A first-order Temporal Logic for Actions

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Abstract

We present a multi-modal action logic with first-order modalities, which contain terms which can be unified with the terms inside the subsequent formulas and which can be quantified. This makes it possible to handle simultaneously time and states. We discuss applications of this language to action theory where it is possible to express many temporal aspects of actions, as for example, beginning, end, time points, delayed preconditions and results, duration and many others. We present tableaux rules for a decidable fragment of this logic.

1 Introduction

Most action theories consider actions being specified by their preconditions and their results. The temporal structure of an action system is then defined by the sequence of actions that occur. A world is conceived as a graph of situations where every link from one node to the next node is considered as an action transition. This yields also a temporal structure of the action space, namely sequences of actions can be considered defining sequences of world states. The action occurs instantly at one moment and its results are true at the "next" moment.

However, the temporal structure of actions can be much more complex and complicated.

- Actions may have a duration.

- The results may be true before the action is completed or after it is finished.

- Actions may have preconditions which have to have been true during some interval preceding the action occurrence.

In order to represent complex temporal structures, underlying actions' occurrences, we have developed an action logic which allows to handle both states and time simultanuously.

We want to be able to express, for instance that action a occurs at moment t if conditions $p_1, \ldots p_n$ have been true during the intervals $i_1, \ldots$ all preceding t.

Here we present an approach where it is possible to represent actions in a complex temporal environment. In reality, actions have sometimes a beginning time, a duration, preconditions which may also have temporal aspects ; and the results may be true only instances after the end of the action performance. For an example, consider the action of calling an elevator, taking place at instance t_1. Depending on the actual situation this action may cause the elevator to move only many instances later, to stop still later, an so on. The action of pressing the button of a traffic light, in order to get green light to traverse the street may result in a switch immediately or after some seconds and in another switch after some more minutes.

In order to represent such issues, we define a modal action logic, where the modalities are terms containing variables which can be quantified. The same variables can occur inside the modalities as well as in the formulas after the modalities, allowing for unification between action term components and logical terms. This language makes it possible to express reasoning on states and the action terms allow to express temporal aspects of the actions.

2   The first-order modal action logic $\mathcal{D}al$

The language of first-order action logic, $\mathcal{L}$ is an extension of the language of classical predicate logic, $\mathcal{L}_0$. $\mathcal{L}_0$ consists of a set of variables $x, y, x_1, y_1, \ldots$, a set $\boldsymbol{F}$ of function symbols F, where $|F| \in \omega$ is the arity of F, a set $\boldsymbol{P}$ of predicate symbols P, including $\top$ and $\bot$, where $|P| \in \omega$ is the arity of P, an equality symbol $=$, the logical symbols $\neg, \wedge, \forall$. Terms and formulas are defined as usual and so are $\exists$ and $\vee$. We denote by V_t the set of all terms of $\mathcal{L}_0$. If ϕ is a formula and x a variable then we say that x is bounded in ϕ, when it occurs in a subformula $\forall x \phi$. x occurs free in ϕ if it occurs in ϕ and is not bounded in ϕ.

Action terms   The language for action operators consists of

- a set $\boldsymbol{A}$ of *action symbols* $a_1, a_2, \ldots$ where $|a| \in \omega$ is the arity of a and such that $\boldsymbol{A} \cap \boldsymbol{P} = \emptyset$

Action terms are built from action symbols and terms of $\mathcal{L}_0$.

- if a is an action symbol of arity n and $t_1, \ldots t_n$ are terms of $\mathcal{L}_0$, then $a(t_1, \ldots t_n)$ is an action term.

An action term is called *grounded* if no variable occurs free in it. The set of grounded action terms is denoted by $\boldsymbol{A}_t$.

Action operators   If a and $a_1, a_2, \ldots a_n$ are action terms, then

- $[a]$ is an action operator

- $[a_1; a_2; \ldots a_n]$ is an action operator

Modal operator □ is the standard modal operator (**S4**)

For $n = 0$, the corresponding action operator is noted $[\varepsilon]$.

An action operator is called *grounded* if all the action terms occurring in it are grounded.

Example: $[a]$, $[a(c_1, c_2, c_3)]$ are grounded, $[a(x, c_2, y)]$ is not grounded.

Action formulas

- If ϕ is a first-order formula and $[A]$ is an action operator, then $[A]\phi$ is an action formula.

- If ϕ is a first-order formula and □ is the modal operator, then $\Box\phi$ is an action formula.

- If ϕ is an action formula and x is a variable, then $\forall x \phi$ is an action formula.

Instantiation If ϕ is a formula and t is a term, then ϕ_t^x is the formula obtained from ϕ by replacing every free occurrence of x by t. If t is the name of an element of a set $\mathcal{O}$ then ϕ_t^x is called $\mathcal{O}$-instance of ϕ.

Example:

$$[a(x, c)](\neg\phi(c, x) \vee \psi(x))_{c_1}^x = [a(c_1, c)](\neg\phi(c, c_1) \vee \psi(c_1))$$
$$[a_1; a_2; a_3(c, y)]P(c, y)_{c_3}^y = [a_1; a_2; a_3(c, c_3)]P(c, c_3)$$

A formula is called *grounded* if there is no variable occurring free in it.

2.1 Semantical Characterization of $\mathcal{D}al$

A $\mathcal{D}al$ structure is defined as a Kripke-type structure, such that the transition relation between worlds depends on grounded action terms.

A $\mathcal{D}al$ structure is a tuple $\mathcal{M} = (\mathcal{W}, \{\mathcal{S}_w : w \in \mathcal{W}\}, \mathcal{A}, \mathbf{R}, \tau)$, where

- $\mathcal{W}$ is a set of *worlds*

- for every $w \in \mathcal{W}$, $\mathcal{S}_w = (\mathcal{O}, \mathcal{F}_w, \mathcal{P}_w)$ is a classical structure, where $\mathcal{O}$ is the set of individual objects (the same set in all worlds), $\mathcal{F}_w$ is a set of functions over $\mathcal{O}$ and $\mathcal{P}_w$ is a set of predicates over $\mathcal{O}$.

- $\mathcal{A}$ is a set of *action functions*, for $f \in \mathcal{A}$, $f : \mathcal{W} \times \underbrace{\mathcal{O} \times \ldots \times \mathcal{O}}_{n} \longrightarrow 2^{\mathcal{W}}, n \in \omega$.

  Action functions will characterize the action operators (every action symbol of arity n in $\mathbf{A}$ will be associeted with an action function of arity $n + 1$).

- $\mathbf{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a binary accessability relation on $\mathcal{W}$, which will characterize the modal operater □. We will write $R(w) = \{w' : (w, w') \in R\}$.

3

- τ is a valuation, $\tau = (\tau_0, \tau_1, \tau_2, \tau_3)$, where τ_0 is a function assigning objects from $\mathcal{O}$ to terms. In order to speak about objects from $\mathcal{O}$, we introduce into the language, for every $o \in \mathcal{O}$, an o-place function symbol (denoted equally o, for simplicity).

  τ_1 is a function assigning, for every world $w \in \mathcal{W}$, functions (from $\mathcal{F}$) to function symbols (from $\boldsymbol{F}$), of the same arity,
  $\tau_1 : \mathcal{W} \times \boldsymbol{\mathcal{F}} \longrightarrow \mathcal{F}$ such that $|\tau_1(w, F)| = |F|$.
  τ_2 is a function assigning, for every world $w \in \mathcal{W}$, predicates to predicate symbols of the same arities,
  $\tau_2 : \mathcal{W} \times \boldsymbol{\mathcal{P}} \longrightarrow \mathcal{P}$, such that $|\tau_2(w, P)| = |P|$.
  τ_3 is a function assigning action functions to action symbols,
  $\tau_3 : \boldsymbol{A} \longrightarrow \mathcal{A}$, such that $|\tau_3(a)| = |a| + 1$

- $\tau_3(a)(w, \tau_0(t_1), \ldots, \tau_0(t_n)) \subseteq R(w)$. If a world can be reached from w by the execution of action $a(t_1, t_2, \ldots, t_n)$ then it is accessible (via the relation R).

τ_0, τ_1, τ_2 and τ_3 define the valuation τ as follows:

- If $F(t_1, t_2, \ldots, t_m)$ is a term then
  $\tau_0(w, F(t_1, t_2, \ldots, t_m)) = \tau_1(w, F)(\tau_0(w, t_1), \tau_0(w, t_2), \ldots, \tau_0(w, t_m))$.

- if P is an n-ary predicate symbol and $t_1, t_2, \ldots, t_n$ are free object variables then
  $\tau(w, Pt_1, t_2, \ldots, t_n) = \top$ iff $(\tau_0(t_1), \ldots \tau_0(t_n)) \in \tau_2(w, P)$

- $\tau(w, t_1 = t_2) = \top$ iff $\tau(w, t_1) = \tau(w, t_2)$

- $\tau(w, \neg\phi) = \top$ iff $\tau(w, \phi) = \bot$

- $\tau(w, \phi \wedge \psi) = \top$ iff $\tau(w, \phi) = \tau(w, \psi) = \top$

- $\tau(w, \forall x \phi) = \top$ iff for every $o \in \mathcal{O}$ $\tau(w, \phi_o^x) = \top$

- $\tau(w, [a(t_1, t_2, \ldots, t_n)]\phi) = \top$ iff for every $w' \in \tau_3(a)(w, \tau_0(t_1), \ldots, \tau_0(t_n))$, $\tau(w', \phi) = \top$

- $\tau(w, \Box\phi) = \top$ iff for every $w' \in R(w)$, $\tau(w', \phi) = \top$

Let ϕ be a formula and $x_1, x_2, \ldots, x_n$ be the free object variables occurring in ϕ. Then $\tau(s, \phi) = t$ iff for every tuple $t_1, t_2, \ldots, t_n$ of ground terms,

$$\tau(w, \phi_{t_1, \ t_2, \ \ldots, \ t_n}^{x_1, \ x_2, \ \ldots, \ x_n}) = t$$

A formula ϕ is called valid in state $w \in \mathcal{W}$ of a $\mathcal{D}al$ -structure $\mathcal{M}$ iff $\tau(w, \phi) = \top$. This is denoted by $\mathcal{M}, w \models \phi$. We also say then that ϕ is satisfiable. A formula ϕ is called valid in a $\mathcal{D}al$ - structure $\mathcal{M}$ with the set of states $\mathcal{W}$, iff ϕ is valid in every $w \in \mathcal{W}$. We denote that by $\mathcal{M} \models \phi$. A formula ϕ is called $\mathcal{D}al$ - valid iff ϕ is valid in every $\mathcal{D}al$ - structure. This is denoted by $\models_{\mathcal{D}al} \phi$. We suppress the index $\mathcal{D}al$, whenever it is clear from the context, in which system we are.

Remark 1 $[a]\bot$ *is satisfiable and we have* $\tau(w, [a]\bot) = \top$ *iff* $\tau_3(a)(w, \tau_0(t_1), \ldots, \tau_0(t_n)) = \emptyset$

2.2 Axioms and inference rules of $\mathcal{D}al$

In addition to the axioms and inference rules of classical first - order logic and those of the system K, which rule the action operators including $[\varepsilon]$, and those of the system and $\mathcal{S}_4$, which rule the operator $\Box$, we have the following axioms and inference rules, (where $[A]$, $[A_1]$ and $[A_2]$ are arbitrary action operators):

$[A1]$ $[A_1; A_2]\alpha \leftrightarrow [A_1][A_2]\alpha$
$[A2]$ $\Box\alpha \rightarrow [A]\alpha$
$[A3]$ $[\varepsilon]\alpha \rightarrow \alpha$
$[A4]$ $\forall x\alpha \rightarrow \alpha_c^x$ for any individual term c of $\mathcal{L}$
$[A5]$ $\forall x[X]\alpha \leftrightarrow [X]\forall x\alpha$ for any modal operator X, with no occurrence of x

$[R1]$ From α infer $\Box\alpha$
$[R2]$ From $\alpha \rightarrow \beta$ infer $\alpha \rightarrow \forall x\beta$ provided x has no free occurrence in α

$\vdash_{\mathcal{D}al}$ is defined as usual, such that $\vdash_{\mathcal{D}al} \phi$ for any instance ϕ of one of the axioms; and $\vdash_{\mathcal{D}al} \psi$, whenever ψ can be inferred from ϕ, for any ϕ, such that $\vdash_{\mathcal{D}al} \phi$ by use of one of the inference rules. Again, we suppress the index $\mathcal{D}al$, whenever it is clear from the context, in which system we are.

2.3 Soundness, Completeness, Decidability

The $\mathcal{D}al$ -logic is sound and complete:

Theorem 1 $\vdash_{\mathcal{D}al} \phi$ *if and only if ϕ is $\mathcal{D}al$ - valid* ($\models_{\mathcal{D}al} \phi$)

The soundness proof is easy and the completeness proof goes along the lines of completeness proofs for modal logics by construction of a canonical model. The proof, which can be found in the appendix, bears several modifications according to the specific language which allows to quantify over terms occurring within modal operators.

$\mathcal{D}al$ is a first order language and therefore undecidable in the general case. But for action logics, we will make use of decidable subsets of $\mathcal{D}al$.

$\mathcal{D}al$ is very close to term modal logic introduced by [1]. Term modal logic allows terms in general as modalities, whereas our action logic only admits action terms. Moreover $\mathcal{D}al$ contains the **S4** modal operator $\Box$ which is not part of term modal logic.

3 Temporal Action Theories

Using $\mathcal{D}al$, we can modelize temporal aspects of dynamic actions. The modal logic allows to define action operators as modalities [3, 11]. The first order logic is used to formulate actions at a more general level. Here, we show an example where in addition to the relative representation of time by the modal operators, it is possible to express time points by terms.

We presuppose a time axis, linearly ordered (dense or continuous or discrete). Given a $\mathcal{D}al$ -structure, we will define a transitive relation on the set of states, $\mathcal{W}$, which will be related to the order on $\mathcal{T}$.

Definition 1 *Let* $\mathcal{M} = (\mathcal{W}, \{\mathcal{S}_w : w \in \mathcal{W}\}, \mathcal{A}, \mathbf{R}, (\tau_0, \tau_1, \tau_2, \tau_3))$, *be a* $\mathcal{D}al$ *-model. Then* $w \prec_0 w'$ *iff* $\exists a \in \mathcal{A}$ *of arity* n *and there are terms* $t_1, \ldots, t_n$, *such that* $w' \in f(w, t_1, \ldots, t_n)$. *Let be* $\preceq$ *the reflexive and transitive closure of* $\prec_0$.

Intuitively, this means that $w \prec w'$ if we can possibly "reach" w' from w by performing actions $a_1, a_2, \ldots, a_n$. Obviously, $\preceq$ is transitive and reflexive. Since we want to "link" worlds of $\mathcal{W}$ to time points in $\mathcal{T}$, which is ordered, $\prec$ must also be antisymmetric. The temporal entrenchment of the states is defined by a homomorphism $time : \mathcal{W} \longrightarrow \mathcal{T}$ from $\mathcal{W}$ into $\mathcal{T}$, where $w \preceq w'$ implies $time(w) \leq time(w')$. Using this construction, actions operators can be defined admitting complex temporal structures, including beginning and ending instances and a duration, which can be 0, when the result is immediate, or $\Delta \in \mathcal{T}$. The preconditions and results of actions can be defined to occur at freely determinable time instances before or after the instance when the action occurs. When an action a occurs in the state w, $time(w)$ gives us the time point at which a occurs. If the duration of the action is Δ, the time point of the resulting state w' is $time(w') = time(w) + \Delta$.

In this particular framework, we define

- *Action terms* as binary action predicates $a(t, d, \overrightarrow{x})$, where t denotes the instance on which a occurs and d denotes the duration of a, i.e. the interval on $\mathcal{T}$ after which the results of a will hold. , $\overrightarrow{x}$ is the sequence of other variables denoting the other entities or objects involved in the action occurrence.

  To give an example, let $\mathcal{T} = \{1, \ldots, 24\}$ be discrete and finite, denoting the hours during one day. Then action $move(t, 3, TGV, Marseille, Paris)$ is the action "train TGV goes from Marseille to Paris, the duration being 3 hours".

- *Action axioms*. An action axiom is characterized by a precondition $\pi(t, \overrightarrow{x})$ and a result $\rho(t + d, \overrightarrow{x})$, where π and ρ are $\mathcal{D}al$ formulas describing all preconditions and results of action a.

  To continue the previous example, the action execution axiom of the move-action is $at(t, x, y) \rightarrow [move(t, d, x, y, z)]at(t + d, x, z)$ (and can be instantiated to $at(6, TGV, Marseille) \rightarrow [move(6, 3, TGV, Marseille, Paris)]at(9, TGV, Paris)$), which means: if x is at y at instance t, then, after moving from y to z, x is at z at instance $t + d$.

The general form of an action law is

$$\pi(t_1, \overrightarrow{x_1}) \rightarrow [a(t, d, \overrightarrow{x_2})]\rho(t_2, \overrightarrow{x_3}), \text{w} here \ \overrightarrow{x_1} \cup \overrightarrow{x_2} \subseteq \overrightarrow{x_3}$$

4 Example

The following example is due to Lewis [5] and has been discussed by Halpern and Pearl in [4] in the framework of a theory of causation. Interestingly, this example defines actions with a complex temporal structure.

Billy and Suzanne throw rocks at a bottle. Suzanne throws first and her rock arrives first. The bottle shatters. When Billy's rock gets to where the bottle used to be, there is nothing there but flying shards of glass. Without Suzanne's throw, the impact of Billy's rock on the intact bottle would have been one of the final steps in the causal chain from Billy's throw to the shattering of the bottle. But, thanks to Suzanne's preempting throw, that impact never happens.

In our formulation, we focalize on the temporal structure of the throw action. We consider that the action occurs along a continuous (or dense) time axis, $[0, \infty[$. We define one action term for "throw", T, and two predicates H for "hits" and BB for "the bottle is broken". The action term $T(t, d, p)$ means that "person p throws a stone to a bottle at instance t and the result of the action (the stone hits its target) occurs at instance $t + d$". The formula $H(t, p)$ means that "the stone thrown by person p hits the bottle at instance t and formula $BB(t)$ means that the bottle is broken at instance t. The intended result of the action is to hit the bottle, but this result can only be achieved if the bottle is still at the intended place and nothing else has been happened to it, namely if it is not broken in the meantime. In this example it is not enough to have the precondition that the bottle is there and not broken at the instance of throwing, but it must be non-broken at the moment when the action is to be completed, just before it is to be hit. Therefore the action law for "throw" has a precondition which must hold after the instance when the action occurs.

Example 1 *The following set of laws represents the framework of this story:*

$$
\begin{array}{ll}
(1) & \neg BB(t + d) \rightarrow [T(t, d, p)]H(t + d, p) \\
(2) & \Box(H(t, p) \rightarrow BB(t + d_1)) \\
(3) & \Box(BB(t) \rightarrow \forall t'(t < t' \rightarrow BB(t'))) \\
(4) & \neg BB(0)
\end{array}
$$

(1) is the action law for successful execution of the throw action, (2) describes the impact of hitting the bottle (d_1 is infinitesimally small) and the general law (3) says that a broken bottle remains broken "forever" [1].

Several scenarios can happen within this framework. Here we discuss the scenario where Suzanne throws at instance 0 and Billy throws some instance later[2].

$$
\begin{array}{ll}
(5) & < T(0, d_s, suzy) > \top \\
(6) & < T(t_1, d_b, billy) > \top
\end{array}
$$

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[1] In this example we focus on the temporal relations between the different instances of throwing (by Suzanne and by Billy), so we neglected other preconditions, as for example having a stone, heavy enough, but not too heavy, having members enabling the person to throw, seeing the object to aim, etc. The throw action defined here is highly abstracted for the purpose of our temporal action theory.

[2] In order to express that action a occurs, we write $[a]\top$, which simply means that action a occurs (even when nothing can be said about its results). It is always possible to throw a stone at a bottle, even if the intended result of hitting cannot be achieved.

We need these "empty results", because

Three cases can then be distinguished:

1. The moment when the bottle can be hit (and broken) after Suzanne's throw ($d_s + d_1$) occurs **before** Billy's stone could possibly hit the bottle $t_1 + d_b$.

   (7) $d_s + d_1 < t_1 + d_b$
   (8) $\Box(BB(d_s + d_1) \rightarrow BB(t_1 + d_b))$ from (3) and (7)
   (9) $\neg BB(d_s)$ by persistency from (4)[3]
   (10) $[T(0, d_s, suzy)]H(d_s, suzy)$ from (1) and (9)
   (11) $[T(0, d_s, suzy)]BB(d_s + d_1)$ from (2), (10), K for the action modality and (A2)
   (12) $[T(0, d_s, suzy)]BB(t_1 + d_b)$ from (11), (8), K and (A2)

   In this scenario, the law $\neg BB(t_1 + d_b) \rightarrow [T(t_1, d_b, billy)]H(t_1 + d_b, billy)$ cannot be used to derive $T(t_1, d_b, billy)]H(t_1 + d_b, billy)$ because $BB(t_1 + d_b)$ holds after Suzanne's throw (12). Billy's stone cannot hit the bottle, because it is already broken when his stone could hit it and we have just $[T(t_1, d_b, billy)]\top$ ((6), Billy has thrown).

2. Billy's stone hits the bottle, which breaks, **before** Suzanne's stone could possibly hit the bottle.

   (13) $t_1 + d_b + d_1 < d_s$
   (14) $\Box(BB(t_1 + d_b + d_1) \rightarrow BB(d_s))$ from (3) and (13)
   (15) $\neg BB(t_1 + d_b)$ by persistency from (4), see (9)
   (16) $[T(t_1, d_b, billy)]H(t_1 + d_b, billy)$ from (1) and (15)
   (17) $[T(t_1, d_b, billy)]BB(t_1 + d_b + d_1)$ from (2), (16), K and (A2)
   (18) $[T(t_1, d_b, billy)]BB(d_s)$ from (14), (17), K and (A2)

   Here, Suzanne's stone, which could hit the bottle at instance d_s, will not hit it since we have $BB(d_s)$ and therefore the precondition $\neg BB(d_s)$ is not more true. The law $\Box(\neg BB(d_s) \rightarrow [T(0, d_s, suzy)]H(d_s, suzy))$ cannot be used to derive $[T(0, d_s, suzy)]H(d_s, suzy)$ because $BB(t_1 + d_b + d_1)$ holds after Billy's throw (17). All we have is $[T(0, d_s, suzy)]\top$ (Suzanne throws).

3. Suzanne's and Billy's stone hit the bottle precisely at the same moment.

   (19) $t_1 + d_b = d_s$
   (20) $\neg BB(t_1 + d_b) \wedge \neg BB(d_s)$ by persistency from (4), see (9)
   (21) $[T(0, d_s, suzy)]H(d_s, suzy)$ like (10)
   (22) $[T(t_1, d_b, billy)]H(t_1 + d_b, billy)$ as (16)
   (23) $[T(0, d_s, suzy)]BB(d_s + d_1)$ from (21)
   (24) $[T(t_1, d_b, billy)]BB(t_1 + d_sb + d_1)$ from (22))

   In this case, both stones hit the bottle which breaks as a result of Suzanne's throw and Billy's throw.

5 Conclusion and Related Work

Modal logic approaches to action theories define a space of states but cannot handle time, neither explicitly not implicitly [2, 6]. In situation calculus [9, 7] reasoning about time was not foreseen, properties change discretely and actions do not have durations. Remember that in situation calculus, there is a starting state, s_0 and for any action a and state s, $do(a, s)$ is a resulting state of s. One can consider that the set of states is given by $\{s : \exists a_1 \ldots a_n s = do(a_n, do(a_{n-1}, \ldots do(a_1, s_0)))\}$. Hence the temporal structure of situation calculus is discrete and branching and does not allow for actions of different duration neither for preconditions or results which become true during the action execution or later after the action is ended.

Javier Pinto has extended situation calculus in order to integrate time [8]. He conserves the framework of situation calculus and introduces a notion of time. Intuitively, every situation s has a starting time and an ending time, where $end(s, a) = start(do(a, s))$ meaning that situation s ends when the succeeding situation $do(a, s)$ is reached. The end of the situation s is the same time point as the beginning of the next situation resulting from the occurrence of action a in s. The obvious asymmetry of the $start$ and end functions is due to the fact that the situation space has the form of a tree whose root is the beginning state s_0. Thus, every state has a unique preceding state but eventually more that one succeeding state.

Paolo Tereziani proposes in [12] a system that can handle temporal constraints between events and temporal constraints between instances of events.

In this present article, we have introduced a new modal logic formalism which can handle simultaneously states and time. We did not address here the problem of the persistency of facts over time (or over the execution of actions), because we wanted to focus on the modal temporal formalism. We have adopted a solution similar to the one presented in [11], i.e. "weak" frame laws are nonmonotonically added to the theory. But this solution is a bit more complicated in the case of our first-order action logic presented in this paper, because we need to restrict ourselves to a decidable subset of $\mathcal{D}al$.

Concerning the implementation, we use a labelled analytic tableaux approach including an abductive mechanism for the weak persistency laws, which will be described in more detail in a following paper.

We will apply this formalism to planning problems where a hybrid approach (states and time) can be very powerful. The idea is to infer temporal constraints from a $\mathcal{D}al$ specification in order to create a plan for a problem

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A   Appendix: Proofs of the theorems

A.1   Soundness and Completeness

Soundness is easy and left tyo the reader. For completeness, in this part we will show that every $\mathcal{D}al$ - valid formula is a theorem of $\mathcal{D}al$. Our proof is along the same lines as [10]. Subsequently, formulas are always grounded.

Definition 2 *A set of formulas, s, is called $\mathcal{D}al$ -inconsistent if it contains a finite subset $\{\phi_1, \phi_2, \ldots, \phi_k\}$ with $\vdash_{\mathcal{D}al} \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Otherwise s is called $\mathcal{D}al$ - consistent (or consistent).*

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Let s be a set of formulas. Then we denote by $V_t(s)$ the set of all terms occurring in formulas of s. We denote by $P(s)$ the set of all $\mathcal{D}al$ - formulas containing only nonlogical symbols (terms, action terms and predicate symbols) of formulas of s.

Definition 3 *A set of formulas s is called complete (or $\mathcal{D}al$ - complete) if*

1. *s is consistent.*

2. *s is maximal, i.e. for all ϕ in $P(s)$ holds: if $\phi \notin s$ then $s \cup \{\phi\}$ is inconsistent.*

3. *s is saturated, i.e. for every existential formula $\exists x \phi \in s$ where x is a variable, there is a formula $\phi_c^x \in s$ for some constant $c \in V_t(s)$.*

Lemma 1 *Every consistent set of formulas can be extended to a complete set of formulas.*

Proof Let s be a consistent set of formulas and let $c_1, c_2, \ldots$ be a sequence of 'new' object variables not in $V_t(s)$. We define $P^\star(s) = P(s) \cup \{\psi : \psi$ is a formula with variables from $c_1, c_2 \ldots\}$. Let $\{\exists x_i \phi_i(x_i)\}_{i=1,\ldots}$ be an enumeration of all existential formulas of $P^*(s)$. Then we form a new set of formulas $s^\sim$, by adding to s all formulas $\exists x_i \phi_i(x_i) \rightarrow \phi(a_i), i = 1, 2, \ldots$ for every existential formula of $P(s)$. It is easy to see that $s^\sim$ is consistent.

$s^\sim$ is extended to a complete (maximal) set of formulas $s^\star$ as follows: Let $\Theta = \{s' : s \subset s'$ and s' consistent$\}$. Let H be a chain in Θ, i.e. $H \subset \Theta$ and if $s_1, s_2 \in H$ then $s_1 \subset s_2$ or $s_2 \subset s_1$. Then $\bigcup H$ is an upper bound of H in Θ, since $s \subset \bigcup H$ for all $s \in H$. It is easy to see that $\bigcup H$ is consistent. Therefore $\bigcup H \in \Theta$ and, by Zorn's Lemma, Θ then has a maximal element, $s^\star$. It is easy to see that $s^\star$ is complete.

    Q.E.D.

The following properties of complete sets are straightforward.

Lemma 2 *Let s be a complete set of formulas. Then*

1. *$\phi \in s$ if and only if $\neg\phi \notin s$*

2. *If $\phi \in s$ and $\psi \in P(s)$ and $\vdash_{\mathcal{D}al} \phi \rightarrow \psi$ then $\psi \in s$*

3. *If $\Box\phi \in s$ then $[a]\phi \in s$*

4. *If $\phi \vee \psi \in s$ then $\phi \in s$ or $\psi \in s$*

5. *If $\phi \vee \psi \in P(s)$ and $\phi \in s$ or $\psi \in s$ then $\phi \vee \psi \in s$*

6. *If $c \in V_t(s)$, then $\phi_c^x \in s$ if and only if $\exists x \phi \in s$*

7. *$t = t \in s$ for any term $t \in V_t(s)$*

8. *If $t = t' \in s$ then $t' = t \in s$*

9. *If $t = t' \in s$ and $t' = t'' \in s$ then $t = t'' \in s$*

The last three items show that $t = t' \in s$ defines an equivalence relation $\sim_s$ over $V_t(s)$, where $t \sim_s t'$ iff $t = t' \in s$. The equivalence class of t according to $\sim_s$ is denoted by $[t]_s$.

Definition 4 *Let s be a set of formulas and $a \in A_t$. Then we define*

1. $s^a = \{\phi : [a]\phi \in s\}$

2. $s^\square = \{\phi : \square\phi \in s\}$

Lemma 3   *1. s^a is inconsistent iff $[a]\bot \in s$*

2. $s^\square \subseteq s^a$

Lemma 4 *Let s be a complete set of formulas and ϕ a formula such that $s^a \cup \{\phi\}$ is consistent. Let $s' = (s^a \cup \{\phi\})^*$ the complete extension of $s^a \cup \{\phi\}$ (which exists by lemma 1). Then $V_t(s) = V_t(s')$ Proof Since s is complete, it is saturated, hence for every existential formula $\exists x \phi \in s$ there is a formula $\phi^x{}_c \in s$ for some $c \in V_t(s)$. The crucial point concerns existential formulas in s^a not in s. But for those formulas we have that $\square \exists x \phi \in s$ and therefore $\exists x \phi \in s^a$, by T. Hence existential formulas in s^a are also existential formulas in s. But this means that all terms in s' are terma in s.*
Q.E.D.

Definition 5 *A set $\mathcal{S}$ of sets of formulas is called* complete *if*
1. Every element s of S is a complete set of formulas.
2. For all $s \in S$ and $\phi \in P(s)$, if $s^a \cup \{\phi\}$ is consistent, then there is $s' \in S$ such that $s^a \cup \{\phi\} \subset s'$

Remark 2 *If $[a]\bot \in s$, then, since s^a is not consistent, it is and not contained in any $s' \in S$.*

Now we construct the canonical model starting from a complete set of formulas, s.

Lemma 5 *For every complete set of formulas s, there is a $\mathcal{D}al$ -structure $\mathcal{M} = (\mathcal{W}, \{\mathcal{S}_w : w \in \mathcal{W}\}, \mathcal{A}, \mathbf{R}, (\tau_0, \tau_1, \tau_2, \tau_3))$, such that $s \in \mathcal{W}$.*
Proof :
For every $\phi \in P(s)$ and every $a \in A_t$ such that $s^a \cup \{\phi\}$ is consistent, we extend $s^a \cup \{\phi\}$ to a complete set of formulas s', which exists by lemma 1. So we form successively sets of formula sets S_0, S_1,... as follows
$S_0 := \{s\}$
$S_{i+1} := \{X : X$ *is a complete extension of* $Y^a \cup \{\phi\}$ *where* $Y \in S_i$ *and* $\phi \in P(s)$ *and* $Y^a \cup \{\phi\}$ *is consistent$\}$*
$\mathcal{W} := \bigcup\{S_i : i \in \omega\}$.

Because of remark 4, $V_t(s) = V_t(s')$ for $s, s' \in \mathcal{W}$. We note $V_t(s) = V_t$, $[t]_s = [t]$ and $\sim_s = \sim$.

For every $w \in \mathcal{W}$, $\mathcal{S}_w = (\mathcal{O}, \mathcal{F}_w, \mathcal{P}_w)$ is a classical structure where the object set $\mathcal{O}$ is defined as the set of $\sim$-equivalence classes over V_t.

- $\mathcal{O} =_{def} \{[t] : t \in V_t\}$

- *For every* $t \in \mathcal{O}$, $\tau_0(w, t) = [t]$

- *for every n-ary function symbol* $F \in \boldsymbol{F}$ *and for every tuple of terms* $t_1, \dots t_n \in V_t$, $\tau_1(w, F)([t_1], \dots [t_n]) = [F(t_1, \dots t_n)]$

- $\mathcal{P}_w$ *is the set of predicates* P^w, *where for every predicate symbol* P *of arity* $n = |P|$, P^w *is the predicate defined by* $(\tau_1(t_1), \dots \tau_1(t_n)) \in P^w$ *if and only if* $P(t_1, \dots t_n) \in w$. *We set* $\tau_2(w, P) = P^w$

- $\mathcal{A}$ *is the set of action functions* a. *For every action symbol* a *of arity* $|a| = m$, *and terms* $t_1, t_2, \dots, t_m$, $a(w, t_1, t_2, \dots, t_m) \subseteq \mathcal{W}$ *is defined by* $w' \in a(w, t_1, t_2, \dots, t_m)$ *if and only if* $w^{a(t_1, t_2, \dots, t_m)} \subseteq w'$

- *We set* $\tau_3(a) = a$

Q.E.D.

We show that $\mathcal{M}$ is a $\mathcal{D}al$ -structure. By the construction, $\mathcal{W}$ is a complete set of sets.

Lemma 6 $\tau_3(a)(w, \tau_0(t_1), \dots, \tau_0(t_n)) \subseteq R(w)$. *Let be* $w' \in \tau_3(a)(w, \tau_0(t_1), \dots, \tau_0(t_n))$. *Then* $w^{a(t_1, t_2, \dots, t_m)} \subseteq w'$ *by construction of the model. But then* $w^\square \subseteq w'$ *by lemma 3, from which follows that* $w' \in R(w)$.

Truth value $\tau(w, \phi)$ for formula ϕ and world w is defined by induction over the construction of formulas as usual.

In order to show that $\tau(s, \phi) = t$ if and only if $\phi \in s$, we need the following lemma:

Lemma 7 *Let* $\mathcal{M} = (\mathcal{W}, \{\mathcal{S}_w : w \in \mathcal{W}\}, \mathcal{A}, \mathbf{R}, (\tau_0, \tau_1, \tau_2, \tau_3))$ *be the* $\mathcal{D}al$ *-structure constructed from a complete formula set* w *according to lemma 5 and* $w \in W$.
Then for every action symbol a *of arity* n, $[a]\phi \in w$ *if and only if*
for every tuple of terms $t_1, t_2, \dots, t_n$ and $w' \in \mathcal{W}$, *if* $w' \in \tau_3(a)(w, t_1, t_2, \dots, t_n)$, *then* $\phi \in w'$.
Proof
If ϕ *is* $\bot$ *then this lemma trivially holds because then* $\tau_3(a)(w, t_1, t_2, \dots, t_n$ *is empty. Let* ϕ *be different from false.*
$(\Rightarrow)$: *if* $[a]\phi \in w$ *then* $\phi \in w^a$ *and therefore for every* $w' \in a(w, t_1, t_2, \dots, t_n)$, $\phi \in w'$ *by the definition of* $\mathcal{M}$.
$(\Leftarrow)$: *Let be* $|a| = n$. *Consider* w^a. *We first show that* $w^a \cup \{\neg\phi\}$ *is inconsistent. Assume for the contrary, that* $w^a \cup \{\neg\phi\}$ *is consistent. Since* $\mathcal{W}$ *is a complete set of formula sets, according to definition 5, there is* $w' \in \mathcal{W}$ *and* $w^a \cup \{\neg\phi\} \subseteq w'$. *From this follows* $w^a \subseteq w'$, *hence* $w' \in a(w, t(|a|_1, |a|_2, \dots, |a|_)$, *by the constuction of* $\mathcal{M}$ *in lemma 5. Since* $\neg\phi \in w'$ *and* w' *is complete,* $\phi \notin w'$, *which contradicts the hypothesis.*
Therefore, $w^a \cup \{\neg\phi\}$ *is inconsistent. Therefore there exist formulas*

$\phi_1, \phi_2, \ldots, \phi_k \in w^a$ *such that* $\vdash \phi_1 \wedge \ldots \wedge \phi_k \rightarrow \phi$ *from which* $\vdash [a](\phi_1 \wedge \ldots \wedge \phi_k) \rightarrow$ $[a]\phi$ *by rule 1 of the logic* $\mathcal{D}al$ *and therefore* $\vdash [a]\phi_1 \wedge \ldots \wedge [a]\phi_k \rightarrow [a]\phi$. *But* $[a]\phi_1, [a]\phi_2, \ldots, [a]\phi_k \in w$, *and therefore* $[a]\phi \in w$, *by the lemma 2. 2.* Q.E.D.

The next lemma concludes our proof.

Lemma 8 *For every closed formula* $\phi \in P(s)$,

$$\tau(w, \phi) = t \text{ if and only if } \phi \in w$$

The proof is straightforward, by induction over formulas using the lemmata 2 and 7.

Proof of the completeness theorem:

Proof Assume for the contrary that there is a $\mathcal{D}al$ valid formula ϕ which is not deducible in $\mathcal{D}al$. Then $\{\neg\phi\}$ is a consistent set of formulas, which can be extended to a complete set of formulae s, by Lemma 1, where $\neg\phi \in s$. By Lemma 5, there is a $\mathcal{D}al$-structure $\mathcal{M} = (\mathcal{W}, \{\mathcal{S}_w : w \in \mathcal{W}\}, \mathcal{A}, \mathbf{R}, (\tau_0, \tau_1, \tau_2, \tau_3))$ where $\mathcal{W}$ is a complete system of sets and $s \in \mathcal{W}$. By Lemma 8, $\tau(s, \neg\phi) = t$ and hence $T(s, \phi) = f$, which contradicts the validity of ϕ. Q.E.D.