Fuzzy Nominal Scales

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ABSTRACT

The main objective of a measurement process is to convert the physical state of an entity into an information entity. At a higher level, relations on physical states are also converted into relations between the information entities. Then it’s possible to manipulate the information entities with these relations instead of manipulating the physical states. Nominal scales are scales where the only relation being converted is an equivalence relation. In most applications of nominal scales, the symbolic representation of a physical state is a symbol. An other approach shows that a physical state can be represented by a fuzzy subset of symbols. This paper presents how a representation with fuzzy subsets of symbols can be used to build a scale. Then, the incidence of the fuzziness on this new kind of nominal scale, called fuzzy nominal scale, is analysed.

Keywords: Measurement theory, Scales, Nominal scales, Fuzzy Scales.

1. INTRODUCTION

1.1. Nominal scales

Let us present the general scale formalism in order to introduce the aim of the paper. Any scale can be defined with a symbolism [1]:

$$C = < X, S, M, R_X, R_S, F> \qquad (1)$$

- where X refers to a set of object states, and R_X is a set of relations on X.
- S refers to a set of information entities [2], and R_S is a set of relations on S.
- M, called the representation, is a mapping from X to S.
- F is a one-to-one mapping with domain R_X and range R_S.

In the case of nominal scales [3], R_S contains only the equality relation on S, and R_X contains an equivalence relation on X denoted ~. F is then simply defined by $F = \{(\sim,=)\}$.

$$\forall(x_i \in X), \forall(x_j \in X), x_i \sim x_j \Leftrightarrow M(x_i) = M(x_j) \qquad (2)$$

The equivalence relation ~ defines a partition of the set of object states, and M associates each item of the partition to an information entity. The general symbolism of a nominal scale is:

$$C = < X, S, M, \{\sim\}, \{=\}, \{(\sim,=)\}> \qquad (3)$$

The easiest way for the definition of a nominal scale is to define a partition and a one-to-one mapping from the partition to the set of information entities.

Depending on the measurement aim, the nature of information entities differs:

- An information entity can be a symbol. This is the simplest case used for example with color measurement.
- An information entity can be a subset of symbols. For example, a subset of predicates can be used as a representation [4].

In this paper we consider the mechanism that represents each object with a fuzzy subset of symbols. This mechanism, called fuzzy description and introduced few years after the fuzzy subset theory in [5], is now usually applied in fuzzy sensors [6][7].

1.2. Fuzzy description

If numerical values are commonly used to represent measurement results, it is now admitted that some applications would better manipulate symbolic values or linguistic terms. For example, a rule-based decision system uses linguistic term in its rules. The advantages of using linguistic terms instead of numeric ones is to reduce the number of symbols.

A measurement result is represented by a fuzzy subset of linguistic terms. The conversion from numerical to linguistic representation is called a linguistic description. To perform a linguistic measurement, it is necessary to clearly specify the relation between linguistic terms and numbers [8].

Let X be the universe of discourse associated with the measurement of a particular physical quantity. In order to linguistically characterise any measurement over X, let $\mathcal{W}$ be a set of words, representative of the physical phenomenon. Denotes R the relation defined on $X \times W$ that formalize the link between items of the universe of discourse and the words of the lexical universe. In order to simplify the management of this link, two mapping are deduced form this relation. The first one is a one-to-one mapping m called the *meaning* of a linguistic terms [5]. It associates any word w of $\mathcal{W}$ with a subset of X i.e. an item of the set $S(X)$ of subsets of X.

$$\begin{array}{ll} m & W \to S(X) \\ \forall w \in W & m(w) = \{x \in X | xRw\} \end{array} \quad (4)$$

The other one is a one-to-one mapping d called the *description* of a measurement.

$$\begin{array}{ll} d & X \to S(W) \\ \forall x \in X & d(x) = \{w \in W | xRw\} \end{array} \quad (5)$$

From the linguistic description point of view, it is interesting to define R such that each object state x is characterized by only one word w: $d(x) = \{w\}$. This means that the range of the mapping m is a partition on X i.e. each $m(w)$ is an equivalence class. Thus, the mapping m defines an equivalence relation ~ on X. Then the description mechanism can be considered as a nominal scale where:

$$C = \langle X, \{\{w\} | (w \in W)\}, d, \{\sim\}, \{=\}, \{(\sim, =)\}\rangle \quad (6)$$

- The universe of discourse is the set of object states.
- Information entities are singletons of the lexical set W.
- M is the description *d*.

If the number of information entities is small then the precision of this symbolism is said low. When object states change in a same equivalence class, the associated information entity does not change. Introducing the vagueness into the relation between object states and information entities is a solution for increasing the precision of a symbolism.

Consider now that the relation R is a fuzzy relation on $X \times W$. It is characterized by its membership function denoted $\mu_R(x,w)$. Then the fuzzy meaning of a word and the fuzzy description of an object state are respectively defined as:

$$\begin{array}{ll} m & X \to FS(W) \\ \forall w \in W, \forall x \in X & \mu_{m(w)}(x) = \mu_R(x, w) \end{array} \quad (7)$$

$$\begin{array}{ll} d & W \to FS(X) \\ \forall w \in W, \forall x \in X & \mu_{d(x)}(w) = \mu_R(x, w) \end{array} \quad (8)$$

Where *FS(M)* is the set of fuzzy subsets of *M*, and *FS(X)* is the set of fuzzy subsets of *X*. These two mappings are simply linked by

$$\forall w \in W, \forall x \in X \qquad \mu_{d(x)}(w) = \mu_{m(w)}(x) \quad (9)$$

As for a crisp description mechanism, it is interesting to define R such that the range of the fuzzy mapping m be a partition on X. Then, a fuzzy equivalence relation ~ on X, also called similarity relation, can be deduced. Several fuzzy partitions can be defined. The preferred partitions used for the description with linguistic terms respect the following constraint:

$$\forall (x \in X), \sum_{w \in W} \mu_{m(w)}(x) = 1 . \quad (10)$$

This constraint reveals the need of exclusion between linguistic terms for the description of states. Based on this partition a similarity relation can be extracted:

$$\mu_{\sim}(x, y) = \sum_{w \in W} (min(\mu_{m(w)}(x), \mu_{m(w)}(y))) \quad (11)$$

This relation is reflexive, symmetric and verifies a weak version of transitivity based on the Lukasiewicz T-norm. With this fuzzy equivalence relation it seems possible to interpret the fuzzy description as a nominal scale.

2. THE FUZZY DESCRIPTION MECHANISM IS A SCALE

In this paper, some answers to the following questions are proposed:

- Can it be possible to interpret the fuzzy description of objects into the general scale formalism?
- If possible, what kind of scale uses a fuzzy description as representation mechanism?

2.1. Considering fuzzy subsets of words as information entities

One solution for the interpretation of the fuzzy description into the scale formalism is to define the representation set as the set of fuzzy subsets of words, and the mapping M as the fuzzy description.

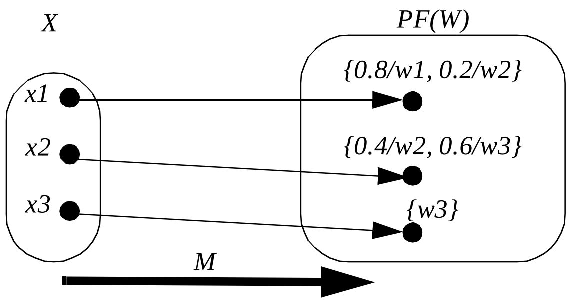

Fig. 1. Scale interpretation

The following symbolism can then be defined:

$$C = \langle X, FS(W), d, \{\sim\}, \{\equiv\}, \{(\sim, \equiv)\}\rangle \quad (12)$$

- where X refers to the set of object states,
- where information entities are fuzzy subsets of the lexical set W,
- the fuzzy description d is the representation mapping,
- the relation ~ is the fuzzy equivalence relation on X defined in (11),
- and ≡ is the equivalence relation on fuzzy subsets of W such that:

$$\forall (x, y) \in X^2 \qquad x \sim y \Leftrightarrow d(x) \equiv d(y) \quad (13)$$

This equivalence relation is only defined on the range of the fuzzy description d. It can be extended to any fuzzy subset A of W that respect (14).

$$\sum_{w \in W} (\mu_A(w)) = 1 \quad (14)$$

Denote *MFS*(*W*) the subset of *FS*(*W*) such that each item of *MFS*(*W*) verifies (14). It's interesting to see that *MFS*(*W*) is the union of ranges of all possible fuzzy descriptions on the lexical set *W*.

$$\forall (A, B) \in MFS(W)^2$$
$$\mu_{\equiv}(A, B) = \sum_{w \in W} min(\mu_A(w), \mu_B(w)) \quad (15)$$

In this solution the symbolisation is precise. This property come from the size of the set of information entities that can be very large. This is also contradictory with the first interest of fuzzy description i.e. reducing the number of information entities in order to simplify the manipulation of these entities.

2.2. Considering fuzzy description as the representation mapping

In an other solution, the information entities are symbols, and *M* is a fuzzy relation from *X* to the lexical set *W*.

$$C = \langle X, W, R, \{\sim\}, \{=\}, \{(\sim, =)\}\rangle \quad (16)$$

- where *X* refers to the same set of object states,
- where information entities are words of the lexical set *W*,
- where *R* the fuzzy relation from *X* to *W* that defines the fuzzy meaning and the fuzzy description (7) and (8).
- the relation ~ is the fuzzy equivalence relation on *X* defined in (11),
- and $=$ is an equivalence relation on items of *W*. This relation can be simply defined such that:

$$\forall (w, z) \in W^2 \qquad w{=}z \Leftrightarrow \{w\}{\equiv}\{z\} \quad (17)$$

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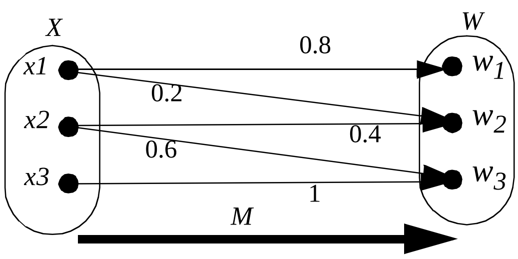

Fig. 2. Scale interpretation

In this case, the symbolisation stays precise, because the equivalence relation on object states is the same than in preceding case. The number of representational symbols is small. This is useful when we need to handle the knowledge associated with object states.

It can be noted that using a crisp partition instead of a fuzzy one induces a usual nominal scale.

3. FUZZY NOMINAL SCALES

To the question «what kind of scale uses a fuzzy description as representation mechanism?», the last symbolism presented seems to let conclude that a fuzzy description is a nominal scale. But we consider that this conclusion is dangerous. Indeed, in the case of a nominal scale, the only relation that can be used on the lexical set is an equivalence relation. But the introduction of fuzzyness into the measurement mechanism induces a new proximity relation on the lexical set. This relation comes from relation between the meanings of words i.e. the items of the fuzzy partition.

As each word *w* belongs to an item of *d*(*X*) it is possible to define an equivalence relation based on the general combination-conjunction:

$$\forall (w, z) \in W^2$$
$$\mu_{=}(w, z) = sup_{A \in d(X)}(min(\mu_A(w), \mu_A(z))) \quad (18)$$

such that

$$\forall w \in W \qquad sup_{A \in d(X)}(\mu_A(w)) = 1 \quad (19)$$

This relation is an equivalence relation that verify a weak transitivity based on Lukasiewicz T-norm. This relation does not verify a strict definition of the transitivity i.e. the sup-min transitivity. It is then considered as a proximity relation. This relation is not the crisp equality relation, and can link different words. Actually it acts as a proximity relation between words.

Introducing a fuzzy description of object states instead of a crisp one induces a proximity relation between items of the fuzzy partition and then induces a proximity relation between information entities that represent these objects states. As symbolisms are used in order to give tools for manipulating information entities instead of their associated object states, a symbolism do not have to introduce relations that not exist between object states.

Using fuzzy description can be an approach for taking into account existing proximity relation between object states. And fuzzy nominal scales can be used when nominal scales are too poor and when an ordinal scale cannot be defined.

4. COLOR MEASUREMENT EXAMPLE

The color measurement is a typical measurement where its seems to be possible to use a scale more informative than a simple nominal scale. Indeed, we can imagine that a proximity relation can exist into a color space. Red color is close to orange color but not close to green color. Unfortunately no existing crisp scale presently includes this proximity relation.

A color measurement is performed with three photometric transducers that recreate the effects of red, green and blue human cones. The set of color measurements is part of the so-called RGB cube. Then a non linear mapping is applied to the RGB cube. This mapping has been chosen in order to obtain a colour information which do not depend on the luminosity. This information, called the chrominance, is included into the chrominance plane. In this example, we choose the C1C2 chrominance plane [6]. Seven typical colors are localized on this plane (Fig. 3.).

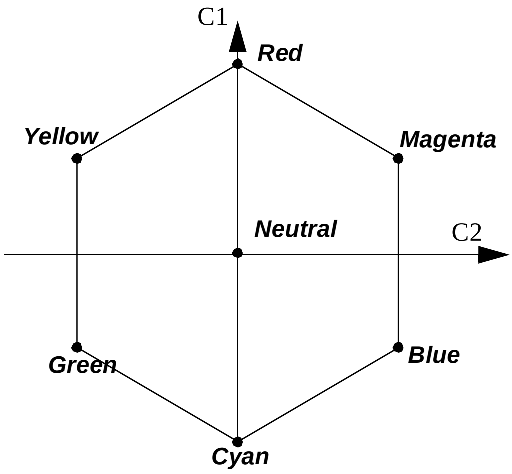

Fig. 3. C1C2 chrominance plane used for color measurements

A fuzzy partition based on an interpolation of human knowledge can be defined on this chrominance plane (see Fig. 3.). The human knowledge is defined by a set of measurement states, With each state associated to a word. The interpolation knowledge is performed with the triangulation method. Other methods like fuzzy C-means, or K-Nearest neighbour that can also be used are not presented in this paper.

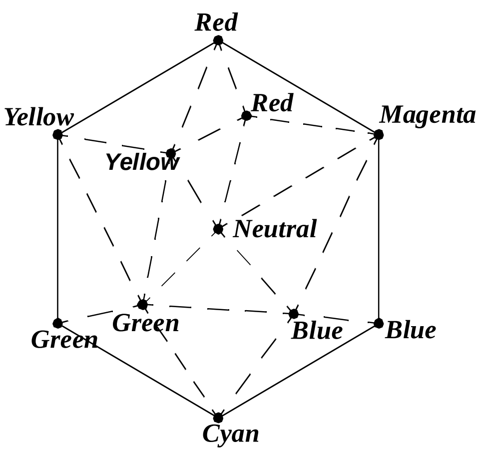

Fig. 4. Human knowledge and triangulation of the chrominance space.

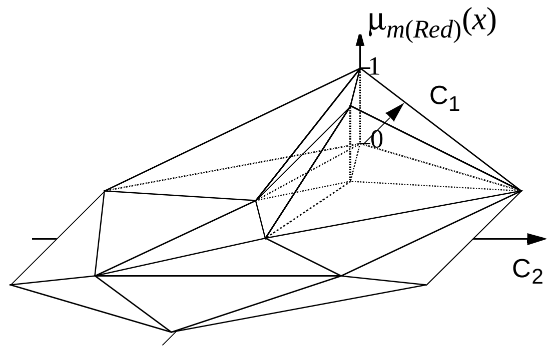

Fig. 5. Fuzzy meaning of the word *Red*.

This fuzzy partition is used for the definition of the fuzzy nominal scale. The proximity relation on the lexical set can then be define (see table 1).

Table1: = relation on color words

=	Red	Mag.	Yell.	Neut.	Green	Cyan	Blue
Red	1	0.5	0.5	0.5	0	0	0
Mag.	0.5	1	0	0.5	0	0	0.5
Yell.	0.5	0	1	0.5	0.5	0	0
Neut.	0.5	0.5	0.5	1	0.5	0	0.5
Green	0	0	0.5	0.5	1	0.5	0.5
Cyan	0	0	0	0	0.5	1	0.5
Blue	0	0.5	0	0.5	0.5	0.5	1

The equivalence/proximity relation = on color words shows that *red* colors are near *yellow* colors. In can be noted that if the lexical set is larger, including for example the *orange* color, then the relation = can differs, and *red* colors can be near *orange* colors but not near *yellow* colors. Actually, the definition of the proximity relation on the lexical set of words depends on the definition of the fuzzy partition. It depends on the fuzzy meaning of each word.

5. CONCLUSION

In this paper, the definition of a fuzzy nominal scale had been presented, and the interest and the danger of using such scale where exposed. Using fuzzy nominal scale for a measurement induces a proximity relation on the representational set that is more informative than the equality. This is an improvement only if this proximity relation represents a real proximity relation between object states. If not, the danger is to introduce a relation that does not have any meaning. As several family of fuzzy equivalence relations can be used, it will be useful to study the properties and the useability of associated fuzzy nominal scales.

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