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Structural aspects of tilings

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Abstract We are interested in the structure of tilings that can be obtained from a given tile sets. We choose to study this structure by comparing the set of finite patterns they contain. We use two different approaches of the same notions: one combinatorial, the other topological. These two approaches provide nice results somehow surprising.

1 Introduction

Tilings are basic models for geometric phenomena of computation: local constraints they formalize happened to be of broad interest in the community since they capture geometric aspects of computation. This phenomenon was discovered in the 60ies when tiling problems happened to be crucial in logics: more specifically, interest shown in tilings drastically increased when Berger proved the undecidability of the so-called domino problem [1] (see also [7] and the well known book [2] for logical aspects). Later, tilings were basic tools for complexity theory (see the nice review of Peter van Emde Boas [15] and some of Leonid Levin's paper such as [11]).

Because of this growing interest for this very simple model, several research tracks were aimed directly on tilings: some people tried to generate the most complex tilings with the most simple constraints (see [14,8,12,5]), others were most interested in structural aspects (see [13,4]).

In this paper we are interested in structural properties of tilings. To analyze and compare what kind of family of tilings is the consequence of a given tile set, we focus on finite patterns they contain. We thus introduce a natural preorder on tilings: a tiling is extracted from another one if all finite patterns that appear in the first one also appear in the later. We develop this combinatorial notion in Section 2.1. This notion can be expressed in terms of topology (subshifts of finite type) and we explain the relations between both these approaches in Section 2.2.

It is important to stress that both these combinatorial and topological approaches have independent merits. Among results we present, different approaches are used for proofs. More specifically, we prove Theorem 6 by a rather subtle combinatoric construction and Theorem 7 with the strong help of topology. These results are our main results: despite its apparent simplicity, Theorem 6 was surprisingly open until now and Theorem 7 gives a nice result involving periodicity in a unique direction.

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Our paper is organized as follows: Section 2 is devoted to definitions (combinatorics, topology) and basic structural remarks. Then Section 3 focuses on our main results. We first prove the existence of minimal and maximal elements in tilings enforced by a tile set. Then we present an analysis in terms of Cantor-Bendixon derivative which provides powerful tools. We study the particular case where tilings are countable and present our main results. We conclude by the presentation of some open problems.

2 Definitions

2.1 Tilings

We present notations and definitions for tilings since several models are used in literature: Wang tiles, geometric frames of rational coordinates, local constraints... all these models are equivalent for our purposes since we consider very generic properties of them (proofs in [3]). We focus our study on tilings of the plane although all our results still hold in higher dimensions.

In our definition of tilings, we first associate a state with each cell of the plane and then impose a local constraint on them. More formally, Q is a finite set, called the *set of states*. A *configuration* c consists of cells of the plane associated with states; thus c is an element of $Q^{\mathbb{Z}^2}$. We denote by $c_{i,j}$ or $c(i,j)$ the state of c at the cell (i,j) .

A tiling is a configuration which satisfies a given finite set of finite constraints everywhere. We may define these constraints as a set of allowed patterns: a configuration is a tiling if around any of its cells we can see one of the allowed patterns:

Definition 1 (patterns). *A pattern P is a finite restriction of a configuration i.e., an element of Q^V for some finite domain V of \mathbb{Z}^2 . A pattern appears in a configuration c (resp. in some other pattern P') if it can be found somewhere in c (resp. in P'); i.e., if there exists a vector $t \in \mathbb{Z}^2$ such that $c(x+t) = P(x)$ on the domain of P (resp. if $P'(x+t)$ is defined for $x \in V$ and $P'(x+t) = P(x)$).*

By language extension we say that a pattern is *absent* in a configuration if it does not appear in it.

Definition 2 (tile-sets and tilings). *A tile-set is a tuple $\tau = (Q, \mathcal{P}_\tau)$ where \mathcal{P}_τ is a finite set of patterns on Q . All the elements of \mathcal{P}_τ are supposed of same domain denoted by V ($\mathcal{P}_\tau \subseteq Q^V$).*

A tiling by τ is a configuration c equal to one of the patterns on all cells: $\forall x \in \mathbb{Z}^2, c|_{V+x} \in \mathcal{P}_\tau$.

We denote by \mathcal{T}_τ the set of tilings by τ .

Notice that we could as well define tile-sets with patterns of different sizes since we consider only finitely many of them.

Throughout the following, it will be more convenient to define tile-sets by the set of their *forbidden patterns*, even if this definition is less common in

literature: a tile-set is then given by a finite set \mathcal{F}_τ of forbidden patterns ($\mathcal{F}_\tau = Q^V \setminus \mathcal{P}_\tau$); a configuration is a tiling if no forbidden pattern appears.

An example of a tile-set is given in Fig. 1. Produced tilings are given in Fig. 2; the meaning of the edges will be explained later. Tilings in Fig. 2 are represented up to shift, with an exception for A_i and B_i where i is an integer that represents the size of the white stripe.

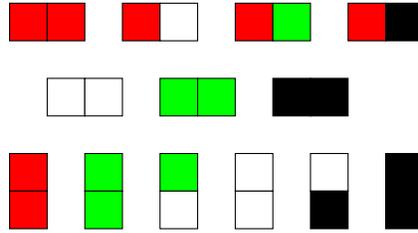


Figure 1. Allowed patterns

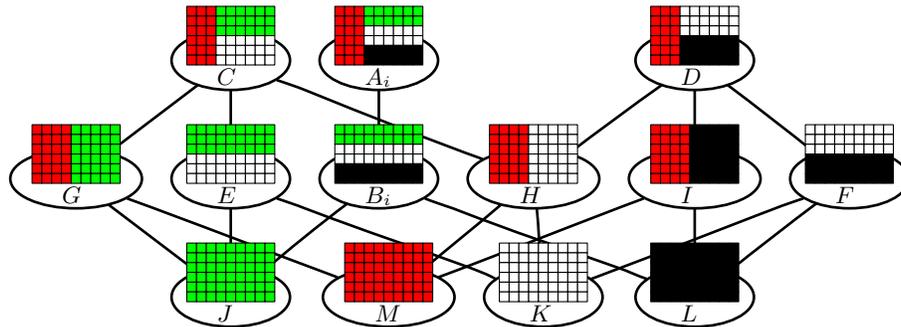


Figure 2. Lattice

The goal of our article is to study the structure of sets of tilings. It is natural to focus on finite parts of them, that is patterns. We thus introduce the following natural preorder:

Definition 3 (Preorder). *Let x, y be two tilings, we say that $x \preceq y$ if any pattern that appears in x also appears in y .*

We say that two tilings x, y are equivalent if $x \preceq y$ and $y \preceq x$. We denote this relation by $x \approx y$. In this case, x and y contain the same patterns. The equivalence class of x is denoted by $\langle x \rangle$. We write $x \prec y$ if $x \preceq y$ and $x \not\approx y$.

Some structural properties of tilings can be seen with the help of this pre-order. The edges in Fig. 2 correspond to the relation \prec .

We choose to distinguish two types of tilings: A tiling x is of *type a* if any pattern that appears in x appears infinitely many times: x is of *type b* if there exists a pattern that appears only once in x . Remark that any tiling is either of *type a* or of *type b*: suppose that there is a pattern that appears only a finite number of times in x ; consider the pattern which is the union of those patterns, this pattern appears only once.

If x is of *type b*, then the only tilings equivalent to x are its shifted: there is a unique way in $\langle x \rangle$ to grow around the unique pattern.

2.2 Topology

In the domain of symbolic dynamics, topology provides both interesting results and is also a nice way to express some combinatorial proofs in a condensed way [9,6]. The benefit of topology is a little more surprising for tilings since they are essentially static objects. Nevertheless, we can get nice results with topology as will be seen in the sequel.

We see the space of configurations $Q^{\mathbb{Z}^2}$ as a metric space in the following way: the distance between two configurations c and c' is 2^{-i} where i is the minimal offset of a point where c and c' differ:

$$d(c, c') = 2^{-\min\{|i|, c(i) \neq c'(i)\}}$$

We could also endow Q with the discrete topology and then $Q^{\mathbb{Z}^2}$ with the product topology, and thus obtain the same topology as the one induced by d .

In this topology, a basis of open sets is given through the patterns: for each pattern P , the set \mathcal{O}_P of all configurations c which contains P in their center (i.e., such that c is equal to P on its domain) is an open set. Furthermore cylinders such defined are also closed (complements as a finite unions of $\mathcal{O}_{P'}$ where P' are patterns of same domain different from P). Thus \mathcal{O}_P 's are clopen.

Proposition 1. $Q^{\mathbb{Z}^2}$ is a compact perfect metric space (a Cantor space).

We say that a set of configurations S is shift-invariant if any shifted version of any of its configuration is also in S ; i.e., if for every $c \in S$, and every $t \in \mathbb{Z}^2$ the configuration c' defined by $c'(x) = c(x+t)$ is also in S .

For a given configuration x , we define the topological closure of shifted forms of x : $\Gamma(x) = \overline{\{\sigma_{i,j}(x), i, j \in \mathbb{Z}\}}$. We remark that $x \preceq y$ if and only if $\Gamma(x) \subseteq \Gamma(y)$, therefore x is minimal for \prec if and only if $\langle x \rangle$ is closed.

As sets of tilings omit a finite number of patterns, they correspond to *subshifts* of finite type. In the sequel, we sometimes use arbitrary subshifts; they correspond to a set of configurations with a possibly infinite set of forbidden patterns.

3 Main results

3.1 Basic structure

Let us first present a few structural result so that we understand better those objects we manipulate in this paper.

First, the existence of minimal classes for \prec is well known.

Theorem 1 (minimal elements). *Every set of tilings contains a minimal class for \prec .*

In the context of tilings, those that belong to minimal classes are often called *quasiperiodic*, while in language theory they are called *uniformly recurrent* or *almost periodic*. Those quasiperiodic configurations admit a nice characterization: any pattern that appears in one of them can be found in any sufficiently large pattern (placed anywhere in the configuration).

For a combinatorial proof of this theorem see [4]. Alternatively, consider a minimal subshift of \mathcal{T}_τ (such a subshift exists, see e.g. [13]). Remark that every tiling in this set is in a minimal class.

The existence of maximal classes of tilings is not trivial and we have to prove it:

Theorem 2 (maximal elements). *Every set of tilings contains a maximal class for \prec .*

Proof. Let us prove that any increasing chain has an upper bound. The theorem is then obtained by Zorn's lemma.

Consider T_i an increasing chain of tiling classes. Consider the set P of all patterns that this chain contains. As the set of all patterns is countable, P is countable too, $P = \{p_i\}_{i \in \mathbb{N}}$.

Now consider two tilings T_i and T_j , any pattern that appears in T_i or T_j appears in $T_{\max(i,j)}$. Thus for all $i \in \mathbb{N}$ there exists a pattern p'_i that contains all p_j , $j \leq i$ and p'_{i-1} . Note that p'_i is correctly tiled by the considered tile set.

The sequence of patterns p'_i grows in size. By shift invariance, we can center each p'_i by superimposing an instance of p'_{i-1} found in p'_i over p'_{i-1} , we can conclude that this sequence has a limit and this limit is a tiling that contains all p_i , hence is an upper bound for the chain T_i . \square

Note that this proof also works when the *set of states* Q and/or the *set of forbidden patterns* \mathcal{F}_τ are countably infinite (neither compactness nor finiteness is assumed). We do not know if there always exists a minimal tiling when Q is infinite.

3.2 Cantor-Bendixson

In this section we use the topological derivative and define Cantor-Bendixson rank and then discuss properties of set of tilings from this viewpoint.

A configuration c is said to be *isolated* in a set of configurations S if there exists a pattern P (of domain V) such that c is the only configuration in S that contains the pattern P in its center ($\forall x \in V, c(x) = P(x)$). We say that P *isolates* c . This corresponds to the topological notion: a point is isolated if there exists an open set that contains only this point. As an exemple, in Fig. 3, the tilings A_i are isolated, the pattern isolating an A_i is the boundary between red, white, black and green parts of it.

The topological derivative of a set S is formed by its elements that are not isolated. We denote it by S' .

If S is a set of tilings, or more generally a subshift, we get some more properties. If P isolates a configuration in S then a shifted form of P isolates a shifted form of this configuration. Any configuration of S that contains P is isolated.

As a consequence, if $S = \mathcal{T}_\tau$, then $S' = \mathcal{T}_{\tau'}$ where τ' forbids the set $\mathcal{F}_\tau \cup \{P \mid P \text{ isolates some configuration in } \mathcal{T}_\tau\}$.

Note that S' is not always a set of tilings, but remains a subshift. Let us examine the example shown in Fig. 3. S' is S minus the classes A_i , however if C, B_i and D are obtained by in a set of tilings (subshift of finite type) then one of the A_i is also obtained. Hence S' is not of finite type in this example.

We define inductively $S^{(\lambda)}$ for any ordinal λ :

- $S^{(0)} = S$
- $S^{(\alpha+1)} = (S^{(\alpha)})'$
- $S^{(\lambda)} = \bigcap_{\alpha < \lambda} S^{(\alpha)}$

Notice that there exists a countable ordinal λ such that $S^{(\lambda+1)} = S^{(\lambda)}$. Indeed, at each step of the induction, the set of forbidden patterns increases, and there is at most countably many patterns. We call the least such ordinal the *Cantor-Bendixson rank* of S .

An element c is of *rank* λ in S if λ is the least ordinal such that $c \notin S^{(\lambda)}$. If no such λ exists, c is of infinite rank. For instance all quasiperiodic configurations are of infinite rank. We write $\rho(x)$ the rank of x .

An example of what Cantor-Bendixson ranks look like is shown in Fig. 3, the first row contains the tilings of rank 1, the second row the ones of rank 2. . .

Ranked tilings have many interesting properties. First of all, as any $\mathcal{T}_\tau^{(\lambda)}$ is shift-invariant, a tiling has the same rank as its shifted forms.

Note that at each step of the inductive definition, the set of isolated points is at most countable (there are less isolated points than patterns). As a consequence, if all tilings are ranked, \mathcal{T}_τ is countable, as a countable union (the Cantor-Bendixson rank is countable) of countable sets.

The converse is also true:

Theorem 3. \mathcal{T}_τ is countable if and only if all tilings are ranked.

Proof. Let λ be the Cantor-Bendixson rank of \mathcal{T}_τ . $\mathcal{T}_\tau^{(\lambda)} = \mathcal{T}_\tau^{(\lambda+1)}$ is a perfect set (no points are isolated). As a consequence, $\mathcal{T}_\tau^{(\lambda)}$ must be either empty or uncountable (classical application of Baire's Theorem).

As \mathcal{T}_τ is countable, $\mathcal{T}_\tau^{(\lambda)} = \emptyset$. □

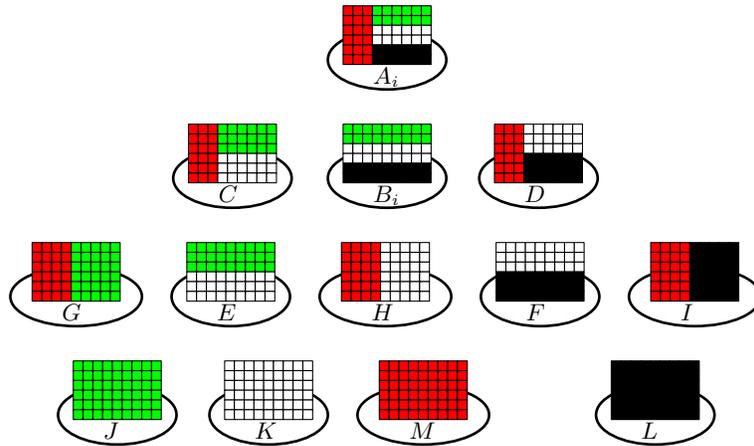


Figure 3. Cantor-Bendixson ranks

As the topology of $Q^{\mathbb{Z}^2}$ has a basis of clopens \mathcal{O}_P , $Q^{\mathbb{Z}^2}$ is a 0-dimensional space, thus any subset of $Q^{\mathbb{Z}^2}$ is also 0-dimensional. As any (non empty) perfect 0-dimensional compact metric space is isomorphic to the Cantor Space and, we also get:

Theorem 4 (Cardinality of tiling spaces). *A set of tilings is either finite, countable or has the cardinality of continuum.*

We now present a link between the preorder \prec and the Cantor-Bendixson rank.

Lemma 1. *Let x and y be two ranked tilings such that $x \prec y$, then $\rho(x) > \rho(y)$.*

Proof. By definition, any pattern that appears in x also appears in y . As a consequence, if P isolates x in $S^{(\lambda)}$, then y cannot be in $S^{(\lambda)}$. \square

As a consequence, tilings of rank 1 (minimal rank) are maximal tilings. If all tilings are ranked, tilings of maximal rank are minimal tilings and thus periodic.

Another consequence is that if all tilings are ranked, there is no infinite increasing chain for \prec because this would give an infinite decreasing chain of ordinals:

Theorem 5. *If \mathcal{T}_τ is countable, there is no infinite decreasing chain for \prec .*

In previous results, we saw that the case where the set of tiling is countable is an interesting particular case. We study this case more precisely in the next section.

3.3 The countable case

We will now deal with tile-sets that generate only countably many tilings. If the number of tilings is finite, the situation is easy: any tiling is periodic.

Our aim is to prove that in the countable case, there exists a tiling c which has exactly one vector of periodicity (such a tiling is sometimes called weakly periodic in the literature).

We split the proof in three steps :

- There exists a tiling which is not minimal.
- There exists a tiling c which is at level 1, that is such that all tilings less than c are minimal.
- Such a tiling has exactly one vector of periodicity.

The Cantor-Bendixson rank will be used only in the last two steps. The first step is a result of independent interest.

Since minimal tilings are quasiperiodic tilings, if the number of tilings is countable, any minimal tiling is periodic. As a consequence, the first step of the proof may be reformulated:

Theorem 6. *If all tilings produced by a tile-set are periodic, then there are only finitely many of them.*

Proof. Suppose that \mathcal{T}_τ is infinite with only periodic configurations. We will prove that there exists a sequence $(M_i)_{i \in \mathbb{N}}$ of patterns such that :

- M_i is a subpattern of M_{i+1} ;
- for any i , there exists an infinite number of elements of \mathcal{T}_τ that contain M_i ;
- any configuration containing M_i has a period greater than i ;
- each M_i is a $m_i \times m_i$ pattern centered at 0;

Take M_0 an empty pattern.

Assuming that M_i is constructed, let us construct M_{i+1} . If M_i has a size of $a \times a$, consider the patterns of size $(a+2(i+1)) \times (a+2(i+1))$ that contain M_i at their center and that are not $i+1$ -periodic. There are infinitely many elements of \mathcal{T}_τ that contain such a pattern : Consider a configuration that contains M_i , shift it so that M_i is at its center; if it does not contain any of those non $(i+1)$ -periodic patterns, then it can be shifted by $i+1$ in both directions with M_i still at its center. Therefore, if it does not contain any of those patterns it is $i+1$ -periodic and since there are only a finite number of $i+1$ -periodic configurations, there are infinitely many configurations that contain one of those new patterns.

Since there is only a finite number of patterns of size $(a+2(i+1)) \times (a+2(i+1))$ that contain M_i and that are not $i+1$ -periodic, one of them is contained in an infinite number of elements of \mathcal{T}_τ , call this pattern M_{i+1} . By construction, any tiling that contains M_i has a period greater than i .

Now to finish the proof, the "limit" of the sequence M_i defines a configuration that is in \mathcal{T}_τ (since it is closed) and that is not i -periodic for any i , a contradiction. \square

This proof does not assume that \mathcal{F}_τ is finite, therefore it is still valid for any shift-invariant closed subset of $Q^{\mathbb{Z}^2}$.

Important remark: *the authors would be interested in a topological proof of this theorem. If any reader can find such a proof please contact the authors.*

Now we prove stronger results about the Cantor-Bendixson rank of \mathcal{T}_τ :

Let α be the Cantor-Bendixson rank of \mathcal{T}_τ . Since $(\mathcal{T}_\tau)^{(\alpha)} = \emptyset$, α cannot be a limit ordinal. Suppose that it is, therefore $\bigcap_{\beta < \alpha} (\mathcal{T}_\tau)^{(\beta)} = \emptyset$ is an empty intersection of closed sets in $Q^{\mathbb{Z}^2}$ therefore by compactness there exists $\gamma < \alpha$ such that $\bigcap_{\beta < \gamma} (\mathcal{T}_\tau)^{(\beta)} = \emptyset$ and therefore \mathcal{T}_τ can not have rank α . Hence α is a successor ordinal, $\alpha = \beta + 1$.

However, we can refine this result :

Lemma 2. *The rank of \mathcal{T}_τ cannot be the successor of a limit ordinal.*

Proof. Suppose that $\beta = \cup_{i < \omega} \beta_i$. Since $(\mathcal{T}_\tau)^{(\beta+1)} = \emptyset$, $(\mathcal{T}_\tau)^{(\beta)}$ is finite (otherwise it would have a non-isolated point by compactness), it contains only periodic tilings.

Let p be the least common multiple of the periods of the tilings in $(\mathcal{T}_\tau)^{(\beta)}$. Let M be the set of patterns of size $2p \times 2p$ that does not admit p as a period. Let x_i be an element that is isolated in $(\mathcal{T}_\tau)^{(\beta_i)}$.

As there is only a finite number of p -periodic tilings, we may suppose w.l.o.g. that no x_i admit p as a period.

For any i , there exists a pattern of M that appears in x_i , let x'_i be the tiling with this pattern at its center. By compactness, one can extract a limit x' of the sequence $(x'_i)_{i \in \mathbb{N}}$, $x' \in (\mathcal{T}_\tau)^{(\beta)}$, x' is by construction in $\bigcap_i (\mathcal{T}_\tau)^{(\beta_i)} = \mathcal{T}_\tau^{(\beta)}$. However, x' does not contain a p periodic pattern at its center, that is a contradiction. \square

We write $\alpha = \lambda + 2$ the rank of \mathcal{T}_τ .

We proved that there exists a non minimal tiling but this is not sufficient to conclude that there exists a tiling at level 1. However, we achieve this as a corollary of the previous lemma: $(\mathcal{T}_\tau)^{(\lambda)}$ is infinite (otherwise $(\mathcal{T}_\tau)^{(\lambda+1)}$ would be empty) and contains a non periodic tiling by theorem 6. This non periodic tiling c is not minimal (otherwise it would be strictly quasiperiodic and then \mathcal{T}_τ would not be countable). Now c is at level 1 : any tiling less than c is in $(\mathcal{T}_\tau)^{(\lambda+1)}$ therefore periodic (hence minimal).

If a tiling x is of *type a* and is ranked, then it has a vector of periodicity: Consider the pattern P that isolates it in the last topological derivative of \mathcal{T}_τ that it belongs to. Since x is of *type a*, this pattern appears twice in it, therefore there exists a shift σ such that $\sigma(x)$ contains P at its center. $x = \sigma(x)$ because P isolates x .

As any tiling of *type a* has a vector of periodicity, it remains to prove that c is of *type a*.

Lemma 3. *c is of type a.*

Proof. Suppose the converse : there exists a pattern P that appears only once in c . Considering the union of this pattern P and a pattern that isolates c , we may assume that P isolates c . c has only a finite number of tilings smaller than itself: they lie in $\mathcal{T}_\tau^{(\lambda+1)}$ which is finite, and are all periodic, say of period p . As P isolates c , none of these tilings contain P .

Consider the patterns of size $2p \times 2p$ of T that are not p -periodic. If those patterns can appear arbitrary far from P then one can extract a tiling from c (thus smaller than c) that is not p -periodic and does not contain P ; this is not possible.

Therefore there is a pattern in c that contains P (thus appears only once) and any other part of c is p -periodic (one can gather all non p -periodic parts of c around P).

This non periodic part could also be inserted elsewhere in c since the tiling rules are of bounded radius. Hence the number of tilings is not countable. \square

c is of type a , c is not periodic, c has a vector of periodicity, therefore our theorem 7 holds :

Theorem 7. *If the set of tilings is countable then one of the tilings has exactly one vector of periodicity.*

4 Open problems

We are interested in proving more precise results for the order \prec for a countable set of tilings : We believe that the order \prec has at most 3 levels, as it is the case in Fig. 2.

We also intend to prove a similar result for uncountable set of tilings; the problem is that we are tempted to think that if the set of tilings is uncountable, then a quasiperiodic tiling must appear. However, this is not true: imagine a tile set that admit a vertical line of white or black cells with red on the left and green on the right. The uncountable part is due to the vertical line that itself contains a quasiperiodic of dimension 1 but not of dimension 2.

A generalization of lemma 2 would be to prove that the Cantor-Bendixson rank of a countable set of tilings cannot be infinite; we know how to construct set of tilings that have an arbitrary finite Cantor-Bendixson rank, but we do not know how to obtain a set of tilings of rank greater than ω . Note that there exists countable compact spaces with Cantor-Bendixson rank greater than ω .

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