Local Sentences and Mahlo Cardinals

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Abstract

Local sentences were introduced by Ressayre in [Res88] who proved certain remarkable stretching theorems establishing the equivalence between the existence of finite models for these sentences and the existence of some infinite well ordered models. Two of these stretching theorems were only proved under certain large cardinal axioms but the question of their exact (consistency) strength was left open in [FR96]. Here, we solve this problem, using a combinatorial result of J. H. Schmerl [Sch74]. In fact, we show that the stretching principles are equivalent to the existence of n-Mahlo cardinals for appropriate integers n. This is done by proving first that for each integer n, there is a local sentence ϕ_n which has well ordered models of order type τ, for every infinite ordinal $\tau > \omega$ which is not an n-Mahlo cardinal.

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Keywords: Local sentences; stretching principles; well ordered models; Mahlo cardinals; spectra of sentences.

1 Introduction

Local sentences were introduced by J.-P. Ressayre who proved some remarkable links between the finite and the infinite model theory of these sentences, [Res88]. A local sentence is a first order sentence which is equivalent to a universal sentence and satisfies some semantic restrictions: closure in its models takes a finite number of steps. Assuming that a binary relation symbol belongs to the signature of a local sentence φ and is interpreted by a linear order in every model of φ, the stretching theorems state that the existence of certain well ordered models of φ is equivalent to the existence of a finite model of φ, generated by some particular kind of indiscernibles, like special, remarkable, semi-monotonic or monotonic ones (see [FR96] for a precise definition). Two of these stretching theorems were only proved under large cardinal axioms but the question of their exact (consistency) strength was left open in [FR96]. We solve here this problem, using a combinatorial result of J. H. Schmerl which characterizes n-Mahlo cardinals in [Sch74]. We show that the two stretching theorems are in fact equivalent to the existence, for each integer n, of an n-Mahlo cardinal. This is done by proving first that for each integer n, there is a local sentence ϕ_n which has some well ordered models of order type τ, for every infinite ordinal $\tau > \omega$ which is not an n-Mahlo cardinal. Using this result we show also that a kind of hanf number μ for local sentences defined by Ressayre in [Res88] is in fact equal to the first ω-Mahlo cardinal, if such a cardinal exists.

The paper is organized as follows. In section 2 we recall some definitions and stretching theorems. In section 3 we prove the existence of a local sentence ϕ_n which has some well ordered models of order type τ, for every infinite ordinal $\tau > \omega$ which is not an n-Mahlo cardinal. In section 4 we solve the question of the exact (consistency) strength of some stretching principles for local sentences.

2 Stretching theorems for local sentences

In this paper the (first order) signatures are finite, always contain one binary predicate symbol $=$ for equality, and can contain both functional and relational symbols.

When M is a structure in a signature Λ, $|M|$ is the domain of M.
If S is a function, relation or constant symbol in Λ, then S^M is the interpretation in the structure M of S.
Notice that, when the meaning is clear, the superscript M in S^M will be sometimes omitted in order to simplify the presentation.

For $X \subseteq |M|$, we define:
$cl^1(X, M) = X \cup \bigcup_{\{f \text{ k-ary function of } \Lambda\}} f^M(X^k) \cup \bigcup_{\{a \text{ constant of } \Lambda\}} a^M$
$cl^{n+1}(X, M) = cl^1(cl^n(X, M), M)$ for an integer $n \geq 1$
and $cl(X, M) = \bigcup_{n \geq 1} cl^n(X, M)$ is the closure of X in M.

The signature of a first order sentence φ, i.e. the set of non logical symbols appearing in φ, is denoted $\mathrm{S}(\varphi)$.

Definition 2.1 *A first order sentence φ is local if and only if:*

(a) $M \models \varphi$ and $X \subseteq |M|$ imply $cl(X, M) \models \varphi$

(b) $\exists n \in \mathbb{N}$ such that $\forall M$, if $M \models \varphi$ and $X \subseteq |M|$, then $cl(X, M) = cl^n(X, M)$, (closure in models of φ takes at most n steps).

For a local sentence φ, n_φ is the smallest integer $n \geq 1$ satisfying (b) of the above definition. In this definition, (a) implies that a local sentence φ is always equivalent to a universal sentence, so we may assume that this is always the case.

Example 2.2 *Let φ be the sentence, (already given in [Fin05]), in the signature $S(\varphi) = \{<, P, i, a\}$, where $<$ is a binary relation symbol, P is a unary relation symbol, i is a unary function symbol, and a is a constant symbol, which is the conjunction of:*

(1) $\forall xyz[(x \leq y \vee y \leq x) \wedge ((x \leq y \wedge y \leq x) \leftrightarrow x = y) \wedge ((x \leq y \wedge y \leq z) \to x \leq z)]$,

(2) $\forall xy[(P(x) \wedge \neg P(y)) \to x < y]$,

(3) $\forall xy[(P(x) \to i(x) = x) \wedge (\neg P(y) \to P(i(y)))]$,

(4) $\forall xy[(\neg P(x) \wedge \neg P(y) \wedge x \neq y) \to i(x) \neq i(y)]$,

(5) $\neg P(a)$.

We now explain the meaning of the above sentences (1)-(5).
Assume that M is a model of φ. The sentence (1) expresses that $<$ is interpreted in M by a linear order; (2) expresses that P^M is an initial segment

of the model M; (3) expresses that the function i^M is trivially defined by $i^M(x) = x$ on P^M and is defined from $\neg P^M$ into P^M. (4) says that i^M is an injection from $\neg P^M$ into P^M and (5) ensures that the element a^M is in $\neg P^M$.
The sentence φ is a conjunction of universal sentences thus it is equivalent to a universal one, and closure in its models takes at most two steps (one adds the constant a in the first step then takes the closure under the function i). Thus φ is a local sentence.
If we consider only the order types of *well ordered* models of φ, we can easily see that φ has a model of order type α, for every finite ordinal $\alpha \geq 2$ and for every infinite ordinal α which is not a cardinal.

The reader may also find many other examples of local sentences in the papers [Res88, FR96, Fin01, Fin04, Fin05]. Notice that local sentences play a role in defining many classes of formal languages of finite or infinite words which are important in theoretical computer science, like the classes of regular or quasirational languages; the latter one forms of rich subclass of the class of context-free languages, see [Res88, Fin01, Fin04].

From now on we shall assume that the signature of local sentences contain a binary predicate $<$ which is interpreted by a linear ordering in all of their models.
We recall the stretching theorem for local sentences. Below, semi-monotonic, special, remarkable, and monotonic indiscernibles are particular kinds of indiscernibles which are precisely defined in [FR96].

Theorem 2.3 ([FR96]) *For each local sentence φ there exists a positive integer N_φ such that*

(A) φ has arbitrarily large finite models if and only if φ has an infinite model if and only if φ has a finite model generated by N_φ indiscernibles.

(B) φ has an infinite well ordered model if and only if φ has a finite model generated by N_φ semi-monotonic indiscernibles.

(C) φ has a model of order type ω if and only if φ has a finite model generated by N_φ special indiscernibles.

(D) φ has well ordered models of unbounded order types in the ordinals if and only if φ has a finite model generated by N_φ monotonic indiscernibles.

(E) φ *has for every infinite cardinal* k *a model of order type* k *if and only if* φ *has a finite model generated by* N_φ *monotonic and remarkable indiscernibles.*

(E') φ has for every infinite cardinal k a model of order type k in which a distinguished predicate P is cofinal if and only if φ has a finite model generated by N_φ monotonic and special indiscernibles belonging to P.

To every local sentence φ and every ordinal α such that $\omega \leq \alpha < \omega^\omega$ one can associate by an effective procedure a local sentence φ_α, a unary predicate symbol P being in the signature $S(\varphi_\alpha)$, such that:

(C_α) φ has a well ordered model of order type α if and only if φ_α has a finite model M generated by N_{φ_α} semi-monotonic indiscernibles belonging to P^M.

It is proved in [FR96] that the integer N_φ can be effectively computed from n_φ and q where

$$\varphi = \forall x_1 \ldots \forall x_q \theta(x_1, \ldots, x_q)$$

and θ is a quantifier free formula. If $v(\varphi)$ is the maximum number of variables of terms of complexity $\leq n_\varphi + 1$ (resulting by at most $n_\varphi + 1$ applications of function symbols) and $v'(\varphi)$ is the maximum number of variables of an atomic formula involving terms of complexity $\leq n_\varphi + 1$ then

$$N_\varphi = max\{3v(\varphi); v'(\varphi) + v(\varphi); q.v'(\varphi)\}.$$

It is also proved in [FR96] that actually some equivalences of this stretching theorem hold only under strong axioms of infinity:

(E) is provable in **ZF** + existence for each integer n of an n-Mahlo cardinal; but not in **ZF**, for it implies the existence of an inaccessible cardinal.

(E') is provable in **ZF** + the scheme asserting for each standard integer n the existence of an n-Mahlo cardinal; and (E') implies the consistency of this scheme.

The question of the exact (consistency) strength of (E) and (E') was left open in [FR96] and is solved in this paper.

3 Infinite spectra of local sentences

We are mainly interested in this paper by *well ordered* models of local sentences, so we now recall the notion of spectrum of a local sentence φ. As usual the class of all ordinals is denoted by **On**.

Definition 3.1 *Let φ be a local sentence; the spectrum of φ is*

$$Sp(\varphi) = \{\tau \in \mathbf{On} \mid \varphi \text{ has a model of order type } \tau\}$$

and the infinite spectrum of φ is

$$Sp_\infty(\varphi) = \{\tau \in \mathbf{On} \mid \tau \geq \omega \text{ and } \varphi \text{ has a model of order type } \tau\}$$

The following result was proved in [FR96]:

Theorem 3.2 *There exists a local sentence ϕ_0 such that*

$$Sp_\infty(\phi_0) = \{\tau > \omega \mid \tau \text{ is not an inaccessible cardinal }\}.$$

We are going firstly to extend this result by proving the following one.

Theorem 3.3 *For each integer $n \geq 1$, there exists a local sentence ϕ_n such that*

$$Sp_\infty(\phi_n) = \{\tau > \omega \mid \tau \text{ is not an } n\text{-Mahlo cardinal }\}.$$

In order to construct the local sentences ϕ_n, we shall use a combinatorial result of J. H. Schmerl which gives a characterization of n-Mahlo cardinals [Sch74]. Notice that in [Sch74], the now usually called n-Mahlo cardinals are just called n-inaccessible.

As usual, the set of subsets of cardinality n of a set X is denoted by $[X]^n$. If C is a partition of $[X]^n$ then $Y \subseteq X$ is C-homogeneous iff every two elements of $[Y]^n$ are C-equivalent, i.e. are in the same set of C.

Definition 3.4 *For an integer $n \geq 1$ and an ordinal α, let $P(n, \alpha)$ be the class of infinite cardinals k which have the following property: Suppose for each $\nu < k$ that C_ν is a partition of $[k]^n$ and card(C_ν) $< k$ then there is $X \subseteq k$ of length α such that for each $\nu \in X$, the set $X - (\nu + 1)$ is C_ν-homogeneous.*

Theorem 3.5 ([Sch74]) *Let $n \geq 1$ be an integer and k be an infinite cardinal. Then $k \in P(n + 2, n + 5)$ if and only if $k = \omega$ or k is an n-Mahlo cardinal.*

We are going to construct a local sentence θ_n such that, for each regular infinite cardinal κ, it holds that:

(θ_n has a model of order type κ) if and only if ($\kappa \notin P(n+2, n+5)$)

So we have to express that, for each $\nu < \kappa$, there is a partition C_ν of $[\kappa]^{n+2}$ with card$(C_\nu) < \kappa$, such that, for all subsets X of κ having $n+5$ elements, there exists $\nu \in X$ such that $X - (\nu + 1)$ is not C_ν-homogeneous.

We shall firstly express by the following sentence ψ_1 that a model M is divided into successive segments. The signature of ψ_1 is $\{<, I, P\}$, where I is a unary function and P is a unary predicate. ψ_1 is the conjunction of :

(1) ($<$ is a linear order),

(2) $\forall yz[y \leq I(y)$ and $(y \leq z \rightarrow I(y) \leq I(z))$ and $(y \leq z \leq I(y) \rightarrow I(z) = I(y))]$,

(3) $\forall y[I(y) = y \leftrightarrow P(y)]$.

In a model M of ψ_1, the function I^M is constant on each of these segments and the image $I^M(x)$ of an element x is the last element of the segment containing x. We have added that the set of the last elements of the successive segments is the subset P^M of $|M|$.
The sentence ψ_1 is equivalent to a universal sentence and closure (under the function I) in its models takes at most one step thus ψ_1 is a local sentence.

If M is a well ordered model of ψ_1 whose order type is a regular cardinal κ, then the set P^M is cofinal in κ so the order type of $(P^M, <^M)$ will be also κ. The set P^M will then be identified with κ and each segment defined by the sentence ψ_1 will be of cardinal smaller than κ (because it is bounded in the model M by the last element of the segment).

We are now going to express, using a $(n+3)$-ary function f, that, for each $\nu < \kappa$, there is a partition C_ν of $[\kappa]^{n+2}$ with card$(C_\nu) < \kappa$. We shall use the following sentence ψ_2 in the signature $S(\psi_2) = \{<, I, P, f\}$, which is the conjunction of :

(1) $\forall \nu y_1 y_2 \ldots y_{n+2}[f(\nu, y_1, y_2, \ldots, y_{n+2}) = f(I(\nu), I(y_1), I(y_2), \ldots, I(y_{n+2}))]$,

(2) $\forall \nu y_1 y_2 \ldots y_{n+2}[(\bigvee_{1 \leq i < j \leq n+2} \neg(I(y_i) < I(y_j))) \rightarrow f(\nu, y_1, y_2, \ldots, y_{n+2}) = I(\nu)]$,

(3) $\forall \nu y_1 y_2 \ldots y_{n+2}[(\bigwedge_{1\leq i<j\leq n+2}(I(y_i) < I(y_j))) \rightarrow I(f(\nu, y_1, y_2, \ldots, y_{n+2})) = I(\nu)]$,

(4) $\forall \nu y_1 y_2 \ldots y_{n+2}[(\bigwedge_{1\leq i<j\leq n+2}(I(y_i) < I(y_j))) \rightarrow \neg P(f(\nu, y_1, y_2, \ldots, y_{n+2}))]$.

If M is a well ordered model of $\psi_1 \wedge \psi_2$ whose order type is a regular cardinal κ, then we have, for each $\nu < \kappa$ (identified with $\nu \in P^M$ so $\nu = I(\nu)$) a partition C_ν of $[\kappa]^{n+2}$ which is defined by the function f. For all elements $y_1 < y_2 < \ldots < y_{n+2}$ and $y'_1 < y'_2 < \ldots < y'_{n+2}$ of κ (so all elements y_i and y'_i are in P^M) the two sets $\{y_1, y_2, \ldots y_{n+2}\}$ and $\{y'_1, y'_2, \ldots y'_{n+2}\}$ are in the same set of C_ν iff

$$f(\nu, y_1, y_2, \ldots, y_{n+2}) = f(\nu, y'_1, y'_2, \ldots, y'_{n+2}).$$

But the elements $f(\nu, y_1, y_2, \ldots, y_{n+2})$ and $f(\nu, y'_1, y'_2, \ldots, y'_{n+2})$ are in the segment of M whose last element is $I(\nu)$ (and they are different from $I(\nu)$) thus it will hold that card$(C_\nu) < \kappa$ because we have seen that each segment defined by ψ_1 will have cardinality smaller than κ.

Notice that items (1) and (2) above ensure that the function f is always defined and that closure in models of $\psi_1 \wedge \psi_2$ takes at most two steps : one takes closure under the function I in one step, then closure under the function f in a second step. So the sentence $\psi_1 \wedge \psi_2$ is local.

We have now to express that, for all subsets X of κ having $n + 5$ elements, there exists $\nu \in X$ such that $X - (\nu + 1)$ is not C_ν-homogeneous.

We shall use the following sentence ψ_3 in the same signature $\{<, I, P, f\}$, which is equal to :

$\forall x_1 \ldots x_{n+5} \in P[\bigwedge_{1\leq i<j\leq n+5} x_i < x_j \rightarrow \phi(x_1 \ldots x_{n+5})]$,

where $\phi(x_1 \ldots x_{n+5})$ is the sentence :

$\bigvee_{\nu\in\{x_1,x_2,x_3\}} \bigvee_{\{y_1,\ldots,y_{n+2}\}\in[\{x_j|1\leq j\leq n+5\}]^{n+2}} \bigvee_{\{y'_1,\ldots,y'_{n+2}\}\in[\{x_j|1\leq j\leq n+5\}]^{n+2}}$
$[\nu < y_1 < \ldots < y_{n+2} \bigwedge \nu < y'_1 < \ldots < y'_{n+2} \bigwedge f(\nu, y_1, y_2, \ldots, y_{n+2}) \neq f(\nu, y'_1, y'_2, \ldots, y'_{n+2})]$.

Consider now the sentence $\theta_n = \bigwedge_{1\leq i\leq 3} \psi_i$. This sentence is local. Moreover if κ is an infinite regular cardinal, then θ_n has a model of order type κ iff for each $\nu < \kappa$, there is a partition C_ν of $[\kappa]^{n+2}$ with card$(C_\nu) < \kappa$, such that for all subsets X of κ having $n+5$ elements, there exists $\nu \in X$ such that $X-(\nu+1)$ is not C_ν-homogeneous.

Recall now that from two local sentences φ_1 and φ_2 we can construct another local sentence $\varphi_1 \cup \varphi_2$ such that $Sp(\varphi_1 \cup \varphi_2) = Sp(\varphi_1) \cup Sp(\varphi_2)$, [FR96].

Consider now the local sentence ϕ_0 given in Theorem 3.2 such that

$$Sp_\infty(\phi_0) = \{\tau > \omega \mid \tau \text{ is not an inaccessible cardinal }\}.$$

The local sentence $\phi_n = \theta_n \cup \phi_0$ is a local sentence and

$$Sp_\infty(\phi_n) = \{\tau > \omega \mid \tau \text{ is not an } n\text{-Mahlo cardinal }\}.$$

This ends the proof of Theorem 3.3. □

We can now determine the kind of Hanf number μ for local sentences defined in [FR96].

Definition 3.6 *The ordinal μ is the smallest ordinal such that, for every local sentence φ, whenever φ has a model of order type μ, then φ has models of order type τ, for each cardinal $\tau \geq \omega$.*

It is proved in [FR96], using the sentence ϕ_0 given by Theorem 3.2, that μ is at least inaccessible. We can now state the folllowing result.

Theorem 3.7 *The ordinal μ is the first ω-Mahlo cardinal, if such a cardinal exists.*

Proof. Ressayre proved in [Res88] that μ exists if an ω-Mahlo cardinal exists and then μ is bounded by this large cardinal. On the other hand, using the sentence ϕ_n given by Theorem 3.3, we can see that μ must be an n-Mahlo cardinal. The ordinal μ is then an ω-Mahlo cardinal because it is an n-Mahlo cardinal, for each integer $n \geq 1$. Thus the ordinal μ is in fact the first ω-Mahlo cardinal, if such a cardinal exists.

4 Strength of the stretching principles

We are going to prove now the following result.

Theorem 4.1 *The statement (E) (respectively, the statement (E')) implies the existence, for each integer n, of an n-Mahlo cardinal.*

Proof. Assume first that (E) is true in a model U of ZF. We know from [FR96] that there exists a local sentence φ such that $Sp_\infty(\varphi) = \{\omega\}$. Consider now the sentence $\Psi_n = \varphi \cup \phi_n$, where ϕ_n is given by Theorem 3.3. Assume that there is no n-Mahlo cardinal in U. Then the sentence Ψ_n has models of order type κ, for each cardinal $\kappa \geq \omega$. So if (E) is true, then Ψ_n would have a finite model M containing N_{Ψ_n} remarkable and monotonic indiscernibles. But M can not satisfy φ because otherwise the stretching $M(\omega + 1)$ would be a well ordered model of φ of order type greater than ω. (recall that by Lemma 8 of [FR96],if the indiscernibles are monotonic, then, for each ordinal α, the stretching $M(\alpha)$ is well ordered) . And M can not satisfy ϕ_n because otherwise the sentence ϕ_n would have a model of order type ω by the equivalence (C) of the Stretching Theorem 2.3. This would lead to a contradiction so we can conclude that U contains some n-Mahlo cardinal for each integer $n \geq 1$.
To prove the corresponding result for (E'), it suffices to add a unary predicate R to the signatures of φ and ϕ_n and to reason in a similar way, replacing the sentence φ by the sentence $\varphi \wedge \forall x R(x)$ and the sentence ϕ_n by the sentence $\phi_n \wedge \forall x R(x)$.

We recall from [FR96] that (E) and (E') can be proved assuming the existence for each integer n of an n-Mahlo cardinal. Thus we can infer the following result.

Corollary 4.2 *The statement (E) (respectively, the statement (E')) is equivalent to the existence, for each integer n, of an n-Mahlo cardinal.*

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