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Lie antialgebras: *prémices*

V. Ovsienko

Abstract

The main purpose of this work is to develop the basic notions of the Lie theory for commutative algebras. We introduce a class of $\mathbb{Z}_2$-graded commutative but not associative algebras that we call "Lie antialgebras". These algebras form a very special class of Jordan superalgebras, they are closely related to Lie (super)algebras and, in some sense, link together commutative and Lie algebras. The main notions we define in this paper are: representations of Lie antialgebras, an analog of the Lie-Poisson bivector (which is not Poisson) and central extensions. We will explain the geometric origins of Lie antialgebras and provide a number of examples. We also classify simple finite-dimensional Lie antialgebras. This paper is a new version of the unpublished preprint [10].

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1 Introduction

Let $(\mathfrak{a}, \cdot)$ be a commutative $\mathbb{Z}_2$-graded algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, that is, $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$, such that $\mathfrak{a}_i \cdot \mathfrak{a}_j \subset \mathfrak{a}_{i+j}$ and for all homogeneous elements x, y one has

$$x \cdot y \;=\; (-1)^{p(x)p(y)}\, y \cdot x, \tag{1.1}$$

where p is the $\mathbb{Z}_2$-valued parity function $p|_{\mathfrak{a}_i} = i$. In particular, the space $\mathfrak{a}_0 \subset \mathfrak{a}$ is a commutative subalgebra while the bilinear map $\mathfrak{a}_1 \times \mathfrak{a}_1 \to \mathfrak{a}_0$ is skew-symmetric.

Typical examples of $\mathbb{Z}_2$-graded commutative algebras are the algebras of differential forms on manifolds, or, more generally, the algebras of functions on supermanifolds; these algebras are of course associative and, in particular, the space $\mathfrak{a}_1$ is an $\mathfrak{a}_0$-module. Examples of commutative but not associative algebras are Jordan algebras.

The algebras considered in this paper are *not associative*, however the subalgebra $\mathfrak{a}_0$ will always be associative. In this sense, we suggest an alternative way to extend the associativity condition of $\mathfrak{a}_0$ to all of the $\mathfrak{a}$. We will try to convince the reader this leads to algebras that have quite remarkable properties.

1.1 The definition

A $\mathbb{Z}_2$-graded commutative algebra $\mathfrak{a}$ is called a *Lie antialgebra* if it satisfies the following identities:

$$\alpha \cdot (\beta \cdot \gamma) \;=\; (\alpha \cdot \beta) \cdot \gamma, \tag{1.2}$$

$$\alpha \cdot (\beta \cdot a) \;=\; \tfrac{1}{2}\,(\alpha \cdot \beta) \cdot a, \tag{1.3}$$

$$\alpha \cdot (a \cdot b) \;=\; (\alpha \cdot a) \cdot b \;+\; a \cdot (\alpha \cdot b)\,, \tag{1.4}$$

$$a \cdot (b \cdot c) \;+\; b \cdot (c \cdot a) \;+\; c \cdot (a \cdot b) = 0 \tag{1.5}$$

where $\alpha, \beta, \gamma \in \mathfrak{a}_0$ and $a, b, c \in \mathfrak{a}_1$. In particular, $\mathfrak{a}_0$ is a (commutative) associative subalgebra of $\mathfrak{a}$.

Weaker form of (1.3) The identity (1.3) needs a special discussion. This identity means that for every $\alpha \in \mathfrak{a}_0$ the operator $2\,\mathrm{ad}_\alpha : a \mapsto 2\,\alpha \cdot a$ defines an *action* of the commutative algebra $\mathfrak{a}_0$ on the space $\mathfrak{a}_1$. Furthermore, the identity (1.3) implies

$$(\alpha \cdot \beta) \cdot a = \alpha \cdot (\beta \cdot a) + \beta \cdot (\alpha \cdot a) \tag{1.6}$$

which is a weaker identity. The identity (1.6) together with an additional assumption that the operators of multiplication by even elements commute with each other, is equivalent to (1.3). It turns out that the identity (1.6) has an independent algebraic meaning.

Axioms of Lie antialgebra and derivations Recall that $D \in \mathrm{End}(\mathfrak{a})$ is a derivation of $\mathfrak{a}$ if for homogeneous $x, y \in \mathfrak{a}$ (i.e., x, y are either purely even or purely odd) one has

$$D(x \cdot y) = D(x) \cdot y + (-1)^{p(D)p(x)} x \cdot D(y). \tag{1.7}$$

This formula then extends by linearity for arbitrary $x, y \in \mathfrak{a}$. The space of all derivations of $\mathfrak{a}$ is a Lie superalgebra denoted by $\mathrm{Der}(\mathfrak{a})$.

Let us associate to every *odd* element $a \in \mathfrak{a}_1$ the operator $T_a : \mathfrak{a} \to \mathfrak{a}$ of *right* multiplication by a

$$T_a(x) = x \cdot a. \tag{1.8}$$

The following observation partly clarifies the definition of Lie antialgebras.

The set of three identities: (1.6), (1.4) and (1.5) is equivalent to the condition that for all $a \in \mathfrak{a}_1$ the operator T_a is a derivation.

Associativity of $\mathfrak{a}_0$ The associativity axiom (1.2) seems quite different from the other three axioms of Lie antialgebras. For instance, it has no interpretation in terms of derivations. It turns out however, that this axiom can be understood as a corollary of the axioms (1.3)–(1.5).

Assume that every even element of $\mathfrak{a}$ is a linear combination of products of odd elements: $\alpha = \sum a_i \cdot a_j$. We will call such a Lie antialgebra *ample*. The following simple statement is obtained in [7].

If a Lie antialgebra $\mathfrak{a}$ is ample, then the identities (1.3), (1.4) and (1.5) imply (1.2).

Note that a similar property holds for Lie superalgebras.

Remark 1.1. It was noticed in [7] that Lie antialgebras are closely related to the Kaplansky superalgebras, see [8]. More precisely, a half-unital Lie antialgebra is a Kaplansky superalgebra.

1.2 Examples

Let us give here a few examples of simple Lie antialgebras.

1. Our first example is a 3-dimensional Lie antialgebra called the *tiny Kaplansky superalgebra*[1] and denoted by K_3. This algebra has the basis $\{\varepsilon; a, b\}$, where ε is even and a, b are odd, satisfying the relations

$$\begin{aligned} &\varepsilon \cdot \varepsilon = \varepsilon \\ &\varepsilon \cdot a = \tfrac{1}{2} a, \quad \varepsilon \cdot b = \tfrac{1}{2} b, \\ &a \cdot b = \tfrac{1}{2} \varepsilon. \end{aligned} \tag{1.9}$$

[1]This algebra was rediscovered in [10] where it was denoted by $\mathrm{asl}(2, \mathbb{K})$, see also [9].

The algebra K_3 is simple, i.e., it contains no non-trivial ideal. The corresponding algebra of derivations is the simple Lie superalgebra osp(1|2), see [1] and Section 2.1. Moreover, this property completely characterizes the algebra K_3.

Theorem 1. *The algebra K_3 is the unique finite-dimensional $\mathbb{Z}_2$-graded commutative algebra such that the corresponding algebra of derivations is isomorphic to* osp(1|2).

This theorem will be proved in Section 4.1.

2. The most interesting example of a Lie antialgebra we know is a simple infinite-dimensional Lie antialgebra with the basis $\left\{\varepsilon_n,\ n\in\mathbb{Z};\quad a_i,\ i\in\mathbb{Z}+\frac{1}{2}\right\}$, where ε_n are even and a_i are odd and satisfy the following relations

$$\begin{aligned} \varepsilon_n\cdot\varepsilon_m &= \varepsilon_{n+m},\\ \varepsilon_n\cdot a_i &= \tfrac{1}{2}\,a_{n+i},\\ a_i\cdot a_j &= (j-i)\,\varepsilon_{i+j}. \end{aligned} \tag{1.10}$$

This algebra is called the *full derivation algebra*[2], see [8]. We will denote it by $\mathcal{AK}(1)$. We will prove that $\mathcal{AK}(1)$ is closely related to the well-known Neveu-Schwarz conformal Lie superalgebra $\mathcal{K}(1)$ namely

$$\mathcal{K}(1)=\mathrm{Der}(\mathcal{AK}(1)).$$

We conjecture that similarly to K_3 the algebra $\mathcal{AK}(1)$ is the unique $\mathbb{Z}_2$-graded commutative algebra satisfying this property.

The Lie antialgebra $\mathcal{AK}(1)$ contains infinitely many copies of K_3 with the basis $\{\varepsilon_0;\ a_i, a_{-i}\}$ for each half-integer i.

3. The subalgebra of $\mathcal{AK}(1)$ with the basis

$$\left\{\varepsilon_0,\ \varepsilon_1,\ \varepsilon_2,\ldots;\ a_{-\frac{1}{2}},\ a_{\frac{1}{2}},\ a_{\frac{3}{2}},\ldots\right\}$$

is also of some interest. This algebra is simple and can be understood as analog of the Lie algebra of (polynomial) vector fields on $\mathbb{R}$. This algebra has interesting non-trivial cohomology studied in [6].

4. Let us finally explain how to construct a large class of examples of Lie antialgebras. Given an arbitrary commutative algebra $\mathfrak{C}$, there always exists an ample Lie antialgebra $\mathfrak{a}$ such that $\mathfrak{a}_0=\mathfrak{C}$. An example is provided by $K_3(\mathfrak{C})=\mathfrak{C}\otimes_{\mathbb{K}}K_3(\mathbb{K})$. This already shows that there are at least as many ample Lie antialgebras as there are commutative algebras. However, this is not the only possibility to realize a commutative algebra as an even par of a Lie antialgebra, so that there are much more Lie antialgebras than commutative algebras.

We will give more concrete examples of Lie antialgebras in Section 4.

[2]It was called in [10] the "conformal Lie antialgebra".

1.3 The main properties

Lie antialgebras and Lie superalgebras and their representations Let V be a $\mathbb{Z}_2$-graded vector space. Consider the structure of Jordan $\mathbb{Z}_2$-graded algebra on $\mathrm{End}(V)$ equipped with the anticommutator $[A, B]_+ = AB + (-1)^{p(a)p(b)}\, BA$. A *representation* of a Lie superalgebra is a homomorphism

$$\chi : \mathfrak{a} \to (\mathrm{End}(V),\, [.,.]_+)$$

such that the *image* $\chi(\mathfrak{a}_0)$ *is commutative*, i.e., $\chi(\alpha)\,\chi(\beta) = \chi(\beta)\,\chi(\alpha)$ for all $\alpha, \beta \in \mathfrak{a}_0$. This differs the notion of representation from the well-known notion of specialization of a Jordan superalgebra. Representations of K_3 were studied in [9], study of representations of $\mathcal{AK}(1)$ is an interesting problem, see Section 2.3 and 3.2 for some examples.

The most interesting property of Lie antialgebras is their relation to Lie superalgebras. With each Lie antialgebra $\mathfrak{a}$, we associate a Lie superalgebra $\mathfrak{g}_\mathfrak{a}$ in the following way. The odd part $(\mathfrak{g}_\mathfrak{a})_1$ coincides with $\mathfrak{a}_1$ while the even part $\mathfrak{g}_0$ is the symmetric tensor square $S^2(\mathfrak{a}_1)_{\mathfrak{a}_0}$, where the tensor product is defined over the commutative algebra $\mathfrak{a}_0$. Note that this construction is completely different from the classical Kantor-Koecher-Tits construction.

- Every representation of a Lie antialgebra $\mathfrak{a}$ is a representation of the corresponding Lie superalgebra $\mathfrak{g}_\mathfrak{a}$.
- The Lie superalgebra $\mathfrak{g}_\mathfrak{a}$ acts on $\mathfrak{a}$, in other words, there is a canonical homomorphism $\mathfrak{g}_\mathfrak{a} \to \mathrm{Der}(\mathfrak{a})$.

One can say that the Lie antialgebra $\mathfrak{a}$ selects a class of representations of $\mathfrak{g}_\mathfrak{a}$ that are also representations of $\mathfrak{a}$. This is an interesting characteristic of representations of the Lie superalgebra $\mathfrak{g}_\mathfrak{a}$.

The properties of the Lie superalgebra $\mathfrak{g}_\mathfrak{a}$ and the universal enveloping algebras $U(\mathfrak{g}_\mathfrak{a})$ and $U(\mathfrak{a})$ will be studied with more details in [7].

Odd "Lie-Poisson" bivector The notion of (odd) Lie-Poisson type bivector is the origin of Lie antialgebras, see Section 2.1. For an arbitrary Lie antialgebra $\mathfrak{a}$, the dual space with *inverse parity,* $\Pi\,\mathfrak{a}^*$, can be equipped with a canonical odd linear bivector field $\Lambda_\mathfrak{a}$, see Section 3.3. Amazingly, the construction makes sense in the case of (purely even) commutative associative algebra, i.e., for $\mathfrak{a} = \mathfrak{a}_0$, but the dual space should be understood as purely odd in this case. The bivector $\Lambda_\mathfrak{a}$ is *not Poisson* in any sense. Its general geometric characteristics is an interesting problem.

Central extensions and cohomology In Section 3.4, we introduce the notion of central extension of a Lie antialgebra. We prove that existence of the unit element $\varepsilon \in \mathfrak{a}_0$ implies that the Lie antialgebra $\mathfrak{a}$ has no non-trivial central extension. Central extensions is a part of the general cohomology theory developed in [6].

Classification In the finite-dimensional case, the classification of simple Lie antialgebras is similar to the classification of commutative division algebras, see Section 4.1. In the infinite-dimensional case, the situation is of course much more complicated. We also classify the Lie antialgebras of rank 1.

2 Lie antialgebras and symplectic geometry

In this section, we show the way Lie antialgebras appear in geometry. Notice that the invariant operations we construct are odd; we recover Lie antialgebra structures using the parity inversion functor.

2.1 The algebra K_3 and the odd bivector Λ

Consider the vector space $\mathbb{K}^{2|1}$ equipped with the standard symplectic form, see [5],

$$\omega = dp \wedge dq + \tfrac{1}{2}\, d\tau \wedge d\tau, \tag{2.1}$$

where p and q are the usual even coordinates on $\mathbb{K}^2$ and τ is the formal Grassmann variable so that $\tau^2 = 0$. An equivalent way to define this symplectic structure is to introduce the Poisson bivector on $\mathbb{K}^{2|1}$:

$$\mathcal{P} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau}. \tag{2.2}$$

which is inverse to the symplectic form: $\mathcal{P} = \omega^{-1}$.

The Lie superalgebra $\mathrm{osp}(1|2)$ is defined as the space of infinitesimal linear transformations preserving the symplectic structure. The bivector (2.2) is the unique (up to a multiplicative constant) even bivector invariant with respect to the action of $\mathrm{osp}(1|2)$.

The odd bivector It turns out that there exists another, *odd*, $\mathrm{osp}(1|2)$-invariant bivector on $\mathbb{K}^{2|1}$.

Proposition 2.1. *There exists a unique (up to a multiplicative constant) odd bivector invariant with respect to the action of* $\mathrm{osp}(1|2)$. *It is give by the formula*

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \mathcal{E} + \tau\, \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}, \tag{2.3}$$

where

$$\mathcal{E} = p\, \frac{\partial}{\partial p} + q\, \frac{\partial}{\partial q} + \tau\, \frac{\partial}{\partial \tau} \tag{2.4}$$

is the Euler field.

Proof. The $\mathrm{osp}(1|2)$-invariance of Λ is a very easy check. Note that we will prove a much stronger statement, see Proposition 2.9.

Let us prove the uniqueness. An arbitrary odd bivector on $\mathbb{K}^{2|1}$ is given by

$$\Lambda = \frac{\partial}{\partial\tau} \wedge A + \tau\, F\, \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q},$$

where A is an even vector field and F is an even function. Let X be an even vector field, one has

$$L_X\Lambda = \frac{\partial}{\partial\tau} \wedge [X, A] + \tau\, X(F)\, \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \tau\, F\, L_X \left(\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} \right).$$

If, furthermore, $X \in \mathrm{osp}(1|2)$, then it preserves the even part $\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$ of the Poisson bivector. The condition $L_X\Lambda = 0$ then implies: $[X, A] = 0$ and $X(F) = 0$.

The even part of $\mathrm{osp}(1|2)$ is a Lie algebra isomorphic to $\mathrm{sl}(2, \mathbb{K})$ and generated by the Hamiltonian vector fields with quadratic Hamiltonians $\langle p^2,\, pq,\, q^2\rangle$. One checks that:

a) an even vector field A commuting with any even element of $\mathrm{osp}(1|2)$ is of the form

$$A = c_1\tau\, \frac{\partial}{\partial\tau} + c_2\, E,$$

where c_1 and c_2 are arbitrary constants and $E = p\, \frac{\partial}{\partial p} + q\, \frac{\partial}{\partial q}$;

b) an even function F such that $X(F) = 0$, where X is an even element of $\mathrm{osp}(1|2)$ is necessary a constant: $F = c_3$.

The odd part of $\mathrm{osp}(1|2)$ is spanned by the following two vector fields:

$$X_{\tau p} = \tau\, \frac{\partial}{\partial q} + p\, \frac{\partial}{\partial\tau}, \qquad X_{\tau q} = -\tau\, \frac{\partial}{\partial p} + q\, \frac{\partial}{\partial\tau}.$$

Applying any of these elements of $\mathrm{osp}(1|2)$ to the bivector Λ as above, one immediately gets $c_1 = c_2 = c_3$. □

The relation of the bivector Λ to the algebra K_3 is as follows. Any bivector defines an algebraic structure on the space of functions. Consider the bilinear operation associated with the bivector (2.3):

$$]F, G[\, := \frac{(-1)^{p(F)}}{2}\, \langle \Lambda, dF \wedge dG\rangle, \tag{2.5}$$

where F and G are arbitrary functions on $\mathbb{K}^{2|1}$, that is, $F = F_0(p, q) + \tau\, F_1(p, q)$. This is of course not a Poisson bracket.

Lemma 2.2. *The space of linear functions on $\mathbb{K}^{2|1}$ equipped with the bracket (2.5) is a Lie antialgebra isomorphic to K_3.*

Proof. One checks that after the *parity inverting* identification $\{\varepsilon;\, a,\, b\} \longleftrightarrow \{\tau;\, p,\, q\}$, the algebra (1.9) coincide with the bracket (2.5) restricted to linear functions. □

We proved that the Lie superalgebra $\mathrm{osp}(1|2)$ preserves the bivector Λ. Since $\mathrm{osp}(1|2)$ acts on $\mathbb{K}^{2|1}$ by linear vector fields, it also preserves the space of linear functions. In other words, $\mathrm{osp}(1|2)$ is the algebra of derivation $\mathrm{osp}(1|2) = \mathrm{Der}\,(K_3))$, cf. [1].

Remark 2.3. The bivector Λ given by (2.3), and the Lie antialgebra K_3 are equivalent structures, they contain the same information. The above lemma provides a parity inverting identification of the dual space:

$$\Pi\, K_3^* \cong (\mathbb{K}^{2|1}, \Lambda).$$

The bivector Λ is therefore analog of the "Lie-Poisson structure" corresponding to K_3, cf. Section 3.3 for a general setting.

An algebraic reformulation A purely algebraic way to reformulate the results of this section is as follows.

Consider the space of polynomials $\mathbb{K}[p, q, \tau]$ equipped with the standard action of the Lie supergroup $\mathrm{OSp}(1|2)$ (or, equivalently, of the Lie superalgebra $\mathrm{osp}(1|2)$). We are looking for $\mathrm{OSp}(1|2)$-invariant bilinear maps $(.,.) : \mathbb{K}[p, q, \tau] \otimes \mathbb{K}[p, q, \tau] \to \mathbb{K}[p, q, \tau]$ satisfying the Leibniz rule in the both arguments, viz

$$(FG, H) = F\,(G, H) + (-1)^{p(G)p(H)}\,(F, H)\,G$$

and similarly in the second argument.

This problem has exactly two solutions.

1. The first operation is even, this is nothing but the standard Poisson bracket. It can be defined at order one by

$$\{p, q\} = 1, \qquad \{p, \tau\} = 0, \qquad \{q, \tau\} = 0, \qquad \{\tau, \tau\} = 1,$$

   and then extended to $\mathbb{K}[p, q, \tau]$ via the Leibniz rule. Polynoms of order 1 span the Heisenberg Lie superalgebra.

2. The second operation is odd, it is defined at order 1 by

$$]p, q[= \tfrac{1}{2}\,\tau, \qquad]p, \tau[= \tfrac{1}{2}\,p, \qquad]q, \tau[= \tfrac{1}{2}\,q, \qquad]\tau, \tau[= \tau$$

   and, again, extends to $\mathbb{K}[p, q, \tau]$ via the Leibniz rule. This is the "antibracket" (2.5), note that the homogeneous polynomials of order 1 form an algebra isomorphic to K_3.

2.2 The full derivation algebra $\mathcal{AK}(1)$

In this section we study the full derivation algebra $\mathcal{AK}(1)$ defined by formula (1.10). We show that $\mathcal{AK}(1)$ is simple and that $\mathrm{Der}(\mathcal{AK}(1))$ is isomorphic to the famous conformal Lie superalgebra $\mathcal{K}(1)$, also known as the (centerless) Neveu-Schwarz algebra.

We also prove that the conformal Lie superalgebra $\mathcal{K}(1)$ is the maximal algebra of vector fields on $\mathbb{R}^{2|1}$ that preserves the bivector (2.3). The algebra $\mathcal{AK}(1)$ can be viewed as the maximal space of functions on $\mathbb{R}^{2|1}$ that form a Lie antialgebra with respect to the bracket (2.5).

The algebra $\mathcal{AK}(1)$ is simple We start with the following

Proposition 2.4. *The relations (1.10) define a structure of a simple Lie antialgebra.*

Proof. The identities (1.2) and (1.3) are evident. Let us prove the identity (1.4). One has to check that

$$\varepsilon_n \cdot (a_i \cdot a_j) = (\varepsilon_n \cdot a_i) \cdot a_j + a_i \cdot (\varepsilon_n \cdot a_j)$$

One obtains in the left-hand-side $\frac{1}{2}(j-i)\,\varepsilon_{i+j+n}$ and in the right-hand-side the sum of two terms: $\frac{1}{4}(j-(i+n))\,\varepsilon_{i+j+n}$ and $\frac{1}{4}(j+n-i)\,\varepsilon_{i+j+n}$, so that the identity (1.4) is satisfied. Furthermore, the identity (1.5) reads:

$$(a_i \cdot a_j) \cdot a_k + (a_j \cdot a_k) \cdot a_i + (a_k \cdot a_i) \cdot a_j = 0.$$

One obtains the sum $\frac{1}{4}(j-i)\,a_{i+j+n} + \frac{1}{4}(k-j)\,a_{i+j+n} + \frac{1}{4}(i-k)\,a_{i+j+n} = 0$. We proved that $\mathcal{AK}(1)$ is, indeed, a Lie antialgebra.

It is quite easy to prove that $\mathcal{AK}(1)$ is simple, see, e.g. [8]. We do not dwell on the details here. □

The conformal Lie superalgebra $\mathcal{K}(1)$ as the algebra of derivations The conformal Lie superalgebra $\mathcal{K}(1)$ has the basis $\left\{x_n,\ n\in\mathbb{Z};\quad \xi_i,\ i\in\mathbb{Z}+\frac{1}{2}\right\}$ satisfying the following commutation relations

$$\begin{aligned} [x_n, x_m] &= (m-n)\,x_{n+m}, \\ [x_n, \xi_i] &= \left(i-\tfrac{n}{2}\right)\xi_{i+n}, \\ [\xi_i, \xi_j] &= x_{i+j}. \end{aligned} \tag{2.6}$$

It contains infinitely many copies of osp(1|2) with the generators $\{x_{-n}, x_0, x_n; \xi_{-\frac{n}{2}}, \xi_{\frac{n}{2}}\}$.

Define the following action of $\mathcal{K}(1)$ on $\mathcal{AK}(1)$:

$$\begin{aligned} x_n(a_i) &= \left(i-\tfrac{n}{2}\right)a_{n+i}, \\ x_n(\varepsilon_m) &= m\,\varepsilon_{n+m}, \\ \xi_i(a_j) &= (j-i)\,\varepsilon_{i+j}, \\ \xi_i(\varepsilon_n) &= a_{i+n}. \end{aligned} \tag{2.7}$$

Note that his formula is well-known and represents the action of the algebra $\mathcal{K}(1)$ on the space of tensor densities of weight $-\frac{1}{2}$, see, e.g., [2, 3, 4] and Section 3.2.

Proposition 2.5. *The action (2.7) preserves the structure of $\mathcal{AK}(1)$.*

Proof. Consider for instance the action of an odd element of $\mathcal{K}(1)$. One has

$$\xi_i\,(a_j \cdot \varepsilon_k) = \tfrac{1}{2}\,\xi_i(a_{j+k}) = \tfrac{1}{2}\,(j+k-i)\,\varepsilon_{i+j+k},$$

together with

$$\xi_i(a_j)\cdot\varepsilon_k=(j-i)\,\varepsilon_{i+j}\cdot\varepsilon_k=(j-i)\,\varepsilon_{i+j+k}$$

and

$$a_j\cdot\xi_i(\varepsilon_k)=a_j\cdot a_{i+k}=\tfrac{1}{2}\,(j-i-k)\,\varepsilon_{i+j+k}.$$

One finally gets:

$$\xi_i\,(a_j\cdot\varepsilon_k)=\xi_i(a_j)\cdot\varepsilon_k-\,a_j\cdot\xi_i(\varepsilon_k)$$

which is precisely the condition of odd derivation, see formula (1.7).

The action of other elements can be checked in the same way. Hence the result. □

We will prove in the end of this section that $\mathcal{K}(1)$ actually coincides with $\mathrm{Der}(\mathcal{AK}(1))$.

Lie antialgebra $\mathcal{AK}(1)$ and symplectic geometry In this section we show that, similarly to K_3, the Lie antialgebra $\mathcal{AK}(1)$ can be obtained from the odd Poisson bivector (2.3).

Consider the bracket (2.5) given by the explicit expression

$$]F,G[=\frac{(-1)^{p(F)}}{2}\left(\frac{\partial F}{\partial\tau}\,\mathcal{E}(G)-(-1)^{p(F)}\,\mathcal{E}(F)\,\frac{\partial G}{\partial\tau}+\tau\left(\frac{\partial F}{\partial p}\,\frac{\partial G}{\partial q}-\frac{\partial F}{\partial q}\,\frac{\partial G}{\partial p}\right)\right). \tag{2.8}$$

One checks that the full antialgebra of functions $C^\infty(\mathbb{R}^{2|1})$ equipped with this bracket is *not* a Lie antialgebra.

Let $\mathcal{F}_\lambda$ be the space of *homogeneous functions of degree* -2λ on $\mathbb{R}^{2|1}$, that is, of the functions satisfying the condition

$$\mathcal{E}(F)=-2\lambda\,F,$$

where $\mathcal{E}$ is the Euler field (2.4). We will allow F to have singularities in codimension 1, for instance, we can consider rational functions.

Note that there is a dense subspace of homogeneous functions on $\mathbb{R}^{2|1}$ that correspond to the space of functions on $\mathbb{R}^{1|1}$. Indeed, given a function $f(x,\xi)=f_0(x)+\xi\,f_1(x)$ in one even and one Grassmann variable, one defines a homogeneous of degree λ function (with singularities at $(p=0)$) by

$$F^\lambda_f(p,q;\tau)=p^\lambda\,f\left(\tfrac{q}{p},\,\tfrac{\tau}{p}\right). \tag{2.9}$$

Proposition 2.6. *The space, $\mathcal{F}_{-\frac{1}{2}}$, of homogeneous of degree 1 functions on $\mathbb{R}^{2|1}$ is a Lie antialgebra with respect to the antibracket (2.8) that contains $\mathcal{AK}(1)$.*

Proof. The space of homogeneous of degree 1 functions is obviously stable with respect to the antibracket (2.8) so that $\mathcal{F}_{-\frac{1}{2}}$ is an algebra. Define a bilinear operation $f\cdot g$ on the space of functions in (x,ξ) by

$$F^1_{f\cdot g}:=\frac{(-1)^{p(f)}}{2}\left\langle\Lambda,dF^1_f\wedge dF^1_g\right\rangle. \tag{2.10}$$

One then easily checks the Lie antialgebra conditions.

Choose the following "Taylor basis":

$$a_i = p\left(\tfrac{q}{p}\right)^{i+\frac{1}{2}}, \qquad \varepsilon_n = \tau\left(\tfrac{q}{p}\right)^{n}$$

and substitute it into the antibracket (2.5). One obtains the commutation relations (1.10), so that the Lie antialgebra $\mathcal{AK}(1)$ is a subalgebra of $\mathcal{F}_{-\frac{1}{2}}$. □

The conformal Lie superalgebra $\mathcal{K}(1)$ also has a symplectic realization.

Proposition 2.7. *The space $\mathcal{F}_{-1}$ of homogeneous of degree 2 functions on $\mathbb{R}^{2|1}$ is a Lie superalgebra with respect to the Poisson bracket (2.2) that contains $\mathcal{K}(1)$.*

Proof. The Poisson bracket of two homogeneous of degree 2 functions is, again, a homogeneous of degree 2 function. Therefore, $\mathcal{F}_{-1}$ is, indeed, a Lie superalgebra.

A homogeneous of degree 2 function can be written in the form (2.9) with $\lambda = 2$. Choosing the basis of the space of all such functions:

$$x_n = \tfrac{p^2}{2}\left(\tfrac{q}{p}\right)^{n+1}, \qquad \xi_i = \tau p\left(\tfrac{q}{p}\right)^{i+\frac{1}{2}}$$

and substituting it into the Poisson bracket (2.2), one immediately obtains the commutation relations (2.6). Therefore, $\mathcal{K}(1)$ is a subalgebra of $\mathcal{F}_{-1}$. □

Remark 2.8. (a) The Lie superalgebra $\mathcal{F}_{-1}$ is a "geometric version" of the conformal Lie superalgebra $\mathcal{K}(1)$, which is a polynomial part of $\mathcal{F}_{-1}$. Similarly, $\mathcal{AK}(1)$ is the polynomial part of the Lie antialgebra $\mathcal{F}_{-\frac{1}{2}}$.

(b) The action (2.7) written in terms of homogeneous functions is, again, given by the standard Poisson bracket (2.2).

Lie superalgebra $\mathcal{F}_{-1}$ as algebra of symmetry

Proposition 2.9. *The Lie superalgebra $\mathcal{F}_{-1}$ is the maximal Lie superalgebra of vector fields that preserves the bivector (2.3).*

Proof. Part 1. Let us first show that $\mathcal{F}_{-1}$ preserves the bivector (2.3). Given a function $H \in \mathcal{F}_{-1}$, the corresponding Hamiltonian vector field is homogeneous of degree 0:

$$[\mathcal{E}, X_H] = 0. \tag{2.11}$$

Consider a more general case, where P is a purely even (independent of τ) Poisson bivector and E a purely even vector field. Assume P is homogeneous of degree -2 with respect to E, that is

$$L_E(P) = -2\,P.$$

Let Λ be the odd bivector field

$$\Lambda = \frac{\partial}{\partial\tau} \wedge \mathcal{E} + \tau \wedge P,$$

where

$$\mathcal{E} = E + \tau \frac{\partial}{\partial \tau}.$$

(Note that in our case $E = p\,\frac{\partial}{\partial p} + q\,\frac{\partial}{\partial q}, \qquad P = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$.)

Let X_H be a Hamiltonian (with respect to the Poisson structure P) vector field satisfying the homogeneity condition (2.11). The Lie derivative of Λ along X_H is as follows:

$$L_{X_H}\Lambda = L_{X_H}\left(\tfrac{\partial}{\partial \tau}\right) \wedge \mathcal{E} + X_H(\tau)\,P + \tau\, L_{X_H}(P)\,.$$

If H is even, the above expression obviously vanishes. Consider now an odd function $H = \tau H_1$, then one gets from (2.2)

$$X_{\tau H_1} = \tau X_{H_1} + H_1 \frac{\partial}{\partial \tau}.$$

Lemma 2.10. *One has*

$$L_{X_{\tau H_1}}(\Lambda) = \langle P \wedge E, dH_1 \rangle \tag{2.12}$$

where d is the de Rham differential.

Proof. Using the obvious expressions

$$\left[X_{\tau H_1}, \tfrac{\partial}{\partial \tau}\right] = X_{H_1}, \qquad L_{X_{\tau H_1}}(P) = -\tfrac{\partial}{\partial \tau} \wedge X_{H_1},$$

one obtains:

$$L_{X_{\tau H_1}}(\Lambda) = X_{H_1} \wedge \mathcal{E} + H_1\,P + \tau \tfrac{\partial}{\partial \tau} \wedge X_{H_1} = X_{H_1} \wedge E + H_1\,P.$$

Finally, using the fact that $E(H_1) = H_1$, one obtains the expression (2.12). □

The even tri-vector $P \wedge E$ obviously vanishes on $\mathbb{R}^{2|1}$, and so we proved that X_H, indeed, preserves the bivector (2.3).

Part 2. Conversely, one has to show that any vector field preserving the bivector (2.3) is a Hamiltonian vector field commuting with $\mathcal{E}$.

If X is a purely even vector field, i.e.,

$$\left[X, \tfrac{\partial}{\partial \tau}\right] = 0 \quad \text{and} \quad X(\tau) = 0,$$

then $L_X(\Lambda) = 0$ implies that X commutes with $\mathcal{E}$ and preserves the even bivector $P = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$, so that X is Hamiltonian.

If X is an odd vector field:

$$X = F_0 \frac{\partial}{\partial \tau} + \tau\, X_0,$$

where F_0 is an even function and X_0 an even vector field, then one obtains explicitly

$$L_X\Lambda = (E(F_0) - F_0)\frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \tau} \wedge (X_{F_0} + [E, X_0]) + (F_0\,P + X_0 \wedge E)\,.$$

The assumption $L_X(\Lambda) = 0$ implies that each of the three summands in this expression vanishes. It follows from the equation $E(F_0) - F_0 = 0$, that F_0 is a homogeneous of degree 1 function. The condition

$$X_{F_0} + [E, X_0] = 0$$

then implies that X_0 is a vector field homogeneous of degree -1, since so is X_{F_0}, and thus $X_0 = X_{F_0}$. We proved that the vector field X is Hamiltonian and $[\mathcal{E}, X] = 0$.

Proposition 2.9 is proved. □

Corollary 2.11. *One has*

$$\mathcal{F}_{-1} = \mathrm{Der}(\mathcal{F}_{-\frac{1}{2}}), \qquad \mathcal{K}(1) = \mathrm{Der}(\mathcal{AK}(1)).$$

Indeed, the first statement is a reformulation of Proposition 2.7 while the second is its algebraic version. The subalgebra $\mathcal{K}(1) \subset \mathcal{F}_{-1}$ corresponds precisely to the space of vector fields preserving the basis of $\mathcal{AK}(1) \subset \mathcal{F}_{-\frac{1}{2}}$.

2.3 Representation of $\mathcal{AK}(1)$ by tangent vector fields

In this section we investigate the relation of the Lie antialgebra $\mathcal{AK}(1)$ to contact geometry. In some sense, $\mathcal{AK}(1)$ provides a way to "integrate" the contact structure.

The contact structure on $S^{1|1}$ The natural projection $\mathbb{R}^{2|1} \setminus \{0\} \longrightarrow S^{1|1}$, equips $\mathbb{R}^{1|1}$ with a structure of $1|1$-dimensional contact manifold, see [2, 4, 3] for recent developments. This contact structure can be defined in terms of a contact 1-form $\alpha = dx + \xi d\xi$, or, equivalently, in terms of an odd vector field[3]

$$D = \frac{1}{2}\left(\frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}\right),$$

since D spans the kernel of α.

The conformal Lie superalgebra $\mathcal{K}(1)$ can be realized as the Lie superalgebra of contact vector fields on $S^{1|1}$. Every contact vector field on $S^{1|1}$ is of the form

$$X_h = h(x, \xi)\, \frac{\partial}{\partial x} + 2\, D\left(h(x, \xi)\right)\, D,$$

where $h(x, \xi) = h_0(x) + \xi\, h_1(x)$ is an arbitrary function. The map $h \mapsto F_h^2$, see (2.9), is an isomorphism between Lie superalgebra of contact vector fields and $\mathcal{F}_{-1}$. The Lie superalgebra of contact vector fields with polynomial coefficients is isomorphic to $\mathcal{K}(1)$.

[3]This vector field is also known in physical literature as "SUSY-structure".

Vector fields tangent to the contact distribution It turns out that there is a similar relation between the algebra $\mathcal{AK}(1)$ and the contact geometry.

A vector field *tangent* to the contact distribution is a vector field proportional to D, that is, $X = f\,D$ for some function $f(x,\xi)$.

Definition 2.12. We introduce the following *anticommutator* of tangent vector fields:

$$[f\,D, g\,D]_+ := f\,D \circ g\,D + (-1)^{(p(f)+1)(p(g)+1)}\, g\,D \circ f\,D. \tag{2.13}$$

Note that the sign in this operation is inverse to that of usual commutator of vector fields.

Remark 2.13. The space of tangent vector fields is not a Lie superalgebra since the Lie bracket of two tangent vector fields is not a tangent vector field, this is equivalent to non-integrability of the contact distribution.

Quite miraculously, that the anticommutator of two tangent vector fields is again a tangent vector field.

Proposition 2.14. *The space of tangent vector fields equipped with the anticommutator (2.13) is a Lie antialgebra that contains the Lie antialgebra $\mathcal{AK}(1)$. More precisely,*

$$[\chi_f, \chi_g]_+ = \chi_{f\cdot g},$$

where $f \cdot g$ is the product (2.5).

Proof. Define a map from $\mathcal{AK}(1)$ to the space of tangent vector fields as follows. To each homogeneous of degree 1 function $F^1_f(p,q,\tau)$, cf. formula (2.9), we associate a tangent vector field by

$$\chi_f = f(x,\xi)\,D. \tag{2.14}$$

Let us calculate the explicit formula of the anticommutator (2.13).

Consider first the case where both of the functions, f and g, are odd, i.e., $f = \xi f_1$ and $g = \xi g_1$. One then has

$$f\,D \circ g\,D + g\,D \circ f\,D = \frac{1}{2}\, fg\, \frac{\partial}{\partial x} + (f\,D(g) + g\,D(f))\,D.$$

The first summand is zero since it contains $\xi^2 = 0$, while the second one is equal to $\xi\,(f_1 g_1)\,D$, so that the anticommutator (2.13) on the odd functions coincides with $f_1 g_1$. This corresponds to the product $f_1 \cdot g_1$ of two elements of the even part $\mathcal{AK}(1)_0$, see formula (1.10).

If $f = \xi f_1$ is odd and $g = g_0$ is even, then the Leibniz rule $D \circ f = D(f) - f\,D$ implies

$$f\,D \circ g\,D + g\,D \circ f\,D = g\,D(f)\,D = \tfrac{1}{2}\, f_1 g_0\, D,$$

so that one gets $f_1 \cdot g_0 = \frac{1}{2}\, f_1 g_0$, accordingly to the $\mathcal{AK}(1)_0$-action on $\mathcal{AK}(1)_1$, cf. (1.10).

If, finally, the both functions f and g are even, i.e., $f = f_0$ and $g = g_0$, then

$$f\,D \circ g\,D - g\,D \circ f\,D = (f\,D(g) - g\,D(f))\,D = \tfrac{1}{2}\,\xi\,(f_0 g_0' - g_0 f_0')\,D,$$

gives the skew-symmetric product $f_0 \cdot g_0 = \frac{1}{2}\,(f_0 g_0' - g_0 f_0')$, on $\mathcal{AK}(1)_1$ with values in $\mathcal{AK}(1)_0$, see formula (1.10). $\square$

Remark 2.15. The map (2.14) is nothing but the bivector (2.3) contracted with the elements of $\mathcal{AK}(1)$. One checks that

$$\chi_f(g) = \frac{(-1)^{p(f)}}{2} \left\langle \Lambda, dF_f^1 \wedge dF_g^0 \right\rangle, \tag{2.15}$$

where F_f^1 and F_g^0 are functions on $\mathbb{R}^{2|1}$ homogeneous of degree 1 and 0, respectively, obtained as the lift of f and g according to (2.9). It is interesting to compare the above formula with (2.10).

We conclude this section with the formula for the product on the space of smooth functions on $S^{1|1}$ that *coincides* with (1.10) on the polynomial basis. The functions are of the form $f(x,\xi) = \alpha(x) + \xi\, a(x)$, the product is given by

$$(\alpha + \xi\, a) \cdot (\beta + \xi\, b) = \alpha\beta + ab' - a'b + \tfrac{1}{2}\xi\,(\alpha b + \beta a)\,. \tag{2.16}$$

This is the C^∞-analog of $\mathcal{AK}(1)$.

2.4 A pair of symplectic forms on $\mathbb{R}^{4|2}$ and the algebras $K_3(\mathbb{C})$ and $\mathcal{AK}(1)_{\mathbb{C}}$

Consider the space $\mathbb{K}^{2n|m}$ with linear symplectic form. There is no analog of the odd bivector (2.3), if $n \geq 2$ or $m \geq 2$, since the only $\mathrm{OSp}(m|2n)$-invariant bivector is the Poisson bivector. In this section we investigate the second special case: $n = m = 2$. It turns out that one has to fix a *pair* of symplectic forms and consider the group of linear transformations preserving both of them.

Note that, if $n \geq 3$ or $m \geq 3$, then one has to consider degenerate linear 2-forms, cf. Section 3.3.

A pair of symplectic forms on $\mathbb{K}^{4|2}$ Consider two *generic* linear symplectic forms on $\mathbb{K}^{4|2}$. There exist linear coordinates $(p_1, q_1, p_2, q_2; \tau_1, \tau_2)$ such that these forms are as follows.

$$\omega_\varepsilon = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \tfrac{1}{2}\,(d\tau_1 \wedge d\tau_1 - d\tau_2 \wedge d\tau_2)$$

and

$$\omega_\sigma = dp_1 \wedge dp_2 - dq_1 \wedge dq_2 + d\tau_1 \wedge d\tau_2.$$

This is a very simple fact of linear algebra.

Note that, in the real case, the forms ω_ε and ω_σ are, respectively, the real and the imaginary parts of the complex 2-form (2.1), where $p, q; \tau$ are the following complex coordinates $p = p_1 + iq_2$, $q = q_1 + ip_2$, $\tau = \tau_1 + i\tau_2$.

Proposition 2.16. *The Lie superalgebra of linear vector fields on $\mathbb{K}^{4|2}$ preserving the two symplectic forms ω_ε and ω_σ is isomorphic to:*

(i) $\mathrm{osp}(1|2,\mathbb{C})$, *if* $\mathbb{K} = \mathbb{R}$;

(ii) $\mathrm{osp}(1|2,\mathbb{C}) \oplus \mathrm{osp}(1|2,\mathbb{C})$, *if* $\mathbb{K} = \mathbb{C}$;

Proof. The Lie superalgebra of linear vector fields on $\mathbb{K}^{4|2}$ preserving the two symplectic forms ω_ε and ω_σ is a subalgebra of $\mathrm{osp}(2|4,\mathbb{K})$ spanned by 6 even Hamiltonian vector fields corresponding to the quadratic Hamiltonians

$$\left\{p_1^2 - q_2^2, \quad p_2^2 - q_1^2, \quad p_1p_2 + q_1q_2, \quad p_1q_1 - p_2q_2, \quad p_1q_2, \quad q_1p_2\right\}$$

and 4 odd bi-Hamiltonian vector fields with the Hamiltonians

$$\left\{p_1\tau_1 - q_2\tau_2, \quad p_2\tau_1 + q_1\tau_2, \quad q_1\tau_1 - p_2\tau_2, \quad q_2\tau_1 + p_1\tau_2\right\}.$$

In the real case, this defines an $\mathrm{osp}(1|2,\mathbb{C})$-action on $\mathbb{R}^{4|2}$. In the complex case, this is $\mathrm{osp}(1|2,\mathbb{C})_{\mathbb{C}} \cong \mathrm{osp}(1|2,\mathbb{C}) \oplus \mathrm{osp}(1|2,\mathbb{C})$. □

Remark 2.17. Note that, in the case (i), $\mathrm{osp}(1|2,\mathbb{C})$ is viewed as a simple Lie superalgebra over $\mathbb{R}$. This algebra of derivations is thus obtained as intersection of two copies of the real symplectic Lie superalgebra:

$$\mathrm{osp}(1|2,\mathbb{C}) = \mathrm{osp}_\varepsilon(1,1\,|\,4) \cap \mathrm{osp}_\sigma(1,1\,|\,4)$$

corresponding to the symplectic forms ω_ε and ω_σ, respectively.

The bivector Λ^C The following bivector is the unique (up to a multiplicative constant) odd $\mathrm{osp}(1|2,\mathbb{C})$-invariant bivector on $\mathbb{K}^{4|2}$:

$$\Lambda^C = \frac{\partial}{\partial\tau_1} \wedge \mathcal{E} + \frac{\partial}{\partial\tau_2} \wedge \mathcal{J} + \tau_1\,\pi_\varepsilon + \tau_2\,\pi_\sigma, \tag{2.17}$$

where

$$\mathcal{E} = \sum_{i=1,2} \left(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} + \tau_i \frac{\partial}{\partial \tau_i} \right)$$

is the Euler vector field and

$$\mathcal{J} = q_2 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_2} - p_1 \frac{\partial}{\partial q_2} + \tau_2 \frac{\partial}{\partial \tau_1} - \tau_1 \frac{\partial}{\partial \tau_2} \tag{2.18}$$

and where the bivectors π_ε and π_σ are

$$\pi_\varepsilon = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}, \qquad \pi_\sigma = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2}.$$

It is now easy to check that the space of linear functions on $(\mathbb{R}^{4|2}, \Lambda^C)$ form a Lie antialgebra isomorphic to $K_3(\mathbb{C})$.

Algebra $\mathcal{AK}(1)_{\mathbb{C}}$ and the pair of symplectic structures on $\mathbb{R}^{4|2}$ Let us now realize the algebras $\mathcal{AK}(1)_{\mathbb{C}}$ and $\mathcal{K}(1)_{\mathbb{C}}$ in terms of real rational harmonic functions on $\mathbb{R}^{4|2}$. This is related to the bi-Hamiltonian formalism defined by the pair of symplectic structures ω_ε and ω_σ.

The Lie antialgebra $\mathcal{AK}(1)_{\mathbb{C}}$ is represented by the homogeneous of degree 1 harmonic functions on $\mathbb{R}^{4|2}$ with the complex structure (2.18). The odd generators of $\mathcal{AK}(1)_{\mathbb{C}}$ are as follows

$$\varepsilon_n = \tfrac{1}{2^n}\operatorname{Re}\tau\left(\tfrac{q}{p}\right)^n, \quad \beta_n = \tfrac{1}{2^n}\operatorname{Im}\tau\left(\tfrac{q}{p}\right)^n,$$

and the even ones are

$$a_n = \tfrac{1}{2^n}\operatorname{Re}p\left(\tfrac{q}{p}\right)^n, \quad b_n = \tfrac{1}{2^n}\operatorname{Im}p\left(\tfrac{q}{p}\right)^n,$$

where p, q, τ are the complex coordinates. One then checks the relations in $\mathcal{AK}(1)_{\mathbb{C}}$.

The conformal Lie superalgebra $\mathcal{K}(1)_{\mathbb{C}}$ is realized by homogeneous bi-Hamiltonian vector fields on $\mathbb{R}^{4|2}$. For each function $F \in C^\infty(\mathbb{R}^{4|2})$, denote by X_F^ε and X_F^σ the Hamiltonian vector fields on $\mathbb{R}^{4|2}$ with respect to the symplectic form ω_ε and ω_σ, respectively.

One checks that the following three conditions are equivalent.

1. F is a homogeneous of degree 2 harmonic function:

$$F = \operatorname{Re}p^2 f\left(\tfrac{q}{p}, \tfrac{\tau}{p}\right),$$

   where f is an arbitrary function.

2. The function F satisfies the relations: $\mathcal{E}(F) = 2F$ and $\mathcal{J}(\mathcal{J}(F)) = -4F$.

3. The Hamiltonian vector fields with the Hamiltonian F commute with E and $\mathcal{J}$:

$$[\mathcal{E}, X_F^\varepsilon] = [\mathcal{E}, X_F^\sigma] = [\mathcal{J}, X_F^\varepsilon] = [\mathcal{J}, X_F^\sigma] = 0$$

   and are bi-Hamiltonian such that

$$X_F^\varepsilon = -\tfrac{1}{2}X_{\mathcal{J}(F)}^\sigma, \qquad X_F^\sigma = \tfrac{1}{2}X_{\mathcal{J}(F)}^\varepsilon.$$

This space of homogeneous harmonic bi-Hamiltonian vector fields is a Lie superalgebra isomorphic to $\mathcal{K}(1)_{\mathbb{C}}$.

Proposition 2.18. *The Lie superalgebra $\mathcal{K}(1)_{\mathbb{C}}$ is the maximal algebra of vector fields on $\mathbb{R}^{4|2}$ preserving the bivector (2.17).*

Proof. Proposition 2.9 implies that $\mathcal{K}(1)_{\mathbb{C}}$, indeed, preserves the bivector Λ^C. The proof of maximality is similar to that of Theorem 2.9, Part 2. We omit here the corresponding straightforward computations. □

3 Elements of the general theory

In this section we investigate the relations between Lie antialgebras and Lie superalgebras. We introduce the notion of representation and module over a Lie antialgebra and show how to extend them to the corresponding Lie superalgebra.

We also define the analog of the Lie-Poisson structure for an arbitrary Lie antialgebra $\mathfrak{a}$. This is an odd linear bivector field on the dual space with inverse parity $\Pi\,\mathfrak{a}^*$. This notion relates Lie antialgebras to geometry.

We finally define and study the notion of central extensions that will later be useful for classification results.

3.1 Relation to Lie superalgebras

In this section we associate a Lie superalgebra $\mathfrak{g}_\mathfrak{a}$ with an arbitrary Lie antialgebra $\mathfrak{a}$. The construction will be important since representations of a Lie antialgebra can always be extended to the corresponding Lie superalgebra.

Definition 3.1. Consider the space $\mathfrak{g}_\mathfrak{a} = (\mathfrak{g}_\mathfrak{a})_0 \oplus (\mathfrak{g}_\mathfrak{a})_1$ defined as follows. The even part, $(\mathfrak{g}_\mathfrak{a})_0$, is the symmetric tensor square of $\mathfrak{a}_1$ over $\mathfrak{a}_0$:

$$(\mathfrak{g}_\mathfrak{a})_0 := \mathfrak{a}_1 \odot_{\mathfrak{a}_0} \mathfrak{a}_1,$$

more precisely,

$$(\mathfrak{g}_\mathfrak{a})_0 = \{(a \otimes b + b \otimes a)/_\sim \mid a, b \in \mathfrak{a}_1\},$$

where the equivalence relation $\sim$ is defined by

$$\alpha \cdot a \otimes b \sim a \otimes \alpha \cdot b$$

for all $a, b \in \mathfrak{a}_1$ and $\alpha \in \mathfrak{a}_0$. The odd part of $(\mathfrak{g}_\mathfrak{a})$, is nothing but the odd part of $\mathfrak{a}$:

$$\mathfrak{g}_1 := \mathfrak{a}_1.$$

The Lie bracket on $\mathfrak{g}_\mathfrak{a}$ is defined by

$$\begin{aligned} [a \odot b\,,\, c \odot d] &= a \cdot (b \cdot c) \odot d + b \cdot (a \cdot c) \odot d - c \cdot (d \cdot a) \odot b - d \cdot (c \cdot a) \odot b, \\ [a \odot b\,,\, c] &= a \cdot (b \cdot c) + b \cdot (a \cdot c)\,, \\ [a\,,\, b] &= a \odot b. \end{aligned} \tag{3.1}$$

Theorem 2. *The space $\mathfrak{g}_\mathfrak{a}$ endowed with the bracket (3.1) is a Lie superalgebra.*

The Jacobi identity and the compatibility with the equivalence $\sim$ for the bracket (3.1) can be checked by a complicated but straightforward calculation that we omit.

The action of $\mathfrak{g}_{\mathfrak{a}}$ on $\mathfrak{a}$ Let us define a Lie superalgebra homomorphism $T : \mathfrak{g}_{\mathfrak{a}} \to \mathrm{Der}(\mathfrak{a})$. The restriction $T\big|_{(\mathfrak{g}_{\mathfrak{a}})_1}$ is naturally defined since the odd part $(\mathfrak{g}_{\mathfrak{a}})_1$ is identified with $\mathfrak{a}_1$ and the odd part $\mathfrak{a}_1$ acts on $\mathfrak{a}$ by derivations, cf. formula (1.8). One then extends this map to the even part $(\mathfrak{g}_{\mathfrak{a}})_0$ in a unique way as follows: $T_{a\odot b} := [T_a, T_b]$. One then has explicitly

$$T_{a\odot b}\, x = (x \cdot a) \cdot b + (x \cdot b) \cdot a \tag{3.2}$$

for odd $a, b \in \mathfrak{a}_1$ and arbitrary $x \in \mathfrak{a}$.

Lemma 3.2. *(i) The map (3.2) is well-defined, that is, it is compatible with the equivalence relation $\sim$.*

(ii) For all $a, b \in \mathfrak{a}_1$, the linear map $T_{a\odot b} \in \mathrm{End}(\mathfrak{a})$ is an even derivation.

Proof. Part (i). The computation is straightforward but we provide here some details in order to show how work the axioms of Lie antialgebra.

One has to check that

$$(x \cdot (\alpha \cdot a)) \cdot b + (x \cdot b) \cdot\ (\alpha \cdot a) = (x \cdot (\alpha \cdot b)) \cdot a + (x \cdot a) \cdot\ (\alpha \cdot b)$$

for every $\alpha \in \mathfrak{a}_0$ and arbitrary x. Consider, for instance, the case where x is odd, that is $x \in \mathfrak{a}_1$. Using (1.3), one obtains

$$(x \cdot b) \cdot\ (\alpha \cdot a) = \alpha \cdot\ ((x \cdot b) \cdot a)$$

while using (1.4) and then (1.5) and again (1.3), one obtains

$$\begin{aligned} (x \cdot (\alpha \cdot a)) \cdot b &= (\alpha \cdot (x \cdot a)) \cdot b - ((\alpha \cdot x) \cdot a) \cdot b \\ &= 2\,\alpha \cdot\ ((x \cdot a) \cdot b) + (a \cdot b) \cdot\ (\alpha \cdot x) + (b \cdot (\alpha \cdot x)) \cdot a \end{aligned}$$

for the terms in the left-hand-side. Similarly, for the right-hand-side one has

$$\begin{aligned} (x \cdot a) \cdot\ (\alpha \cdot b) &= \alpha \cdot\ ((x \cdot a) \cdot b) \\ (x \cdot (\alpha \cdot b)) \cdot a &= 2\,\alpha \cdot\ ((x \cdot b) \cdot a) - ((\alpha \cdot x) \cdot b) \cdot a. \end{aligned}$$

Finally, collecting the terms in the (lhs) $-$ (rhs) one gets

$$\begin{aligned} \alpha \cdot\ ((x \cdot a) \cdot b) - \alpha \cdot\ ((x \cdot b) \cdot a) + (a \cdot b) \cdot\ (\alpha \cdot x) &= \\ -\alpha \cdot ((a \cdot b) \cdot x) + (a \cdot b) \cdot\ (\alpha \cdot x) &= 0, \end{aligned}$$

using (1.5).

The computation for an even element $x \in \mathfrak{a}_0$ is similar.

Part (ii) is obvious since $T_{a\otimes b}$ is a commutator of two derivations. □

Remark 3.3. The above proof makes an extensive use of the axioms (1.3)–(1.5) of Lie antialgebra. The associativity axiom (1.2) is never used, but, as already mentioned, this axiom is also a corollary of (1.3)–(1.5). The complete proof of Theorem 2 is quite similar but much longer and more complicated, see [7].

Example 3.4. In the case of the simple algebras $\mathfrak{a} = K_3$ and $\mathcal{AK}(1)$, the corresponding Lie superalgebras $\mathfrak{g}_{\mathfrak{a}}$ coincide with the respective algebra of derivations:

$$\mathfrak{g}_{K_3} = \mathrm{osp}(1|2), \qquad \mathfrak{g}_{\mathcal{AK}(1)} = \mathcal{K}(1).$$

However, in general this is not the case. The more Lie antialgebra $\mathfrak{a}$ is far of being simple, the more the corresponding Lie superalgebra $\mathrm{gl}(\mathfrak{a})$ is far of $\mathrm{Der}(\mathfrak{a})$.

3.2 Representations and modules

In this section we introduce important notions of representation and of module over Lie antialgebras. Amazingly, these two notions are different for Lie antialgebras. We will investigate these notions in some details for the algebras $K_3(\mathbb{C})$ and $\mathcal{AK}(1)$.

Representations: the definition Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. Consider the Jordan algebra structure on the space $\mathrm{End}(V)$ defined by the anticommutator

$$[A, B]_+ = A\,B + (-1)^{p(A)p(B)} B\,A. \tag{3.3}$$

Note that this operation the sign rule inverse to that of the usual $\mathbb{Z}_2$-graded commutator (the sign is "$-$" if and only if both A and B are odd).

Remark 3.5. Note that the Jordan algebra $(\mathrm{End}(V), [.,.]_+)$ is of course not a Lie antialgebra. However, we use this algebra to define the notion of representation.

Given a Lie antialgebra $\mathfrak{a}$, an (even) linear map $\chi : \mathfrak{a} \to \mathrm{End}(V)$ is called a *representation* of $\mathfrak{a}$, if two conditions are satisfied:

$$\chi_{x\cdot y} = [\chi_x, \chi_y]_+ \,, \tag{3.4}$$

for all $x, y \in \mathfrak{a}$ and the images of even elements are commuting operators:

$$\chi_\alpha\, \chi_\beta = \chi_\beta\, \chi_\alpha \tag{3.5}$$

for all $\alpha, \beta \in \mathfrak{a}_0$.

Remark 3.6. A linear map $\chi : \mathfrak{a} \to \mathrm{End}(V)$ satisfying (3.4), but not necessarily (3.5) is called a *specialization* of a Jordan algebra. This means, a representation is a very particular case of specialization. The condition (3.5) is crucial for us.

Example 3.7. We have already considered a particular case of the anticommutator (3.3), namely the operation (2.13) on the space of vector fields on $S^{1|1}$ tangent to the contact structure. The map (2.14) defines a representation of the full derivation algebra $\mathcal{AK}(1)$ in the space of differential operators on $\mathbb{R}^{1|1}$.

Theorem 3. *Every representation of a Lie antialgebra $\mathfrak{a}$ is naturally a representation of the corresponding Lie superalgebra $\mathfrak{g}_{\mathfrak{a}}$.*

Proof. Given a representation χ of a Lie antialgebra $\mathfrak{a}$, the construction of the corresponding representation of $\mathfrak{g}_{\mathfrak{a}}$ is as follows. The odd part $(\mathfrak{g}_{\mathfrak{a}})_1$ coincides with $\mathfrak{a}_1$, so that the map $\chi|_{(\mathfrak{g}_{\mathfrak{a}})_1}$ is already defined. For the even elements of $\mathfrak{g}_{\mathfrak{a}}$ we take the usual commutator:

$$\chi_{a\odot b} := [\chi_a, \chi_b] = \chi_a \circ \chi_b + \chi_b \circ \chi_a.$$

To prove Theorem 3, one has to show that this is indeed a representation of $\mathfrak{g}_{\mathfrak{a}}$.

The complete proof is again a very complicated but straightforward computation, that we omit. The details are given in [7]. □

The above result shows that representations of $\mathfrak{a}$ is some particular class of representations of $\mathfrak{g}_{\mathfrak{a}}$.

Representations of K_3 Representations of the simple algebra K_3 were studied in [9].

The corresponding Lie superalgebra is osp(1|2), so that representations of K_3 is a particular class of representations of osp(1|2). It turns out that this class is characterized in terms of the classical Casimir element $C \in U(\mathrm{osp}(1|2))$. The following statement is the main result of [9].

There is a one-to-one correspondence between representations of s Lie antialgebra K_3 and the representations of the Lie superalgebra osp(1|2) *such that $\chi(C) = 0$.*

A representation of K_3 is obviously given by one even operator $\mathcal{E} \in \mathrm{End}(V)$ and two odd operators $A, B \in \mathrm{End}(V)$ satisfying the relations

$$\begin{aligned} AB - BA &= \mathcal{E}, \\ A\mathcal{E} + \mathcal{E}A &= A, \\ B\mathcal{E} + \mathcal{E}B &= B, \\ \mathcal{E}^2 &= \mathcal{E}. \end{aligned} \tag{3.6}$$

Among other classification results of [9], let us mention the following simple but beautiful statement.

Up to isomorphism, the operator $\mathcal{E}$ in (3.6) is of the form:

$$\mathcal{E}|_{V_0} = 0, \qquad \mathcal{E}|_{V_1} = \mathrm{Id}. \tag{3.7}$$

The relations (3.6) together with (3.7) are very similar to the Heisenberg canonical commutation relations. The difference is that the operator $\mathcal{E}$ is not the identity but the projector to the odd part of V.

Example: tensor-density representations of $\mathcal{K}(1)$ and $\mathcal{AK}(1)$ Consider the representations of $\mathcal{K}(1)$ called tensor density representations $\mathcal{F}_\lambda$, where $\lambda \in \mathbb{C}$ is the parameter. The basis in $\mathcal{F}_\lambda$ is $\{f_n,\, n \in \mathbb{Z}, \quad \phi_i,\, i \in \mathbb{Z} + \frac{1}{2}\}$ and the action of $\mathcal{K}(1)$ is given by

$$\begin{aligned}
\chi_{x_n}(f_m) &= (m + \lambda n)\, f_{n+m}, \\
\chi_{x_n}(\phi_i) &= \big(i + (\lambda + \tfrac{1}{2})n\big)\, \phi_{n+i}, \\
\chi_{\xi_i}(f_n) &= \big(\tfrac{n}{2} + \lambda i\big)\, \phi_{i+n}, \\
\chi_{\xi_i}(\phi_j) &= f_{i+j}.
\end{aligned}$$

For instance, the adjoint representation of $\mathcal{K}(1)$ is isomorphic to $\mathcal{F}_{-1}$. The $\mathcal{K}(1)$-action (2.7) on $\mathcal{AK}(1)$ is isomorphic to $\mathcal{F}_{-\frac{1}{2}}$ with inverse parity.

It is now very easy to check that the $\mathcal{K}(1)$-module $\mathcal{F}_\lambda$ is a representation of $\mathcal{AK}(1)$ if and only if

$$\lambda = 0 \qquad \text{or} \qquad \lambda = \frac{1}{2}.$$

Note that the modules $\mathcal{F}_0$ and $\mathcal{F}_{\frac{1}{2}}$ are dual to each other, these two modules are known to be special.

Modules over Lie antialgebras: the definition A $\mathbb{Z}_2$-graded vector space V is called an $\mathfrak{a}$-*module* if there is an even linear map $\rho : \mathfrak{a} \to \mathrm{End}(V)$, such that the direct sum $\mathfrak{a} \oplus V$ equipped with the product

$$(a,\, v) \cdot (b,\, w) = \big(a \cdot b\,,\, \rho_a\, w + (-1)^{p(b)p(v)}\, \rho_b\, v\big)\,, \tag{3.8}$$

where $a, b \in \mathfrak{a}$ and $v, w \in V$ are homogeneous elements, is again a Lie antialgebra. We call the Lie antialgebra structure (3.8) a *semi-direct* product and denote it by $\mathfrak{a} \ltimes V$.

Example 3.8. 1). The "adjoint action" $\mathrm{ad} : \mathfrak{a} \to \mathrm{End}(\mathfrak{a})$ defined by

$$\mathrm{ad}_x\, y = x \cdot y,$$

for all $x, y \in \mathfrak{a}$, defines a structure of $\mathfrak{a}$-module on $\mathfrak{a}$ itself. This follows, for instance, from the fact that the tensor product $\mathfrak{C} \otimes \mathfrak{a}$ of a Lie antialgebra $\mathfrak{a}$ with a commutative algebra $\mathfrak{C}$ is again a Lie antialgebra. Indeed, consider $\mathfrak{C} = \mathbb{K}[t]/(t^2)$, then one has $\mathfrak{C} \otimes \mathfrak{a} = \mathfrak{a} \ltimes \mathfrak{a}$.

2). The "coadjoint action" $\mathrm{ad} : \mathfrak{a} \to \mathrm{End}(\mathfrak{a}^*)$ defined by

$$\mathrm{ad}_x\, \varphi = (-1)^{p(x)p(\varphi)} \mathrm{ad}^*_x \varphi,$$

for $x \in \mathfrak{a}$ and $\varphi \in \mathfrak{a}^*$, defines a structure of $\mathfrak{a}$-module on $\mathfrak{a}^*$.

Note that the maps ad and ad^* are *not* a representation of $\mathfrak{a}$ since these maps do not satisfy (3.4).

3.3 The odd Lie-Poisson type bivector

In this section we introduce the notion of canonical odd bivector field $\Lambda_{\mathfrak{a}}$ on the dual space $\Pi\,\mathfrak{a}^*$.

The rank and the pencil of presymplectic forms Given a Lie antialgebra $\mathfrak{a}$, we call the dimension of the odd part $\mathfrak{a}_0$ of a Lie antialgebra $\mathfrak{a}$ the *rank* of $\mathfrak{a}$:

$$\operatorname{rk}\mathfrak{a} := \dim\,\mathfrak{a}_0.$$

Let $\mathfrak{a}$ be a Lie antialgebra of rank r. Fix an arbitrary basis $\{\varepsilon_1,\dots,\varepsilon_r\}$ of $\mathfrak{a}_0$. One obtains a set of r bilinear skew-symmetric (or presymplectic) forms: $\{\omega_1,\dots,\omega_r\}$ on $\mathfrak{a}_1$ by projection on the basic elements:

$$a\cdot b =: \sum_{i=1}^{r}\omega_i(a,b)\,\varepsilon_i, \tag{3.9}$$

for all $a,b\in\mathfrak{a}_1$.

Changing the basis, one obtains linear changes of the corresponding set of skew-symmetric forms. Therefore, the *pencil of presymplectic forms* $\langle\omega_1,\dots,\omega_r\rangle$ is well-defined. To each 2-form ω_i, one associates a bivector $\pi_i\in\wedge^2\mathfrak{a}_1^*$, that we can understand as a constant bivector field on $\mathfrak{a}_1^*$.

The adjoint vector fields To each element $\alpha\in\mathfrak{a}_0$, one associates an even linear operator $A_\alpha:\mathfrak{a}\to\mathfrak{a}$ defined by

$$A_\alpha\,|_{\mathfrak{a}_0} = \mathrm{ad}_\alpha, \qquad A_\alpha\,|_{\mathfrak{a}_1} = 2\,\mathrm{ad}_\alpha. \tag{3.10}$$

These linear operators can be, of course, viewed as *linear vector fields* on $\mathfrak{a}^*$.

We will denote $A_1,\dots,A_r$ the (even) vector fields on $\mathfrak{a}^*$ corresponding to the elements of the basis $\{\varepsilon_1,\dots,\varepsilon_r\}$.

Example 3.9. In the important case where $\mathfrak{a}_0$ contains the unit element ε and the center of $\mathfrak{a}$ is trivial (see Theorem 4 below), one has $A_\varepsilon=\mathcal{E}$, where $\mathcal{E}$ is the Euler vector field on the vector space $\mathfrak{a}^*$, i.e., the generator of the $\mathbb{K}^*$-action by homotheties.

The definition We will use the parity inversion functor Π. Consider the space $\Pi\mathfrak{a}_0^*$ and denote by $(\tau_1,\dots,\tau_r)$ the *Grassmann* coordinates dual to the chosen basis. Choose furthermore arbitrary linear coordinates $(x_1,\dots,x_n)$ on $\Pi\mathfrak{a}_1^*$.

Define the following odd bivector on $\Pi\mathfrak{a}^*$:

$$\Lambda_{\mathfrak{a}} = \sum_{i=1}^{r}\left(\frac{\partial}{\partial\tau_i}\wedge A_i + \tau_i\,\pi_i\right). \tag{3.11}$$

The corresponding antibracket on the space of (polynomial, smooth, etc.) functions on $\Pi\mathfrak{a}^*$ is defined as in (2.5). This antibracket is obviously linear, i.e., the space of linear functions on $\Pi\mathfrak{a}^*$ is stable.

Proposition 3.10. *Linear functions on the space* $(\Pi\mathfrak{a}^*, \Lambda_{\mathfrak{a}})$ *form a Lie antialgebra isomorphic to* $\mathfrak{a}$.

Proof. The antibracket of two even linear functions obviously corresponds to (3.9). The odd linear functions on $\Pi\mathfrak{a}^*$ are linear combinations of $\tau_1, \dots, \tau_r$. The antibracket of an odd and an even linear functions is given by $]\tau_i, \ell[= \frac{1}{2} A_i(\ell)$, where $\ell \in \mathfrak{a}_0$. This corresponds precisely to the adjoint action of ε_i on ℓ.

Finally, the antibracket of two odd linear functions is given by

$$]\tau_i, \tau_j[= \tfrac{1}{2}\left(A_i(\tau_j) + A_j(\tau_i)\right) \leftrightarrow \tfrac{1}{2}\left(\mathrm{ad}_{\varepsilon_i}\varepsilon_j + \mathrm{ad}_{\varepsilon_j}\varepsilon_i\right) = \varepsilon_i \cdot \varepsilon_j.$$

Hence the result. □

Corollary 3.11. *The bivector (3.11) is independent of the choice of the basis.*

Example 3.12. The bivectors (2.3) and (2.17) are precisely the canonical bivectors on $\Pi\, K_3(\mathbb{R})^*$ and $\Pi\, K_3(\mathbb{C})^*$, respectively.

Example 3.13. The bivector (2.3) makes sense in the case of a (purely even) commutative algebra, i.e., where $\mathfrak{a}_1 = \{0\}$. Let $\mathfrak{C}$ be a commutative associative algebra with basis $\{\varepsilon_1, \dots, \varepsilon_r\}$ and structural constants c_{ij}^k such that $\varepsilon_i \cdot \varepsilon_j = \sum_k c_{ij}^k \varepsilon_k$. Let $\tau_1, \dots, \tau_r$ be a set of Grassmann coordinates. The bivector (3.11) is then of the form

$$\Lambda_{\mathfrak{C}} = \sum_{i,j,k} c_{ij}^i \, \tau_k \, \frac{\partial}{\partial \tau_i} \wedge \frac{\partial}{\partial \tau_j}.$$

We understand this linear bivector field as commutative analog of the Lie-Poisson bivector.

The conformal Lie superalgebra The Lie superalgebra $\mathrm{Der}(\mathfrak{a})$ can be viewed as the algebra of *linear* vector fields on $\Pi\mathfrak{a}^*$ preserving the bivector $\Lambda_{\mathfrak{a}}$. It is natural to define (an infinite-dimensional) Lie superalgebra of "conformal derivations" of $\mathfrak{a}$ as the algebra of vector fields on $\Pi\mathfrak{a}^*$ preserving the bivector $\Lambda_{\mathfrak{a}}$. We simply drop the linearity condition.

Example 3.14. The algebra $\mathcal{K}(1)$ is the algebra of conformal derivations of K_3.

The notion of algebra of conformal derivations of $\mathfrak{a}$ deserves a special study. It would be also interesting to understand if there is a general notion of conformal Lie antialgebra. It makes sense to look for such a notion in terms of the algebra of functions on $\Pi\mathfrak{a}^*$ homogeneous with respect to the vector fields $A_1, \dots, A_r$.

3.4 Central extensions

In this section we define the notion of extension of a Lie antialgebra $\mathfrak{a}$ with coefficients in any $\mathfrak{a}$-module. It will be useful for the classification result of Section 4.2. The notion of extension is a part of a general cohomology theory that will be developed in [6].

Let $\mathfrak{a}$ be a Lie antialgebra and V an $\mathfrak{a}$-module. We will consider V as a trivial (or abelian) Lie antialgebra.

Definition 3.15. (a) An exact sequence of Lie antialgebras

$$0 \longrightarrow V \longrightarrow \widetilde{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow 0 \tag{3.12}$$

is called an *abelian extension* of the Lie antialgebra $\mathfrak{a}$ with coefficients in V. As a vector space, $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus V$, and the subspace V is obviously an $\mathfrak{a}$-module.

(b) An extension (3.12) is called *non-trivial* if the Lie antialgebra $\widetilde{\mathfrak{a}}$ is not isomorphic to the semi-direct sum $\mathfrak{a} \ltimes V$.

(c) If the subspace V belongs to the center of $\widetilde{\mathfrak{a}}$, then the extension (3.12) is called a *central extension.*

In this section we develop the notion of central extension. Since any central extension can be obtained by iteration of one-dimensional central extensions, it suffice to consider only the case of one-dimensional central extensions. One then has two possibilities:

$$\dim V = 0|1, \quad \text{or} \quad \dim V = 1|0.$$

We then say that the one-dimensional central extension is of *type I* or of *type II*, respectively.

Central extensions of type I The general form of central extensions of type I is as follows.

Proposition 3.16. *(i) A central extension of type I is defined by (an even) symmetric bilinear map $C : \mathfrak{a} \otimes \mathfrak{a} \to \mathbb{K}^{0|1}$ satisfying the the following conditions:*

$$\begin{aligned} C\,(\alpha,\, \beta \cdot a) \;=\; C\,(\beta,\, \alpha \cdot a) \;&=\; \tfrac{1}{2}\, C\,(\alpha \cdot \beta,\, a) \\ C\,(a \cdot b,\, c) + C\,(b \cdot c,\, a) + C\,(c \cdot a,\, b) \;&=\; 0, \end{aligned} \tag{3.13}$$

for all $\alpha, \beta \in \mathfrak{a}_0$ and $a, b, c \in \mathfrak{a}_1$.

(ii) A central extension is trivial if and only if there exists an even linear function $f : \mathfrak{a}_0 \to \mathbb{K}$ such that

$$C(\alpha, a) = f(\alpha \cdot a) \tag{3.14}$$

for all $\alpha \in \mathfrak{a}_1$ and $a \in \mathfrak{a}_0$.

Proof. Part (i). Given a map C as in (3.13), let us define an antibracket on $\mathfrak{a} \oplus \mathbb{K}$. We fix an element $z \in \mathbb{K}$ and set:

$$x \cdot y = (x \cdot y)_{\mathfrak{a}} + C(x, y)\, z, \qquad x \cdot z = 0, \tag{3.15}$$

for all $x, y \in \mathfrak{a}$. Note that, since the map C is symmetric and even, one has

$$C(x, y) = C\,(x_1, y_0) + C\,(y_1, x_0)\,, \tag{3.16}$$

where $x_1, y_1 \in \mathfrak{a}_1$ and $x_0, y_0 \in \mathfrak{a}_0$. One then easily checks that formula (3.15) defines a structure of a Lie antialgebra if and only if the relations (3.13) are satisfied.

Conversely, a Lie antialgebra structure on $\mathfrak{a}\oplus\mathbb{K}^{0|1}$ such that the subspace $\mathbb{K}^{0|1}$ belongs to the center is obviously of the form (3.15).

Part (ii). In the case where C is as in (3.14), the linear map $\mathfrak{a}\oplus\mathbb{K}^{0|1}\to\mathfrak{a}\oplus\mathbb{K}^{0|1}$ given by $(x,z)\mapsto(x,z+f(x))$ intertwines the structure (3.15) with the trivial direct sum structure. This means that the central extension is trivial.

Conversely, if the extension is trivial, then there exists an intertwining map $\mathfrak{a}\oplus\mathbb{K}^{0|1}\to\mathfrak{a}\oplus\mathbb{K}^{0|1}$ sending the structure (3.15) to the trivial one. This map can, again, be chosen in the form $(x,z)\mapsto(x,z+f(x))$, since a different choice of the embedding of $\mathbb{K}^{0|1}$ does not change the structure. □

We will call a map C satisfying (3.13) a 2-*cocycle of type* I. A 2-cocycle of the form (3.14) will be called a *coboundary*.

Central extensions of type II

Proposition 3.17. *(i) A central extension of type II is defined by an even symmetric bilinear map $C:\mathfrak{a}\otimes\mathfrak{a}\to\mathbb{K}$ satisfying the following identities:*

$$\begin{aligned} C(\alpha,\, a\cdot b) &= C(\alpha\cdot a,\, b)+C(a,\, \alpha\cdot b)\\ C(\alpha\cdot\beta,\,\gamma) &= C(\alpha,\,\beta\cdot\gamma)\,, \end{aligned} \tag{3.17}$$

for all $\alpha,\beta,\gamma\in\mathfrak{a}_0$ and $a,b\in\mathfrak{a}_1$.

(ii) The extension is trivial if and only if there exists an even linear functional $f:\mathfrak{a}_1\to\mathbb{K}$ such that

$$C(x,y)=f(x\cdot y), \tag{3.18}$$

for all $x,y\in\mathfrak{a}$.

Proof. The proof is similar to that of Proposition 3.16. □

We will call an odd map (3.17) satisfying (3.17) a 2-*cocycle of type* II. In the case where it is given by (3.18), the map C is called a coboundary.

Example 3.18. Consider the kernel of the presymplectic pencil of $\mathfrak{a}$:

$$\mathcal{I}=\bigcap_{i=1}^{r}\ker\omega_i,$$

where $r=\operatorname{rk}\mathfrak{a}$, the forms ω_i are defined by (3.9). In other words,

$$\mathcal{I}=\{a\in\mathfrak{a}_1\mid a\cdot b=0,\ \text{for all } b\in\mathfrak{a}_1\}.$$

Proposition 3.19. *The subspace $\mathcal{I}$ is an abelian ideal of $\mathfrak{a}$.*

Proof. By definition, $\mathcal{I}$ is an abelian subalgebra and the bracket of $a \in \mathcal{I}$ with any element $b \in \mathfrak{a}_1$ vanishes. One has to show that $\alpha \cdot a \in \mathcal{I}$, for arbitrary $\alpha \in \mathfrak{a}_0$ and $a \in \mathcal{I}$. Indeed, using the identity (1.4), one obtains

$$(\alpha \cdot a) \cdot b = \alpha \cdot \, (a \cdot b) - a \cdot \, (\alpha \cdot b) = 0$$

since for $a \in \mathcal{I}$ and every $b \in \mathfrak{a}_1$, one has $a \cdot b = 0$. □

It follows that the Lie antialgebra $\mathfrak{a}$ itself is an abelian extension of type I of the quotient-algebra $\mathfrak{a}/\mathcal{I}$:

$$0 \longrightarrow I \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a}/\mathcal{I} \longrightarrow 0.$$

Let us also outline the case where this extension has to be central.

Proposition 3.20. *If $\mathfrak{a}$ is ample, then the ideal $\mathcal{I}$ belongs to the center of $\mathfrak{a}$.*

Proof. The ideal $\mathcal{I}$ belongs to the center of $\mathfrak{a}$ if and only if the action of $\mathfrak{a}_0$ on $\mathcal{I}$ is trivial. Recall that $\mathfrak{a}$ is ample if map $\mathfrak{a}_1 \otimes \mathfrak{a}_1 \to \mathfrak{a}_0$ is surjective. Surjectivity means that for every $\alpha \in \mathfrak{a}_0$ there are $a, b \in \mathfrak{a}_1$ such that $\alpha = a \cdot b$. Using the Jacobi identity (1.5), one obtains for every $c \in \mathcal{I}$:

$$\alpha \cdot c = (a \cdot b) \cdot c = -\,(b \cdot c) \cdot a - (c \cdot a) \cdot b = 0,$$

since both summands in the right-hand-side vanish. □

The Lie antialgebra $\mathfrak{a}$ is therefore a central extension of $\mathfrak{a}/\mathcal{I}$ (of type I).

The case of the unit element Let us consider the case where the associative commutative algebra $\mathfrak{a}_0$ contains the unit element ε, and show the Lie antialgebra $\mathfrak{a}$ has essentially no non-trivial central extensions in this case.

Theorem 4. *If the Lie antialgebra $\mathfrak{a}$ contains the unit element $\varepsilon \in \mathfrak{a}_0$, then $\mathfrak{a}$ is a direct sum:*

$$\mathfrak{a} = \overline{\mathfrak{a}} \oplus Z(\mathfrak{a}), \tag{3.19}$$

of its center and the Lie antialgebra $\overline{\mathfrak{a}} = \mathfrak{a}/Z(\mathfrak{a})$ that has no non-trivial central extensions.

Proof. Let us consider the action of ε on $\mathfrak{a}_1$. The identity (1.3) implies the "half-projector" relation:

$$\mathrm{ad}^1_\varepsilon \circ \mathrm{ad}^1_\varepsilon = \tfrac{1}{2}\,\mathrm{ad}^1_\varepsilon.$$

Therefore, $\mathfrak{a}_1$ is split $\mathfrak{a}_1 = \mathfrak{a}_{1,\frac{1}{2}} \oplus \mathfrak{a}_{1,0}$ to a sum of $\frac{1}{2}$- and 0-eigenspaces, respectively. That is, $\mathrm{ad}_e|_{\mathfrak{a}_{1,\frac{1}{2}}} = \frac{1}{2}\mathrm{Id}$ and $\mathrm{ad}_e|_{\mathfrak{a}_{1,0}} = 0$.

Lemma 3.21. *The space $\mathfrak{a}_{1,0}$ coincides with the center of $\mathfrak{a}$.*

Proof. Let $a \in \mathfrak{a}_{1,0}$, that is, $\varepsilon \cdot a = 0$. One has to show that $x \cdot a = 0$ for all $x \in \mathfrak{a}$.

Let first $x = \alpha$ be an element of $\mathfrak{a}_0$. The identity (1.3) implies

$$\varepsilon \cdot (\alpha \cdot a) = \varepsilon \cdot (\alpha \cdot a) + \alpha \cdot (\varepsilon \cdot a) = (e \cdot \alpha) \cdot a = \alpha \cdot a.$$

But then $\mathrm{ad}^1_\varepsilon \circ \mathrm{ad}^1_\varepsilon = \frac{1}{2}\,\mathrm{ad}^1_\varepsilon$ implies $\alpha \cdot a = 0$.

In the case where $x = b$ is an element of $\mathfrak{a}_0$ the proof is similar. □

Let us show now that $\overline{\mathfrak{a}}$ has no non-trivial central extensions.

Let C be a 2-cocycle of type I on $\mathfrak{a}$. Apply the first identity (3.13) to $\beta = \varepsilon$, where ε is the unit. One has

$$C(\alpha, a) - C(\alpha, \varepsilon \cdot a) = C(\varepsilon, \alpha \cdot a).$$

If $a \in Z(\mathfrak{a})$, this formula implies $C(\alpha, a) = 0$. If a is an element of the $\frac{1}{2}$-eigenspace $\mathfrak{a}_{1,\frac{1}{2}}$ of the unit element ε, then one obtains $C(\alpha, a) = 2\,C(\varepsilon, \alpha \cdot a)$. Therefore, the cocycle C is a coboundary.

Let now C be a 2-cocycle of type II. It can be decomposed into a pair (C_0, C_1) of maps

$$C_0 : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \to \mathbb{K}, \qquad C_1 : \mathfrak{a}_1 \otimes \mathfrak{a}_1 \to \mathbb{K},$$

where C_0 is symmetric and C_1 is skew-symmetric. The first condition (3.17) gives

$$C_0(\varepsilon, a \cdot b[) = C_1(\varepsilon \cdot a, b) + C_1(a, \varepsilon \cdot b),$$

so that $C_1(a, b) = C_0(\varepsilon,]a, b[)$. The second condition (3.17) implies $C_0(\alpha, \beta) = C_0(\varepsilon, \alpha \cdot \beta)$. Therefore, the cocycle C is, again, a coboundary.

Proof of Theorem 4 is complete. □

4 Classification results

In this section, we prove that $K_3(\mathbb{C})$ is the only finite-dimensional complex simple Lie antialgebra. In the real case one has $K_3(\mathbb{R})$ and $K_3(\mathbb{C})$. This means that the situation is similar to the case of commutative algebras. We also prove that K_3 is characterized by the fact that its algebra of derivations is isomorphic to $\mathrm{osp}(1|2)$.

We also obtain a complete classification of finite-dimensional Lie antialgebras of rank 1, i.e., with $\dim \mathfrak{a}_0 = 1$. This, in particular, provides with a number of examples of Lie antialgebras, other that we already considered.

4.1 Classification of finite-dimensional simple Lie antialgebras

We call a Lie antialgebra $\mathfrak{a}$ *simple* if it contains no ideal except for the trivial one and $\mathfrak{a}$ itself. The classification of finite-dimensional simple Lie antialgebras is similar to that of commutative algebras.

Theorem 5. *(i) There exists a unique finite-dimensional complex simple Lie antialgebra.*

(ii) There are two finite-dimensional simple Lie antialgebras over $\mathbb{R}$.

Recall that the only simple finite-dimensional commutative algebras over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C} = \mathbb{R} + i\mathbb{R}$. In the complex case there is only $\mathbb{C}$ itself.

Let us start with the complex case. Let $\mathfrak{a}$ be a simple finite-dimensional Lie antialgebra. We will first assume that the commutative algebra $\mathfrak{a}_0$ has no nilpotent elements. As it is very well known, $\mathfrak{a}_0$ is of the form

$$\mathfrak{a}_0 = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{r \text{ times}}.$$

We will prove that if $\mathfrak{a}$ is simple, then $r = 1$.

Choose a basis $\{\varepsilon_1, \ldots, \varepsilon_r\}$ in $\mathfrak{a}_0$ such that $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$. As in Section 3.3, one associates with each element ε_i a presymplectic form ω_i on $\mathfrak{a}_1$ (see formula 3.9).

The following statement shows that, if $r > 1$, then the algebra $\mathfrak{a}$ contains a non-trivial ideal. Consider the subspace $\ker \omega_1 \subset \mathfrak{a}_1$ consisting of the elements $a \in \mathfrak{a}_1$ such that, for all b one has: $a \cdot b$ is a combination of ε_i with $i \geq 2$. Consider also the following subspace of $\mathfrak{a}$:

$$\mathcal{I} = \ker \omega_1 \oplus \langle \varepsilon_2, \cdots, \varepsilon_r \rangle.$$

Lemma 4.1. *The subspace $\mathcal{I}$ is an ideal of $\mathfrak{a}$.*

Proof. One has to prove that

(a) $\alpha \cdot \mathcal{I} \subset \mathcal{I}$, for an arbitrary $\alpha \in \mathfrak{a}_0$;

(b) $a \cdot \mathcal{I} \subset \mathcal{I}$, for an arbitrary $a \in \mathfrak{a}_1$.

Part (a). Let $a \in \ker \omega_1$, by identity (1.4), one has for an arbitrary $b \in \mathfrak{a}_1$:

$$(\alpha \cdot a) \cdot b = \alpha \cdot (a \cdot b) - a \cdot (\alpha \cdot b).$$

The both terms in the right-hand-side are combinations of ε_i with $i \geq 2$. Therefore, one obtains $\alpha \cdot a \in \ker \omega_1$.

Part (b). Let $\alpha \in \langle \varepsilon_2, \cdots, \varepsilon_r \rangle$, then $\varepsilon_1 \cdot \alpha = 0$. Let $a \in \mathfrak{a}_1$ be arbitrary, one has to prove that, again, $\alpha \cdot a \in \ker \omega_1$. Choose an arbitrary $b \in \mathfrak{a}_1$ and consider again the expression $(\alpha \cdot a) \cdot b$. Since this is an even element of $\mathfrak{a}$, it can be written in the form:

$$(\alpha \cdot a) \cdot b = \sum_{i=1}^{r} c_i \, \varepsilon_i.$$

One has to prove that $c_1 = 0$. One has $\varepsilon_1 \cdot ((\alpha \cdot a) \cdot b) = (\varepsilon_1 \cdot (\alpha \cdot a)) \cdot b + (\alpha \cdot a) \cdot (\varepsilon_1 \cdot b)$. But the first summand in the right-hand-side vanishes. Indeed, by (1.3), one has

$$\varepsilon_1 \cdot (\alpha \cdot a) = \tfrac{1}{2} (\varepsilon_1 \cdot \alpha) \cdot a = 0,$$

since $\varepsilon_1 \cdot \alpha = 0$. In the same way, one obtains

$$\varepsilon_1 \cdot (\varepsilon_1 \cdot ((\alpha \cdot a) \cdot b)) = (\alpha \cdot a) \cdot (\varepsilon_1 \cdot (\varepsilon_1 \cdot b)).$$

However, for the left-hand-side, one obtains using (1.2):

$$\varepsilon_1 \cdot (\varepsilon_1 \cdot ((\alpha \cdot a) \cdot b)) = (\varepsilon_1 \cdot \varepsilon_1) \cdot ((\alpha \cdot a) \cdot b) = \varepsilon_1 \cdot ((\alpha \cdot a) \cdot b) = c_1$$

since $\varepsilon_1 \cdot \varepsilon_1 = \varepsilon_1$; while, for the right-hand-side, one gets using (1.3):

$$(\alpha \cdot a) \cdot (\varepsilon_1 \cdot (\varepsilon_1 \cdot b)) = \tfrac{1}{2}(\alpha \cdot a) \cdot ((\varepsilon_1 \cdot \varepsilon_1) \cdot b) = \tfrac{1}{2}(\alpha \cdot a) \cdot (\varepsilon_1 \cdot b) = \varepsilon_1 \cdot ((\alpha \cdot a) \cdot b) = \tfrac{1}{2} c_1.$$

Therefore, $c_1 = 0$ and so $\alpha \cdot a \in \ker \omega_1$. The result follows. □

Lemma 4.1 implies that $\dim(\mathfrak{a}_0) = 1$. Choose a non-zero element $\varepsilon \in \mathfrak{a}_0$ and denote ω the corresponding 2-form.

Lemma 4.2. *The form ω is of rank 2.*

Proof. Choose a canonical (Darboux) basis $\{a_1 \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m\}$, so that

$$\omega(a_k, b_\ell) = \delta_{k\ell}, \qquad \omega(a_k, a_\ell) = \omega(b_k, b_\ell) = \omega(c_k, .) = 0.$$

where $k, \ell = 1, \dots, n$. Let us show that $n = 1$. First, $n \geq 1$, since otherwise $\omega = 0$ and $\mathfrak{a}_1$ is an ideal. It follows that there are some elements a_k and b_k such that $\varepsilon = a_k \cdot b_k$.

Assume that $n > 1$. The identity (1.5) implies

$$\varepsilon \cdot a_\ell = (a_k \cdot b_k) \cdot a_\ell = -(b_k \cdot a_\ell) \cdot a_k - (a_\ell \cdot a_k) \cdot b_k = 0$$

for any $k \neq \ell$. It follows that $\varepsilon \cdot a = 0$, for any $a \in \mathfrak{a}_1$. Furthermore,

$$\varepsilon \cdot \varepsilon = \varepsilon \cdot (a_k \cdot b_k) = (\varepsilon \cdot a_k) \cdot b_k + a_k \cdot (\varepsilon \cdot b_k) = 0,$$

for any $i = 1, \dots, n$. Therefore, ε belongs to the center of $\mathfrak{a}$; in particular, $\mathfrak{a}$ cannot be simple. This is a contradiction. □

Lemma 4.1 and Lemma 4.2 imply Theorem 5 in the complex case, where the commutative algebra $\mathfrak{a}_0$ has no nilpotent elements. If now $\mathfrak{a}_0 = \mathbb{C}^n \ltimes \mathcal{N}$, where $\mathcal{N}$ is a nilpotent ideal, then the same arguments prove that $n \leq 1$ and $\ker \omega_1 \oplus \mathcal{N}$ is an ideal. Theorem 5 is proved in the complex case.

The real case immediately follows from the complex one. Indeed, let $\mathfrak{a}$ be a real simple Lie antialgebra, the standard arguments show that the complexification $\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ is either simple or the direct sum of two isomorphic simple ideals.

Theorem 5 is proved.

Proof of Theorem 1 Let $\mathfrak{a}$ be a finite-dimensional commutative $\mathbb{Z}_2$-graded algebra such that $\mathrm{Der}(\mathfrak{a}) \cong \mathrm{osp}(1|2)$. As an $\mathrm{osp}(1|2)$-module, $\mathfrak{a}$ is a direct sum of irreducible modules. Recall that finite-dimensional irreducible $\mathrm{osp}(1|2)$-modules are the modules $\mathcal{D}(h/2)$ with positive integer h. These modules are of dimension $2h + 1$ and of highest weight h. It follows that for two elements of $\mathfrak{a}$ such that $x_1 \in \mathcal{D}(h_1/2)$ and $x_2 \in \mathcal{D}(h_2/2)$, one has:

$$x_1 \cdot x_2 \in \mathcal{D}((h_1 + h_2)/2). \tag{4.1}$$

Since $\mathfrak{a}$ is finite-dimensional, there exists a non-zero submodule $\mathcal{D}(h'/2) \subset \mathfrak{a}$ with maximal h'.

If $h' \neq 0$, then (4.1) implies that $\mathcal{D}(h'/2)$ belongs to the center of $Z(\mathfrak{a})$. However, the Lie superalgebra $\mathrm{End}(Z(\mathfrak{a}))$ is a subalgebra of $\mathrm{Der}(\mathfrak{a})$ so that $\mathrm{Der}(\mathfrak{a})$ cannot be isomorphic to $\mathrm{osp}(1|2)$ if $\mathfrak{a}$ has a non-trivial center.

Theorem 1 is proved.

4.2 Lie antialgebras of rank 1

Let us assume that the commutative algebra $\mathfrak{a}_0$ is one-dimensional. There are two different possibilities:

1. $\mathfrak{a}_1 = \mathbb{K}$, as a commutative algebra, so that it contains the unit element denoted by ε, such that $\varepsilon \cdot \varepsilon = \varepsilon$;

2. $\mathfrak{a}_1$ is nilpotent and the only odd generator α satisfies $\alpha \cdot \alpha = 0$.

The structure of the algebra $\mathfrak{a}$ is characterized by one bilinear skew-symmetric form on $\mathfrak{a}_1$ defined by

$$a \cdot b = \omega(a, b)\, \alpha,$$

where α is a (unique up to a constant) non-zero element of $\mathfrak{a}_1$.

Let us first construct several examples of Lie antialgebras of rank 1.

A. The form ω is non-degenerate **(A1)** The $(2n)|1$-dimensional nilpotent Lie antialgebra with basis $\{a_1, b_1, \ldots, a_n, b_n; \alpha\}$ that appeared in the case a) of the above proof is characterized by the relations

$$a_i \cdot b_j = \delta_{ij}\, \alpha, \tag{4.2}$$

where $i, j = 1, \ldots, n$ and all other products of the basic elements vanish. We call it the *Heisenberg antialgebra* and denote it by ah_n. This algebra is ample.

Remark 4.3. Notice, that the relations (4.2) are exactly as those of the standard Heisenberg Lie algebra, but the central element α is odd. As in the usual Lie case, the Heisenberg antialgebra ah_n is a central extension of type II of the abelian Lie antialgebra $\mathbb{K}^{2n}$.

(A2) Another interesting example is a family of Lie antialgebras of dimension $1|2$. The basis of these Lie antialgebras will be denoted by $\{\alpha;\, a, b\}$; the commutation relations are

$$a \cdot b = \alpha, \qquad \alpha \cdot a = \kappa\, b \tag{4.3}$$

where κ is a constant and the other products vanish. If $\kappa = 0$, then this is just the Heisenberg antialgebra ah_1, if $\kappa \neq 0$, then this is a non-trivial deformation of ah_1.

In the case $\mathbb{K} = \mathbb{C}$, all of the Lie antialgebras (4.3) with $\kappa \neq 0$ are isomorphic to each other. We call this algebra *twisted Heisenberg antialgebra* and denote it by $\widetilde{\mathrm{ah}}_1$.

If $\mathbb{K} = \mathbb{R}$, however, the sign of κ is an invariant. We will assume:

$$\kappa = 1, \quad \text{if} \quad \mathbb{K} = \mathbb{C}, \qquad \kappa = \pm 1, \quad \text{if} \quad \mathbb{K} = \mathbb{R}.$$

One thus gets two different Lie antialgebras: $\widetilde{\mathrm{ah}}_1^+$ and $\widetilde{\mathrm{ah}}_1^-$.

We are ready to formulate a partial result.

Proposition 4.4. *The complete list of the real Lie antialgebras of rank 1 with a non-degenerate 2-form ω is as follows:*

$$K_3, \qquad \mathrm{ah}_n, \qquad \widetilde{\mathrm{ah}}_1^+, \qquad \widetilde{\mathrm{ah}}_1^-; \tag{4.4}$$

in the complex case, the Lie antialgebras $\widetilde{\mathrm{ah}}_1^+$ and $\widetilde{\mathrm{ah}}_1^-$ are isomorphic.

Proof. Consider first the case where $\mathfrak{a}_1$ is not nilpotent, i.e., $\alpha \cdot \alpha \neq 0$. We already proved that, in this case, $\mathfrak{a}$ is of dimension $1|2$, see Lemma 4.2. Therefore, $\mathfrak{a} = K_3$.

Assume that $\alpha \cdot \alpha = 0$. If ω is of rank $n > 1$, one proves, in the same way as in Lemma 4.2, that $\alpha \cdot a = 0$ for all a, so that $\mathfrak{a} = \mathrm{ah}_n$.

If, finally, ω is of rank 1, then the identity (1.3) implies that $(\mathrm{ad}^1_\alpha)^2 = 0$. One then easily shows that any such operator is equivalent to ad^1_α in (4.3) up to the area preserving changes of the basis. It follows that $\mathfrak{a} = \widetilde{\mathrm{ah}}_1^+$ or $\widetilde{\mathrm{ah}}_1^-$. □

B. The 2-form ω is identically zero The Lie antialgebra $\mathfrak{a}$ is then non-ample and determined by the operator ad_α.

(B1) If $\mathfrak{a}_1$ contains the unit element ε, then $\mathfrak{a}$ is split into a direct sum (3.19). The centerless Lie antialgebra $\overline{a}$ has the basis $\{\varepsilon;\, a_1, \dots, a_n\}$ with the following set of relations:

$$\varepsilon \cdot \varepsilon = \varepsilon, \qquad \varepsilon \cdot a_i = \tfrac{1}{2}\, a_i, \qquad a_i \cdot a_j = 0.$$

We call this Lie antialgebra the *affine antialgebra* and denote by $\mathrm{aaf}(n)$. One then has $\mathfrak{a} = \mathrm{aaf}(n) \oplus Z$, where Z is the center of $\mathfrak{a}$.

(B2) If $\alpha \cdot \alpha = 0$ for all $\alpha \in \mathfrak{a}_0$, then $\mathrm{ad}^1_\alpha \circ \mathrm{ad}^1_\alpha = 0$. These are very degenerated Lie antialgebras and their classification is equivalent to the classification of nilpotent (of order 2) linear operators. We do not discuss here this problem of linear algebra.

Let us summarize the above considerations.

Proposition 4.5. *A Lie antialgebras of rank 1 with $\omega = 0$ is one of the following two classes:*

$$\mathfrak{a} = \mathrm{aaf}(n) \oplus Z, \qquad \mathfrak{a} \textit{ is of type (B2).} \tag{4.5}$$

C. The "mixed case" $2 < \mathrm{rk}\,\omega < \dim \mathfrak{a}_0$ **(C1)** Define a $1|3$-dimensional Lie antialgebra with basis $\{\alpha;\, a, b, z\}$ and the relations

$$a \cdot b = \alpha, \qquad \alpha \cdot a = z \tag{4.6}$$

and all other products vanish. We denote this Lie antialgebra $\widehat{\mathrm{ah}}_1$. The element z spans the center, so that this is a central extension (of type I) of ah_1.

(C2) Define a 1|4-dimensional Lie antialgebra with the basis $\{\alpha;\, a, b, z_1, z_2\}$ and the relations

$$a \cdot b = \alpha, \qquad \alpha \cdot a = z_1, \qquad \alpha \cdot b = z_2 \tag{4.7}$$

and all other products vanish. We denote this Lie antialgebra $\widehat{\widehat{\mathrm{ah}}}_1$. This is a central extension (of type I) of the above algebra $\widehat{\mathrm{ah}}_1$.

We are now ready to formulate the main statement of this section.

Theorem 6. *A Lie antialgebra of rank 1 is of the form $\mathfrak{a} = \overline{\mathfrak{a}} \oplus Z$, where $\overline{\mathfrak{a}}$ belongs either to the list (4.4), or to the list (4.5), or one of the antialgebras $\widehat{\mathrm{ah}}_1, \widehat{\widehat{\mathrm{ah}}}_1$.*

Proof. We already proved the theorem in the following two cases: the form ω is non-degenerate, or $\omega \equiv 0$. It remains to consider the intermediate case where the 2-form ω is not identically zero but with a non-trivial kernel: $\mathcal{I} = \ker \omega \neq \{0\}$. The space $\mathcal{I}$ is then an abelian ideal (see Proposition 3.19) and, furthermore, belongs to the center (see Proposition 3.20). We summarize this in a form of a

Lemma 4.6. *The Lie antialgebra $\mathfrak{a}$ is a central extension of $\mathfrak{a}/\mathcal{I}$.*

To complete the classification, one now has to classify the central extensions of type I of the antialgebras with non-degenerate form ω, that is, of (4.2) and (4.3).

Lemma 4.7. *The Lie antialgebras ah_n with $n \geq 2$ and $\widetilde{\mathrm{ah}}_1$ (resp. $\widetilde{\mathrm{ah}}_1^+, \widetilde{\mathrm{ah}}_1^-$) have no non-trivial central extensions of type I.*

Proof. Let C be a 2-cocycle of type I on ah_n with $n \geq 2$. One has

$$C\,(\alpha,\, a_i) = C\,(a_j \cdot b_j,\, a_i) = -C\,(b_j \cdot a_i,\, a_j) - C\,(a_i \cdot a_j,\, b_j) = 0$$

(from the second identity (3.13)) for all $i \neq j$. Similarly, $C(\alpha, b_i) = 0$ for all i. Therefore, C is identically zero.

For the Lie antialgebra $\widetilde{\mathrm{ah}}_1$ and an arbitrary 2-cocycle C of type I, one has

$$C\,(\alpha, b) = \tfrac{1}{\kappa}\, C\,(\alpha,\, \alpha \cdot a) = \tfrac{1}{2\kappa}\, C\,(\alpha \cdot \alpha,\, a) = 0.$$

The cocycle C is then defined by its values on α and a. Let us show that the corresponding extension is trivial. Let $C(\alpha, a) = cz$, where z is an arbitrary generator of the center and c arbitrary constant. Set $b' = b + \frac{c}{\kappa}\, z$. In the basis $\{\alpha;\, a, b', z\}$, the cocycle C' vanishes. $\square$

Lemma 4.8. *The algebra ah_1 has a unique non-trivial central extension of type I.*

Proof. Indeed, let C be a 2-cocycle of type I on ah_1. It is given by the formula

$$C(\alpha, a) = c_1\, z, \qquad C(\alpha, b) = c_2\, z,$$

where z is an element of the center and c_1 and c_2 are arbitrary constants. If $c_1 \neq 0$, then choose another element of the basis: $b' = b + \frac{c_2}{c_1}\, a$. One obtains $C(\alpha, b) = 0$. Now taking $z' = c_1 z$, one gets precisely the Lie antialgebra $\widehat{\mathrm{ah}}_1$. This Lie antialgebra is not isomorphic to ah_1 so that the extension is, indeed, non-trivial. □

Lemma 4.9. *The algebra* $\widehat{\widehat{\mathrm{ah}}}_1$ *is the unique non-trivial central extension of type I of* $\widehat{\mathrm{ah}}_1$.

Proof. It is similar to the proof of Lemma 4.8. □

In the same way, one proves that the Lie antialgebra $\widehat{\widehat{\mathrm{ah}}}_1$ has no non-trivial central extensions of type I. We thus classified all the non-trivial central extensions of the Lie antialgebras of rank 1 with non-degenerate form ω.

Theorem 6 is proved. □

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References

[1] G. Benkart, A. Elduque, *A new construction of the Kac Jordan superalgebra*, Proc. Amer. Math. Soc. **130** (2002), 3209–3217.

[2] C. Conley, *Conformal symbols and the action of contact vector fields over the superline*, J. Reine Angew. Math., **633** (2009), 115–163.

[3] C. Duval, J-P. Michel, *On the projective geometry of the supercircle: the unified construction of the super cross-ratio and Schwarzian derivative*, Int. Math. Res. Not., 2008, no. 14.

[4] H. Gargoubi, N. Mellouli, V. Ovsienko, *Differential operators on supercircle: conformally equivariant quantization and symbol calculus*, Lett. Math. Phys. **79** (2007), 51–65.

[5] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Lecture Notes in Math., **570**, Springer, Berlin, 1977.

[6] P. Lecomte, V. Ovsienko, *Alternated Hochschild Cohomology*, in preparation.

[7] S. Leidwanger, S. Morier-Genoud, *Universal enveloping algebras of Lie antialgebras, and representations*, Algebr. Represent. Theory **13** (2010).

[8] K. McCrimmon, *Kaplansky superalgebras*, J. Algebra, **164** (1994), 656–694.

[9] S. Morier-Genoud, *Representations of* asl_2, Intern. Math. Res. Notices., 2009.

[10] V. Ovsienko, *Lie antialgebras*, arXiv:0705.1629.

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