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Ten-dimensional super-Yang–Mills with nine off-shell supersymmetries

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Abstract

After adding 7 auxiliary scalars to the d=10 super-Yang–Mills action, 9 of the 16 supersymmetries close off-shell. In this paper, these 9 supersymmetry generators are related by dimensional reduction to scalar and vector topological symmetry in $\mathcal{N}$=2 d=8 twisted super-Yang–Mills. Furthermore, a gauge-invariant superspace action is constructed for d=10 super-Yang–Mills where the superfields depend on 9 anticommuting θ variables.

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1 Introduction

The off-shell field content of 10-dimensional super-Yang–Mills theory has an excess of seven fermionic degrees of freedom as compared to the number of gauge invariant bosonic degrees of freedom. To balance this mismatching, it was proposed in [1] to add to the supersymmetry transformation laws a set of seven auxiliary scalar fields G_a, together with a $\sum_a G_a^2$ term in the action. In order for the algebra to close off-shell, the parameters associated with the supersymmetry transformations must obey some identities. However, there is no linear solution to these identities, and thus no conventional supersymmetric formulation which permits the algebra to completely close.

It has been demonstrated in [1] that it is impossible to construct more than nine consistent solutions of these identities. Thus, only nine supersymmetry generators can generate an algebra that closes off-shell. These nine supersymmetry generators in ten-dimensional super-Yang–Mills are related to the octonionic division algebra in the same manner that the supersymmetry generators in three-, four-, and six-dimensional super-Yang–Mills are related to the real, complex, and quaternionic division algebras. However, the non-associativity of octonions makes the ten-dimensional supersymmetry algebra more complicated than in the other dimensions.

On the other hand, the $\mathcal{N} = 2$ twisted 8-dimensional super-Yang–Mills theory, which is a particular dimensional reduction of the 10-dimensional theory, has been determined in [2] by the invariance under a subalgebra of the maximal Yang–Mills supersymmetry. This subalgebra is small enough to close independently of equations of motion with a finite set of auxiliary fields, and yet is large enough to determine the Yang–Mills supersymmetric theory. It is also made of nine generators. The latter can be geometrically understood and constructed as scalar and vector topological Yang–Mills symmetries. This 8-dimensional topological symmetry can be built independently of the notion of supersymmetry, but, surprisingly, the latter symmetry with 16 generators can be fully recovered at the end of the construction.

The aim of this paper is to make a bridge between the results of [1] and [2]. We will find that in 10-dimensional flat space with Lorentz group $SO(1,9)$ reduced to $SO(1,1) \times Spin(7)$, the supersymmetry algebra can be twisted such that the 10-dimensional super-Yang–Mills theory is determined by a supersymmetry algebra with 9 generators, which is related by dimensional reduction to the twisted $\mathcal{N} = 2$ 8-dimensional super-Yang–Mills theory. Reciprocally, the extended curvature equation of the $\mathcal{N} = 2$ 8-dimensional supersymmetric theory can be "oxidized" into an analogous 10-dimensional equation

that determines the supersymmetry algebra and 10-dimensional super-Yang–Mills action. We argue that the largest symmetry group that can preserve an off-shell subalgebra of supersymmetry is $SO(1,1)\times Spin(7)$, and we obtain the most general $SO(1,1)\times Spin(7)$ covariant solution of the identities defined in [1]. The supersymmetry algebra that we derive is exactly the one obtained by the twist operation.

We then define a superspace involving nine Grassmann θ variables such that the off-shell supersymmetry subalgebra acts in a manifest way on the super-Yang–Mills superfields. Using these off-shell superfields, a superspace action is constructed which reproduces the ten-dimensional super-Yang–Mills action including the seven auxiliary scalar fields G_a. Although this superspace action is manifestly invariant under only a $Spin(7)\times SO(1,1)$ subgroup of $SO(9,1)$, it is manifestly invariant under nine supersymmetries as well as gauge transformations. This can be compared with the light-cone superspace action for ten-dimensional super-Yang–Mills which is manifestly invariant under eight supersymmetries and an $SO(8)\times SO(1,1)$ (or $U(4)\times SO(1,1)$) subgroup of $SO(9,1)$, but is not manifestly invariant under gauge transformations.

2 Ten dimensional supersymmetric Yang–Mills with auxiliary fields

The Poincaré supersymmetric Yang–Mills theory in ten dimensional Minkowski space contains a gauge field A_μ ($\mu=1,\cdots 10$) and a sixteen-component Majorana–Weyl spinor Ψ, with values in the Lie algebra of some gauge group. In order to balance the gauge-invariant off-shell degrees of freedom, one can introduce a set of scalar fields G_a ($a=1,\cdots 7$) which count for the 7 missing bosonic degrees of freedom [1]. The Lagrangian is given by

$$\mathcal{L}=Tr\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{2}(\bar{\Psi}\hat{\Gamma}^\mu D_\mu\Psi)+8G_aG_a\} \tag{1}$$

where $\hat{\Gamma}^\mu$ are the ten-dimensional gamma matrices. As shown in [1], the action (1) is invariant under the following supersymmetry transformations, which depend on the ordinary Majorana–Weyl parameter ϵ and on seven other spinor parameters v_a

$$\delta A_\mu = i\bar{\epsilon}\hat{\Gamma}_\mu\Psi$$

$$\delta\Psi = \hat{\Gamma}^{\mu\nu}F_{\mu\nu}\epsilon+4G_av_a \tag{2}$$

$$\delta G_a = -\frac{i}{4}\bar{v}_a\hat{\Gamma}^\mu D_\mu\Psi \tag{3}$$

The commuting spinor parameters v_a must be constrained as follows

$$\bar{v}_a \hat{\Gamma}_\mu \epsilon = \bar{v}_a \, \hat{\Gamma}_\mu v_b - \delta_{ab} \bar{\epsilon} \hat{\Gamma}_\mu \epsilon = 0 \tag{4}$$

The transformations (2) generate a closed algebra modulo gauge transformations and equations of motion

$$\{\delta, \hat{\delta}\} \approx -2i\bar{\epsilon}\hat{\Gamma}^\mu \hat{\epsilon}\, \partial_\mu - 2i\delta^{\text{gauge}}(\bar{\epsilon}\hat{\Gamma}^\mu A_\mu \hat{\epsilon}) \tag{5}$$

and close independently of equations of motion when

$$(\hat{\epsilon}, \hat{v_a}) \tag{6}$$

is some linear combination of $(\hat{\Gamma}^{\mu\nu}\epsilon, \hat{\Gamma}^{\mu\nu} v_a)$. To recover conventional supersymmetry transformations, one must have a solution for v in (4) that is linear in ϵ. This in turn will give a realisation of (5) which, thanks to (6), will effectively hold off-shell.

Using octonionic notations and light-cone coordinates, a solution was found for the v's and ϵ in [1] that preserves nine supersymmetries. This solution is only covariant under $SO(1,1) \times Spin(7) \subset SO(1,9)$. In fact, in order to define the v's as linear combinations of ϵ, we must reduce the covariance to a subgroup H that admits a 7-dimensional representation. Moreover, since the maximal sub-algebra that can be closed off-shell contains 9 supersymmetry generators, the Majorana–Weyl spinor representation of $Spin(1,9)$ must decompose into $\mathbf{7} + \mathbf{9}$ of H. The biggest subgroup of $SO(1,9)$ that satisfies these criteria is $SO(1,1) \times Spin(7)$.

2.1 Light-cone variables

The choice of light-cone variables implies a reduction of the Lorentz group as

$$SO(1,9) \to SO(8) \times SO(1,1) \tag{7}$$

where the spinor $\Psi \in \mathbf{16}_+$ of $SO(1,9)$ decomposes into one chiral and one antichiral spinor of $Spin(8)$, $\Psi \to \lambda_1 \oplus \lambda_2 \in \mathbf{8}_+^{\mathbf{-1}} \oplus \mathbf{8}_-^{\mathbf{1}}$, as well as $\epsilon \to \epsilon_1 \oplus \epsilon_2$ and $v_a \to v_{a1} \oplus v_{a2}$. The connection $A_\mu \in \mathbf{10}$ of $SO(1,9)$ decomposes according to $A_\mu \to A_i \oplus A_+ \oplus A_- \in \mathbf{8}_\mathbf{v}^\mathbf{0} \oplus \mathbf{1}^\mathbf{2} \oplus \mathbf{1}^{\mathbf{-2}}$ of $SO(8) \times SO(1,1)$, where the superscripts denote the eigenvalue associated with the $SO(1,1)$ factor and $A_\pm = A_0 \pm A_9$.

We can consider a gamma matrix algebra of $Cl(1,9)$ in terms of gamma matrices of $Cl(0,8)$

$$\begin{aligned} \hat{\Gamma}_0 &= (i\sigma_2) \otimes \Gamma_9 \\ \hat{\Gamma}_i &= \sigma_2 \otimes \Gamma_i \qquad (i = 1 \ldots 8) \\ \hat{\Gamma}_9 &= \sigma_1 \otimes 1 \end{aligned} \tag{8}$$

and $\hat{\Gamma}_{11}\Psi = \sigma_3 \otimes 1\,\Psi = \Psi$. These matrices obey $[\hat{\Gamma}_\mu, \hat{\Gamma}_\nu] = 2\,\eta_{\mu\nu}$, with the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$ and $[\Gamma_i, \Gamma_j] = 2\,\delta_{ij}$. The decomposition $SO(1,9) \to SO(8) \times SO(1,1)$ is performed by taking $A_\pm = A_0 \pm A_9$ and by projecting Ψ to $\hat{\Gamma}_+\Psi \to \lambda_2$ and $\hat{\Gamma}_-\Psi \to \lambda_1$, with $\Gamma_9\lambda_1 = \lambda_1$ and $\Gamma_9\lambda_2 = -\lambda_2$.

The transformations laws (2) read

$$\begin{aligned}
\delta A_i &= -i\bar{\epsilon}\Gamma_i\lambda \\
\delta A_+ &= -2\bar{\epsilon}_2\lambda_2 \\
\delta A_- &= 2\bar{\epsilon}_1\lambda_1 \\
\delta\lambda_1 &= (F_{ij}\Gamma_{ij} + \frac{1}{2}F_{+-})\epsilon_1 + iF_{i-}\Gamma_i\epsilon_2 + 4G_a v_{a1} \\
\delta\lambda_2 &= (F_{ij}\Gamma_{ij} - \frac{1}{2}F_{+-})\epsilon_2 - iF_{i+}\Gamma_i\epsilon_1 + 4G_a v_{a2} \\
\delta G_a &= \frac{i}{4}\bar{v}_a\Gamma_i D_i\lambda + \frac{1}{4}\bar{v}_{a1}D_+\lambda_1 - \frac{1}{4}\bar{v}_{a2}D_-\lambda_2
\end{aligned} \tag{9}$$

together with the constraints

$$\bar{v}_a\Gamma_i\epsilon = \bar{v}_{a1}\epsilon_1 = \bar{v}_{a2}\epsilon_2 = 0 \tag{10}$$

$$\bar{v}_{a1}v_{b1} = \delta_{ab}\bar{\epsilon}_1\epsilon_1 \qquad \bar{v}_a\Gamma_i v_b = \delta_{ab}\bar{\epsilon}\Gamma_i\epsilon \qquad \bar{v}_{a2}v_{b2} = \delta_{ab}\bar{\epsilon}_2\epsilon_2$$

The Lagrangian is given by

$$\begin{aligned}
\mathcal{L} = Tr\Big(&-\frac{1}{4}(F^{ij})(F_{ij}) - \frac{1}{4}(F^{i+})(F_{i+}) - \frac{1}{4}(F^{i-})(F_{i-}) - \frac{1}{8}(F^{+-})(F_{+-}) \\
&-\frac{i}{2}\bar{\lambda}\Gamma_i D_i\lambda - \frac{1}{2}\bar{\lambda}_1 D_+\lambda_1 + \frac{1}{2}\bar{\lambda}_2 D_-\lambda_2 + 8(G_a)^2\Big).
\end{aligned} \tag{11}$$

2.2 Twisted variables

In [1], the light-cone projections ϵ_1 and ϵ_2 of the spinor parameter are expressed in terms of octonions and the imaginary components of ϵ_1 are set to zero, that is ϵ_1 is taken real. Formally, the reality constraint on the supersymmetry parameter ϵ_1 implies the decomposition of the corresponding representation $\mathbf{8}_+ \to \mathbf{1} \oplus \mathbf{7}$ associated to the inclusion $Spin(7) \subset Spin(8)$[1]

$$\begin{aligned}
\epsilon_1 \in \mathbf{8}_+ &\to \mathbf{1} \oplus \mathbf{7} \\
\epsilon_2 \in \mathbf{8}_- &\to \quad \mathbf{8}
\end{aligned} \tag{12}$$

[1] A discussion of various solutions of (4) can be found in [4] together with their invariance groups. In particular, a solution is presented preserving nine supersymmetries, but with a reduction of $SO(1,9) \to G_2 \times SO(1,1)$.

where **1**,**7** and **8** define the scalar, vectorial respectively spinorial representations of $Spin(7)$. The reality constraint is then equivalent to retaining just the singlet part of ϵ_1, isolating 9 supersymmetries.

For expressing the decomposition $SO(8) \to Spin(7)$, it is convenient to introduce projectors onto the irreducible representations of $Spin(7)$. In order to do so, we use the spinor ζ scalar of $Spin(7)$. We take it chiral and of norm 1 and define

$$(4!)\bar{\zeta}\Gamma_{ijkl}\zeta \equiv \Omega_{ijkl} \quad \text{with} \quad \bar{\zeta}\zeta = 1, \tag{13}$$

where Ω_{ijkl} stands for the octonionic $Spin(7)$ invariant 4-form. It can be used to construct orthogonal projectors to decompose the adjoint representation **28** of $Spin(8)$ into the irreducible $Spin(7)$ ones $\mathbf{21} \oplus \mathbf{7}$

$$\begin{aligned} P^{+}_{ij}{}^{kl} &\equiv \frac{3}{4}(\delta^{kl}_{ij} + \frac{1}{6}\Omega^{kl}_{ij}) \\ P^{-}_{ij}{}^{kl} &\equiv \frac{1}{4}(\delta^{kl}_{ij} - \frac{1}{2}\Omega^{kl}_{ij}) \end{aligned} \tag{14}$$

The supersymmetry parameter can then be expressed as

$$\begin{aligned} \epsilon_1 &= \bar{\omega}\zeta + \Gamma_{ij}\nu^{ij}\zeta \\ \epsilon_2 &= i\Gamma_i\varepsilon^i\zeta \end{aligned} \tag{15}$$

where $\nu_{ij} = P^{-}_{ij}{}^{kl}\nu_{kl}$. This provides us with a decomposition of the supersymmetry generators as an antiselfdual tensorial charge δ_{ij}, a scalar charge δ_0 and a vectorial charge δ_i, of which we will retain just the scalar and vectorial ones.

2.3 Decoupling and resolution of the constraints

In terms of $Spin(7)$ representations, the relations (10) together with the explicit expression for the supersymmetry paramaters (15) read

$$\begin{gathered} \bar{v}^{ij}_1(\bar{\omega} + \nu^{kl}\Gamma_{kl})\zeta = 0, \quad \bar{v}^{ij}_1\Gamma_m i\varepsilon^k\Gamma_k\zeta + \bar{v}^{ij}_2\Gamma_m(\bar{\omega} + \nu^{kl}\Gamma_{kl})\zeta = 0, \quad \bar{v}^{ij}_2 i\varepsilon^k\Gamma_k\zeta = 0 \\ \bar{v}_{1ij}v^{kl}_1 = 8P^{-}_{ij}{}^{kl}(\bar{\omega}^2 + 2|\nu|^2), \;\; \bar{v}_{ij}\Gamma_m v^{kl} = 16iP^{-}_{ij}{}^{kl}(\bar{\omega}\varepsilon_m + 4\varepsilon^n\nu_{mn}), \;\; \bar{v}_{2ij}v^{kl}_2 = -8P^{-}_{ij}{}^{kl}|\varepsilon|^2 \end{gathered} \tag{16}$$

In order to find the most general covariant solution for v_{ij} that is linear in the supersymmetry parameters, we consider

$$\begin{aligned} v^{ij}_1 &= a\bar{\omega}\Gamma^{ij}\zeta + e\nu^{ij}\zeta + f\nu_{kl}\Gamma^{ijkl}\zeta \\ v^{ij}_2 &= b\varepsilon^k\Gamma_k\Gamma^{ij}\zeta + c\varepsilon^{[i}\Gamma^{j]-}\zeta \end{aligned} \tag{17}$$

from which we can show, remembering that all terms, but $\bar{\zeta}\zeta$ and $\bar{\zeta}\Gamma_{ijkl}\zeta$, vanish, that (16) is verified for the solution $\nu^{ij}=0$, $a=-ib=2$ and $c=0$, that is

$$\begin{array}{ll} \epsilon_1=\bar{\omega}\zeta & v_1^{ij}=2\bar{\omega}\Gamma^{ij}\zeta \\ \epsilon_2=i\Gamma_i\varepsilon^i\zeta & v_2^{ij}=2i\varepsilon^k\Gamma_k\Gamma^{ij}\zeta \end{array}$$

As expected, this solution provides us with a set of nine components parameterized by $\bar{\omega}$ and ε^i, which form the maximal set of supersymmetry generators that can generate an off-shell algebra.

The fields of the theory decompose according to

$$\begin{aligned} \lambda_1 \in \mathbf{8}_+ &\rightarrow \mathbf{1}\oplus\mathbf{7} \\ \lambda_2 \in \mathbf{8}_-,\ A_i \in \mathbf{8_v} &\rightarrow \quad \mathbf{8} \end{aligned} \tag{18}$$

and G_a is reexpressed in terms of $G^-_{ij}\in\mathbf{7}$ as $G_{8a}=G_a$ and $G_{ab}=C_{ab}{}^cG_c$, where $C_{ab}{}^c$ are the structure constants of the imaginary octonions. One has explicitly

$$\begin{aligned} \lambda_1 &= \eta\zeta+\Gamma_{ij}\chi^{ij}\zeta \\ \lambda_2 &= i\Gamma_i\psi^i\zeta \end{aligned} \tag{19}$$

and

$$\begin{aligned} \eta &= \bar{\zeta}\lambda_1 \\ \chi_{ij} &= -\frac{1}{2}\bar{\zeta}\Gamma_{ij}\lambda_1 \\ \psi_i &= -i\bar{\zeta}\Gamma_i\lambda_2 \end{aligned} \tag{20}$$

where $\chi_{ij}=P^-_{ij}{}^{kl}\chi_{kl}$. The supersymmetry transformations are now generated by

$$\delta^{Susy}=\bar{\omega}\delta_0+\varepsilon^i\delta_i \tag{21}$$

We display here the resulting transformation laws in a form which is more convenient with respect to the approach related to theories of cohomological type (called BRSTQFTs in the terminology of [7]) of the next section. That is, we redefine $G_{ij}\rightarrow G_{ij}-P^{-kl}_{ij}F_{kl}$ and

redefine some of the fields by scale factors to get

$$\begin{aligned}
\delta_0 A_i &= \psi_i \\
\delta_0 A_+ &= 0 \\
\delta_0 A_- &= \eta \\
\delta_0 \psi_i &= -F_{i+} \\
\delta_0 \eta &= F_{+-} \\
\delta_0 \chi_{ij} &= G_{ij} \\
\delta_0 G_{ij} &= D_+ \chi_{ij}
\end{aligned} \tag{22}$$

$$\begin{aligned}
\delta_i A_j &= -\delta_{ij}\eta - \chi_{ij} \\
\delta_i A_+ &= -\psi_i \\
\delta_i A_- &= 0 \\
\delta_i \psi_j &= F_{ij} + G_{ij} + \delta_{ij} F_{+-} \\
\delta_i \eta &= F_{i-} \\
\delta_k \chi_{ij} &= 8 P^{-\,l}_{ijk} F_{l-} \\
\delta_k G_{ij} &= D_k \chi_{ij} - 8 P^{-\,l}_{ijk} (D_l \eta - D_- \psi_l)
\end{aligned} \tag{23}$$

The algebra closes independently of the equations of motion as

$$\begin{aligned}
\delta_0^2 &= \partial_+ + \delta^{\text{gauge}}(A_+) \\
\delta_{\{i}\delta_{j\}} &= \delta_{ij}(\partial_- + \delta^{\text{gauge}}(A_-)) \\
\{\delta_0, \delta_i\} &= \partial_i + \delta^{\text{gauge}}(A_i)
\end{aligned} \tag{24}$$

and the action becomes

$$\begin{aligned}
S = \int_M d^{10}x \, \mathrm{Tr} \, \Big(& \frac{1}{2} G^{ij}(F_{ij} + \frac{1}{4} G_{ij}) - \chi^{ij}(D_i \psi_j + \frac{1}{8} D_+ \chi_{ij}) + \eta D_i \psi^i + \\
& + (F^i_-)(F_{i+}) - \psi^i D_- \psi_i + (F_{+-})^2 - \eta D_+ \eta \Big).
\end{aligned} \tag{25}$$

The formal dimensional reduction on the "Minkowski torus" consists trivially here to neglect the non-zero modes of the operators $\partial_\pm$. Doing so we recover the eight-dimensional cohomological action and its twisted supersymmetry algebra, obtained in [2] by twisting the eight-dimensional theory. In the next section, we discuss this link to eight-dimensional cohomological theory in more details.

3 Link with eight-dimensional Yang–Mills BRSTQFT

In this section, we will directly obtain the light-cone twisted subalgebra of 10-dimensional super-Yang–Mills of the last section. It will close off-shell by construction and will be inspired by the analogous known subalgebra of maximal twisted[2] supersymmetry in 8 dimensions.

The eight-dimensional algebra has been built [2] from the scalar and vector topological symmetries with $9 = 1 + 8$ generators that can be algebraically constructed. These 9 generators build a maximally closed and consistent sector of the twisted $\mathcal{N} = 2$, $d = 8$ Yang–Mills supersymmetry. Invariance under this subalgebra completely determines the Yang–Mills supersymmetric action. The full on-shell supersymmetry is recovered in this way and one can then interpret the invariance of the action under the 7 additional supercharges as accidental.

3.1 The 8-dimensional BRSTQFT formula

The nine 8-dimensional supersymmetry generators can be encoded in a graded differential operator Q which depends on nine twisted supersymmetry parameters consisting of one scalar $\bar{\omega}$ and one eight-dimensional vector ε.

Using the notation of [2], Q satisfies the horizontality condition in eight dimensions

$$\begin{aligned}\hat{F}_8 &\equiv (d + Q - \bar{\omega} i_\varepsilon)(A + c) + (A + c)^2 \\ &= F + \bar{\omega}\psi + \delta(\epsilon)\eta + i_\varepsilon \chi + \bar{\omega}^2 \Phi + |\varepsilon|^2 \bar{\Phi}. \end{aligned} \tag{26}$$

In Eq. (26), all fields are forms taking values in the Lie algebra of the gauge group. For example, $A = A_i dx^i$ is the Yang–Mills connection where $i = 1$ to 8, Φ and $\bar{\Phi}$ are scalars, and $F = dA + AA$ is a two-form. Furthermore, $\Psi = \Psi_i dx^i$ is a 1-form, $\chi = \frac{1}{2}\chi_{ij} dx^i dx^j$ is an antiselfdual 2-form with seven independent components, and η is a scalar field where $(\Psi_i, \chi_{ij}, \eta)$ are twisted Fermi spinors. Moreover, d is the usual exterior differential $d = \partial_i dx^i$, i_v is the contraction operator along the vector v, $i_\varepsilon dx^i = \varepsilon^i$, and $\delta(\varepsilon) \equiv \varepsilon^i \delta_{ij} dx^j$.

Finally, the anticommuting scalar field c is a shadow field. It plays an important role by closing the supersymmetry without field-dependent gauge transformations in the right-hand-side of commutators, and, eventually, for quantizing the theory [3] . There

[2]Here, the word twist means the mapping between forms and spinors that is allowed reducing the $SO(8)$ covariance down to $Spin(7)$ [7].

is no need at this stage to introduce a Faddeev–Popov ghost. Note that all fields and operators have a grading that is the sum of shadow number and ordinary form degree.

The closure of Q is ensured by the Bianchi identity, which also determines the action of the symmetry on the fields on the right-hand-side of Eq. (26). By expanding the equation

$$\begin{aligned}&(d+Q-\bar{\omega}i_\varepsilon)(F+\bar{\omega}\psi+\delta(\epsilon)\eta+i_\varepsilon\chi+\bar{\omega}^2\Phi+|\varepsilon|^2\bar{\Phi})\\&+[A+c,\,F+\bar{\omega}\psi+\delta(\epsilon)\eta+i_\varepsilon\chi+\bar{\omega}^2\Phi+|\varepsilon|^2\bar{\Phi}]=0,\end{aligned}\tag{27}$$

one finds that the action of the operator Q on the fields can be decomposed into a gauge transformation with parameter c and a supersymmetry transformation δ^{Susy} with 9 twisted parameters $\bar{\omega}$ and ε as

$$Q=\delta^{Susy}-\delta^{\text{gauge}}(c)=\bar{\omega}\delta_0+\varepsilon^i\delta_i-\delta^{\text{gauge}}(c).\tag{28}$$

The off-shell closure of δ^{Susy} follows from the identity $Q^2=\bar{\omega}\mathcal{L}_\varepsilon$. Notice that no gauge transformation is involved in this equation.

3.2 Light-cone 10-dimensional equation

We may understand Eq. (26) as a light-cone projection of an analogous equation in 10 dimensions. In order to determine the subalgebra of the 10-dimensional theory, we "oxidize" this equation by introducing light-cone modes (∂_+,∂_-) and redefining $(\Phi,\bar{\Phi})\to(A_+,A_-)$ in such a way that

$$Spin(7)\times\mathbb{R}_+^*\cong Spin(7)\times SO(1,1)\subset SO(8)\times SO(1,1)\subset SO(1,9).\tag{29}$$

In this way, we can interpret the scalar fields in the right-hand-side of Eq. (26) as elements of a connexion in 10 dimensions. They can be carried to the left-hand-side of the horizontality condition (26), which thus appears as the dimensional reduction of the 10-dimensional condition

$$\begin{aligned}\hat{F}_{10}&\equiv(d+Q-\bar{\omega}i_\varepsilon-\bar{\omega}^2i_+-|\varepsilon|^2i_-)(A+c)+(A+c)^2\\&=F+\bar{\omega}(\psi+\eta dx^-)+(\delta(\epsilon)\eta+i_\varepsilon\chi+i_\varepsilon\psi\,dx^+).\end{aligned}\tag{30}$$

Eq. (30) has the Bianchi identity

$$\begin{aligned}&(d+Q-\bar{\omega}i_\varepsilon-\bar{\omega}^2i_+-|\varepsilon|^2i_-)(F+\bar{\omega}(\psi+\eta dx^-)+(\delta(\epsilon)\eta+i_\varepsilon\chi+i_\varepsilon\psi\,dx^+))\\&\qquad+[A+c,F+\bar{\omega}(\psi+\eta dx^-)+(\delta(\epsilon)\eta+i_\varepsilon\chi+i_\varepsilon\psi\,dx^+)]=0,\end{aligned}\tag{31}$$

which insures that $(d+Q-\bar{\omega}i_\varepsilon-\bar{\omega}^2 i_+-|\varepsilon|^2 i_-)^2=0$.

By expansion according to the various gradings, we obtain 10-dimensional transformation laws for all fields, which exactly reproduce those described earlier and determined by a mere twist of supersymmetry transformations. The self-dual 2 form auxiliary fields G_{ij} with seven degrees of freedom is now introduced here in the standard TQFT way, by solving the degenerate equations $\delta_0\chi_{ij}+\delta_i\Psi_j+\cdots=0$. As a consequence of the Bianchi identity, the algebra of generators δ_0 and δ_i closes independently of any equations of motion, as expressed in the preceding section. Let us stress again the relevance of the shadow c for supressing field dependent gauge transformations in the commutators of supersymmetries.

The 10-dimensional action (assuming no higher-derivative terms) is completely determined from the Q-invariance with the nine parameters $\bar{\omega}$ and ε_i. As in [2], one can show that the most general Q-invariant expression, which is independent of ε_i and contains no higher order derivative terms, can be written either as a δ_0-exact or as a δ_i-exact functional up to a topological term

$$\begin{aligned} S &= \delta_0\mathbf{Z}^{(-1)}-\frac{1}{8}\int_M d^{10}x\,\mathrm{Tr}\,\left(\Omega^{ijkl}F_{ij}F_{kl}\right)\\ &= \varepsilon^i\delta_i\mathbf{Z}^{(+1)}+\frac{1}{8}\int_M d^{10}x\,\mathrm{Tr}\,\left(\Omega^{ijkl}F_{ij}F_{kl}\right) \end{aligned} \tag{32}$$

where $\mathbf{Z}^{(-1)}$ and $\mathbf{Z}^{(+1)}$ are completely fixed respectively by the δ_i and δ_0 symmetries, i.e. $\delta_i\mathbf{Z}^{(-1)}=\delta_0\mathbf{Z}^{(+1)}=0$.

As will be shown in the following section, this matches the Lagrangian obtained by twist in (25). We have thus obtained an off-shell formulation of ten-dimensional super-Yang–Mills from the eight-dimensional Yang–Mills BRSTQFT.

4 Toward a superspace formulation in twisted fermionic variables

The fact that we are able to obtain a subalgebra that closes without the use of the equations of motion suggests that there should exist an off-shell superspace formulation of ten dimensional super-Yang–Mills. Since there are nine off-shell supersymmetry generators, it is natural to define a superspace with nine anticommuting variables. Let us define the reduced superspace with vector coordinates θ^i (spinor representations of $Spin(7)$) and

scalar coordinate θ. We define superspace derivatives

$$\hat{\nabla} = \frac{\partial}{\partial\theta} - \theta\partial_+, \qquad \hat{\nabla}_i = \frac{\partial}{\partial\theta^i} - \theta\partial_i - \theta_i\partial_-,$$
$$\partial_+, \qquad \partial_-, \qquad \partial_i, \tag{33}$$

which obey

$$\hat{\nabla}^2 = -\partial_+, \qquad \hat{\nabla}_{\{i}\hat{\nabla}_{j\}} = -\delta_{ij}\partial_-, \qquad \{\hat{\nabla}, \hat{\nabla}_i\} = -\partial_i, \tag{34}$$

with all other commutators equal to zero. For each of the superspace derivatives, we introduce a corresponding gauge connection superfield and define the covariant superderivatives

$$\nabla = \hat{\nabla} + \mathbb{C}, \qquad \nabla_i = \hat{\nabla}_i + \Gamma_i,$$
$$\mathcal{D}_+ = \partial_+ + \mathbb{A}_+, \qquad \mathcal{D}_- = \partial_- + \mathbb{A}_-, \qquad \mathcal{D}_i = \partial_i + \mathbb{A}_i. \tag{35}$$

To reduce the number of degrees of freedom, one needs to constrain the supercurvature associated to the connection superfields. The usual $SO(9,1)$-covariant constraint for $\mathcal{N} = 1$ $D = 10$ super-Yang–Mills superfields is $\{\nabla_\alpha, \nabla_\beta\} + 2\gamma^m_{\alpha\beta}\mathcal{D}_m = 0$, which puts the superfields on-shell. However, in the reduced superspace, the analogous constraints are

$$\nabla^2 + \mathcal{D}_+ = 0, \qquad \{\nabla_i, \nabla_j\} + 2\delta_{ij}\mathcal{D}_- = 0,$$
$$\{\nabla, \nabla_i\} + \mathcal{D}_i = 0, \tag{36}$$

It is remarkable that the resolution of these constaints no longer imply the equations of motion, in contrast with the case of the full superspace constraints, that can only be written on-shell.

As usual, the commutator of a fermionic covariant derivative and a bosonic covariant derivative gives a fermionic gauge-covariant superfield. The Bianchi identities imply that the symmetric part of $[\nabla_i, \mathcal{D}_j]$ is proportional to δ_{ij}. In order to obtain the right supermultiplet we furthermore impose the constraint that the antisymmetric part of $[\nabla_i, \mathcal{D}_j]$ is antiselfdual

$$P^{+kl}_{ij}[\nabla_k, \mathcal{D}_l] = 0 \tag{37}$$

We therefore define the gauge-covariant superfields $\mathbf{\Psi}_i$, $\mathbf{\eta}$ and $\mathbf{\chi}_{ij} = P^{-kl}_{ij}\mathbf{\chi}_{kl}$ that correspond to the commutators

$$[\nabla, \mathcal{D}_i] \equiv \mathbf{\Psi}_i = -[\nabla_i, \mathcal{D}_+], \quad [\nabla, \mathcal{D}_-] \equiv \mathbf{\eta}, \quad [\nabla, \mathcal{D}_+] = 0,$$
$$[\nabla_i, \mathcal{D}_-] = 0, \quad [\nabla_i, \mathcal{D}_j] = -\delta_{ij}\mathbf{\eta} - \mathbf{\chi}_{ij}. \tag{38}$$

The constraints and their Bianchi identities imply that $\boldsymbol{\Psi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\chi}$ satisfy

$$\nabla_{\{i}\boldsymbol{\Psi}_{j\}} + \delta_{ij}\nabla\boldsymbol{\eta} = 0, \qquad \nabla_k\boldsymbol{\chi}_{ij} + 8\, P^{-\ l}_{ijk}\nabla_l\boldsymbol{\eta} = 0. \tag{39}$$

Furthermore, the commutators of bosonic covariant derivative give the superderivative of these fermionic superfields as

$$\begin{aligned} [\mathcal{D}_i, \mathcal{D}_-] &\equiv \mathbb{F}_{i-} = \nabla_i\boldsymbol{\eta}, \qquad & [\mathcal{D}_-, \mathcal{D}_+] &\equiv \mathbb{F}_{-+} = \nabla\boldsymbol{\eta}, \\ [\mathcal{D}_i, \mathcal{D}_+] &\equiv \mathbb{F}_{i+} = \nabla\boldsymbol{\Psi}_i, \qquad & [\mathcal{D}_i, \mathcal{D}_j] &\equiv \mathbb{F}_{ij} = \nabla\boldsymbol{\chi}_{ij} - \nabla_{[i}\boldsymbol{\Psi}_{j]}. \end{aligned} \tag{40}$$

The superfields $\mathbb{C}$ and $\boldsymbol{\Gamma}_i$ have expansion of the form

$$\mathbb{C} = c + \theta^i c_i + \cdots \qquad \boldsymbol{\Gamma}_i = \gamma_i + \theta^j\gamma_{ij} + \cdots \tag{41}$$

The transformation laws are such that, c can be identified as the shadow of [3] and $\gamma_i\kappa^i$ as an analogous field introduced in [2], in the context of the topological vector symmetry, for κ a constant vector field.

In order to concretely realize the abstract algebra defined by the above equations, one must determine the action of δ_0 and δ_i on all components of $\mathbb{C}$ and $\boldsymbol{\Gamma}_i$, which satisfy the relevant commutation relations. We have checked this non trivial property, both in component formalism and directly in superfield formalism [8], and prove thereby that the above constraints that hold off-shell can be solved and that the solution corresponds to the supermultiplet of ten dimensional super-Yang–Mills in its twisted formulation.

4.1 Super-Yang–Mills action in superspace

To write a superspace action in terms of these constrained superfields, first note that the component action of (25) can be written as a δ_0-exact functional as long as we neglect instantons

$$S = \delta_0\mathbf{Z}^{(-1)} \tag{42}$$

with $\mathbf{Z}^{(-1)}$ completely fixed by the δ_i symmetry, i.e. $\delta_i\mathbf{Z}^{(-1)} = 0$ where

$$\mathbf{Z}^{(-1)} = \int_M d^{10}x\, \mathrm{Tr}\, \Big(\frac{1}{2}\,\chi^{ij}(F_{ij} + \frac{1}{4}G_{ij}) + F_{-i}\psi^i + \eta F_{+-}\Big) \tag{43}$$

Moreover, defining $\delta(\varepsilon) = \varepsilon^i\delta_i$, the action can be expressed as a $\delta_0\delta(\varepsilon)$-exact term as

$$S = \delta_0\delta(\varepsilon)\int_M d^{10}x\, \frac{1}{|\varepsilon|^2}\mathscr{F} \tag{44}$$

with

$$\mathscr{F} = \mathrm{Tr}\left(\frac{1}{4}\,\varepsilon_i\Omega^{ijkl}(A_jF_{kl} - \frac{2}{3}A_jA_kA_l) + \varepsilon_i(-\delta^{ij}\eta - \chi^{ij})\psi_j\right). \tag{45}$$

Note that $\mathscr{F}$ is completely constrained by the condition that its $\delta(\varepsilon)$ variation is independent of ε.

This situation is reminiscent of the case of harmonic superspace [5, 6] where harmonic coordinates allow the construction of manifestly supersymmetric actions using a reduced superspace. In this case, one does not have harmonic variables but one can nevertheless write the above action in reduced superspace as

$$S = \int_M d^{10}x\,\nabla\nabla_i\,\mathbf{K}^i \equiv \int_M d^{10}x\,\int d\theta d\theta_i\,\mathbf{K}^i \tag{46}$$

where

$$\mathbf{K}^i = \mathrm{Tr}\left(\frac{1}{4}\,\Omega^{ijkl}(\mathbb{A}_j\mathbb{F}_{kl} - \frac{2}{3}\mathbb{A}_j\mathbb{A}_k\mathbb{A}_l) - (\delta^{ij}\boldsymbol{\eta} + \boldsymbol{\chi}^{ij})\boldsymbol{\Psi}_j\right). \tag{47}$$

Since the $\delta(\varepsilon)$ variation of $\frac{1}{|\varepsilon|^2}\mathscr{F}$ is independent of ε, one learns that

$$\nabla_i\mathbf{K}_j + \nabla_j\mathbf{K}_i = \delta_{ij}f + \frac{\partial}{\partial x^\mu}h^\mu_{ij} \tag{48}$$

for some f and h^μ_{ij}. Using (48), it is straightforward to show that (46) is independent of θ and θ^i and is therefore invariant under all nine supersymmetries.

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