



HAL
open science

The Minnaert bubble: a new approach

Martin Devaud, Thierry Hocquet, Jean-Claude Bacri, Valentin Leroy

► **To cite this version:**

Martin Devaud, Thierry Hocquet, Jean-Claude Bacri, Valentin Leroy. The Minnaert bubble: a new approach. *European Journal of Physics*, 2008, 29 (6), pp.1263. 10.1088/0143-0807/29/6/014. hal-00145867

HAL Id: hal-00145867

<https://hal.science/hal-00145867>

Submitted on 14 May 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

The Minnaert bubble: a new approach

Martin Devaud,* Thierry Hocquet,[†] and Jean-Claude Bacri[‡]
*Laboratoire Matière et Systèmes Complexes, Université Paris 7,
10 rue Alice Domont et Léonie Duquet, 75013 PARIS, France, and CNRS UMR 7057*

Valentin Leroy[§]
Laboratoire Ondes et Acoustique, Université Paris 7 - ESPCI, 10 rue Vauquelin, 75005 PARIS, France
(Dated: May 11, 2007)

We propose a new *ab initio* introduction to the well known Minnaert pulsating bubble, beginning with a detailed discussion of the radial movements of an air bubble in water. In unbounded water, the {air – water} system has a continuum of eigenmodes, some of them correspond to regular Fabry-Pérot resonances. A singular resonance, the lowest one, is shown to coincide with that of Minnaert. In bounded water, the eigenmodes' spectrum is discrete, with a finite fundamental frequency. A spectacular quasi-locking of the latter occurs if it happens to exceed the Minnaert frequency, which provides an unforeseen one-bubble alternative version of the famous “hot chocolate effect”. In the (low) frequency domain in which sound propagation inside the bubble reduces to a simple inflation/deflation, the – light – air bubble can be “dressed” by the outer water pressure forces, and is turned into the – heavy – Minnaert bubble. Thanks to this unexpected renormalization process, we demonstrate that the Minnaert bubble definitely behaves like a true harmonic oscillator of the mass-spring type, but with a damping term and a forcing term in apparent disagreement with those commonly admitted in literature. At last, in order to tell the water's motion associated with water compressibility (*i.e.* the sound) from the simple incompressible accompaniment of the bubble's breathing, we introduce a new picture analogous to the electromagnetic radiative picture in Coulomb gauge, which naturally leads us to split the water displacement in an instantaneous and a retarded parts. The Minnaert renormalized mass of the dressed bubble is then recovered.

PACS numbers: 43.20.+g, 47.55.D-, 47.55.dd

I. INTRODUCTION

Since the pioneering paper by Minnaert¹, a wealth of literature has been devoted to the acoustic bubble problem. To make an idea of the infatuation about this question from the Acoustics community, one may simply consider that the large review of this topic by Leighton² already contains, in its 1997 issue, not less than about 1500 references. Roughly, this topic can be taken up from a twofold view-point. (*i*) The *application* view-point first. An air bubble in water is a powerful sound scatterer, which makes it a deciding actor in numerous acoustic situations. From the murmur of the brooks to the noise of the oceans, from the acoustic fog-impeded propagation of SONAR beams to the turning to account of bubbles in the blood stream as contrast agent in medical ultrasonic imaging, few aspects, if any, of the sound propagation in bubbly liquids have not been put forward or used here or there yet. (*ii*) The acoustic bubble is a fascinating object from the *theoretical* view-point *too*. It raises indeed several fundamental questions, some of them are also encountered in the electrodynamic domain. As a consequence, well-tried methods and concepts of the latter domain can be adapted to the acoustic bubble. It is precisely our approach in this paper, which is deliberately devoted to a theoretical questioning of some basic statements about the bubble. In this connection, our purpose is *inter alia* to show that the Minnaert bubble behaving like a harmonic oscillator at the isentropic approximation is in fact a highly nontrivial issue. Aiming at proposing

an accurate meaning to this statement, we discuss several aspects of the problem as the mass renormalization of the “naked” bubble, the radiative damping and the external forcing of the bubble. In this respect, we make a few remarks about the role played by water compressibility in the single-bubble Minnaert oscillation problem. With this aim, our paper is wilfully progressive, and organized as follows.

In section II, we consider the radial eigenmodes of an infinite extent of (compressible) water surrounding an air bubble. Considering the amplitude of the extrapressure at the air-water boundary, we find that some of these eigenmodes – more precisely an infinite discrete series of them – present resonances. The fundamental resonance corresponds to a wavelength which is much larger than the bubble's radius, and is found to coincide with the Minnaert resonance. We then briefly sketch the modifications that should be introduced to take capillarity and dissipation into account. We finally consider the experimental case of a finite extent of surrounding water, thus involving reverberation effects at the outer boundary. The bubble is then placed at the center of a spherical cavity. We reexamine the radial eigenmodes and the existence of resonances among them. A spectacular lowering of the fundamental eigenmode's frequency is observed if the radii ratio of the water cavity and the air bubble is not large enough, thus entailing an exotic and original “hot chocolate effect”³.

In section III, we consider the same problem as in section II, but from the point of view of the *mere* air bubble.

The role of water is then simply to impose a dynamical boundary condition at the bubble's surface. We derive simple expressions for the pressures undergone and radiated by this surface, and we relate them to the resonances found in section II.

In section IV, we focus on the low frequency domain, more precisely the domain in which sound propagation can be neglected within the air cavity. The motion of air inside this cavity is then described, by means of a single dynamical variable, as a uniform inflation/deflation movement. In this “breathing” approximation, we derive a motion equation for the bubble that turns out to be that of a harmonic oscillator. In this respect, we parallel our acoustic problem with its electromagnetic equivalent and show that the pressure forces exerted by water onto the bubble's surface are quite analogous to the Abraham-Lorentz forces undergone by an accelerated electric charge. We are then led to distinguish between the “naked” bubble, with a light mass (due to air's inertia), and the “dressed” bubble, with a heavy renormalized mass (due to water's inertia). We also discuss the corresponding damping terms accounting for the acoustic radiation as well as those – viscous and thermal – accounting for dissipation.

In section V we address the following question: when the Minnaert bubble pulsates, the surrounding water moves accordingly back and forth; which part of this water motion should then be regarded as “sound” ? We propose and discuss an answer to this question and we seize this opportunity to interpret the mass renormalization term introduced in section IV.

II. RADIAL EIGENMODES OF A PULSATING BUBBLE

A. One bubble in an infinite extent of water

Let us consider, in an infinite extent of water, an air bubble with equilibrium radius R and center located at the origin of coordinates. At equilibrium pressure P_0 and temperature T_0 , air and water have mass densities ρ_a and ρ_w , and adiabatic compressibilities χ_a and χ_w . Throughout this paper, we shall use exclusively lagrangian dynamical variables: the element of fluid (air or water) which stood at equilibrium at point \vec{r} (and under pressure P_0) is located at time t at position $\vec{r} + \vec{u}(\vec{r}, t)$ and under total pressure $P(\vec{r}, t) = P_0 + p(\vec{r}, t)$. Moreover, we consider here only radial motions: $\vec{u}(\vec{r}, t) = u(r, t) \vec{e}_r$, where $\vec{e}_r = \vec{r}/r$, and $p(\vec{r}, t) = p(r, t)$. Furthermore, by care of simplicity, we shall assume that both air and water undergo transformations small enough for the linear theory to be valid. We shall make two further simplifications, ignoring (i) any air-water surface tension effect (then considering that extrapressure $p(r, t)$ is continuous at $r = R$) and (ii) any thermal or viscous dissipation (then regarding the small air or water transformations as isentropic). Both of these approximations are discussed

in subsection II B. As a consequence, we have

$$p(r, t) = -\frac{1}{\chi} \operatorname{div} \vec{u}(r, t). \quad (2.1a)$$

Then, $c = 1/\sqrt{\rho\chi}$ standing for the sound velocity in air (c_a) or water (c_w), and using Newton's law

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = -\overrightarrow{\operatorname{grad}} p, \quad (2.1b)$$

the d'Alembert's propagation equation $\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p$ is recovered.

In this section, we shall consider monochromatic solutions of this equation of the form

$$p(r, t) = \Re\{\bar{p}(r)e^{-i\omega t}\}, \quad u(r, t) = \Re\{\bar{u}(r)e^{-i\omega t}\}, \quad (2.2)$$

where $\bar{p}(r)$ and $\bar{u}(r)$ are complex amplitudes. The d'Alembert's equation then reads $(\Delta + q^2)\bar{p}(r) = 0$, where $q = \omega/c$ is the wave number. Owing to the spherical symmetry, the laplacian operator Δ is reduced to $\frac{1}{r} \frac{d^2}{dr^2} r$, and we get

$$\left(\frac{d^2}{dr^2} + q^2\right)(r\bar{p}) = 0 \\ \rightsquigarrow \bar{p}(r) = \frac{1}{r}(\alpha \sin qr + \beta \cos qr). \quad (2.3)$$

In the air domain ($r \leq R$), to avoid singularity at $r = 0$, coefficient β should be zero, so that (2.3) reduces to

$$r \leq R: \quad \bar{p}(r) = \bar{p}_a(r) = \frac{A}{r} \sin q_a r. \quad (2.4a)$$

On the other hand, in the water domain ($r \geq R$), the solution (2.3) can be rewritten, for the sake of convenience, in the form

$$r \geq R: \quad \bar{p}(r) = \bar{p}_w(r) = \frac{W}{r} \sin(q_w(r - R) + \varphi), \quad (2.4b)$$

with $\varphi \in [0, \pi]$. From the above equations (2.4a,b), the displacement amplitude $\bar{u}(r)$ can be derived using Newton's law ($\bar{u} = \frac{1}{\rho\omega^2} \frac{d\bar{p}}{dr}$). Then, writing that extrapressure p and displacement u should be continuous at the air-water boundary ($r = R$), we get the relations

$$A \sin x_a = W \sin \varphi, \quad (2.5a)$$

$$\frac{A}{\rho_a} (\sin x_a - x_a \cos x_a) = \frac{W}{\rho_w} (\sin \varphi - x_w \cos \varphi), \quad (2.5b)$$

with $x_a = q_a R$ and $x_w = q_w R$. Thus, for a given angular frequency ω , wave numbers q_a and q_w – and consequently angles x_a and x_w – are fixed. Then, dividing (2.5b) by (2.5a), we are left with

$$\frac{1}{\rho_a} (1 - x_a \cot x_a) = \frac{1}{\rho_w} (1 - x_w \cot \varphi), \quad (2.6)$$

from which we get

$$\begin{aligned}\sin \varphi &= \frac{x_w}{\sqrt{x_w^2 + \left(1 + \frac{\rho_w}{\rho_a}(x_a \cot x_a - 1)\right)^2}}, \\ \cos \varphi &= \frac{1 + \frac{\rho_w}{\rho_a}(x_a \cot x_a - 1)}{\sqrt{x_w^2 + \left(1 + \frac{\rho_w}{\rho_a}(x_a \cot x_a - 1)\right)^2}}.\end{aligned}\quad (2.7)$$

As a conclusion, for any angular frequency ω , one can find a monochromatic solution satisfying the boundary conditions (2.5): the {air-water} system has a continuum of eigenmodes.

One nevertheless may wonder whether there exists some particular values of ω for which “something happens” at $r = R$. To see this, let us fix the amplitude W of the standing wave in water, and look for the values of ω that maximize the pressure amplitude $\frac{A}{R} \sin x_a$ (see (2.4a)) at $r = R$. Allowing for (2.5a), $\sin \varphi$ should then be equal to unity: this means that $r = R$ corresponds to an antinode for the standing pressure wave $\bar{p}_w(r)$ displayed in (2.4b). Then $\cos \varphi = 0$ and, according to (2.7),

$$1 + \frac{\rho_w}{\rho_a}(x_a \cot x_a - 1) = 0 \rightsquigarrow \tan x_a = \frac{x_a}{1 - \frac{\rho_a}{\rho_w}}. \quad (2.8)$$

The above equation (2.8) has no analytical solution. However, it can be graphically solved as shown in figure 1 where functions $\tan x_a$ and $x_a/(1 - \frac{\rho_a}{\rho_w})$ are plotted versus x_a . In this figure, we have set $\rho_a/\rho_w = 1.293 \cdot 10^{-3}$, which corresponds to $P_0 = 1$ atm.

As displayed by the figure, “regular” intersections of the straight line with the $\tan x_a$ curve are found for $x_a \simeq 4.49, 7.73, \dots$, very close to the $\frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{(2n+1)\pi}{2}$ abscissae of the vertical asymptots of $\tan x_a$. In addition to these regular intersections, and due to the fact that the $1/(1 - \frac{\rho_a}{\rho_w})$ slope of the straight line is slightly larger than unity (*i.e.* the slope of the tangent to the curve $\tan x_a$ in $x_a = 0$), there is a “singular” intersection with $x_a \simeq 0.0623 \ll 1$, as displayed by the zoom in figure 1; it corresponds to the fundamental (radial) resonance of the {air-water} system. Using expansion $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^7)$, angle x_a for this fundamental resonance can be easily derived:

$$\begin{aligned}x_a \left(1 + \frac{x_a^2}{3} + \frac{2x_a^4}{15}\right) &\simeq \frac{x_a}{1 - \frac{\rho_a}{\rho_w}} \\ \rightsquigarrow x_a^2 &\simeq \frac{3\rho_a}{\rho_w} \left(1 - \frac{1}{5} \frac{\rho_a}{\rho_w}\right),\end{aligned}\quad (2.9a)$$

associated with the angular frequency

$$\omega_0 = \frac{c_a x_a}{R} \simeq \left(1 - \frac{1}{10} \frac{\rho_a}{\rho_w}\right) \omega_M \simeq \omega_M, \quad (2.9b)$$

where

$$\omega_M = \sqrt{\frac{3}{\rho_w \chi_a R^2}} \quad (2.9c)$$

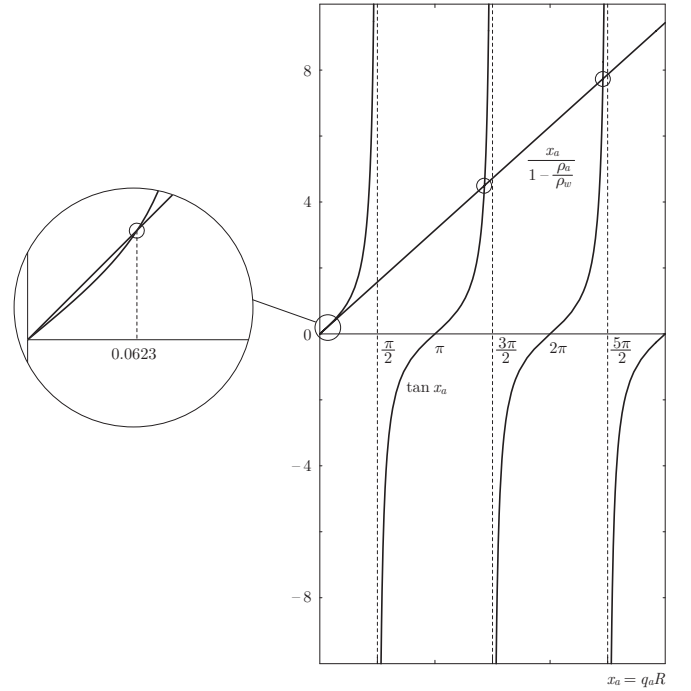


FIG. 1: Graphic resolution of equation (2.8): functions $\tan x_a$ and $x_a/(1 - \frac{\rho_a}{\rho_w})$ are plotted versus x_a . Regular intersections are found for $x_a \simeq 4.49, 7.73, \dots$, very close to $\frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{(2n+1)\pi}{2}$. In addition, since the slope $1/(1 - \frac{\rho_a}{\rho_w})$ of the straight line is slightly larger than unity, a singular resonance (the fundamental one) is found for $x_a \simeq 0.0623 \ll 1$, as displayed by the zoom (in which the difference between the slopes has been deliberately exaggerated, for the sake of visibility).

is known as the Minnaert angular frequency.

The above results deserve a few comments. The resonances found when solving equation (2.8) are *a priori* of the Péro-Fabry type: they can be understood in terms of constructive interferences of a radial pressure wave bouncing inside the bubble between its center and its surface. If the air-water interface’s radius of curvature (*i.e.* the bubble’s radius) R is large compared to the acoustic wavelengths (*i.e.* if $x_a = q_a R, x_w = q_w R \gg 1$), the situation corresponds to the plane wave configuration of a 1-D Péro-Fabry device. In this case, the determination of the resonances is very simple : a pressure wave originating in $\vec{r} = 0$ is dephased by $q_a R = x_a$ when it reaches the bubble’s surface. After one reflection on this surface (with no dephasing since water’s acoustic impedance is larger than air’s one) and a there-and-back center \leftrightarrow surface additional travel, this same wave is dephased at $r = R$ by $3x_a - \pi$ ($-\pi$ when bouncing in $r = 0$). Both paths constructively interfere at $r = R$ provided that $3x_a - \pi = x_a + 2n\pi$, *i.e.* $x_a = \frac{(2n+1)\pi}{2}$. As can be checked from figure 1, this is what is roughly found for the “regular” resonances, corresponding to $n = 1, 2, 3, \dots$ (the higher the value of integer n , the smaller the dis-

crepancy). But what about the “singular” resonance, for which $x_a \ll 1$? It is hardly credible that such a small dephasing may yield constructive interferences. In fact it does. This paradox is removed by considering the geometry of the problem: since the wavelengths (in air and in water) are no longer small compared to the air-water interface’s radius of curvature, the air-water reflection becomes *dispersive*, involving a dephasing of the pressure wave which is no longer zero but tends towards π in the $x_a, x_w \ll 1$ limit. In this dephasing originates the spectacular shift of the $n = 0$ resonance from $x_a = \frac{\pi}{2}$ (as would be the case in 1-D geometry) to $x_a \simeq 0.06$.

In figure 2, we have displayed $\sin \varphi = \frac{R}{W} \bar{p}(R)$ as a function of x_a , with the same ratio ρ_a/ρ_w as in figure 1. Considering the $\frac{1}{\sqrt{2}}$ -linewidth of the peaks (*i.e.* the half-height linewidth of $\sin^2 \varphi$) defined as $\delta = x_a^+ - x_a^-$, where x_a^\pm are solutions of the equations (see (2.7))

$$1 + \frac{\rho_w}{\rho_a} (x_a^\pm \cot x_a^\pm - 1) = \mp x_w, \quad (2.10)$$

we find, after a straightforward rough calculation,

$$\delta \simeq 2 \frac{\rho_a}{\rho_w} \frac{c_a}{c_w} \quad \text{for the “regular” peaks,} \quad (2.11a)$$

$$\delta \simeq 3 \frac{\rho_a}{\rho_w} \frac{c_a}{c_w} \quad \text{for the “singular” peak.} \quad (2.11b)$$

In this figure 2, the water compressibility has its actual value ($\chi_w \simeq 10^{-10} \text{ Pa}^{-1}$). If water is assumed to be incompressible, the positions of the peaks are unchanged (equation (2.8) is independent of χ_w), but their widths are then zero: $\chi_w = 0 \rightsquigarrow c_w = \infty \rightsquigarrow \delta = 0$, hence figure 2 looks like a comb (strictly speaking, the {air-water} system has then a discrete series of eigenmodes).

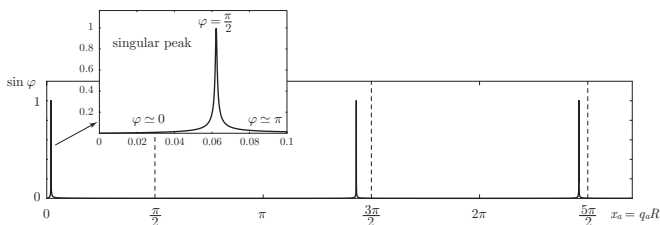


FIG. 2: $\sin \varphi$ as a function of $x_a = q_a R$. In addition to regular peaks for $x_a \simeq \frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{(2n+1)\pi}{2}$, there is a singular peak for $x_a \simeq 0.06 \ll 1$ corresponding to the Minnaert resonance. The compressibility of water is taken into account and the peaks have a finite linewidth, as calculated in the text. If water was assumed to be incompressible, the curve would just be a comb.

It is worth emphasizing that, in the case of the singular Minnaert resonance, we have $x_a = q_a R \ll 1$, so that the air pressure in the bubble is quasi homogeneous:

$$\bar{p}_a(r) = \frac{A}{r} \sin q_a r \simeq A q_a. \quad (2.12a)$$

Accordingly, the radial displacement $\bar{u}(r)$ reads

$$\begin{aligned} \bar{u}(r) = \bar{u}_a(r) &= \frac{1}{\rho_a \omega_0^2} \frac{d\bar{p}_a}{dr} \\ &= \frac{A}{\rho_a \omega_0^2 r^2} [q_a r \cos q_a r - \sin q_a r] \\ &\simeq -\chi_a \frac{A q_a}{3} r, \end{aligned} \quad (2.12b)$$

which corresponds to a simple “breathing” (inflation/deflation) of the bubble. This breathing motion will be the master hypothesis of section IV. Observe by the way that equation (2.1a) is satisfied by approximate formulas (2.12a,b).

To end this subsection, it is interesting to list here the conditions that should be satisfied by two media, respectively labelled “a” (inner) and “w” (outer) for a Minnaert-like resonance to be possible. (i) First, as displayed by condition (2.8), one should have $\rho_a \ll \rho_w$: a drop of water surrounded by air will *not* give the Minnaert resonance. (ii) Second, for the Minnaert angular frequency ω_M to be telled from zero, the width of the singular peak (see the zoom in figure 2) should be smaller than the corresponding x_a , thus implying $\delta \ll x_a$, or equivalently (see (2.9a) and (2.11b)), $3\chi_w \ll \chi_a$. Observe by the way that this latter inequality implies that, at the Minnaert frequency, one has $x_w = R\omega_M/c_w = \sqrt{3\chi_w/\chi_a} \ll 1$. In other words, condition (ii) means that the bubble’s radius is much smaller than the acoustic wavelength in *both* media (a and w).

Let us now, in the next subsection, seriatim sketch how the above results are changed when surface tension and dissipation are considered.

B. Capillarity and dissipation effects

Let σ stand for the air-water surface tension, $P_a(R, t)$ and $P_w(R, t)$ for the (total) pressures at $r = R$ on both sides of the air-water interface. Naturally, we have

$$P_a(R, t) - P_w(R, t) = \frac{2\sigma}{R + u(R, t)}. \quad (2.13)$$

The equilibrium pressure inside the bubble is then $P'_0 = P_0 + \frac{2\sigma}{R}$. Consequently, the equilibrium mass density ρ'_a and isentropic compressibility χ'_a of the air read

$$\rho'_a = \rho_a \left(1 + \frac{2\sigma}{RP_0}\right), \quad \chi'_a = \chi_a \left(1 + \frac{2\sigma}{RP_0}\right)^{-1}. \quad (2.14)$$

(Note that the sound velocity $c_a = 1/\sqrt{\rho_a \chi_a}$, which depends only on temperature, is unaffected by the capillarity-induced $P_0 \rightsquigarrow P'_0$ change of the bubble’s equilibrium inner pressure). With $\sigma = 7 \cdot 10^{-2} \text{ J}\cdot\text{m}^{-2}$ and under atmospheric pressure $P_0 = 10^5 \text{ Pa}$, the ratio $2\sigma/RP_0$

equals unity for $R = 1.4 \mu\text{m}$: corrections (2.14) can be omitted for bubble's radii R larger than 0.1 mm.

In addition to the equilibrium pressure discontinuity, the air-water surface tension produces at the interface a discontinuity of the extrapressure which reads, at first order in u/R and allowing for (2.13),

$$p_a(R, t) - p_w(R, t) = -\frac{2\sigma}{R^2}u(R, t). \quad (2.15)$$

Equations (2.5) and (2.8) should be accordingly modified. All calculations carried out, taking capillarity into account results in a simple shift of the resonances. Condition (2.8) becomes indeed

$$1 + \left(\frac{\rho_w}{\rho_a} + \frac{2\sigma}{R} \frac{\chi'_a}{x_a^2} \right) (x_a \cot x_a - 1) = 0. \quad (2.16)$$

One observes that the above condition (2.16) substantially departs from (2.8) for *small* values of x_a (*i.e.* small values of ω). It is thus particularly interesting to examine how the *lowest* resonance, namely the Minnaert one, is shifted by capillarity. For $x_a \ll 1$, a straightforward calculation gives, at first order in ratio σ/RP_0 :

$$x_a^2 = \frac{3\rho'_a}{\rho_w} \left(1 - \frac{2\sigma}{3} \frac{\chi'_a}{R} \right) \\ \rightsquigarrow \omega_0 = \omega_M \left(1 + \frac{\sigma}{RP_0} \frac{3\gamma - 1}{3\gamma} \right). \quad (2.17)$$

where $\gamma = C_P/C_V$ is the specific heats ratio of air. In conclusion, the effects of surface tension are negligible, as well as concerns equilibrium state or dynamics, if condition $\sigma/RP_0 \ll 1$ is satisfied. Henceforth we shall consider bubble radii larger than 0.1 mm and ignore capillarity.

Taking dissipation effects into account is a far more involved affair⁴. Both basic equations (2.1a) and (2.1b) are indeed no longer available and should be completed; moreover, they should be associated with a third equation ruling heat transfer, as (briefly) recalled below.

Let $S_0(\vec{r}) d^3r$ be the *equilibrium* entropy of the fluid (air or water) element volume d^3r located at position \vec{r} . At time t in the course of motion, the entropy of this same element volume (now located at position $\vec{r} + \vec{u}(\vec{r}, t)$) reads $(S_0(\vec{r}) + s(\vec{r}, t)) d^3r$. Let $T(\vec{r}, t) = T_0 + \theta(\vec{r}, t)$ be its temperature. The entropy field $s(\vec{r}, t)$ was, by definition, zero in the framework of the isentropic approximation (*ii*) in subsection II A; it should now be considered. In order to allow for non isentropicity, straightforward thermodynamical considerations lead to generalize (2.1a) in

$$p(r, t) = -\frac{1}{\chi} \left(\text{div } \vec{u}(r, t) - \frac{s}{S_0} \right). \quad (2.18a)$$

On the other hand, in order to allow for viscosity, equation (2.1b) should be substituted by the Navier-Stokes

equation

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = -\vec{\text{grad}} p \\ + \eta \vec{\Delta} \frac{\partial \vec{u}}{\partial t} + \left(\eta + \frac{\zeta}{3} \right) \vec{\text{grad}} \text{div} \frac{\partial \vec{u}}{\partial t}, \quad (2.18b)$$

where η and ζ respectively stand for the so-called first and second viscosity coefficients. Furthermore, the (linearized) heat equation reads

$$\frac{\partial s}{\partial t} = \frac{\Lambda}{T_0} \Delta \theta, \quad (2.18c)$$

where Λ is the fluid's (air or water) thermal conductivity. Looking for monochromatic solutions of the above set of equations (2.18) of the form (2.2) with in addition

$$s(r, t) = \Re\{\bar{s}(r)e^{-i\omega t}\}, \quad \theta(r, t) = \Re\{\bar{\theta}(r)e^{-i\omega t}\} \quad (2.19)$$

is doubtless an interesting issue. It would nevertheless divert us from the purpose of the present paper. Therefore, we postpone this discussion to the end of section IV (devoted to the mere Minnaert bubble) where we summarize Prosperetti's approach⁵ and approximations to take dissipation into account, and provisionally ignore the related effects.

C. Finite size effects

In fact, in any laboratory experiment, the bubble is placed in an aquarium, *i.e.* is immersed in a finite extent of water. It is therefore worth wondering how the above results are affected by such a confinement⁶. It is of course not possible to find an analytic answer to this question for any aquarium shape. It is nevertheless possible, as explained hereafter, in the – not very realistic – framework of the radial symmetry.

Let us thus now consider our same air bubble, but surrounded by a finite extent of water bounded by a concentric sphere with radius L ($L > R$). No extrapressure is applied at the boundary $r = L$. We are concerned again in the eigenmodes of this {air-finite water} system. The problem is rather simple: since, for $0 < r < L$, the equations of motion are unchanged, solutions (2.4) still apply. The extrapressure \bar{p}_w should just vanish at $r = L$, implying $\sin[q_w(L - R) + \varphi] = 0$, with φ still given by (2.7). This condition is fulfilled for a discrete series of values ω_p of the angular frequency ω satisfying

$$\frac{\omega_p}{c_w} (L - R) + \varphi(\omega_p) = p\pi, \quad (p = 1, 2, \dots). \quad (2.20)$$

The above ω_p can be also regarded as the eigen angular frequencies of the *water* cavity comprised between the bubble's surface (with a $(\pi + 2\varphi)$ -dephasing of the pressure wave at $r = R$, as explained in (3.9a)) and the

water's outer boundary (with a π -dephasing of the pressure wave at $r = L$). Not surprisingly, the ω_p are c_w -dependent. Allowing for (2.7), condition (2.20) can also read (compare with (2.8))

$$1 + \frac{\rho_w}{\rho_a} (x_a \cot x_a - 1) = -x_w \cot(q_w(L - R)). \quad (2.21)$$

Since $\varphi(\omega_p) \in [0, \pi]$, the solution ω_p of equation (2.20) lies within the interval $[(p-1)\frac{c_w\pi}{L-R}, p\frac{c_w\pi}{L-R}]$, and thus, for $L \rightarrow \infty$, the continuum is recovered. Moreover, the pressure wave in water associated with eigenmode p reads (see (2.4b)) $\bar{p}_w(r) = \frac{W}{r} \sin[\frac{\omega_p}{c_w}(r-R) + \varphi(\omega_p)]$. It consequently displays an antinode (pressure loop) at $r = R$ provided that $\varphi(\omega_p) = \frac{\pi}{2}$, in which case $\cot(\frac{\omega_p}{c_w}(L-R)) = 0$ and equation (2.21) reduces to (2.8). If the maximum distance between two consecutive eigen angular frequencies is small compared to the linewidth of function $\sin \varphi$ displayed in equations (2.11), namely if one has

$$\frac{c_w\pi}{L-R} \ll \frac{c_a}{R} \frac{\rho_a c_a}{\rho_w c_w} \rightsquigarrow \frac{\pi R}{L-R} \ll \frac{\chi_w}{\chi_a}, \quad (2.22)$$

then we are sure to find all the resonances foreshadowed by equation (2.8) among the set of eigen angular frequencies satisfying (2.20). It is noteworthy that the above condition (2.22) is extremely severe. With $\chi_w \simeq 10^{-10} \text{ Pa}^{-1}$ and, at atmospheric pressure, $\chi_a \simeq 7 \times 10^{-6} \text{ Pa}^{-1}$, condition (2.22) is satisfied for $\frac{L}{R} \gg 2 \times 10^5$: an aquarium with the depth of an ocean would be necessary to be sure not to miss any resonance of a 1 mm radius bubble!

Another interesting limit-case is that of "small" aquariums. Let us consider equation (2.20) again and look for the fundamental mode $p = 1$. We have to determine ω_1 satisfying the condition $\frac{\omega_1}{c_w}(L-R) = \pi - \varphi(\omega_1)$. If $L-R$ is small enough, one finds the curious following result. Suppose that $\frac{\omega_1}{c_w}(L-R) \ll \frac{\pi}{2}$, *i.e.* that the wavelength (in water) is much larger than $4(L-R)$. Then $\varphi(\omega_1) \simeq \pi$ (and consequently ω_1 is larger than the Minnaert resonance angular frequency displayed in (2.9b), as can be checked from the zoom in figure 2). Equation (2.21) can also be simplified in

$$1 + \frac{\rho_w}{\rho_a} (x_a \cot x_a - 1) \simeq -\frac{R}{L-R} \rightsquigarrow \tan x_a \simeq \frac{x_a}{1 - \frac{\rho_a}{\rho_w} \frac{L}{L-R}}, \quad (2.23)$$

which is exactly equation (2.8) with $\frac{\rho_a}{\rho_w} \frac{L}{L-R}$ substituted for $\frac{\rho_a}{\rho_w}$. Provided that $\frac{\rho_a}{\rho_w} \frac{L}{L-R}$ remains much smaller than unity, the expansion in equation (2.9a) still holds and we calculate a L -dependent Minnaert angular frequency

$$\omega_1(L) \simeq \omega_M(L) = \sqrt{\frac{L}{L-R}} \omega_M \quad (> \omega_M). \quad (2.24)$$

The above result will be re-interpreted in section V (see equations (5.6b), (5.7)). Observe that, although it has

been derived here assuming $\frac{\omega_1(L-R)}{c_w} \ll \frac{\pi}{2}$, one has $\omega_M(L) \rightarrow \omega_M$ with $\frac{L}{R} \rightarrow \infty$. Nevertheless, it should be kept in mind that $\omega_1(L)$ is *but* the angular frequency of the fundamental *mode* of the {bubble-finite water} system. In the very sense precised above in this paper, it is *not* a *resonance*: there is no loop of the pressure wave $\bar{p}_w(r)$ at $r = R$, but rather a quasi-node (since $\varphi(\omega_1) \simeq \pi$); consequently the amplitude $r\bar{p}_w(r)$ of the pressure wave increases like $\sin(\frac{\omega_1}{c_w}(r-R)) \simeq \frac{\omega_1}{c_w}(r-R)$ when one departs from the bubble. Let us recall again that this situation occurs when $\frac{\omega_1(L-R)}{c_w} \simeq \frac{\omega_M}{c_w} \sqrt{L(L-R)} < \frac{\pi}{2}$, *i.e.* (see (2.9c)) for $\frac{L}{R} < \frac{\pi}{2} \sqrt{\frac{\chi_a}{3\chi_w}} \simeq 240$ at atmospheric pressure. In this connection, it should be emphasized that experimental setups designed to illustrate the Minnaert *resonance* will, if miniaturized too much, rather exhibit the {aquarium + bubble} fundamental *mode*, with the misleading quasi-coincidence displayed in (2.24) of their both angular frequencies. In this respect, it is fascinating how a small amount of air can lower the fundamental frequency of a water cavity. As an illustration, let us take $L = 5 \text{ cm}$. Without any bubble, one would have $\omega_1 = \frac{\pi c_w}{L} \simeq 10^5 \text{ s}^{-1}$. With a $R = 1 \text{ mm}$ air bubble at the center of the cavity, one has $\omega_1 \simeq \omega_M \simeq 2 \times 10^4 \text{ s}^{-1}$, *i.e.* a lowering of more than two octaves. Although the geometry and the boundary conditions are not the same, a cylindrical mug with height and diameter equal to 10 cm will *roughly*, when filled up with water, behave like our 5 cm-radius spherical cavity. And a single small 1mm-radius air bubble at its center will produce a spectacular non-standard "hot chocolate effect". Let us recall that the "standard" hot chocolate effect, as described by Crawford³, relies upon the bubble-induced lowering of the effective Wood velocity⁷ in bubbly liquids. A remarkable property of this Wood velocity is that it depends only of the air volume fraction (not of the bubbles' sizes), provided that the sound frequency is much lower than the individual bubbles' Minnaert frequency, which is just *not* the case here.

We shall henceforth assume that the $\frac{L}{R}$ ratio is large enough for the infinite water approximation to be valid. In the next section III, we integrate out the water degrees of freedom and consider the same problem as in this section II, but from the mere bubble's point of view.

III. HOW THE BUBBLE FEELS THE WATER

Since there is no air-water action at a distance, the only outer parameter that rules the dynamics of the air bubble is the water extrapressure $p_w(R, t)$ upon its surface. As a matter of fact, using the d'Alembert's equation, which reads in the present case of spherical symmetry

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial r^2} \right) (r p(r, t)) = 0, \quad (3.1)$$

and the boundary conditions (no singularity of $p_a(0, t)$ and continuity $p_a(R, t) = p_w(R, t)$), one can in principle derive the air's motion, provided of course that one knows $p_w(R, t)$. It is our aim in the present section III to determine $p_w(R, t)$ (subsection III A) and to use the result to illustrate the resonance phenomenon (subsection III B).

A. Determining $p_w(R, t)$

Let us denote the common radial displacement of air and water at the bubble's surface by $\xi(t)$:

$$u_a(R, t) = u_w(R, t) = \xi(t). \quad (3.2)$$

In the water, the most general solution of the d'Alembert's equation ((3.1) with $c = c_w$ and $p = p_w$) is the sum of an incoming and an outgoing waves, which can read

$$p_w(r, t) = p_{\text{in}}(r, t) + p_{\text{out}}(r, t), \quad (3.3a)$$

where

$$\begin{aligned} p_{\text{in}}(r, t) &= \frac{1}{r} g\left(t + \frac{r-R}{c_w}\right), \\ p_{\text{out}}(r, t) &= \frac{1}{r} f\left(t - \frac{r-R}{c_w}\right), \end{aligned} \quad (3.3b)$$

with g and f any functions of time. Using Newton's law (2.1b) at $r = R$, one gets

$$\begin{aligned} \rho_w \ddot{\xi} &= -(\text{grad } p_w)_{r=R} \\ &= \frac{1}{R^2} \left\{ \left(1 - \frac{R}{c_w} \frac{\partial}{\partial t}\right) g(t) + \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right) f(t) \right\}. \end{aligned} \quad (3.4)$$

(i). To begin with, let us suppose that there is no pressure wave coming from the infinite. Function g should then be taken equal to zero. Solving equation (3.4) for f is straightforward and one gets symbolically^{8,12}

$$\begin{aligned} f(t) &= \rho_w R^2 \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right)^{(-1)} \ddot{\xi}(t) \\ &\rightsquigarrow p_w(r, t) = p_{\text{rad}}(r, t), \end{aligned} \quad (3.5a)$$

where

$$p_{\text{rad}}(r, t) = \frac{\rho_w R^2}{r} \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right)^{(-1)} \ddot{\xi}\left(t - \frac{r-R}{c_w}\right) \quad (3.5b)$$

is the outgoing spherical pressure wave radiated by the pulsating bubble. Not surprisingly, p_{rad} is determined by $\ddot{\xi}$ and its time-derivatives.

(ii). Let us now consider another situation. We assume that there is a pressure wave p_{in} coming from the infinite (function g is thus nonzero in equations (3.3)-(3.4)), but that the bubble's surface is motionless

(function $\xi(t)$ is thus zero $\forall t$). Solving equation (3.4) for f gives

$$\begin{aligned} f(t) &= - \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right)^{(-1)} \left(1 - \frac{R}{c_w} \frac{\partial}{\partial t}\right) g(t) \\ &\rightsquigarrow p_w(r, t) = p_{\text{refl}}(r, t), \end{aligned} \quad (3.6a)$$

where

$$\begin{aligned} p_{\text{refl}}(r, t) &= \\ &= - \frac{R}{r} \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right)^{(-1)} \left(1 - \frac{R}{c_w} \frac{\partial}{\partial t}\right) p_{\text{in}}\left(R, t - \frac{r-R}{c_w}\right) \end{aligned} \quad (3.6b)$$

is the reflected wave on a rigid spherical obstacle with radius R .

(iii). Combining the above situations (i) and (ii), we finally consider the general case of a pressure wave coming from the infinite onto a pulsating bubble. Solving equation (3.4) for f , we derive the "superposition theorem"

$$p_{\text{out}}(r, t) = p_{\text{rad}}(r, t) + p_{\text{refl}}(r, t), \quad (3.7)$$

which is by the way a quite intuitive result.

To conclude this subsection's study, we can now propose a compact expression of the full pressure at $r = R$. Using (3.5b) and (3.6b) to determine $p_{\text{out}}(r, t)$, we get, all calculations carried out,

$$\begin{aligned} p_w(R, t) &= \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t}\right)^{(-1)} \left\{ \rho_w R \ddot{\xi}(t) \right. \\ &\quad \left. + \frac{2R}{c_w} \frac{\partial p_{\text{in}}(R, t)}{\partial t} \right\}. \end{aligned} \quad (3.8)$$

We shall use the above (exact) result in the next section subsection III B and in section IV.

B. Back to the resonances

Let us now consider again the radial eigenmodes of the {air-infinite water} system. Solution (2.4b) can be regarded as the superposition of two monochromatic spherical waves: an incoming wave with complex amplitude $\bar{p}_{\text{in}}(r) = -\frac{W}{2ir} e^{-i(q_w(r-R)+\varphi)}$ and an outgoing wave $\bar{p}_{\text{out}}(r) = \frac{W}{2ir} e^{i(q_w(r-R)+\varphi)}$. At $r = R$, the outgoing wave is thus dephased by an angle $\pi + 2\varphi$ with respect to the incoming wave:

$$\bar{p}_{\text{out}}(R) = -e^{2i\varphi} \bar{p}_{\text{in}}(R). \quad (3.9a)$$

From (3.6b) and the definition (2.5b) of x_w , one also derives the dephasing between the reflected wave and the incoming wave:

$$\bar{p}_{\text{refl}}(R) = - \frac{1 + ix_w}{1 - ix_w} \bar{p}_{\text{in}}(R) = -e^{2i\theta} \bar{p}_{\text{in}}(R), \quad (3.9b)$$

with $\tan \theta = x_w$. By the way, one can check from expression (2.7) that phase angles φ and θ do coincide at the limit $\chi_a \rightarrow 0$ (then $x_a \rightarrow 0$ and $\tan \varphi \rightarrow x_w$), as expected for the reflection onto a rigid sphere.

Using (3.5b) at $r = R$ and (2.5b) again, one derives the expression of the radiated wave:

$$\begin{aligned} \bar{p}_{\text{rad}}(R) &= \frac{W \sin \varphi - x_w \cos \varphi}{R(1 - ix_w)} \\ &= (e^{2i\theta} - e^{2i\varphi}) \bar{p}_{\text{in}}(R). \end{aligned} \quad (3.9c)$$

Adding expressions (3.9b) and (3.9c) gives (3.9a), as expected from the “superposition theorem” (3.7). We have displayed the complex amplitudes $\bar{p}_{\text{in}}(R)$, $\bar{p}_{\text{out}}(R)$, $\bar{p}_{\text{refl}}(R)$ and $\bar{p}_{\text{rad}}(R)$ in figure 3. At a resonance, angle φ is equal to $\frac{\pi}{2}$, and then the incoming and outgoing pressure waves are in phase at $r = R$, so that the bubble’s surface corresponds to an antinode of the total pressure wave $p_w(r)$.

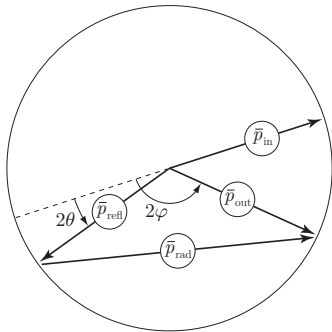


FIG. 3: Complex amplitudes of the incoming (\bar{p}_{in}) and outgoing (\bar{p}_{out} , \bar{p}_{refl} and \bar{p}_{rad}) waves at $r = R$, illustrating the “superposition theorem” (3.7) in the monochromatic regime. Observe that the incoming wave and the outgoing wave are in phase at $r = R$ if angle φ is equal to $\frac{\pi}{2}$. In this case, there is an antinode of the total pressure wave $\bar{p}_w(r)$ at the bubble’s surface, which is referred to as a “resonance” in the present paper.

One can add a further remark about the resonances. Let us suppose that, instead of a monochromatic wave, a wave packet is coming from $r = \infty$. This wave packet can read

$$p_{\text{in}}(r, t) = -\frac{1}{2ir} \int d\omega \mathcal{W}(\omega) e^{-i[q_w(\omega)(r-R) + \varphi(\omega)]} e^{-i\omega t}, \quad (3.10a)$$

while the corresponding outgoing wave packet is

$$p_{\text{out}}(r, t) = \frac{1}{2ir} \int d\omega \mathcal{W}(\omega) e^{i[q_w(\omega)(r-R) + \varphi(\omega)]} e^{-i\omega t}. \quad (3.10b)$$

Let us suppose, for the sake of simplicity, that the spectral density $\mathcal{W}(\omega)$ is real. Using the stationary phase condition to determine the position of the wave packet’s

center, we get

$$\begin{aligned} -\frac{dq_w}{d\omega}(r - R) - \frac{d\varphi}{d\omega} - t &= 0 \\ \rightsquigarrow r - R &= -c_w \left(t + \frac{d\varphi}{d\omega} \right) \end{aligned} \quad (3.11a)$$

for the incoming packet, which consequently reaches the bubble’s surface at time $-\frac{d\varphi}{d\omega}$, and

$$\begin{aligned} \frac{dq_w}{d\omega}(r - R) + \frac{d\varphi}{d\omega} - t &= 0 \\ \rightsquigarrow r - R &= c_w \left(t - \frac{d\varphi}{d\omega} \right) \end{aligned} \quad (3.11b)$$

for the outgoing packet, that leaves the bubble’s surface at time $\frac{d\varphi}{d\omega}$. Thus, the wave packet “spends” a time of order $\Delta t = 2\frac{d\varphi}{d\omega} (> 0)$ in the bubble. This time is negligible out of resonance where the curve $\varphi(\omega)$ ($\simeq 0$ or π) is flat. It is substantial at resonance where $\varphi(\omega)$ jumps from 0 to π within an interval $\Delta\omega \simeq \frac{\chi_w c_w}{\chi_a R}$ (see (2.11)): thus we have $\Delta t \simeq \frac{2\pi}{\Delta\omega} \simeq \frac{\chi_a 2R}{\chi_w c_w}$, much longer than $\frac{2R}{c_w}$ (the corresponding travel time in absence of bubble), which characterizes the resonance as well.

IV. THE MINNAERT BUBBLE

Our aim in this section is to implement the derivation scheduled in introduction and show that, at the isentropic approximation, the Minnaert bubble can be regarded as a one-degree of freedom harmonic oscillator, with a mass, a stiffness, a radiative viscous damping rate and an external forcing term. We begin with the calculations of the total kinetic and (elastic) potential energies of the air bubble. We then derive a motion equation for the “naked” bubble (4.6a), exact but hardly utilizable. This equation is turned, through an unexpectedly exact transformation, in a motion equation for the “dressed” bubble (4.7), readily utilizable.

As suggested by approximated formulas (2.12a,b), one can describe the dynamics of the Minnaert bubble in the following way. Let $R + \xi(t)$ be the bubble’s radius at time t . The radial displacement of air simply reads

$$\vec{u}_a(r, t) = \xi(t) \vec{\lambda}_a(\vec{r}), \quad \text{where } \vec{\lambda}_a(\vec{r}) = \frac{\vec{r}}{R} \quad (r \leq R). \quad (4.1)$$

The above expression of the air displacement is referred to as the “breathing approximation” throughout this paper. As a consequence of (4.1), the total kinetic energy of the air inside the bubble is

$$E_k = \frac{1}{2} \int_{\text{bubble}} \rho_a d^3r \left(\frac{\partial \vec{u}_a}{\partial t} \right)^2 = \frac{1}{2} m \dot{\xi}^2, \quad (4.2a)$$

with

$$m = \int_{\text{bubble}} \rho_a d^3r \lambda_a^2 = \rho_a \frac{4}{5} \pi R^3. \quad (4.2b)$$

According to (2.1a) and (4.1), the (homogeneous) extra-pressure inside the bubble is

$$p_a = -\frac{1}{\chi_a} \operatorname{div} \vec{u}_a = -\frac{3\xi}{\chi_a R}. \quad (4.3)$$

The classical potential energy of the air inside the bubble thus reads

$$\begin{aligned} E_p &= -\int_0^\xi 4\pi R^2 d\xi' p_a(\xi') = \frac{1}{2} \chi_a p_a^2 \times \frac{4}{3} \pi R^3 \\ &= \frac{1}{2} K \xi^2, \end{aligned} \quad (4.4a)$$

with

$$K = \int_{\text{bubble}} d^3 r \frac{(\operatorname{div} \vec{\lambda}_a)^2}{\chi_a} = \frac{12\pi R}{\chi_a}. \quad (4.4b)$$

We have introduced the mass m and the stiffness K . We should now establish the motion equation ruling the dynamical variable ξ . In order to do it, let us come back to Newton's law for the air; multiplying both hand sides of (2.1b) by $\vec{\lambda}_a$, and integrating over the bubble's volume, we get

$$\begin{aligned} \int_{\text{bubble}} \rho_a d^3 r \vec{\lambda}_a \cdot \frac{\partial^2 \vec{u}_a}{\partial t^2} &= - \int_{\text{bubble}} d^3 r \vec{\lambda}_a \cdot \overrightarrow{\operatorname{grad}} p \\ &= \int_{\text{bubble}} d^3 r (p \operatorname{div} \vec{\lambda}_a - \operatorname{div}(p \vec{\lambda}_a)). \end{aligned} \quad (4.5a)$$

Then, using (2.1a), (4.1) and the Ostrogradsky's theorem, and allowing for definitions (4.2b) and (4.4b), we obtain

$$m \ddot{\xi} + K \xi = - \oint_{\text{bubble}} p d^2 r = -4\pi R^2 p(r=R, t). \quad (4.5b)$$

The above motion equation is that of a harmonic oscillator with mass m and stiffness K , submitted to the external (pressure) force $4\pi R^2 p(r=R, t)$. Observe that m involves the mere air's inertia (see (4.2b)): equation (4.5b) rules the time-evolution of what could be called the "naked" bubble. Using expression (3.8) of $p_w(R, t)$ ($= p(R, t)$), we get

$$\begin{aligned} m \ddot{\xi} + K \xi &= -M \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t} \right)^{(-1)} \left\{ \ddot{\xi}(t) \right. \\ &\quad \left. + \frac{2}{\rho_w c_w} \frac{\partial p_{\text{in}}(R, t)}{\partial t} \right\}, \end{aligned} \quad (4.6a)$$

with

$$M = \rho_w 4\pi R^3. \quad (4.6b)$$

It is noteworthy that the $-M \left(1 + \frac{R}{c_w} \frac{\partial}{\partial t} \right)^{(-1)} \ddot{\xi}$ term in the right-hand side of equation (4.6a) much looks like an Abraham-Lorentz force, as that undergone by accelerated charged particles in electrodynamics⁹.

Although exact, equation (4.6a) is nevertheless of poor help to describe the dynamics of the bubble because, even for a free motion, *i.e.* if no external incoming pressure wave is applied ($p_{\text{in}}(R, t) = 0$), its right-hand side is nonzero, and even large compared to $m \ddot{\xi}$ as we shall see. In the aim of finding a motion equation the right-hand side of which effectively vanishes when $p_{\text{in}}(R, t) = 0$, we can transform equation (4.6a) in the following way. First we apply $\left(1 + \frac{R}{c_w} \frac{\partial}{\partial t} \right)$ to both hand sides; then we remove the remaining mass-renormalization term $-M \ddot{\xi}$ and incorporate it with the left-hand side. We get *in fine*

$$\begin{aligned} (m+M) \ddot{\xi} + \frac{RK}{c_w} \dot{\xi} + K \xi + \frac{R}{c_w} m \ddot{\xi} \\ = - \frac{2M}{\rho_w c_w} \frac{\partial p_{\text{in}}(R, t)}{\partial t}. \end{aligned} \quad (4.7)$$

The above equation (4.7) is, as far as we know, unpublished. It is that of a harmonic oscillator with mass $m+M$, stiffness K , and a twofold "friction" mechanism associated with the odd-order time-derivatives of ξ (namely $\dot{\xi}$ and $\ddot{\xi}$). This oscillator, that we shall henceforth call the "dressed" bubble, undergoes an external force if – and *only if*, contrary to the naked bubble – a wave pressure comes from the infinite (see the right-hand side of (4.7)). Observe by the way that this external force term is proportional to the time-derivative of $p_{\text{in}}(R, t)$. As can be checked by comparing expressions (4.2b) and (4.6b), the "dressed" mass $m+M$ of the bubble is roughly thousandfold its "naked" mass m : in the acoustic bubble case, the renormalization of the mass due to water's inertia should by no means be regarded as a small effect or treated perturbatively as could be done in the electrodynamics domain. On the other hand, the "friction" terms are in fact *radiative* terms: they take the emission of spherical acoustic waves by the dressed bubble into account, and they naturally vanish in the incompressible water limit (*i.e.* for $c_w \rightarrow \infty$), in which case equation (4.7) reduces to

$$(m+M) \ddot{\xi} + K \xi = 0. \quad (4.8)$$

According to the above equation (4.8), the free angular frequency reads

$$\begin{aligned} \omega_0 &= \sqrt{\frac{K}{m+M}} \\ &\simeq \sqrt{\frac{K}{M}} \left(1 - \frac{m}{M} \right) = \omega_M \left(1 - \frac{1}{10} \frac{\rho_a}{\rho_w} \right) \simeq \omega_M, \end{aligned} \quad (4.9)$$

which occurs to coincide with expression (2.9b) of the singular resonance's angular frequency of the {air-water} system (note that this coincidence does not hold for higher orders in the ratio ρ_a/ρ_w).

Taking again water compressibility into account, it is easy to determine, at this angular frequency ω_0 , the ratio of the radiative terms in the left-hand side of (4.7). Since

$\ddot{\xi} \simeq -\omega_0^2 \xi$, and using (4.9), we have

$$\left| \frac{\frac{R}{c_w} m \ddot{\xi}}{\frac{RK}{c_w} \dot{\xi}} \right| \simeq \frac{m \omega_0^2}{K} = \frac{m}{m+M} \ll 1. \quad (4.10)$$

Dividing equation (4.7) by $m+M$ and neglecting the $\ddot{\xi}$ term, we get

$$\ddot{\xi} + \Gamma \dot{\xi} + \omega_0^2 \xi = -\frac{2}{\rho_w c_w} \frac{M}{M+m} \frac{\partial p_{\text{in}}(R, t)}{\partial t}, \quad (4.11a)$$

with ω_0 given by (4.9) and

$$\Gamma = \frac{RK}{(m+M)c_w} = \frac{R\omega_0^2}{c_w} \ll \omega_0. \quad (4.11b)$$

Equation (4.11a) describes the motion of a (forced) harmonic oscillator with an effective *weak* radiative damping rate Γ . Due to the damping, the resonance angular frequency is still (roughly) ω_0 , but its linewidth is finite. In this respect, it is very instructive to calculate the relative linewidth $\frac{\Gamma}{\omega_0}$ of this resonance. Owing to (2.9c) and (4.11b), we get

$$\frac{\Gamma}{\omega_0} = \frac{R\omega_0}{c_w} \simeq \frac{R\omega_M}{c_w} = \sqrt{\frac{3\chi_w}{\chi_a}} \sim 6.5 \cdot 10^{-3} \quad (4.12a)$$

at $P_0 = 1$ bar. Now, if we consider the relative linewidth of the singular peak in figure 2, we find, using (2.9c) again and (2.11b),

$$\frac{\delta}{x_a} = 3 \frac{\rho_a}{\rho_w} \frac{c_a}{c_w} \frac{c_a}{R\omega_0} \simeq \frac{3}{\chi_a \rho_w c_w R \omega_M} = \sqrt{\frac{3\chi_w}{\chi_a}}. \quad (4.12b)$$

Results (4.9) and (4.12a,b) fairly show that the Minnaert model sketched in this section IV is relevant to describe the pulsating bubble physics in the “low frequency” domain, *i.e.* the frequency range where condition $x_a \ll 1$ is fulfilled. Furthermore it is readily checked that, within this low frequency domain, expressions (2.7) of $\sin \varphi$ and $\cos \varphi$ simplify in

$$\begin{aligned} \sin \varphi &\simeq \frac{\Gamma \omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2}}, \\ \cos \varphi &\simeq \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2}}, \end{aligned} \quad (4.13)$$

which are well known from the forced harmonic oscillator basic theory: in a monochromatic regime with angular frequency ω , angle φ is thus the dephasing between the complex amplitudes of ξ and of the forcing term in the right-hand side of (4.11a).

We would add a further remark. In experimental situations, it is unusual that an air bubble is excited by means of a spherical (and moreover concentric) acoustic wave. In practise, a monochromatic plane wave of the type $\bar{p}(z) = p_0 e^{iq_w z}$ is more likely to be used. Now,

this plane wave can be expanded on the spherical waves basis¹⁰:

$$\begin{aligned} p_0 e^{iq_w z} &= p_0 e^{iq_w r \cos \theta} \\ &= p_0 \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(q_w r) P_\ell(\cos \theta), \end{aligned} \quad (4.14)$$

where the j_ℓ are the spherical Bessel’s functions and the P_ℓ are the Legendre’s polynomials. In fact, in the “low frequency” domain (as defined above), we have $q_w R \ll 1$. Now, only $j_0(\rho) = \frac{\sin \rho}{\rho}$ is non negligible for $\rho \ll 1$. Consequently, only the $\ell = 0$ wave component in expansion (4.14) will significantly affect the bubble. Said in other words: as far as the wavelength is large compared with its radius R , the bubble makes no difference between a plane wave and a spherical wave. As a conclusion, the motion equation of an air bubble located at the origin of coordinates and undergoing the monochromatic pressure wave $p_0 e^{iq_w z}$ will read (see (4.11a))

$$(\omega_0^2 - \omega^2 - i\omega\Gamma) \bar{\xi} \simeq -\frac{p_0}{\rho_w R} e^{-iq_w R}. \quad (4.15a)$$

Expanding the above $e^{-iq_w R}$ term up to first order in $q_w R$, we finally find, using (4.11b),

$$\bar{\xi} \simeq -\frac{p_0}{\rho_w R} \frac{1}{\omega_0^2 - \omega^2 - i\frac{R\omega^3}{c_w}}, \quad (4.15b)$$

with the ω^3 radiative damping term usually found in literature^{2,5}. Observe that the discrepancy between the radiative damping rates $\Gamma = \frac{R\omega_0^2}{c_w}$ and $\Gamma(\omega) = \frac{R\omega^2}{c_w}$ is in fact but apparent: the viscous constant rate Γ is found whenever the incident pressure wave is appreciated at $r = R$ (see (4.11a) or (4.15a)); the ω -dependent rate $\Gamma(\omega)$ is found whenever this incident pressure is appreciated at $r = 0$ (see (4.15b) and, more generally, the scattering functions used in the literature).

How is the above equation modified when dissipation is taken into account? As already mentioned in subsection IIB about the general motion, an exact answer to this question would imply an *exact solution* of equations (2.18) (under the circumstances, in the $x_a \ll 1$ limit and in monochromatic regime). This delicate enterprise, important though it may be in other respects, is outside the scope of the present study. To manage it, Prosperetti proposes a detailed discussion,⁵ using the following simplifications.

- (i). The water is assumed to be incompressible as concerns viscosity effects: the Navier-Stokes equation (2.18b) thus reduces to (2.1b) in the bulk; at the interface ($r = R$), the viscous stress results in an additional extrapressure $p^{\text{vis}} = 4\eta_w \frac{\dot{\xi}}{R}$ which *in fine* finds expression in a viscous damping rate $\Gamma^{\text{vis}} = \frac{4\eta_w}{\rho_w R^2}$.
- (ii). Due to the water’s high heat capacity, the extratemperature $\theta(r, t)$ is neglected for $r \geq R$: the

water is regarded as a heat reservoir with temperature T_0 .

(iii). Any viscosity of the gas is neglected : equation (2.1b) is available for $r < R$.

With the above approximations, it can be shown that equation (2.18a) can be heuristically simplified in $\bar{p}(r) = -\frac{1}{\bar{\chi}_a(\omega)} \operatorname{div} \vec{u}(r)$, with $\bar{\chi}_a(\omega) = \chi_a \frac{\gamma}{\kappa(\omega)}$ ($\kappa(\omega)$ is the so-called polytropic exponent). Accordingly, the resonance frequency ω_0 is now frequency-dependent and reads $\omega_0^2(\omega) = \frac{\kappa(\omega)}{\gamma} \omega_M^2$: the Minnaert bubble is, strictly speaking, *no longer a harmonic oscillator* in the sense that its stiffness $K(\omega) = \frac{12\pi R}{\bar{\chi}_a(\omega)}$ is frequency-dependent (observe that its mass $M = 4\pi\rho_w R^3$ is unaffected by heat transfers in the gas). Moreover, thermal effects yield an additional frequency-dependent thermal damping, so that the left-hand side of equation (4.15a) should read *in fine* $\omega_0^2(\omega) - \omega^2 - i\omega\Gamma^{\text{eff}}(\omega)$, with $\Gamma^{\text{eff}}(\omega) = \Gamma + \Gamma^{\text{vis}} + \Gamma^{\text{th}}(\omega)$. For a bubble with radius $R = 1$ mm, one has $\Gamma = 260$ s⁻¹, $\Gamma^{\text{vis}} = 7.2 \cdot 10^{-2}$ s⁻¹, $\Gamma^{\text{th}}(\omega_M) = 540$ s⁻¹ : viscosity is fairly negligible, and the damping is dominated by the thermal process. Nevertheless, in the next section, we shall ignore dissipation again for the sake of simplicity.

V. THE ACOUSTIC RADIATIVE PICTURE APPROACH

As discussed in the previous section, 99.9% of the dressed bubble's mass is due to water's inertia. The dressed bubble is consequently, at least partially, "made of water". It is thus tempting to raise the naive question: what part of the water is devoted to the dressing of the bubble, and what part to the propagation of sound? Our aim in this last section is to provide an answer to this puzzling question. We propose an alternative description of the Minnaert bubble, including a possible departure from the breathing approximation for the air motion, as well as a physical interpretation of the mass-renormalization term. To begin with, we introduce an *ad hoc* splitting of the displacement field $\vec{u}(\vec{r}, t)$.

(i). If the breathing approximation was exact, the air displacement would read like (4.1). In order to take a possible departure from this behaviour into account, let us introduce a further degree of freedom for the air and set

$$\vec{u}_a(\vec{r}, t) = \xi(t)\vec{\lambda}_a(\vec{r}) + \vec{u}_{ar}(\vec{r}, t) \quad (r \leq R) \quad (5.1)$$

(\vec{u}_{ar} thus vanishes at the breathing approximation).

(ii). If the water was incompressible, its displacement would read

$$\vec{u}_{wi}(\vec{r}, t) = \xi(t)\lambda_w(\vec{r}), \quad \text{where } \vec{\lambda}_w(\vec{r}) = \frac{R^2}{r^2} \vec{e}_r \quad (r \geq R). \quad (5.2a)$$

In order to take water compressibility into account, let us write the water displacement in the form

$$\vec{u}_w(\vec{r}, t) = \vec{u}_{wi}(\vec{r}, t) + \vec{u}_{wr}(\vec{r}, t) \quad (r \geq R) \quad (5.2b)$$

(\vec{u}_{wr} thus vanishes in the incompressible water limit).

The displacement field \vec{u}_r (\vec{u}_{ar} for $r \leq R$, \vec{u}_{wr} for $r \geq R$) will be referred to as the "radiative" displacement field. It is noteworthy that the above splitting of the *water* displacement in an (instantaneous) "elastostatic" field $\xi(t)\vec{\lambda}_w(\vec{r})$ and a "radiative" field $\vec{u}_{wr}(\vec{r}, t)$ is the acoustic transposition of the well known splitting of the total electric field in an electrostatic part $\vec{E}_e = -\nabla V$ and a radiative part $\vec{E}_r = -\frac{\partial \vec{A}}{\partial t}$ in the electromagnetism domain. Let us recall that the electrostatic field is instantaneous in the Coulomb gauge ($\operatorname{div} \vec{A} = 0$), also called "radiation gauge". Hence our coining the term "radiative". Of course, our splitting of the *air* displacement meets no electromagnetic equivalent, since the interior of a charged particle does not make sense.

Now, let us consider the lagrangian describing the radial motion of the {air-water} system:

$$L = \int_0^\infty 4\pi r^2 dr \left[\frac{1}{2} \rho \left(\frac{\partial \vec{u}}{\partial t} \right)^2 - \frac{1}{2\chi} (\operatorname{div} \vec{u})^2 \right], \quad (5.3)$$

with $(\rho, \chi) = \begin{cases} (\rho_a, \chi_a) & \text{for } r < R, \\ (\rho_w, \chi_w) & \text{for } r > R. \end{cases}$

With the splittings (5.1) and (5.2b), the above lagrangian (5.3) also reads exactly

$$L = L_b + L_r + L_{br}, \quad (5.4)$$

with

$$L_b = \frac{1}{2}(m + M)\dot{\xi}^2 - \frac{1}{2}K\xi^2, \quad (5.5a)$$

$$L_r = \int_0^\infty 4\pi r^2 dr \left\{ \frac{1}{2} \rho \left(\frac{\partial \vec{u}_r}{\partial t} \right)^2 - \frac{1}{2\chi} (\operatorname{div} \vec{u}_r)^2 \right\}, \quad (5.5b)$$

$$L_{br} = \dot{\xi} \int_0^\infty 4\pi r^2 dr \rho \vec{\lambda}_w \cdot \frac{\partial \vec{u}_r}{\partial t}. \quad (5.5c)$$

Let us comment these three terms.

– Lagrangian L_b describes the free motion of the Minnaert bubble as a harmonic oscillator with mass $m + M$ and stiffness K . We should nevertheless emphasize the following point: if M is fairly well the same mass $\rho_w 4\pi R^3$ introduced in equations (4.6), it no longer corresponds here to an Abraham-Lorentz coefficient as in section IV, but to the value of the integral

$$M = \int_R^\infty 4\pi r^2 dr \rho_w \vec{\lambda}_w^2(r). \quad (5.6a)$$

In other words, $\frac{1}{2}M\dot{\xi}^2$ is the total kinetic energy of the water surrounding the bubble in the incompressible water limit. Hence the very intuitive physical meaning of the mass renormalization put forward by Minnaert himself¹. In this respect, it should be remarked that, in the case of a finite extent of water considered in subsection II A, integral (5.6a) should be substituted by

$$\begin{aligned} M(L) &= \int_R^L 4\pi r^2 dr \rho_w \bar{\lambda}_w^2(r) \\ &= \left(1 - \frac{R}{L}\right)M \quad (< M), \end{aligned} \quad (5.6b)$$

which gives for the Minnaert harmonic oscillator the L -dependent angular frequency

$$\omega_M(L) = \sqrt{\frac{K}{M(L)}} = \sqrt{\frac{L}{L-R}}\omega_M \quad (> \omega_M) \quad (5.7)$$

to be compared with expression (2.24).

- Lagrangian L_r describes the free radial motions of air (interval $[0, R]$) or water (interval $[R, \infty[$), these motions being uncoupled *de facto* from each other by the rigid boundary condition at $r = R$: $u_{ar}(R, t) = u_{wr}(R, t) = 0, \forall t$.
- Lagrangian L_{br} couples the Minnaert bubble and the radiative field. Observe that, due to the time-derivatives $\dot{\xi}$ and $\frac{\partial \vec{u}_r}{\partial t}$, this coupling is of the inertial type.

From lagrangian L , one can derive the equations of motion, as well for the Minnaert bubble as for the radiative sound field.

- For the Minnaert bubble, we have

$$(m + M)\ddot{\xi} + K\xi = - \int_0^\infty 4\pi r^2 dr \rho \vec{\lambda} \cdot \frac{\partial^2 \vec{u}_r}{\partial t^2}. \quad (5.8)$$

The fact that the right-hand side of the above equation is an integral is easily understandable: according to splittings (5.1) and (5.2b), the fluid motion is “shared out” among the Minnaert motion and the radiative motion; consequently, whereas the air bubble is strictly confined to the $0 < r < R$ domain, the Minnaert bubble stretches to infinity.

- For the radiative sound field, we have

$$\frac{\partial^2 \vec{u}_r}{\partial t^2} - c^2 \overrightarrow{\text{grad}}(\text{div } \vec{u}_r) = - \ddot{\xi} \vec{\lambda}. \quad (5.9)$$

As expected from the electromagnetic analogy with the radiation of a moving charge, the source term of the excitation of the radiative acoustic field \vec{u}_r is proportional to the second time-derivative of dynamical variable ξ . Observe too that, in perfect symmetry with the above-discussed excitation of

the bubble by the sound field, this source-term ($-\ddot{\xi}\vec{\lambda}$) is not localized at the air-water boundary but spreads inside the air and dies out in the water like $\bar{\lambda}_w(\vec{r})$, *i.e.* like $1/r^2$.

We would finish this discussion with a further remark. Let us come back to expression (3.5b) of the water extrapressure $p_{\text{rad}}(r, t)$. If water was incompressible, implying $c_w = \infty$, equation (3.5b) would simply read

$$p_w(r, t) = p_{wi}(r, t) = \rho_w \frac{R^2}{r} \ddot{\xi}(t). \quad (5.10a)$$

It is then tempting to parallel the splitting (5.2b) of the water displacement, and introduce the radiative extrapressure p_{wr} defined by

$$p_w(r, t) = p_{wi}(r, t) + p_{wr}(r, t). \quad (5.10b)$$

We have then, allowing for (5.2a),

$$\rho_w \frac{\partial^2 \vec{u}_{wi}}{\partial t^2} = \rho_w \ddot{\xi} \frac{R^2}{r^2} \vec{e}_r = - \overrightarrow{\text{grad}} p_{wi}, \quad (5.10c)$$

and consequently (Newton’s law):

$$\rho_w \frac{\partial^2 \vec{u}_{wr}}{\partial t^2} = - \overrightarrow{\text{grad}} p_{wr}. \quad (5.10d)$$

In this respect, it is remarkable that Newton’s law (2.1b) applies *separately* for each couple (\vec{u}_{wi}, p_{wi}) , (\vec{u}_{wr}, p_{wr}) . This is due to the fact that extrapressure p_{wi} is purely inertial, *i.e.* archimedean, the water acceleration $\ddot{\xi}\vec{\lambda}_w$ standing for a (local) pseudo gravity.

VI. CONCLUSION

The {air bubble-water} system has a continuum of radial eigenmodes: for any given angular frequency ω , one can build (subsection II A) a spherical standing wave satisfying the equation of motion and the boundary conditions. For a discrete series of values of ω , this standing wave has, in the water side, a pressure antinode at $r = R$ (bubble’s surface). Throughout this paper, these particular modes are referred to as resonances. They can all the same be understood as the resonances of an air-water spherical Pérot-Fabry cavity. Due to this spherical symmetry, the fundamental resonance’s angular frequency ω_0 is spectacularly lowered with respect to what would be expected in a 1-D plane geometry, and is found to coincide with the so-called Minnaert angular frequency. We have also considered (subsection II C) the case of an air-bubble with a finite extent of surrounding water, and discussed the conditions for the infinite water approximation to be available. Within the framework of this approximation, we have derived (subsection III A) an exact expression for the pressure exerted by the water onto the bubble’s surface, and interpreted the resonances from

the point of view of the water (subsection III B). At low enough frequencies, the acoustic wavelength (in the air) is large compared to the bubble's radius, so that the motion of the air can be approximated by a homogeneous inflation/deflation of the bubble. At this breathing approximation, the bubble's dynamics is described by a single dynamical variable ξ , which is shown to be ruled by a harmonic oscillator motion equation. We have shown how the "naked" bubble can be "dressed" by the pressure forces exerted on its surface by the outer water, and we finally conclude that the Minnaert bubble is a har-

monic oscillator slightly different from the one generally considered in the literature (section IV). In order to interpret the mass renormalization term that appears with the dressing of the naked bubble, we have outlined (section V) an alternative picture, which consists in splitting the fluid (air or water) displacements in two components, as currently done in electrodynamics when splitting the full electric field in an electrostatic (coulombian) and a radiative parts. Besides, this "radiative" picture can be readily extended to the study of many-bubble clouds.¹¹

* Electronic address: devaud@ccr.jussieu.fr

† Electronic address: Thierry.Hocquet@upmc.fr

‡ Electronic address: jcbac@ccr.jussieu.fr

§ Electronic address: valeroy77@yahoo.fr

¹ M. Minnaert, *Phil. Mag.* **6**, 235 (1933).

² T. Leighton, *The Acoustic Bubble* (Academic Press, London, 1997).

³ F. S. Crawford, *Am. J. Phys.* **50**, 398 (1982).

⁴ C. Devin, *J. Acoust. Soc. America* **31**, 1654 (1959).

⁵ A. Prosperetti, *J. Acoust. Soc. Am.* **61**, 17 (1977).

⁶ T. Leighton, P. White, C. Morfey, J. Clarke, G. Heald, H. Dumbrell, and K. Holland, *J. Acoust. Soc. America* **112**, 1366 (2002).

⁷ A. Wood, *A Textbook of sound* (Bell, London, 1932).

⁸ J. D. Templin, *Am. J. Phys.* **67**, 407 (1999).

⁹ J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, 1999).

¹⁰ E. Merzbacher, *Quantum mechanics* (John Wiley and Sons, 1998).

¹¹ V. Leroy, M. Devaud, T. Hocquet, and J.-C. Bacri, *Eur. Phys. J. E* **17**, 189 (2005).

¹² Differential operator $(1 + \frac{R}{c_w} \frac{\partial}{\partial t})^{(-1)}$ can be regarded as the symbolic summation of the infinite series $1 - \frac{R}{c_w} \frac{\partial}{\partial t} + (\frac{R}{c_w})^2 \frac{\partial^2}{\partial t^2} + \dots$