w)$ contains $(e, w(h))$ and $(f, w_0w(h))$.

By Lemma 3.8, (ii), $\mathfrak{s} \times \mathfrak{s}$ is contained in $\mathcal{X}_{\mathfrak{g}}$. Hence there exists an irreducible component of $\mathcal{X}_{\mathfrak{g}}$ which contains $\mathfrak{s} \times \mathfrak{s}$. But $\mathfrak{s} \times \mathfrak{s}$ contains (h, e) , which is a smooth point of $\mathcal{X}_{\mathfrak{g}}$. So $\chi(w)$ contains $\mathfrak{s} \times \mathfrak{s}$ if $w(h)$ is colinear to h , that is to say $w = 1$ or $w = w_0$. \square

Remark 3.13. — When \mathfrak{g} is simple of type $B_{\ell}, C_{\ell}, D_{2\ell}, E_7, E_8, F_4$ or G_2 , w_0 is equal to -1 . So in these cases, for any w in $W_G(\mathfrak{h})$, $\chi(w)$ contains $(w(h), f)$ and $(f, w(h))$ since $\chi(w)$ is a bicone.

A quick computation shows that $\mathcal{X}_{\mathfrak{g}}$ has $|W_G(\mathfrak{h})|/2$ irreducible components when \mathfrak{g} is equal to $\mathfrak{sl}_n(\mathbb{C})$ for $n = 2, 3$.

3.4. — The goal of this subsection is to prove Corollary 3.17. This corollary will be crucial for the study of the dimension of $\mathcal{X}_{\mathfrak{g}}$ (see Section 4). We denote by κ the map

$$G \times \mathfrak{b}_- \times \mathfrak{b} \longrightarrow \mathfrak{g} \times \mathfrak{g}, (g, x, y) \mapsto (g(x), g(y)).$$

Lemma 3.14. — *Let x be a regular element of \mathfrak{g} which belongs to \mathfrak{h} .*

- i) *The map κ is a submersion at $(\mathbf{1}_{\mathfrak{g}}, x, x)$.*
- ii) *There exists an open neighborhood V of (x, x) in $\mathfrak{b}_- \times \mathfrak{b}$ such that the restriction of κ to $G \times V$ is a smooth morphism to an open subset of $\mathfrak{g} \times \mathfrak{g}$.*
- iii) *Let \mathcal{X} be a G -invariant, irreducible, closed subset of $\mathfrak{g} \times \mathfrak{g}$ which contains (x, x) . Then for any irreducible component X of the intersection of \mathcal{X} and $\mathfrak{b}_- \times \mathfrak{b}$ which contains (x, x) , $\kappa(G \times X)$ is dense in \mathcal{X} and X has dimension $\dim \mathcal{X} - 2(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$.*

Proof. — i) We use on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ the Killing form of $\mathfrak{g} \times \mathfrak{g}$,

$$((v_1, v_2), (w_1, w_2)) \mapsto \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle.$$

The tangent map of κ at $(\mathbf{1}_{\mathfrak{g}}, x, x)$ is the map

$$\mathfrak{g} \times \mathfrak{b}_- \times \mathfrak{b} \longrightarrow \mathfrak{g} \times \mathfrak{g}, (u, v, w) \mapsto ([u, x] + v, [u, x] + w).$$

So the orthogonal of its image is the subset of elements (v_1, v_2) of $\mathfrak{g} \times \mathfrak{g}$ such that v_1 is orthogonal to \mathfrak{b}_- , v_2 is orthogonal to \mathfrak{b} and,

$$\langle [u, x], v_1 \rangle + \langle [u, x], v_2 \rangle = 0,$$

for any u in \mathfrak{g} . The latter condition is equivalent to the equality:

$$[x, v_1] + [x, v_2] = 0.$$

As x is a regular element in \mathfrak{g} which belongs to \mathfrak{h} , the subspace of elements (v_1, v_2) of $\mathfrak{u}_- \times \mathfrak{u}$ which satisfy the above equality is equal to $\{0\}$. But \mathfrak{u}_- and \mathfrak{u} are the orthogonal of \mathfrak{b}_- and \mathfrak{b} respectively in \mathfrak{g} . Hence the tangent map of κ at $(\mathbf{1}_{\mathfrak{g}}, x, x)$ is surjective.

ii) By (i), there exists an open neighborhood V of (x, x) in $\mathfrak{b}_- \times \mathfrak{b}$ such that κ is a submersion at $(\mathbf{1}_{\mathfrak{g}}, y, z)$ for any (y, z) in V . As the subset of elements in $G \times \mathfrak{b}_- \times \mathfrak{b}$ at which κ is a submersion, is invariant by left translations of elements in G , κ is a submersion at (g, y, z) for any (g, y, z) in $G \times V$. So the restriction of κ to $G \times V$ is a smooth morphism from $G \times V$ to an open subset of $\mathfrak{g} \times \mathfrak{g}$.

iii) Let Y be the intersection of \mathcal{X} and V . As \mathcal{X} is G -invariant, $\kappa(G \times Y)$ is equal to the intersection of \mathcal{X} and $\kappa(G \times V)$. By (ii), this intersection is a non empty open subset of \mathcal{X} since \mathcal{X} contains (x, x) . Moreover, the fiber of κ at any element of $\kappa(G \times V)$ is equidimensional of dimension

$$\dim \mathfrak{g} + 2b_{\mathfrak{g}} - 2\dim \mathfrak{g} = \text{rk } \mathfrak{g}.$$

Hence Y is equidimensional of dimension

$$\dim \mathcal{X} + \text{rk } \mathfrak{g} - \dim \mathfrak{g} = \dim \mathcal{X} - 2(b_{\mathfrak{g}} - \text{rk } \mathfrak{g}),$$

since \mathcal{X} is irreducible. In particular, any irreducible component X of the intersection of \mathcal{X} and $\mathfrak{b}_- \times \mathfrak{b}$ which contains (x, x) has dimension $\dim \mathcal{X} - 2(b_{\mathfrak{g}} - \text{rk } \mathfrak{g})$ and $\kappa(G \times X)$ is dense in \mathcal{X} . \square

We recall that for w in $W_G(\mathfrak{h})$, $\mathfrak{b}_{0,w}$ is the subspace generated by $w(h)$ and \mathfrak{u} . We denote by $\mathfrak{b}_{0,w,-}$ the subspace generated by $w(h)$ and \mathfrak{u}_- .

Lemma 3.15. — i) *The nullvariety in $\mathfrak{b}_- \times \mathfrak{b}$ of the functions $(x, y) \mapsto q_i(x)$, $(x, y) \mapsto q_i(y)$ where $i = 2, \dots, \text{rk } \mathfrak{g}$, is the union of the subspaces $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w'}$ where w, w' run through $W_G(\mathfrak{h})$.*

ii) *The intersection of $\mathcal{X}_{\mathfrak{g}}$ and $\mathfrak{h} \times \mathfrak{h}$ is the union of the bicones generated by the elements $(w(h), w(h))$ where w is in $W_G(\mathfrak{h})$.*

iii) *Let \mathcal{X} be an irreducible component of $\mathcal{X}_{\mathfrak{g}}$ whose the image by ϖ_1 is equal to $\mathfrak{X}_{\mathfrak{g}}$. Let X be an irreducible component of the intersection of \mathcal{X} and $\mathfrak{b}_- \times \mathfrak{b}$*

whose the intersection with the diagonal of $\mathfrak{h} \times \mathfrak{h}$ is not equal to $\{0\}$. Then there exists w in $W_G(\mathfrak{h})$ such that X is contained in $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w}$.

Proof. — i) Since $\mathfrak{X}_{\mathfrak{g}}$ is the nullvariety of $q_2, \dots, q_{\text{rk } \mathfrak{g}}$, according to Lemma 3.3, (ii), the assertion (i) is an easy consequence of Lemma 3.7.

ii) By (i), the intersection of $\mathcal{X}_{\mathfrak{g}}$ and $\mathfrak{h} \times \mathfrak{h}$ is contained in the union of the bicones generated by the elements $(w(h), w'(h))$ where w, w' are in $W_G(\mathfrak{h})$. As this intersection is invariant under the diagonal action of $W_G(\mathfrak{h})$ in $\mathfrak{h} \times \mathfrak{h}$, we have to prove that for w in $W_G(\mathfrak{h})$, $(h, w(h))$ belongs to $\mathcal{X}_{\mathfrak{g}}$ if and only if $w(h)$ is colinear to h . But by Lemma 3.8, (iii), for such w , $w(h)$ belongs to $\mathfrak{b}_{0,1_{\mathfrak{h}}}$. So $w(h)$ is colinear to h .

iii) As \mathcal{X} is an irreducible component of $\mathcal{X}_{\mathfrak{g}}$, \mathcal{X} is invariant under the action of $\text{GL}_2(\mathbb{C})$. So the intersection of \mathcal{X} and $\mathfrak{h} \times \mathfrak{h}$ contains $(h, 0)$ and (h, h) since h is in the image of \mathcal{X} by ϖ_1 . By (i), as X is irreducible, there exist w and w' in $W_G(\mathfrak{h})$ such that X is contained $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w'}$. But by (ii), $(w(h), w'(h))$ belongs to $\mathcal{X}_{\mathfrak{g}}$ if and only if $w'(h)$ is colinear to $w(h)$. Since the intersection of X with the diagonal of $\mathfrak{h} \times \mathfrak{h}$ is not equal to $\{0\}$, X is contained in $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w}$. \square

Lemma 3.16. — *Let \mathcal{X} be an irreducible component of $\mathcal{X}_{\mathfrak{g}}$ and let X be an irreducible component of the intersection of \mathcal{X} and $\mathfrak{b}_- \times \mathfrak{b}$ which contains $(w(h), w(h))$, for some w in $W_G(\mathfrak{h})$. Let X' be the intersection of X and $\mathcal{Z}_{\mathfrak{g}}$:*

$$X' = \{(x, y) \in X \mid \langle x, y \rangle = 0 \text{ and } y \in \mathfrak{N}_{\mathfrak{g}}\} .$$

We suppose that there is an irreducible component Y of X' satisfying the two following conditions:

- 1) $\varpi_1(Y)$ contains a semisimple element of \mathfrak{g} ,
- 2) $\varpi_2(Y)$ contains a regular nilpotent element of \mathfrak{g} .

Then the sets Y, X', X and \mathcal{X} have a nonempty intersection with $\Omega_{\mathfrak{g}}$. In particular, X has dimension $\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 3$ and \mathcal{X} has dimension $3(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$.

Proof. — By Lemma 3.15, (iii), X is contained in $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w}$. Let Y be an irreducible component of X' satisfying conditions (1) and (2). By Condition (1), the subset Y_1 of elements (x, y) in Y such that x is semisimple is a nonempty open subset of Y since the subset of semisimple elements of \mathfrak{g} which belongs to $\mathfrak{b}_{0,w,-}$ is an open dense subset of $\mathfrak{b}_{0,w,-}$. On the other hand, by Condition (2), the subset Y_2 of elements (x, y) in Y such that y is a regular nilpotent element of \mathfrak{g} is a nonempty open subset of Y since the subset of regular nilpotent elements of \mathfrak{g} which belongs to \mathfrak{u} is an open dense subset of \mathfrak{u} . Let (x, y) be in the intersection of Y_1 and Y_2 . A semisimple element of $\mathfrak{b}_{0,w,-}$ is written up the form $aw(h) + u$ with u in \mathfrak{u}_- and a in $\mathbb{C} \setminus \{0\}$. Since Y is a bicone, we deduce that there are u in \mathfrak{u}_- and v regular element in \mathfrak{u} such that $(w(h) + u, v)$ belongs to Y . Let us suppose that $(w(h) + u, v)$ doesn't

belong to $\Omega_{\mathfrak{g}}$. We expect a contradiction. Then there exists t in \mathbb{C} such that $t(w(h)+u)+v$ is not a regular element of \mathfrak{g} since $w(h)+u$ is a regular element. As $\mathfrak{X}_{\mathfrak{g}} \setminus \mathfrak{g}_{\text{reg}}$ is contained in $\mathfrak{N}_{\mathfrak{g}}$, we obtain

$$\langle t(w(h)+u)+v, t(w(h)+u)+v \rangle = 0 .$$

But the left hand side of this equality is equal to $t^2 \langle h, h \rangle$ since $(t(w(h)+u), v)$ belongs to X' . Hence t is equal to 0 and $t(w(h)+u)+v$ is equal to v since $\langle h, h \rangle$ is not equal to 0. We have thus obtained the expected contradiction, since v is regular. At last, Y has a nonempty intersection with $\Omega_{\mathfrak{g}}$. As Y is contained in X' , X and \mathcal{X} , these subsets have a nonempty intersection with $\Omega_{\mathfrak{g}}$. In particular, according to Lemma 3.11, (ii), \mathcal{X} has dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$. In addition, by Lemma 3.14, (iii), X has dimension $\dim \mathcal{X} - 2(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) = \mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 3$. \square

Corollary 3.17. — *Let \mathcal{X} be an irreducible component of $\mathcal{X}_{\mathfrak{g}}$. We denote by \mathcal{X}' the intersection of \mathcal{X} and $\mathcal{Z}_{\mathfrak{g}}$. Let us suppose that the following conditions are fulfilled:*

- 1) *the image of any irreducible component of \mathcal{X}' by ϖ_1 is equal to $\mathfrak{X}_{\mathfrak{g}}$,*
- 2) *the image of any irreducible component of \mathcal{X}' by ϖ_2 is equal to $\mathfrak{N}_{\mathfrak{g}}$.*

Then \mathcal{X}' and \mathcal{X} have a non empty intersection with $\Omega_{\mathfrak{g}}$. In particular, \mathcal{X} has dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$.

Proof. — By Lemma 3.15, (iii), there exists w in $W_G(\mathfrak{h})$ such that the intersection of \mathcal{X} and $\mathfrak{b}_- \times \mathfrak{b}$ has an irreducible component X which contains $(w(h), w(h))$ and is contained in $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w}$. Let us denote by X' the subset of elements (x, y) of X such that y is nilpotent and $\langle x, y \rangle$ is equal to 0. As X is an irreducible component of the intersection of \mathcal{X} and $\mathfrak{b}_{0,w,-} \times \mathfrak{b}_{0,w}$ which contains $(w(h), w(h))$, $\kappa(G \times X)$ is dense in \mathcal{X} , by Lemma 3.9, (iii). Moreover, $\kappa(G \times X')$ is the intersection of \mathcal{X}' and $\kappa(G \times X)$. As $\kappa(G \times X)$ contains a dense open subset of \mathcal{X} , $\kappa(G \times X')$ contains a nonempty open subset of \mathcal{X}' . So the closure of $\kappa(G \times X')$ in $\mathfrak{g} \times \mathfrak{g}$ contains an irreducible component of \mathcal{X}' . Consequently, there exists an irreducible component Y of X' such that the closure of $\kappa(G \times Y)$ in $\mathfrak{g} \times \mathfrak{g}$ is an irreducible component ξ of \mathcal{X}' .

Let us suppose that $\varpi_2(Y)$ does not contain any regular element. As the subset of elements of \mathfrak{u} which are not regular in \mathfrak{g} is a finite union of ideals of \mathfrak{b} of codimension 1 in \mathfrak{u} , there exists an ideal \mathfrak{u}_1 of \mathfrak{b} contained in \mathfrak{u} , which does not contain any regular element and such that $\varpi_2(Y)$ is contained in \mathfrak{u}_1 , since $\varpi_2(Y)$ is irreducible. Let us recall that the contracted product $G \times_{\mathbf{B}} \mathfrak{b}$ is defined as the quotient of $G \times \mathfrak{b}$ under the right action of \mathbf{B} given by $(g, x).b = (gb, b^{-1}(x))$. We denote by π the map from $G \times_{\mathbf{B}} \mathfrak{b}$ to \mathfrak{g} such that $g(x)$ is the image by π of the image of (g, x) in $G \times_{\mathbf{B}} \mathfrak{b}$. Then the image of $G \times_{\mathbf{B}} \mathfrak{u}_1$ in $G \times_{\mathbf{B}} \mathfrak{b}$ is closed since \mathfrak{u}_1 is \mathbf{B} -invariant. Hence the image of $G \times_{\mathbf{B}} \mathfrak{u}_1$ by

the map $(g, x) \mapsto g(x)$ is closed in \mathfrak{g} , since π is a proper morphism. Moreover, this image is strictly contained in $\mathfrak{N}_{\mathfrak{g}}$ since it does not contain any regular nilpotent element. This is impossible since it contains $\varpi_2(\xi)$ which is equal to $\mathfrak{N}_{\mathfrak{g}}$ by condition (2). Hence $\varpi_2(Y)$ contains a regular nilpotent element of \mathfrak{g} .

Let us suppose that $\varpi_1(Y)$ does not contain any semisimple element. Then $\varpi_1(Y)$ is contained in \mathfrak{u}_- . So, $\varpi_1(\xi)$ is contained in the image of $G \times \mathfrak{u}_-$ by the map $(g, x) \mapsto g(x)$, which is equal to $\mathfrak{N}_{\mathfrak{g}}$. But, this contradicts Condition (1). As a result, Y satisfies Conditions (1) and (2) of the Lemma 3.16. Therefore, by Lemma 3.16, \mathcal{X}' and \mathcal{X} have a non empty intersection with $\Omega_{\mathfrak{g}}$. Moreover, by Lemma 3.11, (ii), \mathcal{X} has dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$. \square

4. Dimension of the nilpotent bicone via motivic integration

We prove in this section the main result of this note. As before, we suppose that \mathfrak{g} is simple. We use the notations of previous sections. The goal of this section is the following theorem:

Theorem 4.1. — i) *The principal bicone of \mathfrak{g} is a reduced complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g} + 1)$.*

ii) *The subscheme $\mathcal{Y}_{\mathfrak{g}}$ is a reduced complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 2$. Moreover, any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ is the intersection of $\mathcal{Y}_{\mathfrak{g}}$ with an irreducible component of $\mathcal{X}_{\mathfrak{g}}$.*

iii) *The subscheme $\mathcal{Z}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$.*

iv) *The nilpotent bicone $\mathcal{N}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$.*

Let us describe briefly our approach to prove Theorem 4.1. We plan to apply Corollary 3.17. More precisely, we wish to prove that any irreducible component of maximal dimension of $\mathcal{X}_{\mathfrak{g}}$ satisfies the conditions (1) and (2) of Corollary 3.17. Then, we will deduce Theorem 4.1, (i), from Lemma 3.8, (i). The statements (ii), (iii) and (iv) of Theorem 4.1 will be roughly consequences of statement (i). The main point is therefore to study the images by ϖ_1 and ϖ_2 of the irreducible components of maximal dimension of $\mathcal{X}_{\mathfrak{g}}$, $\mathcal{Y}_{\mathfrak{g}}$, $\mathcal{Z}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}}$. To do that, we are going to consider jet schemes of $\mathfrak{X}_{\mathfrak{g}}$ and $\mathfrak{N}_{\mathfrak{g}}$ and arguments from motivic integration.

4.1. — We review in this subsection the definition and some properties of jet schemes. Let X be a complex algebraic variety. For any nonnegative integer m and any x in X , a m -order jet at x of X is a morphism from $\mathcal{O}_{X,x}$ to the quotient ring $\mathbb{C}[t]/t^{m+1}\mathbb{C}[t]$. The subset $J_m(X)$ of m -order jets has a natural

structure of scheme. The canonical projections

$$\pi_{l,m} : J_l(X) \longrightarrow J_m(X) ,$$

for $l \geq m$, induce a projective system $(J_m(X), \pi_{l,m})$. The *space of the arcs*

$$J_\infty(X) := \operatorname{proj} \lim_m J_m(X)$$

is the projective limit of the system $(J_m(X), \pi_{l,m})$. The subset $J_\infty(X)$ has a natural structure of scheme. We denote by $\pi_{\infty,m}$ the canonical projection from $J_\infty(X)$ to $J_m(X)$:

$$\pi_{\infty,m} : J_\infty(X) \longrightarrow J_m(X) .$$

For ν in $J_\infty(X)$ and φ a regular function on X , the order of the serie $\varphi \circ \nu$ will be denoted by $\operatorname{ord}(\nu \circ \varphi)$. If \mathcal{I} is an ideal of \mathcal{O}_X , we denote by $\operatorname{ord}(\mathcal{I}, \nu)$ the smallest integer $\operatorname{ord}(\nu \circ \varphi)$, where φ runs through $\mathcal{I}_{\pi_{\infty,0}(\nu)}$, the ideal generated by \mathcal{I} in the local ring $\mathcal{O}_{X, \pi_{\infty,0}(\nu)}$. The function $\nu \mapsto \operatorname{ord}(\mathcal{I}, \nu)$ is denoted by $F_{\mathcal{I}}$. By [6](Theorem 2.1), $F_{\mathcal{I}}$ is semialgebraic. As a matter of fact, for any nonnegative integer m , we can define an analogous function from $J_m(X)$ to $\{1, \dots, m\}$. We denote it by the same symbol. When X is affine and I is the space of global sections of \mathcal{I} , we denote by F_I the function $F_{\mathcal{I}}$. When \mathcal{I} is the ideal of definition of a closed subset Z of X , we denote by F_Z the function $F_{\mathcal{I}}$. In particular, when \mathcal{I} is the ideal of definition of a divisor D of X , we denote by F_D the function $F_{\mathcal{I}}$.

In [25], M. Mustața proves the following result, conjectured by David Eisenbud and Edward Frenkel:

Theorem 4.2 (Mustața). — *If X is locally a complete intersection variety, then $J_m(X)$ is irreducible of dimension $\dim X(m+1)$ for all $m \geq 1$ if and only if X has rational singularities.*

Remark 4.3. — The nilpotent cone $\mathfrak{N}_{\mathfrak{g}}$ is a complete intersection and by [15](Theorem A), it has rational singularities. In addition, by Corollary 3.4, $\mathfrak{X}_{\mathfrak{g}}$ is a complete intersection and has rational singularities. As a consequence, Theorem 4.2 can be applied to $\mathfrak{N}_{\mathfrak{g}}$ or $\mathfrak{X}_{\mathfrak{g}}$. In the appendix of [25], D. Eisenbud and E. Frenkel apply Theorem 4.2 to $\mathfrak{N}_{\mathfrak{g}}$ to extend results of Kostant in the setting of jet schemes.

We conclude this subsection with two lemmas useful in Subsection 4.6. We denote by θ the map

$$\mathbb{C} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g} , (t, x, y) \mapsto (x, y + tx) .$$

Lemma 4.4. — *Let \mathcal{X} be a G -invariant, irreducible, closed bicone of $\mathfrak{g} \times \mathfrak{g}$. We suppose that $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$ and we suppose that $\theta(\mathbb{C} \times \mathcal{X})$ is*

contained in \mathcal{X} . Let \mathcal{X}_1 and \mathcal{X}_2 be two irreducible components of the nullvariety of $p_{1,1,1}$ in \mathcal{X} . If $\varpi_1(\mathcal{X}_1)$ and $\varpi_1(\mathcal{X}_2)$ are equal to $\mathfrak{X}_{\mathfrak{g}}$, then \mathcal{X}_1 is equal to \mathcal{X}_2 .

Proof. — We can suppose that $p_{1,1,1}$ is not identically null on \mathcal{X} and we suppose that $\varpi_1(\mathcal{X}_1)$ and $\varpi_1(\mathcal{X}_2)$ are equal to $\mathfrak{X}_{\mathfrak{g}}$. In particular, \mathcal{X}_1 and \mathcal{X}_2 have codimension 1 in \mathcal{X} . For x semisimple element of $\mathfrak{X}_{\mathfrak{g}}$ and y in \mathfrak{g} such that (x, y) belongs to the nullvariety of $p_{1,1,1}$ in \mathcal{X} , $p_{1,1,1}(x, x + y)$ is not equal to 0. Indeed, by Lemma 3.3, (i), $\langle x, x \rangle$ is not equal to 0. Moreover, by hypothesis, for $i = 1, 2$, $\theta(\mathbb{C} \times \mathcal{X}_i)$ is an irreducible constructible subset of \mathcal{X} which strictly contains \mathcal{X}_i . Hence for $i = 1, 2$, $\theta(\mathbb{C} \times \mathcal{X}_i)$ contains a dense open subset of \mathcal{X} . As a result, for any (t, x, y) in a certain dense open subset of $\mathbb{C} \times \mathcal{X}_1$, x is semisimple and there exists (t', x', y') in $\mathbb{C} \times \mathcal{X}_2$ such that $\theta(t', x', y')$ is equal to $\theta(t, x, y)$. From this equality we deduce the equalities:

$$x' = x, \quad y' - y = (t - t')x.$$

But $\langle x, x \rangle$ is not equal to 0 since x is a semisimple element of $\mathfrak{X}_{\mathfrak{g}}$. In addition x is orthogonal to y and y' . So t' is equal to t and (x, y) is equal to (x', y') . As a consequence \mathcal{X}_1 is equal to \mathcal{X}_2 . \square

Let $T\mathfrak{X}_{\mathfrak{g}}$ be the tangent space of $\mathfrak{X}_{\mathfrak{g}}$ and let $T\mathfrak{N}_{\mathfrak{g}}$ be the tangent space of $\mathfrak{N}_{\mathfrak{g}}$. Then $T\mathfrak{X}_{\mathfrak{g}}$ and $T\mathfrak{N}_{\mathfrak{g}}$ are closed subsets of $\mathfrak{g} \times \mathfrak{g}$. Moreover, $T\mathfrak{X}_{\mathfrak{g}}$ and $T\mathfrak{N}_{\mathfrak{g}}$ are naturally in bijection with $J_1(\mathfrak{X}_{\mathfrak{g}})$ and $J_1(\mathfrak{N}_{\mathfrak{g}})$ respectively. Let $T'\mathfrak{X}_{\mathfrak{g}}$ be the nullvariety of $p_{1,1,1}$ in $T\mathfrak{X}_{\mathfrak{g}}$ and let $T''\mathfrak{X}_{\mathfrak{g}}$ be the nullvariety of $p_{1,0,2}$ in $T'\mathfrak{X}_{\mathfrak{g}}$.

Lemma 4.5. — i) *The subsets $T\mathfrak{X}_{\mathfrak{g}}$ and $T\mathfrak{N}_{\mathfrak{g}}$ are closed, irreducible bicones, invariant under G and their images by ϖ_1 are equal to $\mathfrak{X}_{\mathfrak{g}}$ and $\mathfrak{N}_{\mathfrak{g}}$ respectively. Moreover, their dimensions are $4(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 2$ and $4(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$ respectively.*

ii) *The intersection of $\varpi_1^{-1}(\mathfrak{N}_{\mathfrak{g}})$ and $T'\mathfrak{X}_{\mathfrak{g}}$ is equal to $T\mathfrak{N}_{\mathfrak{g}}$.*

iii) *The subset $T'\mathfrak{X}_{\mathfrak{g}}$ is irreducible of dimension $4(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$. Moreover, $\varpi_1(T'\mathfrak{X}_{\mathfrak{g}})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$.*

iv) *The image of any irreducible component of $T''\mathfrak{X}_{\mathfrak{g}}$ by ϖ_1 is equal to $\mathfrak{X}_{\mathfrak{g}}$.*

Proof. — i) As $\mathfrak{X}_{\mathfrak{g}}$ and $\mathfrak{N}_{\mathfrak{g}}$ are G -invariant closed cones, $T\mathfrak{X}_{\mathfrak{g}}$ and $T\mathfrak{N}_{\mathfrak{g}}$ are G -invariant closed bicones. By Theorem 4.2 and Remark 4.3, $J_1(\mathfrak{X}_{\mathfrak{g}})$ is irreducible and has dimension $2\dim \mathfrak{X}_{\mathfrak{g}}$. Since $T\mathfrak{X}_{\mathfrak{g}}$ and $J_1(\mathfrak{X}_{\mathfrak{g}})$ are canonically isomorphic, $T\mathfrak{X}_{\mathfrak{g}}$ is irreducible and has dimension $4(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 2$. In the same way, by Theorem 4.2 and Remark 4.3, $J_1(\mathfrak{N}_{\mathfrak{g}})$ is irreducible and has dimension $2\dim \mathfrak{N}_{\mathfrak{g}}$. Since $T\mathfrak{N}_{\mathfrak{g}}$ and $J_1(\mathfrak{N}_{\mathfrak{g}})$ are canonically isomorphic, $T\mathfrak{N}_{\mathfrak{g}}$ is irreducible and has dimension $4(\mathfrak{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$.

ii) For x in $G.e$ and y in \mathfrak{g} , (x, y) belongs to $T\mathfrak{N}_{\mathfrak{g}}$ if and only if y belongs to $[x, \mathfrak{g}]$ since $G.e$ is dense in $\mathfrak{N}_{\mathfrak{g}}$. In particular, for any x in $G.e$ and any y in \mathfrak{g} such that (x, y) belongs to $T\mathfrak{N}_{\mathfrak{g}}$, (x, y) belongs to $T'\mathfrak{X}_{\mathfrak{g}}$ since x is orthogonal to $[x, \mathfrak{g}]$. But by (i), the intersection of $T\mathfrak{N}_{\mathfrak{g}}$ and $\varpi_1^{-1}(G.e)$ is dense in $T\mathfrak{N}_{\mathfrak{g}}$. Then $T\mathfrak{N}_{\mathfrak{g}}$ is contained in $T'\mathfrak{X}_{\mathfrak{g}}$. As $\mathfrak{X}_{\mathfrak{g}}$ is the nullvariety in \mathfrak{g} of $q_2, \dots, q_{\text{rk } \mathfrak{g}}$,

for any x in $\mathfrak{X}_{\mathfrak{g}}$, the subspace of elements y of \mathfrak{g} such that (x, y) belongs to $T\mathfrak{X}_{\mathfrak{g}}$ is the intersection of the kernels of the differentials at x of $q_2, \dots, q_{\text{rk } \mathfrak{g}}$. By definition, for $i = 2, \dots, \text{rk } \mathfrak{g}$,

$$q_i = \begin{cases} p_i, & \text{if } d_i \text{ is odd} \\ p_1(h)^{d_i/2} p_i - p_i(h) p_1^{d_i/2}, & \text{otherwise} \end{cases}.$$

So for x in $\mathfrak{N}_{\mathfrak{g}}$, the subset of elements y of \mathfrak{g} such that (x, y) belongs to $T\mathfrak{X}_{\mathfrak{g}}$, is the intersection of the kernels of the differentials at x of $p_2, \dots, p_{\text{rk } \mathfrak{g}}$. Therefore, the intersection of $T'\mathfrak{X}_{\mathfrak{g}}$ and $\varpi_1^{-1}(\mathfrak{N}_{\mathfrak{g}})$ is contained in $T\mathfrak{N}_{\mathfrak{g}}$, since $\mathfrak{N}_{\mathfrak{g}}$ is the nullvariety of $p_1, \dots, p_{\text{rk } \mathfrak{g}}$ in \mathfrak{g} , whence we deduce (ii)

iii) As $\mathfrak{X}_{\mathfrak{g}}$ is a cone, $T\mathfrak{X}_{\mathfrak{g}}$ contains the diagonal of $\mathfrak{X}_{\mathfrak{g}} \times \mathfrak{X}_{\mathfrak{g}}$. So $T\mathfrak{X}_{\mathfrak{g}}$ strictly contains $T'\mathfrak{X}_{\mathfrak{g}}$. Then by (i), $T'\mathfrak{X}_{\mathfrak{g}}$ has dimension $4(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) + 1$. Moreover, $T'\mathfrak{X}_{\mathfrak{g}}$ is an equidimensional, closed bicone, invariant under the action of G since $p_{1,1,1}$ is bihomogeneous and G -invariant. Hence by (ii) and Lemma 1.3, the image by ϖ_1 of any irreducible component of $T'\mathfrak{X}_{\mathfrak{g}}$ is equal to $\mathfrak{X}_{\mathfrak{g}}$. As $\mathfrak{X}_{\mathfrak{g}}$ is a cone, for any t in \mathbb{C} and any (x, y) in $T\mathfrak{X}_{\mathfrak{g}}$, $(x, y + tx)$ belongs to $T\mathfrak{X}_{\mathfrak{g}}$. Then by Lemma 4.4, $T'\mathfrak{X}_{\mathfrak{g}}$ is irreducible.

iv) By (iii), $T''\mathfrak{X}_{\mathfrak{g}}$ is equidimensional of dimension at least $4(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$, since $T''\mathfrak{X}_{\mathfrak{g}}$ is the nullvariety of $p_{1,0,2}$ in $T'\mathfrak{X}_{\mathfrak{g}}$. Moreover, since $p_{1,0,2}$ is bihomogenous and G -invariant, $T''\mathfrak{X}_{\mathfrak{g}}$ is a G -invariant, closed, bicone. As (e, h) belongs to $T'\mathfrak{X}_{\mathfrak{g}}$, $T''\mathfrak{X}_{\mathfrak{g}}$ is strictly contained in $T'\mathfrak{X}_{\mathfrak{g}}$ and $T''\mathfrak{X}_{\mathfrak{g}}$ is equidimensional of dimension $4(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g})$. In addition, as (e, h) belongs to $T\mathfrak{N}_{\mathfrak{g}}$ but not to $T''\mathfrak{X}_{\mathfrak{g}}$, we deduce from (ii) and Lemma 3.3, (i), that the intersection of $T''\mathfrak{X}_{\mathfrak{g}}$ and $\varpi_1^{-1}(\mathfrak{N}_{\mathfrak{g}})$ is equidimensional of dimension $4(\text{b}_{\mathfrak{g}} - \text{rk } \mathfrak{g}) - 1$. Hence, by Lemma 1.3, the image of any irreducible component of $T''\mathfrak{X}_{\mathfrak{g}}$ is equal to $\mathfrak{X}_{\mathfrak{g}}$. \square

4.2. — We recall in this subsection some facts about motivic integration introduced by M. Kontsevich. The construction of motivic integrals for smooth spaces is due to Kontsevich [17] and was generalized by Denef and Loeser to singular spaces in [6] and [7]. We refer to [23] and [6] for more explanations.

We denote by \mathcal{V} the category of complex algebraic varieties, $K_0(\mathcal{V})$ its Grothendieck ring, \mathbb{L} the class of \mathbb{A}^1 in $K_0(\mathcal{V})$. For X in \mathcal{V} , its class in $K_0(\mathcal{V})$ is denoted by $[X]$ and the map $X \mapsto [X]$ extends naturally to the category of constructible subsets of algebraic varieties. The localization $K_0(\mathcal{V})[\mathbb{L}^{-1}]$ at \mathbb{L} of the ring $K_0(\mathcal{V})$ will be denoted by \mathbf{M} . For m in \mathbb{Z} , let $F^m \mathbf{M}$ be the subgroup of \mathbf{M} generated by the elements $[X]\mathbb{L}^{-r}$ where $r - \dim X$ is bigger than m . Then we get a decreasing filtration of the ring \mathbf{M} and we denote by $\hat{\mathbf{M}}$ its separated completion.

Proposition 4.6 ([6](Definition-Proposition 3.2))

Let X be a variety of pure dimension d . There is a well defined subalgebra \mathbf{B}_X of the Boolean algebra of subsets of $J_\infty(X)$ and a map μ_X from \mathbf{B}_X to $\hat{\mathbf{M}}$ satisfying the following properties:

- 1) for any nonnegative integer m and any constructible subset C in $J_m(X)$, the cylinder $\pi_{\infty,m}^{-1}(C)$ is in \mathbf{B}_X and $\mu_X(\pi_{\infty,m}^{-1}(C))$ is equal to $[C]\mathbb{L}^{-(m+1)d}$,
- 2) if Y is a closed subvariety in X of dimension strictly smaller than d , then for any element A in \mathbf{B}_X , contained in $J_\infty(Y)$, $\mu_X(A)$ is equal to 0,
- 3) let A_1, A_2, \dots be a sequence of elements in \mathbf{B}_X whose union A is in \mathbf{B}_X , then the sequence $\mu_X(A_1), \mu_X(A_2), \dots$ converges to $\mu_X(A)$ in $\hat{\mathbf{M}}$,
- 4) if A and B are in \mathbf{B}_X , A is contained in B and $\mu_X(B)$ belongs to the closure of $F^m\mathbf{M}$ in $\hat{\mathbf{M}}$ then $\mu_X(A)$ belongs to the closure of $F^m\mathbf{M}$ in $\hat{\mathbf{M}}$.

Let A be in \mathbf{B}_X . For φ a map from A to $\hat{\mathbf{M}}$ such that $\varphi^{-1}(\{a\})$ is in \mathbf{B}_X for any a in A , we say that φ is *integrable on A with respect to μ_X* if the sum

$$\sum_a \mu_X(\varphi^{-1}(\{a\}))a$$

converges in $\hat{\mathbf{M}}$. In particular, this means that the sum has at most countably many terms $\mu_X(\varphi^{-1}(\{a_1\}))a_1, \mu_X(\varphi^{-1}(\{a_2\}))a_2, \dots$ and that for $i = 1, 2, \dots$, $\mu_X(\varphi^{-1}(\{a_i\}))a_i$ belongs to the closure of $F^{m_i}\mathbf{M}$ in $\hat{\mathbf{M}}$ where m_1, m_2, \dots is a sequence of integers which goes to $-\infty$ when i goes to ∞ . This sum is called *the integral of φ on A with respect to μ_X* and we write:

$$\int_A \varphi d\mu_X := \sum_a \mu_X(\varphi^{-1}(\{a\}))a .$$

For ψ a function from A to $\mathbb{N} \cup \{+\infty\}$ such that $\psi^{-1}(\{n\})$ is in \mathbf{B}_X for any n in $\mathbb{N} \cup \{+\infty\}$, the map $\mathbb{L}^{-\psi}, x \mapsto \mathbb{L}^{-\psi(x)}$, from A to $\hat{\mathbf{M}}$, is integrable if and only the sum

$$\sum_{n \in \mathbb{N}} \mu_X(A \cap \psi^{-1}(\{n\}))\mathbb{L}^{-n}$$

is convergent in $\hat{\mathbf{M}}$. In this case, this sum is called *the integral of $\mathbb{L}^{-\psi}$* and we set

$$\int_A \mathbb{L}^{-\psi} d\mu_X := \sum_{n \in \mathbb{N}} \mu_X(A \cap \psi^{-1}(\{n\}))\mathbb{L}^{-n} .$$

Let us recall that, for a non irreducible variety, its dimension is the biggest dimension of its irreducible components. Let u and v be two indeterminate

variables. For any smooth projective variety X , we denote by $h(X)$ the Hodge-Deligne polynomial of X which is the element of $\mathbb{Z}[u, v]$:

$$h(X) := \sum_{k \in \mathbb{N}} \sum_{\substack{(p,q) \in \mathbb{N}^2 \\ p+q=k}} (-1)^k h_{p,q}(\mathbb{H}_c^k(X; \mathbb{C})) u^p v^q$$

where $h_{p,q}(\mathbb{H}_c^k(X; \mathbb{C}))$ are the Hodge-Deligne numbers of X . The map h factors through the ring $K_0(\mathcal{V})$. So we have a morphism h from $K_0(\mathcal{V})$ to $\mathbb{Z}[u, v]$ such that $h(X)$ is equal to $h([X])$ for any smooth projective variety. In particular, $h(\mathbb{L})$ is equal to uv . What is important for us is that $h([X])$ is a polynomial of degree $2\dim X$ and whose term of degree $2\dim X$ is $c(uv)^{\dim X}$, where c is the number of irreducible components of dimension $\dim X$. By continuity, the morphism h extends uniquely as a morphism from $\hat{\mathbf{M}}$ to the ring $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]]$.

If (x, y) belongs to $\mathcal{X}_{\mathfrak{g}}$ or $\mathcal{N}_{\mathfrak{g}}$, then the arc $t \mapsto x + ty$ is an element of $J_{\infty}(\mathcal{X}_{\mathfrak{g}})$ or $J_{\infty}(\mathcal{N}_{\mathfrak{g}})$ respectively. So $\mathcal{X}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}}$ can be identified to subsets of $J_{\infty}(\mathcal{X}_{\mathfrak{g}})$ and $J_{\infty}(\mathcal{N}_{\mathfrak{g}})$ respectively. But, as $\mathcal{X}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}}$ are strictly contained in \mathfrak{g} , the motivic measure of the so obtained subsets is null by Proposition 4.6, (2). That's why we need to make a subtler construction in Subsection 4.5 to use arguments from motivic integration.

4.3. — We prove in this subsection a technical lemma which will be used in the proof of Proposition 4.9. Let t be a positive integer and let N be an integer bigger than t . Let $b_1, \dots, b_t, c_2, \dots, c_t$ be nonnegative integers and let m, a_2, \dots, a_t be positive integers. We set $a_1 = 1$ and $c_1 = 0$. For k a nonnegative integer, for J a nonempty subset of $\{1, \dots, t\}$ and for J' a nonempty subset of J , we denote by $\psi_{k,J,J'}$, $\psi_{J,J'}$ the affine functionals on \mathbb{Q}^J

$$\begin{aligned} (\alpha_i, i \in J) &\longmapsto (N - |J'|)(m + 1) + \sum_{i \in J} (b_i + 1)\alpha_i - \sum_{i \in J \setminus J'} \alpha_i + k \sum_{i \in J} c_i \alpha_i, \\ (\alpha_i, i \in J) &\longmapsto (N - |J'|)(m + 1) + \sum_{i \in J} (b_i + 1)\alpha_i - \sum_{i \in J \setminus J'} \alpha_i, \end{aligned}$$

respectively. For p a positive integer, we denote by $A_{J,J',p}$ the subset:

$$\begin{aligned} A_{J,J',p} &:= \{(\alpha_i, i \in J) \in \mathbb{N}^J \mid \sum_{i \in J} a_i \alpha_i \geq p \text{ and } \alpha_i \geq m + 1, i \in J' \\ &\quad \text{and } 1 \leq \alpha_i \leq m, i \in J \setminus J'\}. \end{aligned}$$

Lemma 4.7. — For k a positive integer, J a nonempty subset of $\{1, \dots, t\}$ and J' a nonempty subset of J , we denote by $\nu_{k,J,J',p}$ and $\nu_{J,J',p}$ the minimal values of $\psi_{k,J,J'}$ and $\psi_{J,J'}$ respectively on $A_{J,J',p}$.

i) Let k be a positive integer and let p be an integer bigger than $m + 1$. If J is equal to $\{1\}$, then $\nu_{J,J',p}$ and $\nu_{k,J,J',p}$ are equal to $p(b_1 + 1) + (N - 1)(m + 1)$.

ii) We suppose that $b_1 + 1$ is strictly smaller than $\frac{b_i + 1}{a_i}$ for any $i = 2, \dots, t$. Then for p an integer bigger than $m + 1$, $\nu_{k,J,J',p}$ is bigger than $\nu_{k,\{1\},\{1\},p}$ for any positive integer k , any nonempty subset J of $\{1, \dots, t\}$ and any nonempty subset J' of J .

Proof. — i) Let us suppose that J is equal to $\{1\}$. Then J' is equal to $\{1\}$ and $A_{J,J',p}$ is equal to $p(\mathbb{N} \setminus \{0\})$. Hence $\nu_{k,J,J',p}$ and $\nu_{J,J',p}$ are equal to $p(b_1 + 1) + (N - 1)(m + 1)$.

ii) Let k be a positive integer, let J be a nonempty subset of $\{1, \dots, t\}$ and let J' be a nonempty subset of J . As c_1, \dots, c_t are nonnegative integers, for any $(\alpha_i, i \in J)$ in \mathbb{N}^J ,

$$\psi_{k,J,J'}(\alpha_i, i \in J) \geq \psi_{J,J'}(\alpha_i, i \in J) .$$

Hence $\nu_{k,J,J',p}$ is bigger than $\nu_{J,J',p}$. So by (i), we have to prove that $\nu_{J,J',p}$ is bigger than $p(b_1 + 1) + (N - 1)(m + 1)$. In other words, we have to prove that

$$(N - |J'|)(m + 1) + \sum_{i \in J'} (b_i + 1)\alpha_i + \sum_{i \in J \setminus J'} b_i \alpha_i - p(b_1 + 1) - (N - 1)(m + 1)$$

is a nonnegative integer since p is bigger than $m + 1$.

By hypothesis, for $i = 2, \dots, t$, b_i is bigger than $a_i(b_1 + 1)$. Then for any $(\alpha_i, i \in J)$ in $(\mathbb{N} \setminus \{0\})^J$, we have the relations:

$$\begin{aligned} \sum_{i \in J} (b_i + 1)\alpha_i - \sum_{i \in J \setminus J'} \alpha_i &= \sum_{i \in J \setminus \{1\}} b_i \alpha_i + (b_1 + 1)\alpha_1 + \sum_{i \in J \setminus \{1\}} \alpha_i - \sum_{i \in J \setminus J'} \alpha_i \\ &\geq (b_1 + 1) \sum_{i \in J} a_i \alpha_i + \sum_{i \in J' \setminus \{1\}} \alpha_i , \end{aligned}$$

since $a_1 = 1$. As a consequence, we get:

$$\begin{aligned} (N - |J'|)(m + 1) &+ \sum_{i \in J} (b_i + 1)\alpha_i - \sum_{i \in J \setminus J'} \alpha_i \\ &- p(b_1 + 1) - (N - 1)(m + 1) \\ &\geq \sum_{i \in J' \setminus \{1\}} \alpha_i - (|J'| - 1)(m + 1) \\ &\geq (|J' \setminus \{1\}| - |J'| + 1)(m + 1) , \end{aligned}$$

as soon as $(\alpha_i, i \in J)$ belongs to $A_{J,J',p}$, whence the statement. \square

4.4. — Let V be a finite dimensional vector space. For (x, y) in $V \times V$, we denote by $\nu_{x,y}$ the arc $t \mapsto x + ty$ of $J_\infty(V)$. Let m be in \mathbb{N} . We denote by $\nu_{x,y,m}$ the image of $\nu_{x,y}$ by the canonical projection $\pi_{\infty,m}^V$ from $J_\infty(V)$ to $J_m(V)$. As V is a vector space, there is a canonical injection from $J_m(V)$ into $J_{m+1}(V)$. Moreover, this injection is a closed immersion. The first projection from $V \times V$ to V is denoted by ϖ_1 .

Let \mathbf{K} be a connected closed subgroup of $\mathrm{GL}(V)$, let X be an irreducible closed cone in V . We suppose that X is a complete intersection in V with rational singularities and that X is a finite union of \mathbf{K} -orbits. We study in this subsection various properties of particular subsets of $J_\infty(V)$. Let N be the dimension of X and let r be the codimension $\mathrm{codim}(V, X)$ of X in V .

The main technic used in [25] to prove [25](Theorem 3.2) is motivic integration, as developed by Kontsevitch, Denef and Loeser, and Batyrev. We are going to use this technic to prove Proposition 4.9. Precisely, we closely follow the approach of the proof of [25](Theorem 3.2).

We denote by Z the union of \mathbf{K} -orbits in X which are not of biggest dimension. Let B_X be the blowing up of V whose center is X and let

$$\tau : B_X \longrightarrow V$$

be the canonical morphism from B_X to V . As X is a complete intersection in V , $\tau^{-1}(X)$ is an integral divisor on B_X and is locally a complete intersection. Moreover, there exists a regular action of \mathbf{K} on B_X for which τ is a \mathbf{K} -equivariant morphism. By the theorem of embedded desingularization of Hironaka [16], there exists a desingularization $(Y, \tilde{\tau})$ of B_X such that $(\tau \circ \tilde{\tau})^{-1}(X)$ is a divisor with normal crossings. Let us denote by γ the morphism $\tau \circ \tilde{\tau}$. Moreover, we can find $(Y, \tilde{\tau})$ such that there exists a regular action of \mathbf{K} on Y for which $\tilde{\tau}$ is an equivariant morphism, and $(\tau \circ \tilde{\tau})^{-1}(X)$ is a \mathbf{K} -invariant divisor. Denoting by E_1, \dots, E_t the irreducible components of $\gamma^{-1}(X)$, we can assume that the following conditions are fulfilled:

- a) E_1 is the only prime divisor dominating X ,
- b) the divisor $\gamma^{-1}(X)$ is equal to $\sum_{i=1}^t a_i E_i$,
- c) the discrepancy W of γ is equal to $\sum_{i=1}^t b_i E_i$,
- d) a_1 is equal to 1 and $b_1 + 1$ is equal to $\mathrm{codim}(V, X) = r$,
- e) $\gamma^{-1}(Z)$ is contained in the union of E_2, \dots, E_t ,

since $\tau^{-1}(X)$ is an integral divisor on B_X and \mathbf{K} has finitely many orbits in X . By condition (b), E_1, \dots, E_t are \mathbf{K} -invariant. Moreover, by condition (e), Z is the image by γ of the union of E_2, \dots, E_t . So there exist nonnegative integers c_2, \dots, c_t , such that $\gamma^{-1}(Z)$ is the divisor equal to $\sum_{i=2}^t c_i E_i$. For $m \in \mathbb{N} \cup \{\infty\}$, let

$$\gamma_m : J_m(Y) \longrightarrow J_m(V)$$

be the morphism from $J_m(Y)$ to $J_m(V)$ induced by γ . In the remainder of this subsection, for $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$ with $m \geq n$, the projection $\pi_{m,n}$ refers to the canonical projection $J_m(Y) \rightarrow J_n(Y)$.

For any subset J in $\{1, \dots, t\}$, we set:

$$E_J^0 := \bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i .$$

Let m be a positive integer, let S be a nonempty constructible subset of $J_m(V)$ and let J be a subset in $\{1, \dots, t\}$. For any family $\alpha : i \mapsto \alpha_i$ of positive integers indexed by J , we denote by J_α the subset of elements i of J such that α_i is bigger than $m + 1$ and we set:

$$T_\alpha := \{ \nu \in \pi_{m,0}^{-1}(E_J^0) \cap \gamma_m^{-1}(S) \mid F_{E_i}(\nu) = \alpha_i, i \in J \setminus J_\alpha \} .$$

Lemma 4.8. — *Let q be an integer bigger than m and let $i \mapsto \alpha_i$ be a family of positive integers, indexed by a nonempty subset J of $\{1, \dots, t\}$. We set:*

$$S_{\alpha_i, i \in J}^q := \{ \nu \in \pi_{q,0}^{-1}(E_J^0) \cap (\gamma_m \circ \pi_{q,m})^{-1}(S) \mid F_{E_i}(\nu) = \alpha_i, i \in J \} .$$

and we suppose that $S_{\alpha_i, i \in J}^q$ is nonempty. Then $S_{\alpha_i, i \in J}^q$ is a locally trivial fibration over T_α whose fiber is isomorphic to

$$\mathbb{C}^{(q-m)(N-|J_\alpha|)} \times (\mathbb{C} \setminus \{0\})^{|J_\alpha|} \times \mathbb{C}^{q|J_\alpha| - \sum_{i \in J_\alpha} \alpha_i} .$$

Proof. — We follow the proof of [5](Proposition 2.5). Since the divisor $\gamma^{-1}(X) = \sum_{i=1}^t a_i E_i$ on Y has only simple normal crossing, for any y in Y , there exists a neighborhood U of y in Y with global coordinates z_1, \dots, z_N on U for which a local defining equation for $\gamma^{-1}(X)$ is given by

$$(3) \quad g = z_1^{a_1} \cdots z_{j_y}^{a_{j_y}} ,$$

for some $j_y \leq N$. We cover $Y = \bigcup U$ by finitely many charts on which $\gamma^{-1}(X)$ has a local equation of the form (3), and we lift to cover $J_q(Y) = \bigcup \pi_{q,0}^{-1}(U)$. Hence $S_{\alpha_i, i \in J}^q$ is covered by the subsets

$$U_\alpha^q := \bigcap_{i \in J} F_{E_i}^{-1}(\alpha_i) \cap \pi_{q,0}^{-1}(U) \cap \pi_{q,m}^{-1}(\gamma_m^{-1}(S)) .$$

As α_i is at least 1, the subset U_α^q is contained in $\pi_{q,0}^{-1}(E_J^0)$. Thus, the subsets U_α^q are open subsets of $S_{\alpha_i, i \in J}^q$. Let U_α^q be a such open subset of $S_{\alpha_i, i \in J}^q$ which is nonempty. If J is not contained in $\{1, \dots, j_y\}$, then $E_J^0 \cap U$ is empty, and so U_α^q too. So J is contained in $\{1, \dots, j_y\}$.

For $\nu \in \pi_{q,0}^{-1}(U)$, we can view ν as a N -tuple $(f_1(\nu), \dots, f_N(\nu))$ of polynomials of degree at most q with zero constant term. We continue to reason as in the proof of [5](Proposition 2.5). We see that $\nu \in F_{E_i}^{-1}(\alpha_i)$ if and only if

the truncation of $f_i(\nu)$ to degree α_i is of the form $c_{\alpha_i} z^{\alpha_i}$ where c_{α_i} is different from 0. Then we obtain $N - |J|$ polynomials of degree q with zero constant term, and, for each $j \in J$, a polynomial of the form

$$f_j(\nu) = 0 + \cdots + 0 + c_{\alpha_j} z^{\alpha_j} + c_{\alpha_{j+1}} z^{\alpha_{j+1}} + \cdots + c_{\alpha_q} z^{\alpha_q} ,$$

for c_{α_j} in $\mathbb{C} \setminus \{0\}$ and for c_k in \mathbb{C} , for any k strictly bigger than j . So, when we cut the m first terms of each polynomial $f_j(\nu)$, the space of all such N -tuples so obtained is isomorphic to

$$\mathbb{C}^{(q-m)(N-|J_\alpha|)} \times (\mathbb{C} \setminus \{0\})^{|J_\alpha|} \times \mathbb{C}^{q|J_\alpha| - \sum_{i \in J_\alpha} \alpha_i} .$$

As a consequence, U_α^q is isomorphic to

$$(\pi_{m,0}^{-1}(U) \cap T_\alpha) \times \mathbb{C}^{(q-m)(N-|J_\alpha|)} \times (\mathbb{C} \setminus \{0\})^{|J_\alpha|} \times \mathbb{C}^{q|J_\alpha| - \sum_{i \in J_\alpha} \alpha_i} ,$$

whence the lemma. \square

We denote by $\pi_{\infty,m}^V$ the canonical projection from $J_\infty(V)$ to $J_m(V)$. For S a constructible subset of $J_m(V)$, we set, for p at least $m+1$:

$$S_{\infty,m,p} = \{ \nu \in (\pi_{\infty,m}^V)^{-1}(S) \mid F_X(\nu) \geq p \} .$$

Proposition 4.9. — *Let m be a positive integer and let S be a constructible subset of $J_m(V)$. We set:*

$$\mathbf{I}_{m,p,k} = h \left(\int_{S_{\infty,m,p}} \mathbb{L}^{-kF_Z(\nu)} d\mu_V(\nu) \right) ,$$

for $p > m$ and $k \geq 0$. We suppose that the image by $\pi_{m,0}$ of any irreducible component of maximal dimension of S is dense in X . Then the highest degree term of $\mathbf{I}_{m,p,k}$ does not depend on k for p big enough.

Proof. — By [6](Lemma 3.3), we have

$$(4) \quad \mathbf{I}_{m,p,k} = h \left(\int_{\gamma_\infty^{-1}(S_{\infty,m,p})} \mathbb{L}^{-FW(\nu) - k \sum_{i=2}^t c_i F_{E_i}(\nu)} d\mu_Y(\nu) \right) .$$

As X has rational singularities, the canonical injection from its canonical module to its dualizing module is an isomorphism by [12](Satz 1.1). But its canonical module is locally free of rank 1 since X is Gorenstein as a complete intersection in V [3](Theorem 3.3.7 and Proposition 3.1.20). So X has canonical singularities. Moreover, by [25](Theorem 2.1), b_i is bigger than ra_i for $i = 1, \dots, t$. Then by condition (d) above, $b_1 + 1$ is strictly smaller than $b_i + 1/a_i$ for $i = 2, \dots, t$. Thus, we are in the conditions of Lemma 4.7, (ii).

For any subset J in $\{1, \dots, t\}$ and for any family $i \mapsto \alpha_i$ of nonnegative integers indexed by J , we set:

$$S_{\alpha_i, i \in J} := \{ \nu \in \pi_{\infty,0}^{-1}(E_J^0) \cap (\gamma_m \circ \pi_{\infty,m})^{-1}(S) \mid F_{E_i}(\nu) = \alpha_i, i \in J \} .$$

For any integer q bigger than m and α_i , for any $i \in J$, $S_{\alpha_i, i \in J}$ is the inverse image by $\pi_{\infty, q}$ of the subset

$$S_{\alpha_i, i \in J}^q := \{ \nu \in J_q(Y) \mid \nu \in \pi_{q, 0}^{-1}(E_J^0) \cap (\gamma_m \circ \pi_{q, m})^{-1}(S) \\ \text{and } F_{E_i}(\nu) = \alpha_i, i \in J \} .$$

By Lemma 4.8, the subset $S_{\alpha_i, i \in J}^q$ is a locally trivial fibration over

$$T_\alpha := \{ \nu \in \pi_{m, 0}^{-1}(E_J^0) \cap \gamma_m^{-1}(S) \mid F_{E_i}(\nu) = \alpha_i, i \in J \setminus J_\alpha \} .$$

Moreover, the fiber is isomorphic to

$$\mathbb{C}^{(q-m)(N-|J_\alpha|)} \times (\mathbb{C} \setminus \{0\})^{|J_\alpha|} \times \mathbb{C}^{q|J_\alpha| - \sum_{i \in J_\alpha} \alpha_i} .$$

By Proposition 4.6, (1), $h \circ \mu_Y(S_{\alpha_i, i \in J}) = h([S_{\alpha_i, i \in J}^q])(uv)^{-N(q+1)}$. Then we deduce from [5](Theorem 3.2, (iii)) the relation:

$$(5) \quad h \circ \mu_Y(S_{\alpha_i, i \in J}) = h([T_\alpha])(uv - 1)^{|J_\alpha|} \\ (uv)^{-|J_\alpha| - (N-|J_\alpha|)(m+1) - \sum_{i \in J_\alpha} \alpha_i} .$$

As the subset $\gamma^{-1}(X)$ is equal to the union of E_1, \dots, E_t , for any p , the subset $\gamma_\infty^{-1}(S_{\infty, m, p})$ is given by the equality:

$$\gamma_\infty^{-1}(S_{\infty, m, p}) = \bigcup_{\substack{J \subset \{1, \dots, t\} \\ J \neq \emptyset}} \bigcup_{\substack{(\alpha_i, i \in J) \in (\mathbb{N} \setminus \{0\})^{|J|} \\ \sum_{i \in J} \alpha_i \alpha_i \geq p}} S_{\alpha_i, i \in J} .$$

We set:

$$M := 1 + m \sum_{i=1}^t a_i .$$

In particular, when p is bigger than M , J_α is not empty as soon as $\sum_{i \in J} a_i \alpha_i$ is at least p . For any p bigger than M and any nonnegative integer k , we have the equality:

$$\mathbf{I}_{m, p, k} = \sum_{\substack{J \subset \{1, \dots, t\} \\ J \neq \emptyset}} \sum_{\substack{\alpha_i, i \in J \\ \sum_{i \in J} a_i \alpha_i \geq p, \alpha_i \geq 1}} \mathbf{S}_{J, \alpha_i, i \in J, k}$$

where

$$(6) \quad \mathbf{S}_{J, \alpha_i, i \in J, k} := h([T_\alpha])(uv)^{-(N-|J_\alpha|)(m+1) + \sum_{i \in J \setminus J_\alpha} \alpha_i} \\ (uv - 1)^{|J_\alpha|} (uv)^{-|J_\alpha|} (uv)^{-\sum_{i \in J} \alpha_i (b_i + 1) - k \sum_{i \in J \cap \{2, \dots, t\}} c_i \alpha_i} .$$

For J nonempty subset in $\{1, \dots, t\}$, we set:

$$\mathbf{S}'_{J, p, k} = \sum_{\substack{\alpha_i, i \in J \\ \sum_{i \in J} a_i \alpha_i \geq p, \alpha_i \geq 1}} \mathbf{S}_{J, \alpha_i, i \in J, k} .$$

Thus $\mathbf{I}_{m,p,k}$ is the sum of $\mathbf{S}'_{J,p,k}$. For any J , the highest degree terms in $\mathbf{S}'_{J,p,k}$ are the terms $\mathbf{S}_{J,\alpha_i,i \in J,p,k}$ for which the number

$$\dim T_\alpha - (N - |J_\alpha|)(m+1) + \sum_{i \in J \setminus J_\alpha} \alpha_i - \sum_{i \in J} \alpha_i (b_i + 1) - k \sum_{i \in J \cap \{2, \dots, t\}} c_i \alpha_i$$

is maximal. In particular, for $J = \{1\}$, the highest degree term of $\mathbf{S}'_{J,p,k}$ does not depend on k .

By hypothesis, the image by $\pi_{m,0}$ of the irreducible components of maximal dimension of $\gamma_m^{-1}(S)$ is dense in E_1 . Therefore, for J different from $\{1\}$, the degree of

$$h([\pi_{m,0}^{-1}(E_J^0) \cap \gamma_m^{-1}(S)])$$

is strictly smaller than the degree of the same expression with $J = \{1\}$. As a consequence, when J is different from $\{1\}$, for any family $\alpha : i \mapsto \alpha_i$ of positive integers indexed by J , the degree of $h([T_\alpha])$ is strictly smaller than the degree of

$$h([\pi_{m,0}^{-1}(E_{\{1\}}^0) \cap \gamma_m^{-1}(S)]) .$$

Hence by Lemma 4.7 and relation (6), when p is big enough, the highest degree term of $\mathbf{I}_{m,p,k}$ is the highest degree term of $\mathbf{S}'_{\{1\},p,k}$. In particular, it does not depend on k . \square

4.5. — We keep the notations of the previous subsection. In this subsection, for $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, $\pi_{m,n}$ refers to the canonical projection from $J_\infty(V)$ to $J_m(V)$. We construct in this subsection particular subsets in $J_m(X)$, for $m \geq 1$, and we apply the results of Subsection 4.4. Let T be a closed bicone of $X \times V$ satisfying the two following conditions:

- 1) T is \mathbf{K} -invariant under the diagonal action of \mathbf{K} in $V \times V$,
- 2) for any (x, y) in T , y is a tangent vector of X at x .

We define by induction on m a subset C_m of $J_m(V)$.

$m = 1$: we denote by C_1 the image of T by the map $(x, y) \mapsto \nu_{x,y,1}$.

Let us suppose that the subset C_m of $J_m(V)$ is defined. Let C'_{m+1} be the image of C_m by the canonical injection from $J_m(V)$ into $J_{m+1}(V)$. For any p bigger than $m+1$, we set

$$C'_{\infty,m+1,p} := \{\nu \in \pi_{\infty,m+1}^{-1}(C'_{m+1}) \mid F_X(\nu) \geq p\} .$$

Then we set

$$C_{m+1} := \bigcap_{p \geq m+1} \overline{\pi_{\infty,m+1}(C'_{\infty,m+1,p})} ,$$

where \overline{C} denotes the closure of the subset C of $J_{m+1}(V)$. Thus, C_{m+1} is a closed subset of $J_{m+1}(V)$.

By induction, the subsets C_m are closed subsets of $J_m(V)$, for any positive integer m . For m bigger than 1, we set

$$D_m = \{(x, y) \in X \times V \mid \nu_{x,y,m} \in C_m\} .$$

Lemma 4.10. — i) Let m be in \mathbb{N} bigger than 2. For p big enough, C_m is the closure of $\pi_{\infty,m}(C'_{\infty,m,p})$ in $J_m(V)$.

ii) For any m bigger than 1, the subset C_m is the image of D_m by the map $(x, y) \mapsto \nu_{x,y,m}$. Moreover, C_m is contained in $J_m(X)$.

iii) For any m bigger than 1, C_m is invariant under the canonical action of \mathbf{K} in $J_m(V)$.

iv) If $\varpi_1(T)$ is equal to X , then for any m bigger than 1, the image by $\pi_{m,0}$ of C_m is equal to X .

Proof. — i) By definition, the sequence

$$\pi_{\infty,m}(C'_{\infty,m,m}), \pi_{\infty,m}(C'_{\infty,m,m}), \dots$$

is nonincreasing. As C_{m-1} is closed in $J_{m-1}(V)$, C'_m is closed in $J_m(V)$ since the canonical injection from $J_{m-1}(V)$ to $J_m(V)$ is a closed immersion. So for any p , $C'_{\infty,m,p}$ is a semi-algebraic subset of $J_{\infty}(V)$. Hence by [6](Proposition 2.3), $\pi_{\infty,m}(C'_{\infty,m,p})$ is a constructible subset of $J_m(V)$ for $p = m, m+1, \dots$. By noetherianity, since the sequence

$$\overline{\pi_{\infty,m}(C'_{\infty,m,m})}, \overline{\pi_{\infty,m}(C'_{\infty,m,m+1})}, \dots$$

is nonincreasing, it is stationary. Hence for p big enough, C_m is the closure of $\pi_{\infty,m}(C'_{\infty,m,p})$ in $J_m(V)$.

ii) Let us first remark that, for any m bigger than 1, the image of D_m by the map $(x, y) \mapsto \nu_{x,y,m}$ is clearly contained in C_m , by definition of D_m . In addition, for any m bigger than 2, C_m is contained in C'_m , by construction of C_m . In fact, for $p \geq m$, $\pi_{\infty,m}(C'_{\infty,m,p})$ is contained in C'_m so its closure too, since C'_m is closed as image in $J_m(V)$ of the closed subset C_{m-1} by a closed immersion.

It remains to prove that, for any m bigger than 1, C_m is contained in the image of D_m by the map $(x, y) \mapsto \nu_{x,y,m}$ and that C_m belongs to $J_m(X)$. We prove the statements by induction on m . By definition, C_1 is the image of T by the map $(x, y) \mapsto \nu_{x,y,1}$ and C_1 is contained in $J_1(X)$ by condition (2). So D_1 is equal to T , whence the two statements for $m = 1$. Let us suppose that C_m is the image of D_m by the map $(x, y) \mapsto \nu_{x,y,m}$ and let us suppose that C_m is contained in $J_m(X)$, for some m bigger than 1. Then C'_{m+1} is the image of D_m by the map $(x, y) \mapsto \nu_{x,y,m+1}$. By definition, (x, y) belongs

to D_{m+1} if and only if $\nu_{x,y,m+1}$ belongs to C_{m+1} . As C_{m+1} is contained in C'_{m+1} , we deduce that C_{m+1} is contained in the image of D_{m+1} by the map $(x, y) \mapsto \nu_{x,y,m+1}$. In addition, for any p at least $m+2$, $\pi_{\infty,m+1}(C'_{\infty,m+1,p})$ is contained in $J_{m+1}(X)$, by definition of $C'_{\infty,m+1,p}$. Hence C_{m+1} is contained in $J_{m+1}(X)$ since $J_{m+1}(X)$ is closed in $J_{m+1}(V)$.

iii) We prove the statement by induction on m . As T is \mathbf{K} -invariant by condition (1), C_1 is \mathbf{K} -invariant. Let us suppose that m is bigger than 2 and C_m is invariant under the action of \mathbf{K} . Then C'_{m+1} is invariant under the action of \mathbf{K} in $J_{m+1}(V)$. So $\pi_{\infty,m+1}^{-1}(C'_{m+1})$ is invariant under the canonical action of \mathbf{K} in $J_{\infty}(V)$. As X is \mathbf{K} -invariant, the function F_X is \mathbf{K} -invariant. So for any p bigger than $m+1$, $C'_{\infty,m+1,p}$ is invariant under \mathbf{K} . Hence $\pi_{\infty,m+1}(C'_{\infty,m+1,p})$ and C_{m+1} are invariant under \mathbf{K} .

iv) We suppose that $\varpi_1(T)$ is equal to X . Then for any x in X , $(x, 0)$ belongs to T since T is a closed bicone. As a consequence for any positive integer m , C_m contains $\nu_{x,0,m}$. Hence $\pi_{m,0}(C_m)$ is equal to X . \square

As in Subsection 4.4, for any constructible subset of $J_m(V)$, any $m \geq 1$ and any $p \geq m+1$, we set:

$$(7) \quad S_{\infty,m,p} := \{\nu \in \pi_{\infty,m}^{-1}(S) \mid F_X(\nu) \geq p\} .$$

Lemma 4.11. — *Let m, p be in \mathbb{N} with p at least $m+1$, and let S be a constructible subset of $J_m(V)$ contained in $J_m(X)$. We denote by d the dimension of S and by c the number of irreducible components of S of maximal dimension. Then the highest degree term of $h \circ \mu_V(S_{\infty,m,p})$ is $c(uv)^{d-(N-r)(m+1)-pr}$.*

Proof. — First of all, let us remark that for ν in $J_{\infty}(V)$ and for p positive integer, $\pi_{\infty,p-1}(\nu)$ belongs to $J_{p-1}(X)$ if and only if $F_X(\nu)$ is bigger than p . Let S_p be the inverse image of S by the canonical projection from $J_{p-1}(X)$ to $J_m(X)$. Then $S_{\infty,m,p}$ is equal to $\pi_{\infty,p-1}^{-1}(S_p)$. So by Proposition 4.6, (1), we have:

$$(8) \quad h \circ \mu_V(S_{\infty,m,p}) = h([S_p])(uv)^{-Np} .$$

Let $S_{\infty,m}^X$ and $S_{\infty,m,p}^X$ be the inverse images of S and S_p by the canonical projections from $J_{\infty}(X)$ to $J_m(X)$ and $J_{p-1}(X)$ respectively. Then by Proposition 4.6, (1), applied to μ_X , we have:

$$\mu_X(S_{\infty,m}^X) = h([S])(uv)^{-(N-r)(m+1)} , \mu_X(S_{\infty,m,p}^X) = h([S_p])(uv)^{-(N-r)p} .$$

As the two subsets are equal, we deduce the equality:

$$h([S_p]) = h([S])(uv)^{(N-r)(p-m-1)} .$$

Hence from equality (8), we deduce

$$h \circ \mu_V(S_{\infty,m,p}) = h([S])(uv)^{-(N-r)(m+1)-pr} .$$

Then the highest degree term of $h \circ \mu_V(S_{\infty, m, p})$ is $c(uv)^{d-(N-r)(m+1)-pr}$ since $c(uv)^d$ is the highest degree term of $h([S])$. \square

Proposition 4.12. — i) For m big enough, we have

$$D_m = \{(x, y) \in T \mid x + ty \in X, \forall t \in \mathbb{C}\} .$$

ii) We suppose that the image by ϖ_1 of any irreducible component of maximal dimension of T is equal to X . Then, for any m bigger than 1, the image by ϖ_1 of any irreducible component of maximal dimension of D_m is dense in X .

Proof. — i) Let m be an integer such that $m - 1$ is strictly bigger than the degree of any element of a generating family of the ideal of definition of X in the algebra of polynomial functions on V . Let (x, y) be in T . If $x + ty$ belongs to X for any t in \mathbb{C} , then $F_X(\nu_{x, y})$ is equal to ∞ . In particular, C_m contains $\nu_{x, y, m}$ and D_m contains (x, y) . Conversely, let us suppose that D_m contains (x, y) . Then T contains (x, y) since T contains D_i for any positive integer i . Moreover, for any φ in the ideal of definition of X , the function $t \mapsto \varphi(x + ty)$ is divisible by t^m . So by the choice of m , $\varphi(x + ty)$ is equal to 0, for any t in \mathbb{C} . In other words, $x + ty$ belongs to X for any t in \mathbb{C} .

ii) By Lemma 4.10, (ii), the statement is equivalent to the following statement:

for any positive integer m , the image by $\pi_{m, 0}$ of any irreducible component of maximal dimension of C_m is dense in X .

We prove this statement by induction on m . It is true for $m = 1$ by hypothesis and by definition of C_1 . We suppose m at least 2 and the statement true for $m - 1$. Let d be the dimension of C_m and let B_m be the union of irreducible components of C_m whose image by $\pi_{m, 0}$ is dense in X . By Lemma 4.10, (iv), B_m is not empty. We denote by c the number of irreducible components of dimension d of C_m , d' the dimension of B_m and by c' the number of irreducible components of dimension d' of B_m . It is enough to prove the equalities $d = d'$, $c = c'$. In fact, in this case, any irreducible component of dimension d of C_m is an irreducible component of B_m .

For p bigger than $m + 1$, we set:

$$(9) \quad B_{\infty, m, p} := \{\nu \in \pi_{\infty, m}^{-1}(B_m) \mid F_X(\nu) \geq p\} .$$

As m is at least 2, C_m is contained in C'_m , and so $B_{\infty, m, p}$ is contained in $C'_{\infty, m, p}$. The image by $\pi_{m, 0}$ of any irreducible component of B_m is dense in X . Hence, by Proposition 4.9, the highest degree term of the element

$$\mathbf{I}'_{m, p, k} := h\left(\int_{B_{\infty, m, p}} \mathbb{L}^{-kF_Z(\nu)} d\mu_V(\nu)\right)$$

of $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]$ does not depend on k , for p big enough. On the other hand, by induction hypothesis, the image by $\pi_{\infty, m-1}$ of any irreducible component of C_{m-1} is dense in X , so the image by $\pi_{\infty, m}$ of any irreducible component of C'_m is dense in X . Hence by Proposition 4.9, the highest degree term of the element

$$\mathbf{I}_{m,p,k} := h\left(\int_{C'_{\infty,m,p}} \mathbb{L}^{-kF_Z(\nu)} d\mu_V(\nu)\right)$$

of $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]$ does not depend on k , for p big enough. At last, by Lemma 4.10, (i), we may choose p big enough such that the following conditions hold:

- 1) C_m is the closure of $\pi_{\infty, m}(C'_{\infty, m, p})$,
- 2) the highest degree terms of $\mathbf{I}_{m,p,k}$ and $\mathbf{I}'_{m,p,k}$ do not depend on k .

By Lemma 4.10, (ii), C_m and so B_m are contained in $J_m(X)$. So by Lemma 4.11, the highest degree term of $h \circ \mu_V(B_{\infty, m, p})$ is $c'(uv)^{d' - pr - (N-r)(m+1)}$. Therefore, by condition (2) with $k = 0$ in $\mathbf{I}'_{m,p,k}$, we obtain that the highest degree term of the element $\mathbf{I}'_{m,p,k}$ of $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]$ is $c'(uv)^{d' - (N-r)(m+1) - pr}$ for any k . On the other hand, using condition (1), we can easily check that the relation

$$(10) \quad C'_{\infty, m, p} = \{\nu \in \pi_{\infty, m}^{-1}(\pi_{\infty, m}(C'_{\infty, m, p})) \mid F_X(\nu) \geq p\}$$

holds. In addition, by condition (1) again, $h([C_m])$ and $h([\pi_{\infty, m, p}(C'_{\infty, m, p})])$ have the same highest degree term. Moreover, by Lemma 4.10, (ii), C_m is contained in $J_m(X)$ so $\pi_{\infty, m}(C'_{\infty, m, p})$ too. Hence, by Lemma 4.11, the highest degree term of $h \circ \mu_V(C'_{\infty, m, p})$ is $c(uv)^{d - (N-r)(m+1) - pr}$. Then by condition (2) with $k = 0$ in $\mathbf{I}_{m,p,k}$, we obtain that the highest degree term of the element $\mathbf{I}_{m,p,k}$ in $\mathbb{Z}[u, v][[u^{-1}, v^{-1}]$ is $c(uv)^{d - (N-r)(m+1) - pr}$ for any k . Now, as $F_Z(\nu)$ is positive for any ν in $C'_{\infty, m, p} \setminus B_{\infty, m, p}$, we get

$$\lim_{k \rightarrow +\infty} h\left(\int_{C'_{\infty, m, p} \setminus B_{\infty, m, p}} \mathbb{L}^{-kF_Z(\nu)} d\mu_V(\nu)\right) = 0.$$

Hence for k big enough, the highest degree term of $\mathbf{I}_{m,p,k}$ is $c'(uv)^{d' - (N-r)(m+1) - pr}$. As a consequence, $d = d'$ and $c = c'$. \square

4.6. — In this subsection, we apply Proposition 4.12 to suitable subsets V , X , T and \mathbf{K} in order to prove Theorem 4.1.

Corollary 4.13. — i) Let \mathcal{X} be an irreducible component of maximal dimension of $\mathcal{X}_{\mathfrak{g}}$. Then $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$.

ii) Let \mathcal{X} be an irreducible component of maximal dimension of $\mathcal{Y}_{\mathfrak{g}}$. Then $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$.

iii) Let \mathcal{X} be an irreducible component of maximal dimension of $\mathcal{Z}_{\mathfrak{g}}$. Then $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$.

iv) Let \mathcal{X} be an irreducible component of maximal dimension of $\mathcal{N}_{\mathfrak{g}}$, then $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{N}_{\mathfrak{g}}$.

Proof. — i) Let V be the space \mathfrak{g} , let \mathbf{K} be the subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by G and its homotheties and let X be the subvariety $\mathfrak{X}_{\mathfrak{g}}$ of \mathfrak{g} . By Corollary 3.4, $\mathfrak{X}_{\mathfrak{g}}$ is an irreducible closed normal cone and a complete intersection in \mathfrak{g} with rational singularities. Moreover, \mathbf{K} has finitely many orbits in $\mathfrak{X}_{\mathfrak{g}}$. Let T be the bicone $T\mathfrak{X}_{\mathfrak{g}}$ of $\mathfrak{X}_{\mathfrak{g}} \times \mathfrak{g}$. By Lemma 4.5, (i), T is irreducible and its image by ϖ_1 is equal to $\mathfrak{X}_{\mathfrak{g}}$. Moreover, by Proposition 4.12, (i), for m big enough, D_m is equal to $\mathcal{X}_{\mathfrak{g}}$. Hence by Proposition 4.12, (ii), $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$ since $\varpi_1(\mathcal{X})$ is closed by Lemma 1.3.

ii) We use V, \mathbf{K}, X as in (i). Let T be the bicone $T'\mathfrak{X}_{\mathfrak{g}}$ of $\mathfrak{X}_{\mathfrak{g}} \times \mathfrak{g}$. By Lemma 4.5, (iii), T is irreducible and its image by ϖ_1 is equal to $\mathfrak{X}_{\mathfrak{g}}$. Moreover, by Proposition 4.12, (i), for m big enough, D_m is equal to $\mathcal{Y}_{\mathfrak{g}}$. Hence by Proposition 4.12, (ii), $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$ since $\varpi_1(\mathcal{X})$ is closed by Lemma 1.3.

iii) We use V, \mathbf{K}, X as in (i). Let T be the bicone $T''\mathfrak{X}_{\mathfrak{g}}$ of $\mathfrak{X}_{\mathfrak{g}} \times \mathfrak{g}$. By Lemma 4.5, (iv), the image by ϖ_1 of any irreducible component of T is equal to $\mathfrak{X}_{\mathfrak{g}}$. Moreover, by Proposition 4.12, (i), for m big enough, D_m is equal to $\mathcal{X}_{\mathfrak{g}}$. Hence by Proposition 4.12, (ii), $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$ since $\varpi_1(\mathcal{X})$ is closed by Lemma 1.3.

iv) Let V be the space \mathfrak{g} , let \mathbf{K} be the subgroup G of $\mathrm{GL}(\mathfrak{g})$ and let X be the subvariety $\mathfrak{N}_{\mathfrak{g}}$ of \mathfrak{g} . The cone $\mathfrak{N}_{\mathfrak{g}}$ is an irreducible closed normal cone and G has finitely many orbits in $\mathfrak{N}_{\mathfrak{g}}$. Moreover, it is a complete intersection in \mathfrak{g} and by [15](Theorem A), $\mathfrak{N}_{\mathfrak{g}}$ has rational singularities. Let T be the bicone $T\mathfrak{N}_{\mathfrak{g}}$ of $\mathfrak{N}_{\mathfrak{g}} \times \mathfrak{g}$. By Lemma 4.5, (i), T is irreducible and its image by ϖ_1 is equal to $\mathfrak{N}_{\mathfrak{g}}$. Moreover, by Proposition 4.12, (i), for m big enough, D_m is equal to $\mathcal{N}_{\mathfrak{g}}$. Hence by Proposition 4.12, (ii), $\varpi_1(\mathcal{X})$ is equal to $\mathfrak{N}_{\mathfrak{g}}$ since $\varpi_1(\mathcal{X})$ is closed by Lemma 1.3. \square

Let d be the dimension of $\mathcal{X}_{\mathfrak{g}}$.

Proposition 4.14. — *Let \mathcal{X} be an irreducible component of dimension d of $\mathcal{X}_{\mathfrak{g}}$.*

- i) *The dimension of $\mathcal{Y}_{\mathfrak{g}}$ is equal to $d - 1$.*
- ii) *The intersection of \mathcal{X} and $\mathcal{Y}_{\mathfrak{g}}$ is irreducible and has dimension $d - 1$. Moreover, this intersection is stable under the involution $(x, y) \mapsto (y, x)$.*
- iii) *The intersection of \mathcal{X} and $\mathcal{Z}_{\mathfrak{g}}$ is an union of irreducible components of $\mathcal{Z}_{\mathfrak{g}}$ of maximal dimension. Moreover, $\mathcal{Z}_{\mathfrak{g}}$ has dimension $d - 2$.*
- iv) *The intersection of $\mathcal{N}_{\mathfrak{g}}$ and \mathcal{X} is equidimensional of dimension $d - 3$.*
- v) *The dimension of $\mathcal{N}_{\mathfrak{g}}$ is equal to $d - 3$.*

Proof. — i) Let \mathcal{X}' be an irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ of maximal dimension. As $\mathcal{Y}_{\mathfrak{g}}$ is contained in $\mathcal{X}_{\mathfrak{g}}$, \mathcal{X}' is contained in an irreducible component \mathcal{X}'' of

$\mathcal{X}_{\mathfrak{g}}$. Moreover, \mathcal{X}' is an irreducible component of the nullvariety of $p_{1,1,1}$ in \mathcal{X}'' . So \mathcal{X}' has codimension at most 1 in \mathcal{X}'' . By Corollary 4.13, (ii), $\varpi_1(\mathcal{X}')$ is equal to $\mathfrak{X}_{\mathfrak{g}}$. Hence \mathcal{X}' contains $(h, 0)$ since it is a closed bicone. As \mathcal{X}'' is an irreducible component of $\mathcal{X}_{\mathfrak{g}}$, it is invariant under the action of $\mathrm{GL}_2(\mathbb{C})$. Hence \mathcal{X}'' contains (h, h) and strictly contains \mathcal{X}' . So \mathcal{X}' has dimension $\dim \mathcal{X}'' - 1$. In particular, $\dim \mathcal{X}'$ is at most $d - 1$. As any irreducible component of the nullvariety of $p_{1,1,1}$ in \mathcal{X} is contained in $\mathcal{Y}_{\mathfrak{g}}$, $\mathcal{Y}_{\mathfrak{g}}$ has dimension $d - 1$.

ii) As $\mathcal{Y}_{\mathfrak{g}}$ is stable under the involution $(x, y) \mapsto (y, x)$ and \mathcal{X} is stable under the action of $\mathrm{GL}_2(\mathbb{C})$, their intersection is stable under the involution $(x, y) \mapsto (y, x)$. By (i), any irreducible component of the intersection of \mathcal{X} and $\mathcal{Y}_{\mathfrak{g}}$ has dimension $d - 1$. Hence by (i), Corollary 4.13, (ii) and Lemma 4.4, the intersection of $\mathcal{Y}_{\mathfrak{g}}$ and \mathcal{X} is irreducible since $\theta(\mathbb{C} \times \mathcal{X})$ is contained in \mathcal{X} .

iii) Let \mathcal{Y} be the intersection of $\mathcal{Y}_{\mathfrak{g}}$ and \mathcal{X} and let \mathcal{Z} be the intersection of \mathcal{X} and $\mathcal{Z}_{\mathfrak{g}}$. By definition, \mathcal{Z} is the nullvariety of $p_{1,0,2}$ in \mathcal{Y} . Moreover, by (ii) and Corollary 4.13, (ii), $\varpi_2(\mathcal{Y})$ is equal to $\mathfrak{X}_{\mathfrak{g}}$. Hence by (i) and (ii), any irreducible component of \mathcal{Z} has dimension $d - 2$. Let \mathcal{Z}' be an irreducible component of $\mathcal{Z}_{\mathfrak{g}}$ of maximal dimension. Then $\dim \mathcal{Z}'$ is at least $d - 2$. Let us suppose that $\dim \mathcal{Z}'$ is strictly bigger than $d - 2$. As $\mathcal{Z}_{\mathfrak{g}}$ is contained in $\mathcal{Y}_{\mathfrak{g}}$, \mathcal{Z}' is an irreducible component of maximal dimension of $\mathcal{Y}_{\mathfrak{g}}$ since $\mathcal{Y}_{\mathfrak{g}}$ has dimension $d - 1$. By (ii), \mathcal{Z}' is the intersection of $\mathcal{Y}_{\mathfrak{g}}$ and an irreducible component of dimension d of $\mathcal{X}_{\mathfrak{g}}$. So by (ii) and Corollary 4.13, (ii), $\varpi_2(\mathcal{Z}')$ is equal to $\mathfrak{X}_{\mathfrak{g}}$. As it is impossible, $\mathcal{Z}_{\mathfrak{g}}$ has dimension $d - 2$ and \mathcal{Z} is a union of irreducible components of dimension $d - 2$ of $\mathcal{Z}_{\mathfrak{g}}$.

iv) Let \mathcal{Y} and \mathcal{Z} be as in (iii). As $\mathcal{N}_{\mathfrak{g}}$ is contained in $\mathcal{Z}_{\mathfrak{g}}$, the intersection of \mathcal{X} and $\mathcal{N}_{\mathfrak{g}}$ is contained in \mathcal{Z} . By (iii) and Corollary 4.13, (iii), each irreducible component of \mathcal{Z} is not contained in $\mathcal{N}_{\mathfrak{g}}$ and has dimension $d - 2$. Hence the intersection of \mathcal{X} and $\mathcal{N}_{\mathfrak{g}}$ is equidimensional of dimension $d - 3$ since this intersection is the nullvariety of $p_{1,2,0}$ in \mathcal{Z} .

v) As $\mathcal{N}_{\mathfrak{g}}$ is contained in $\mathcal{Z}_{\mathfrak{g}}$, any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ is contained in an irreducible component of $\mathcal{Z}_{\mathfrak{g}}$. So, $\mathcal{N}_{\mathfrak{g}}$ has dimension at most $d - 2$, by (iii). Moreover, by (iii) and Corollary 4.13, (iii), any irreducible component of dimension $d - 2$ of $\mathcal{Z}_{\mathfrak{g}}$ is not contained in $\mathcal{N}_{\mathfrak{g}}$. Hence the maximal dimension of the irreducible components of $\mathcal{N}_{\mathfrak{g}}$ is at most $d - 3$. So, by (iv), $\mathcal{N}_{\mathfrak{g}}$ has dimension $d - 3$. \square

Corollary 4.15. — i) *The subscheme $\mathcal{X}_{\mathfrak{g}}$ is equidimensional of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$. Moreover, any irreducible component of $\mathcal{X}_{\mathfrak{g}}$ has a nonempty intersection with the intersection of $\Omega_{\mathfrak{g}}$ and $\mathcal{Z}_{\mathfrak{g}}$.*

ii) *The subscheme $\mathcal{Y}_{\mathfrak{g}}$ is equidimensional of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) + 2$. Moreover, any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ has a nonempty intersection with $\Omega_{\mathfrak{g}}$.*

Proof. — i) Let \mathcal{X} be an irreducible component of dimension d of $\mathcal{X}_{\mathfrak{g}}$. Let \mathcal{Z} and \mathcal{N} be the intersections of \mathcal{X} with $\mathcal{Z}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}}$ respectively. By Proposition 4.14, (iii) and (iv), \mathcal{Z} and \mathcal{N} are equidimensional of dimension $d - 2$ and $d - 3$ respectively. Hence by Proposition 4.14, (v), \mathcal{N} is a union of irreducible components of maximal dimension of $\mathcal{N}_{\mathfrak{g}}$. As any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ is stable under the action of $\mathrm{GL}_2(\mathbb{C})$, the image by ϖ_2 of any irreducible component of \mathcal{N} is equal to $\mathfrak{N}_{\mathfrak{g}}$ by Corollary 4.13, (iv). Moreover, each irreducible component of \mathcal{Z} contains an irreducible component of \mathcal{N} since \mathcal{N} is the nullvariety of $p_{1,2,0}$ in \mathcal{Z} . Hence the image by ϖ_2 of any irreducible component of \mathcal{Z} is equal to $\mathfrak{N}_{\mathfrak{g}}$. By Proposition 4.14, (iii), each irreducible component of \mathcal{Z} is an irreducible component of $\mathcal{Z}_{\mathfrak{g}}$ of maximal dimension. Hence by Corollary 4.13, (iii), the image by ϖ_1 of any irreducible component of \mathcal{Z} is equal to $\mathfrak{X}_{\mathfrak{g}}$. So \mathcal{Z} satisfies Conditions (1) and (2) of Corollary 3.17. Hence by Corollary 3.17, \mathcal{Z} has a nonempty intersection with $\Omega_{\mathfrak{g}}$ and \mathcal{X} has dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$. But by Lemma 3.7, (i), any irreducible component of $\mathcal{X}_{\mathfrak{g}}$ has dimension at least $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$. Hence $\mathcal{X}_{\mathfrak{g}}$ is equidimensional of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$.

ii) By Proposition 4.14, (ii) and Lemma 3.9, (ii), $\mathcal{Y}_{\mathfrak{g}}$ is equidimensional of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) + 2$. Moreover, any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ is the intersection of $\mathcal{Y}_{\mathfrak{g}}$ and an irreducible component of $\mathcal{X}_{\mathfrak{g}}$. Hence by (i) and Corollary 3.17, any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ has a nonempty intersection with $\Omega_{\mathfrak{g}}$. \square

We can now give the proof of Theorem 4.1:

Proof. — i) By Corollary 4.15, (i), Lemma 3.8, (i) and Lemma 3.11, (ii), $\mathcal{X}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g} + 1)$ and any irreducible component of $\mathcal{X}_{\mathfrak{g}}$ contains a regular element. So $\mathcal{X}_{\mathfrak{g}}$ is generically reduced and by [24](Ch. 8, §23), the scheme $\mathcal{X}_{\mathfrak{g}}$ is reduced.

ii) By Corollary 4.15, (ii), Lemma 3.9, (ii) and Lemma 3.11, (ii), $\mathcal{Y}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) + 2$ and any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ contains a regular element. So $\mathcal{Y}_{\mathfrak{g}}$ is generically reduced and by [24](Ch. 8, §23), the scheme $\mathcal{Y}_{\mathfrak{g}}$ is reduced. Moreover, by Proposition 4.14, (ii), any irreducible component of $\mathcal{Y}_{\mathfrak{g}}$ is the intersection of $\mathcal{Y}_{\mathfrak{g}}$ and an irreducible component of $\mathcal{X}_{\mathfrak{g}}$.

iii) By Proposition 4.14, (iii), Corollary 4.15, (i) and Lemma 3.9, (iii), $\mathcal{Z}_{\mathfrak{g}}$ is equidimensional of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) + 1$. Hence $\mathcal{Z}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g}) + 1$.

iv) As $\mathcal{N}_{\mathfrak{g}}$ is the nullvariety of $\mathrm{b}_{\mathfrak{g}} + \mathrm{rk} \mathfrak{g}$ polynomial functions, any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ has dimension at least $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$. Hence by Corollary 4.15, (i) and Proposition 4.14, (v), $\mathcal{N}_{\mathfrak{g}}$ is a complete intersection of dimension $3(\mathrm{b}_{\mathfrak{g}} - \mathrm{rk} \mathfrak{g})$. \square

Then Theorem 1.2 deduces from Theorem 4.1, (i) and Corollary 4.13, (i) and Theorem 1.1 deduces from Theorem 4.1, (iv) and Corollary 4.13, (iv). We recall here Theorem 1.1 given in the introduction:

Theorem 4.16. — *The nilpotent bicone is a complete intersection of dimension $3b_{\mathfrak{g}} - 3\mathrm{rk}\mathfrak{g}$. Moreover, the images of any irreducible component of $\mathcal{N}_{\mathfrak{g}}$ by the first and second projections from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} are equal to the nilpotent cone.*

We will prove in Section 6 that $\mathcal{N}_{\mathfrak{g}}$ is not reduced.

5. Applications to invariant theory

In this section, we present various applications of previous results to invariant theory. In this section, \mathfrak{g} is supposed to be simple.

5.1. — We denote by $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the subalgebra of $\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{g})$ generated by the 2-order polarizations of elements of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Since the polynomials $p_1, \dots, p_{\mathrm{rk}\mathfrak{g}}$ generate $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomials $p_{i,m,n}$ where $i = 1, \dots, \mathrm{rk}\mathfrak{g}$, $m + n = d_i$ generate $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as graded algebra. Moreover, the morphism σ introduced in Subsection 2.4 is the morphism of affine varieties whose comorphism is the canonical injection from $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ into $\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{g})$. Then, by Proposition 2.12, we can state:

Proposition 5.1. — *The subalgebra $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra in $b_{\mathfrak{g}} + \mathrm{rk}\mathfrak{g}$ variables.*

Now, as a consequence of Theorem 1.1, we obtain:

Theorem 5.2. — *The morphism σ is faithfully flat. Equivalently, the extension $\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{g})$ of $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is faithfully flat.*

Proof. — As $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is generated by homogeneous functions, the fiber at 0 of the morphism σ has maximal dimension. On the other hand, by Proposition 5.1, $\mathrm{pol}_2\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra in $b_{\mathfrak{g}} + \mathrm{rk}\mathfrak{g}$ variables. So σ is an equidimensional morphism and by [24](Ch. 8, Theorem 21.3), σ is a flat morphism. In particular by [14](Ch. III, Exercise 9.4), it is an open morphism whose image contains 0. So σ is surjective. Hence σ is faithfully flat, according to [24](Ch. 3, Theorem 7.2). \square

5.2. — The *nullcone* of \mathfrak{g} , that we denote by $\mathcal{V}_{\mathfrak{g}}$ is the nullvariety in $\mathfrak{g} \times \mathfrak{g}$ of the augmentation ideal of the ideal of $\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{g})$ generated by $(\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathrm{S}(\mathfrak{g}))^{\mathfrak{g}}$. The nullcone plays a leading part in invariant theory. It is studied in [31], [28], [27], and recently in [20] and [21]. By [20], $\mathcal{V}_{\mathfrak{g}}$ is irreducible and has dimension $3(b_{\mathfrak{g}} - \mathrm{rk}\mathfrak{g})$. Moreover, N. Wallach and H. Kraft conjecture in [21]

that $\mathcal{V}_{\mathfrak{g}}$ is an irreducible component of $\mathcal{N}_{\mathfrak{g}}$. Obviously, Theorem 1.1 confirms that conjecture, and we can state:

Theorem 5.3. — *The nullcone $\mathcal{V}_{\mathfrak{g}}$ is an irreducible component of the nilpotent bicone $\mathcal{N}_{\mathfrak{g}}$.*

5.3. — Let k be in \mathbb{N}^* and let us denote by \mathfrak{g}^k the k -th cartesian power of \mathfrak{g} . We extend the previous notions to \mathfrak{g}^k . Let $S(\mathfrak{g}^k)$ be the symmetric algebra of \mathfrak{g}^k and let $S(\mathfrak{g}^k)^{\mathfrak{g}}$ be the subalgebra of its invariant elements under the diagonal action of G in \mathfrak{g}^k . Then we denote by $\mathcal{V}_{\mathfrak{g}}^{(k)}$ the nullvariety in \mathfrak{g}^k of the augmentation ideal $S(\mathfrak{g}^k)_{+}^{\mathfrak{g}}$ of the ideal generated by $S(\mathfrak{g}^k)^{\mathfrak{g}}$. For φ in $S(\mathfrak{g})$, the k -order polarizations $\varphi_{i_1, \dots, i_k}$ of φ are defined by the relation:

$$\varphi(t_1x_1 + \dots + t_kx_k) = \sum_{(i_1, \dots, i_k) \in \mathbb{N}^k} t_1^{i_1} \dots t_k^{i_k} \varphi_{i_1, \dots, i_k}(x_1, \dots, x_k),$$

for (x_1, \dots, x_k) in \mathfrak{g}^k and (t_1, \dots, t_k) in \mathbb{C}^k . Then, we denote by $\text{pol}_k S(\mathfrak{g})^{\mathfrak{g}}$ the subalgebra of $S(\mathfrak{g}^k)$ generated by the k -order polarizations of elements in $S(\mathfrak{g})^{\mathfrak{g}}$. Then $\text{pol}_k S(\mathfrak{g})^{\mathfrak{g}}$ is an homogeneous subalgebra of $S(\mathfrak{g}^k)$ and its augmentation ideal $\text{pol}_k S(\mathfrak{g})_{+}^{\mathfrak{g}}$ is generated by the k -order polarizations of the elements of $S(\mathfrak{g})_{+}^{\mathfrak{g}}$. We denote by $\mathcal{N}_{\mathfrak{g}}^{(k)}$ the nullvariety in \mathfrak{g}^k of the ideal $\text{pol}_k S(\mathfrak{g})_{+}^{\mathfrak{g}}$. Obviously, $\mathcal{V}_{\mathfrak{g}}^{(1)} = \mathcal{N}_{\mathfrak{g}}^{(1)} = \mathfrak{N}_{\mathfrak{g}}$, $\mathcal{V}_{\mathfrak{g}}^{(2)} = \mathcal{V}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}}^{(2)} = \mathcal{N}_{\mathfrak{g}}$.

In [27], the *index of polarization of \mathfrak{g}* , denoted by $\text{polind}(\mathfrak{g})$, is defined as being the upper bound of positive integers k such that $\mathcal{V}_{\mathfrak{g}}^{(k)} = \mathcal{N}_{\mathfrak{g}}^{(k)}$. Clearly, the inequality $\text{polind}(\mathfrak{g}) \geq 1$ always holds. We wish now to establish an equality.

As before, the element e is a regular nilpotent element of \mathfrak{g} .

Lemma 5.4. — *Let x be in \mathfrak{g} satisfying $(\text{ad } x)^2(e) = 0$. Then $(e, [x, e])$ belongs to $\mathcal{N}_{\mathfrak{g}}$.*

Proof. — Since $(\text{ad } x)^2(e) = 0$, we have $\exp(\text{tad } x)(e) = e + t[x, e]$, for all t in \mathbb{C} . As $\mathfrak{N}_{\mathfrak{g}}$ is a closed cone in \mathfrak{g} , the element $se + t[x, e]$ belongs to $\mathfrak{N}_{\mathfrak{g}}$, for all (s, t) in \mathbb{C}^2 . Then $(e, [x, e])$ belongs to $\mathcal{N}_{\mathfrak{g}}$. \square

We recall here an important result in invariant theory, due to R. W. Richardson [27]:

Lemma 5.5 (Richardson). — *The element (x, y) of $\mathfrak{g} \times \mathfrak{g}$ belongs to the nullcone $\mathcal{V}_{\mathfrak{g}}$ if and only if there exists a Borel subalgebra with nilradical \mathfrak{n} such that x and y belong to \mathfrak{n} .*

The following proposition has already been noticed in [27](Theorem 3.16). We provide here a shorter proof.

Proposition 5.6. — *We have:*

$$\text{polind}(\mathfrak{g}) = \begin{cases} \infty, & \text{if } \mathfrak{g} \text{ is isomorphic to } \mathfrak{sl}_2(\mathbb{C}) \\ 1, & \text{otherwise .} \end{cases}$$

Proof. — To start with, suppose that \mathfrak{g} has dimension 3. The elements x in \mathfrak{g} such that the subspace generated by x and e is contained in the nilpotent cone of \mathfrak{g} are colinear to e . Hence $\mathcal{V}_{\mathfrak{g}}^{(2)} = \mathcal{N}_{\mathfrak{g}}$, and $\mathcal{N}_{\mathfrak{g}}$ is irreducible. Then for k bigger than 2 and x_1, \dots, x_k in \mathfrak{g} , the subspace generated by x_1, \dots, x_k is contained in $\mathfrak{N}_{\mathfrak{g}}$ if and only if its dimension is smaller than 1. So $\mathcal{V}_{\mathfrak{g}}^{(k)}$ is equal to $\mathcal{N}_{\mathfrak{g}}^{(k)}$ for any k .

Suppose now that \mathfrak{g} has dimension strictly bigger than 3. As $d_1, \dots, d_{\text{rk } \mathfrak{g}}$ is a nondecreasing sequence, $-2(d_{\text{rk } \mathfrak{g}} - 1)$ is the smallest eigenvalue of $\text{ad } h$. Let v be a non zero eigenvector of $\text{ad } h$ of eigenvalue $-2(d_{\text{rk } \mathfrak{g}} - 1)$ and let us set $v_0 = [v, e]$. Then $(\text{ad } v)^2(e)$ is an eigenvector of $\text{ad } h$ of eigenvalue $-4(d_{\text{rk } \mathfrak{g}} - 1) + 2$. As \mathfrak{g} is a simple algebra of dimension strictly bigger than 3, $d_{\text{rk } \mathfrak{g}}$ is strictly bigger than 2 and $-4(d_{\text{rk } \mathfrak{g}} - 1) + 2$ is strictly smaller than $-2(d_{\text{rk } \mathfrak{g}} - 1)$. So $(\text{ad } v)^2(e)$ is equal to 0. Hence, by Lemma 5.4, (e, v_0) belongs to $\mathcal{N}_{\mathfrak{g}}$. Let us prove now that (e, v_0) doesn't belong to $\mathcal{V}_{\mathfrak{g}}$. As v_0 is an eigenvector of $\text{ad } h$ of negative eigenvalue, \mathfrak{b} does not contain v_0 . Since \mathfrak{b} is the unique Borel subalgebra which contains e , we deduce from Lemma 5.5 that (e, v_0) does not belong to $\mathcal{V}_{\mathfrak{g}}$. At last, $\mathcal{V}_{\mathfrak{g}}$ is strictly contained in $\mathcal{N}_{\mathfrak{g}}$. As a result, for any k bigger than 2, $\mathcal{V}_{\mathfrak{g}}^{(k)}$ is strictly contained in $\mathcal{N}_{\mathfrak{g}}^{(k)}$ since $\mathcal{V}_{\mathfrak{g}} \times \{0_{\mathfrak{g}^{k-2}}\}$ and $\mathcal{N}_{\mathfrak{g}} \times \{0_{\mathfrak{g}^{k-2}}\}$ are the intersections of $\mathcal{V}_{\mathfrak{g}}^{(k)}$ and $\mathcal{N}_{\mathfrak{g}}^{(k)}$ with $\mathfrak{g} \times \mathfrak{g} \times \{0_{\mathfrak{g}^{k-2}}\}$ respectively. \square

Remark 5.7. — The nullcone has a natural structure of scheme. This scheme is irreducible. Moreover, it is reduced if and only if \mathfrak{g} has dimension 3. When \mathfrak{g} has dimension 3, $\text{pol}_2 \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is equal to $(\mathcal{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{g}))^{\mathfrak{g}}$ and the ideal in $\mathcal{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{g})$ generated by the augmentation ideal in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is prime. Let us suppose that \mathfrak{g} has dimension strictly bigger than 3. If this scheme were reduced, the extension $\mathcal{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{g})$ of $(\mathcal{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{g}))^{\mathfrak{g}}$ would be flat by arguments used in [19]. But this extension is not equidimensional since the G -orbit in $\mathfrak{g} \times \mathfrak{g}$ in general position has dimension $2b_{\mathfrak{g}} - \text{rk } \mathfrak{g}$ and this integer is strictly smaller than $3(b_{\mathfrak{g}} - \text{rk } \mathfrak{g})$.

6. Additional properties of the nilpotent bicone

In this section, we give additional results about the nilpotent bicone $\mathcal{N}_{\mathfrak{g}}$. In the first subsection, we prove that $\mathcal{N}_{\mathfrak{g}}$ is not reduced. By Theorem 5.3 and Proposition 5.6, $\mathcal{N}_{\mathfrak{g}}$ is not irreducible except for very particular cases. In the second subsection, we give an estimation of the number of its irreducible components.

As before, we fix a principal \mathfrak{sl}_2 -triple (e, h, f) in \mathfrak{g} and we use the notations of Subsection 1.3.

6.1. — According to [24](§23), a Noetherian ring A is reduced if and only if A satisfies conditions (R_0) and (S_1) . By Theorem 1.1, the subscheme $\mathcal{N}_{\mathfrak{g}}$ of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal I generated by $\text{pol}_2\mathbb{S}(\mathfrak{g})_{\neq}^{\mathfrak{g}}$ is a complete intersection in $\mathfrak{g} \times \mathfrak{g}$. Therefore the quotient ring $(\mathbb{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{g}))/I$ is Cohen-Macaulay, according to [24](Ch. 8, Theorem 21.3).

Theorem 6.1. — *The scheme $\mathcal{N}_{\mathfrak{g}}$ is not reduced.*

Proof. — By which goes before we have to find an irreducible component of $\mathcal{N}_{\mathfrak{g}}$ which does not contain any smooth points of $\mathcal{N}_{\mathfrak{g}}$. Hence, according to Proposition 2.12, Lemma 3.11, (ii) and Theorem 5.3, it is enough to prove that the nullcone $\mathcal{V}_{\mathfrak{g}}$ has an empty intersection with the open subset $\Omega_{\mathfrak{g}}$. Let us suppose that this intersection is not empty and we expect a contradiction. Let (x, y) be in the intersection of $\mathcal{V}_{\mathfrak{g}}$ and $\Omega_{\mathfrak{g}}$. By Lemma 5.5, we can suppose that x and y belongs to \mathfrak{u} , since the Borel subalgebras of \mathfrak{g} are conjugate under G . By Lemma 2.5, (ii), for any (a, b) in $\mathbb{C}^2 \setminus \{(0, 0)\}$, the centralizer of $ax + by$ is contained in \mathfrak{u} . Therefore, it results from Lemma 2.1 that the subspace $\mathfrak{V}'(x, y)$ is contained in \mathfrak{u} . Moreover, by Lemma 2.6, (iii), $\mathfrak{V}'(x, y)$ is equal to $\mathfrak{V}(x, y)$. It is impossible since by Lemma 2.6, (i), $\mathfrak{V}(x, y)$ has dimension $\mathfrak{b}_{\mathfrak{g}}$ since (x, y) belongs to $\Omega_{\mathfrak{g}}$. \square

6.2. — We denote by Φ the map

$$G \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}, (g, x, y) \mapsto (g(x), g(y)).$$

For a subset X of $\mathfrak{g} \times \mathfrak{g}$, the closure in $\mathfrak{g} \times \mathfrak{g}$ of the image of $G \times X$ by Φ is denoted by X^{Φ} . We recall that for Y subset of $\mathfrak{g} \times \mathfrak{g}$ and x in \mathfrak{g} , we denote by Y_x the subset of elements y in \mathfrak{g} such that (x, y) belongs to Y .

Lemma 6.2. — *Let Y be a G -invariant closed bicone of $\mathfrak{g} \times \mathfrak{g}$ and let x_0 be a nilpotent element of \mathfrak{g} . We suppose that $\varpi_1(Y)$ is the closure in \mathfrak{g} of $G.x_0$ and that there exists a subvariety T of G such that the map $g \mapsto g(x_0)$ is an isomorphism from T to an open subset of $G.x_0$ containing x_0 .*

- i) *The subset Y is equal to $(\{x_0\} \times Y_{x_0})^{\Phi}$.*
- ii) *If Z is an irreducible component of Y , then Z_{x_0} is irreducible.*

iii) The map $X \mapsto (\{x_0\} \times X)^\Phi$ is a bijection from the subset of irreducible components of Y_{x_0} to the subset of irreducible components of Y .

Proof. — i) By definition, $\varpi_1(Y)$ is the closure of the orbit Y'_1 of x_0 under the action of G . So Y'_1 is a dense open subset in $\varpi_1(Y)$. Then the subset of elements of Y whose first component is in Y'_1 is a dense open subset in Y . As Y is a G -invariant bicone, Y_{x_0} is a cone and Y' is the image by Φ of $G \times \{x_0\} \times Y_{x_0}$. So Y is equal to $(\{x_0\} \times Y_{x_0})^\Phi$.

ii) Let Z be an irreducible component of Y . Then Z is a G -invariant closed bicone. We denote by Ω the image of T by the map $g \mapsto g(x_0)$ and we denote by \tilde{Z} the subset of elements (x, y) in Z such that x is in Ω . Then \tilde{Z} is an open subset of Z . In particular, it is irreducible. Let τ be the inverse map of the map $g \mapsto g(x_0)$ from T to Ω . Then the map

$$\tilde{Z} \longrightarrow T \times Z_{x_0}, (x, y) \mapsto (\tau(x), \tau(x)^{-1}(y)),$$

is an isomorphism. So Z_{x_0} is irreducible.

iii) Let X be an irreducible component of Y_{x_0} . Then $(\{x_0\} \times X)^\Phi$ is irreducible as the closure of the image of an irreducible variety by a regular map. If Z is an irreducible component of Y which contains $(\{x_0\} \times X)^\Phi$ then Z_{x_0} contains X . So by (ii), X is equal to Z_{x_0} and Z is equal to $(\{x_0\} \times X)^\Phi$ since $\Phi(G \times \{x_0\} \times X)$ is open in Z . We then deduce that the map $Z \mapsto Z_{x_0}$ is the inverse of the map $X \mapsto (\{x_0\} \times X)^\Phi$. \square

Let us recall that \mathbf{B}_- is the normalizer in G of the Borel subalgebra $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{u}_-$.

Lemma 6.3. — *Let \mathbf{U} be the unipotent radical of the normalizer of \mathfrak{b} in G . Then there exists a closed subset Z of \mathbf{U} such that the map*

$$\mathbf{B}_- \times Z \rightarrow G.e, (g, k) \mapsto gk(e),$$

is an isomorphism onto an open subset V of $G.e$ containing e .

Proof. — As $\mathbf{B}_-\mathbf{U}$ is an open subset of G , its image by the map $g \mapsto g(e)$ from G to $G.e$ is an open subset V containing e . Moreover, the map

$$\mathbf{B}_- \times \mathbf{U}.e \rightarrow V, (g, x) \mapsto g(x)$$

is an isomorphism since \mathbf{U} contains the stabilizer of e in G and $\{\mathbf{1}_{\mathfrak{g}}\}$ is the intersection of \mathbf{B}_- and \mathbf{U} . By [29](Ch. I, Part. II, §3), there exists a complement \mathfrak{m} of $\mathfrak{g}(e)$ in \mathfrak{u} and a basis v_1, \dots, v_m of \mathfrak{m} such that the map

$$\mathbb{C}^m \rightarrow \mathbf{U}.e, (t_1, \dots, t_m) \mapsto \exp(t_1 \text{ad } v_1) \cdots \exp(t_m \text{ad } v_m)(e)$$

is an isomorphism. Then the map

$$\mathbb{C}^m \rightarrow \mathbf{U}, (t_1, \dots, t_m) \mapsto \exp(t_1 \text{ad } v_1) \cdots \exp(t_m \text{ad } v_m)$$

is proper and its image Z is closed in \mathbf{U} . Moreover, the map

$$\mathbf{B}_- \times Z \rightarrow V, (g, k) \mapsto gk(e)$$

is an isomorphism. \square

Let $\mathcal{N}_{\mathfrak{g},e}$ be the subset of elements y of \mathfrak{g} such that (e, y) is in $\mathcal{N}_{\mathfrak{g}}$. From Theorem 1.1, Lemma 6.2, (iii) and Lemma 6.3, we deduce the following result:

Corollary 6.4. — *The map $X \mapsto (\{e\} \times X)^\Phi$ is a bijection from the set of irreducible components of $\mathcal{N}_{\mathfrak{g},e}$ onto the set of irreducible components of $\mathcal{N}_{\mathfrak{g}}$.*

According to Subsection 3.3, $\mathcal{N}_{\mathfrak{g},e}$ has a natural structure of scheme. The following result is also a corollary of Theorem 1.1.

Corollary 6.5. — *The subset $\mathcal{N}_{\mathfrak{g},e}$ is a complete intersection of dimension $b_{\mathfrak{g}} - \text{rk } \mathfrak{g}$.*

Proof. — As $\mathcal{N}_{\mathfrak{g},e}$ is the nullvariety in \mathfrak{g} of $b_{\mathfrak{g}}$ polynomial functions, the dimension of any irreducible component of $\mathcal{N}_{\mathfrak{g},e}$ is at least $b_{\mathfrak{g}} - \text{rk } \mathfrak{g}$. Let Y be an irreducible component of $\mathcal{N}_{\mathfrak{g}}$. By Theorem 1.1, since the orbit of e under G is an open subset of $\mathfrak{N}_{\mathfrak{g}}$, Y_e has dimension $b_{\mathfrak{g}} - \text{rk } \mathfrak{g}$. On the other hand, by Corollary 6.4, Y_e is an irreducible component of $\mathcal{N}_{\mathfrak{g},e}$ and any irreducible component of $\mathcal{N}_{\mathfrak{g},e}$ is obtained in this way, whence the corollary. \square

Let $\mathfrak{p} = \mathfrak{l}_{\mathfrak{p}} \oplus \mathfrak{u}_{\mathfrak{p}}$ be a parabolic subalgebra of \mathfrak{g} containing \mathfrak{b} , with Levi part $\mathfrak{l}_{\mathfrak{p}}$ and nilradical $\mathfrak{u}_{\mathfrak{p}}$. We denote by $\varpi_{\mathfrak{l}_{\mathfrak{p}}}$ and $\varpi_{\mathfrak{u}_{\mathfrak{p}}}$ the projections from \mathfrak{p} to $\mathfrak{l}_{\mathfrak{p}}$ and $\mathfrak{u}_{\mathfrak{p}}$ respectively.

Lemma 6.6. — *The map $Y \mapsto Y \oplus \mathfrak{u}_{\mathfrak{p}}$ is a one-to-one correspondence between the set of irreducible components of $\mathcal{N}_{\mathfrak{l}_{\mathfrak{p}}, \varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)}$ and the set of irreducible components of $\mathcal{N}_{\mathfrak{g},e}$ contained in \mathfrak{p} .*

Proof. — Let Y be an irreducible component of $\mathcal{N}_{\mathfrak{l}_{\mathfrak{p}}, \varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)}$. Then $Y \oplus \mathfrak{u}_{\mathfrak{p}}$ is contained in the intersection of $\mathcal{N}_{\mathfrak{g},e}$ and \mathfrak{p} . As e is a regular nilpotent element of \mathfrak{g} , $\varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)$ is a regular nilpotent element of $\mathfrak{l}_{\mathfrak{p}}$. So by Corollary 6.5, Y has dimension $b_{\mathfrak{l}_{\mathfrak{p}}} - \text{rk } \mathfrak{g}$. Then $Y \oplus \mathfrak{u}_{\mathfrak{p}}$ has dimension

$$b_{\mathfrak{l}_{\mathfrak{p}}} - \text{rk } \mathfrak{g} + \dim \mathfrak{u}_{\mathfrak{p}} = b_{\mathfrak{g}} - \text{rk } \mathfrak{g} .$$

According to Corollary 6.5, we deduce that $Y \oplus \mathfrak{u}_{\mathfrak{p}}$ is an irreducible component of $\mathcal{N}_{\mathfrak{g},e}$. Thus, the above map is well-defined. It is clearly injective. Let us prove that it is surjective.

Let Z be an irreducible component of $\mathcal{N}_{\mathfrak{g},e}$ contained in \mathfrak{p} . Then $\varpi_{\mathfrak{l}_{\mathfrak{p}}}(Z)$ is contained in an irreducible component of $\mathcal{N}_{\mathfrak{l}_{\mathfrak{p}}, \varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)}$. Let Y be such an irreducible component. Then $Y \oplus \mathfrak{u}_{\mathfrak{p}}$ is an irreducible subset of $\mathcal{N}_{\mathfrak{g},e}$ which contains Z . So, Z is equal to $Y \oplus \mathfrak{u}_{\mathfrak{p}}$. \square

Let $N_{\mathfrak{g}}$ be the set of irreducible components of $\mathcal{N}_{\mathfrak{g},e}$. We denote by $\Upsilon_{\mathfrak{g}}$ the set of proper parabolic subalgebras strictly containing \mathfrak{b} . Let $N'_{\mathfrak{g}}$ be the subset of $N_{\mathfrak{g}}$ of irreducible components of $\mathcal{N}_{\mathfrak{g},e}$ which are not contained in any element of $\Upsilon_{\mathfrak{g}}$.

Lemma 6.7. — *If \mathfrak{g} has dimension 3, then $|N'_{\mathfrak{g}}| = 0$. Otherwise, $|N'_{\mathfrak{g}}|$ is at least 1.*

Proof. — If \mathfrak{g} has dimension 3, then $\Upsilon_{\mathfrak{g}}$ is an empty set. Suppose now that the dimension of \mathfrak{g} is strictly bigger than 3. Then $\Upsilon_{\mathfrak{g}}$ is not empty. Let $\alpha^{\#}$ be the biggest positive root of \mathcal{R}_+ . Then, by Lemma 5.4, $\mathcal{N}_{\mathfrak{g},e}$ contains $[e, \mathfrak{g}^{-\alpha^{\#}}]$. So it is enough to prove that $[e, \mathfrak{g}^{-\alpha^{\#}}]$ is not contained in \mathfrak{p} for any \mathfrak{p} in $\Upsilon_{\mathfrak{g}}$. For α in \mathcal{R} , we fix a nonzero element x_{α} in \mathfrak{g}^{α} . The elements x_{α} may be chosen in a such way that $e = \sum_{\alpha \in \Pi} x_{\alpha}$. Then we have

$$(11) \quad [x_{-\alpha^{\#}}, e] = \sum_{\alpha \in \Pi} b_{\alpha} x_{-(\alpha^{\#} - \alpha)}$$

where b_{α} is in $\mathbb{C} \setminus \{0\}$ for all α in Π such that $\alpha^{\#} - \alpha$ is a root. Suppose that the set of \mathfrak{p} in $\Upsilon_{\mathfrak{g}}$ such that $[x_{-\alpha^{\#}}, e]$ belongs to \mathfrak{p} is nonempty. Then there is β in Π such that $[x_{-\alpha^{\#}}, e]$ belongs to the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{b} \oplus \sum_{\alpha \in \mathcal{R}_+^S} \mathfrak{g}_{-\alpha},$$

where \mathcal{R}^S is the root subsystem of \mathcal{R} generated by $S = \Pi \setminus \{\beta\}$ and where \mathcal{R}_+^S is the intersection of \mathcal{R}_+ and \mathcal{R}^S . Let n_{γ} , $\gamma \in \Pi$ be the coordinates of $\alpha^{\#}$ in the basis Π . Then for α in Π , we have

$$(12) \quad \alpha^{\#} - \alpha = (n_{\alpha} - 1)\alpha + \sum_{\substack{\gamma \in \Pi \\ \gamma \neq \alpha}} n_{\gamma} \gamma .$$

Let us suppose that α is not equal to β . By equality (12), if $\alpha^{\#} - \alpha$ were a root, then n_{β} would be equal to 0, since $[x_{-\alpha^{\#}}, e]$ belongs to \mathfrak{p} . But n_{β} is at least 1. So $\alpha^{\#} - \alpha$ is not a root for any α in $\Pi \setminus \{\beta\}$. Consequently $\alpha^{\#} - \beta$ is a root, and necessarily $\alpha^{\#} - \beta$ is in \mathcal{R}_+^S , by equality (11). Hence n_{β} is equal to 1, by equality (12). This discussion proves that the simple root β satisfies the two following conditions:

- 1) for all α in $\Pi \setminus \{\beta\}$, $\alpha^{\#} - \alpha$ is not a root,
- 2) $\alpha^{\#} - \beta$ is a root and $n_{\beta} = 1$.

A quick look on the classification of root systems proves that any simple root does not satisfy both conditions (1) and (2). We have so obtained the expected result. \square

Recall that $N_{\mathfrak{g}}$ is the set of irreducible components of $\mathcal{N}_{\mathfrak{g},e}$.

Proposition 6.8. — *The number of irreducible components of $\mathcal{N}_{\mathfrak{g}}$ is equal to $|N_{\mathfrak{g}}|$ and $|N_{\mathfrak{g}}|$ satisfies the following recursive relation:*

$$|N_{\mathfrak{g}}| = |N'_{\mathfrak{g}}| + \left(\sum_{\mathfrak{p} \in \Upsilon_{\mathfrak{g}}} \prod_{\substack{\text{simple factor} \\ \text{of } [\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}}]}} |N'_{\mathfrak{l}}| \right) + 1 .$$

Proof. — By Corollary 6.4, the number of irreducible components of $\mathcal{N}_{\mathfrak{g}}$ is $|N_{\mathfrak{g}}|$. If \mathfrak{g} has dimension 3, then $\mathcal{N}_{\mathfrak{g},e}$ is the line generated by e , whence $|N_{\mathfrak{g}}| = 1$. We suppose that \mathfrak{g} has dimension strictly bigger than 3. By Theorem 5.3, Corollary 6.4 and Lemma 5.5, \mathfrak{u} is an irreducible component of $\mathcal{N}_{\mathfrak{g},e}$. Thus,

$$(13) \quad N_{\mathfrak{g}} = N'_{\mathfrak{g}} \cup \left(\bigcup_{\mathfrak{p} \in \Upsilon_{\mathfrak{g}}} N_{\mathfrak{g}}^{(\mathfrak{p})} \right) \cup \{\mathfrak{u}\} ,$$

where, for \mathfrak{p} in $\Upsilon_{\mathfrak{g}}$, $N_{\mathfrak{g}}^{(\mathfrak{p})}$ is the set of irreducible components of $\mathcal{N}_{\mathfrak{g}}$ contained in \mathfrak{p} . By Lemma 6.6, the set $N_{\mathfrak{g}}^{(\mathfrak{p})}$ is in one-to-one correspondence with the set $N_{\mathfrak{l}_{\mathfrak{p}}}$. Let \mathfrak{p} be an element of $\Upsilon_{\mathfrak{g}}$ and let $\mathfrak{l}_1, \dots, \mathfrak{l}_m$ be the simple factors of $\mathfrak{l}_{\mathfrak{p}}$. As remarked in Subsection 1.3, the irreducible components of $\mathcal{N}_{\mathfrak{l}_{\mathfrak{p}}, \varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)}$ are a product of irreducible components of $\mathcal{N}_{\mathfrak{l}_i, e_i}$, where e_i is the component of $\varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)$ on \mathfrak{l}_i for $i = 1, \dots, m$. Let $Y = Y_1 \times \dots \times Y_m$ be an irreducible component of $\mathcal{N}_{\mathfrak{l}_{\mathfrak{p}}, \varpi_{\mathfrak{l}_{\mathfrak{p}}}(e)}$ where Y_i is an irreducible component of $\mathcal{N}_{\mathfrak{l}_i, e_i}$ for all $i = 1, \dots, m$. If there exists i such that Y_i is contained in an element of $\Upsilon_{\mathfrak{l}_i}$, then Y is contained in an element of $\Upsilon_{\mathfrak{l}_{\mathfrak{p}}}$, and therefore $Y \oplus \mathfrak{p}_{\mathfrak{u}}$ is contained in an element of $\Upsilon_{\mathfrak{g}}$ strictly contained in \mathfrak{p} , so different from \mathfrak{p} . As a consequence, we have:

$$\left| \bigcup_{\mathfrak{p} \in \Upsilon_{\mathfrak{g}}} N_{\mathfrak{g}}^{(\mathfrak{p})} \right| = \sum_{\mathfrak{p} \in \Upsilon_{\mathfrak{g}}} \prod_{\substack{\text{simple factor} \\ \text{of } [\mathfrak{l}_{\mathfrak{p}}, \mathfrak{l}_{\mathfrak{p}}]}} |N'_{\mathfrak{l}}| ,$$

whence the proposition according to relation (13). \square

Remark 6.9. — Considering the formula established in Proposition 6.8, we see that the number $|N_{\mathfrak{g}}|$ becomes considerably big while the dimension of \mathfrak{g} grows up. For example we have:

$$|N_{\mathfrak{sl}_2}| = 1 , |N_{\mathfrak{sl}_3}| \geq 2 , |N_{\mathfrak{sl}_4}| \geq 4 , |N_{\mathfrak{sl}_5}| \geq 7 , |N_{\mathfrak{sl}_6}| \geq 12 , \dots .$$

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