



# Lie antialgebras

Valentin Ovsienko

► To cite this version:

| Valentin Ovsienko. Lie antialgebras. 2007. hal-00145724v3

**HAL Id: hal-00145724**

**<https://hal.science/hal-00145724v3>**

Preprint submitted on 13 Aug 2007 (v3), last revised 14 Oct 2010 (v6)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Lie antialgebras

V. Ovsienko

HATKE

## Abstract

We introduce a new class of algebras that we call Lie antialgebras. The subject is situated in between commutative algebra, symplectic/contact geometry and Lie superalgebra theory. We define the odd Lie-Poisson structure on the space dual to a Lie antialgebra and study the notion of central extensions. We classify simple finite-dimensional Lie antialgebras.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Simple Lie antialgebras</b>	<b>5</b>
2.1	Introducing $\mathfrak{asl}(2)$ and the odd bivector . . . . .	5
2.2	Lie antialgebra $\mathfrak{asl}(2, \mathbb{C})$ and a pair of symplectic structures . . . . .	7
<b>3</b>	<b>Conformal Lie antialgebra <math>\mathcal{AK}(1)</math></b>	<b>9</b>
3.1	Relation to symplectic geometry . . . . .	10
3.2	Conformal Lie superalgebra $\mathcal{K}(1)$ as the algebra of symmetry . . . . .	11
3.3	Representation of $\mathcal{AK}(1)$ by tangent vector fields . . . . .	14
3.4	$\mathcal{AK}(1)_{\mathbb{C}}$ and $\mathcal{K}(1)_{\mathbb{C}}$ : the bi-Hamiltonian formalism . . . . .	15
<b>4</b>	<b>Elements of the general theory</b>	<b>16</b>
4.1	Representations and modules . . . . .	17
4.2	The odd Lie-Poisson bivector . . . . .	18
4.3	Algebras of symmetry . . . . .	20
4.4	Central extensions . . . . .	21
4.5	Kernel of the presymplectic pencil . . . . .	23
4.6	Restrictive role of the unit . . . . .	23
<b>5</b>	<b>Classification results</b>	<b>25</b>
5.1	Proof of Theorem 1 . . . . .	25
5.2	Lie antialgebras of rank 1 . . . . .	27

# 1 Introduction

We call *antialgebra* a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$  with an *odd* product, that is a bilinear map from  $\mathfrak{a} \otimes \mathfrak{a}$  to  $\mathfrak{a}$  changing the parity: if  $a \in \mathfrak{a}_i$  and  $b \in \mathfrak{a}_j$  with  $i, j \in \{0, 1 \mid \text{mod} 2\}$ , then  $a \cdot b \in \mathfrak{a}_{i+j+1}$ . Such structures are well-known and often appear in the context of Poisson geometry. One example is the Buttin bracket (or the “Poisson antibracket”), see, e.g., [1, 13, 14]. Another one is the usual Schouten bracket on the space of skew-symmetric contravariant tensors in the Batalin-Vilkovisky formalism, see [12] for a survey. All these notions are related to graded Lie superalgebras.

We introduce a new class of antialgebras.

**Definition 1.1.** An antialgebra  $\mathfrak{a}$  is called a *Lie antialgebra* if its product, denoted by  $] , [$ :  $\mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}$ , is skew-symmetric:

$$]x, y[ = -(-1)^{p(x)p(y)} ]y, x[, \quad (1.1)$$

where  $p$  is the parity function, and satisfies the following properties.

1. The *odd* part  $\mathfrak{a}_1$  is a *commutative associative* algebra, i.e., one has

$$] \alpha, ] \beta, \gamma [[ = ] \alpha, \beta [, \gamma [ \quad (1.2)$$

for all  $\alpha, \beta, \gamma \in \mathfrak{a}_1$ .

2. The map  $] , [ : \mathfrak{a}_1 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_0$  satisfies the relation

$$] \alpha, ] \beta, a [[ = ] \beta, ] \alpha, a [[ = \frac{1}{2} ] \alpha, \beta [, a [ \quad (1.3)$$

for all  $\alpha, \beta \in \mathfrak{a}_1$  and  $a \in \mathfrak{a}_0$ . In other words, the map  $\rho : \mathfrak{a}_1 \rightarrow \text{End}(\mathfrak{a}_0)$  defined by

$$\rho_\alpha(a) := 2 ] \alpha, a [$$

is an *action* of the commutative algebra  $\mathfrak{a}_1$  on  $\mathfrak{a}_0$ :

$$\rho_\alpha \circ \rho_\beta = \rho_\beta \circ \rho_\alpha = \rho_{] \alpha, \beta [}.$$

The skew-symmetric map  $] , [ : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$  satisfies

3. the Leibniz identity:

$$] \alpha, ] b, c [[ = ] \alpha, b [, c [ + ] b, ] \alpha, c [[ \quad (1.4)$$

for all  $\alpha \in \mathfrak{a}_1$  and  $b, c \in \mathfrak{a}_0$ .

4. the Jacobi-type identity:

$$] a, ] b, c [[ + ] b, ] c, a [[ + ] c, ] a, b [[ = 0 \quad (1.5)$$

for all  $a, b, c \in \mathfrak{a}_0$ .

A Lie antialgebra is, therefore, nothing but a commutative associative algebra  $\mathfrak{a}_1$ , together with an  $\mathfrak{a}_1$ -module  $\mathfrak{a}_0$ , equipped with a skew-symmetric bilinear form with coefficients in  $\mathfrak{a}_1$ . This bilinear form has to satisfy two extra conditions, namely (1.4) and (1.5). The situation can, of course, be considered purely from the commutative algebra viewpoint. The main purpose of this paper is to show, however, that the notion of Lie antialgebra has a strong similarity with Lie algebras (and Lie superalgebras). This suggests another, “Lie algebraic” approach. We make first steps in this direction.

The notion of Lie antialgebra seems to be non-equivalent to any known algebraic structure. In particular, the usual parity-inverting functor

$$\Pi : \mathfrak{a}_0 \oplus \mathfrak{a}_1 \rightarrow \mathfrak{a}_1 \oplus \mathfrak{a}_0$$

transforming an antialgebra structure to a usual algebra (which is non-associative with additional properties) does not make much sense here. The reason is as follows. The Lie superalgebra “of symmetry”,  $\mathfrak{g}_{\mathfrak{a}} = \text{Der}(\mathfrak{a})$ , acts on  $\mathfrak{a}$  and preserves the algebraic structure. The condition of  $\mathfrak{g}_{\mathfrak{a}}$ -invariance is, of course, understood in the graded sense:

$$\varrho_x ]a, b[ = (-1)^{p(x)} ]\varrho_x a, b[ + (-1)^{p(x)(1+p(a))} ]a, \varrho_x b[, \quad (1.6)$$

where  $x \in \mathfrak{g}_{\mathfrak{a}}$ ,  $a, b \in \mathfrak{a}$ ,  $\varrho : \mathfrak{g}_{\mathfrak{a}} \rightarrow \text{End}(\mathfrak{a})$  is the action and  $p$  is the parity function. The sign rule in (1.6) is due to the fact that  $]\cdot, \cdot[$  is an odd operation. The functor  $\Pi$  then has no  $\mathfrak{g}_{\mathfrak{a}}$ -invariant meaning. This notion of algebra of symmetry is important since it often allows to characterize the initial Lie antialgebra  $\mathfrak{a}$  as the unique Lie antialgebra preserved by the  $\mathfrak{g}_{\mathfrak{a}}$ -action.

The origin of Lie antialgebras is related to symplectic geometry. The first examples come from the standard symplectic  $\mathbb{Z}_2$ -graded space (over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) of dimension  $2|1$ . It turns out that there is an odd bivector  $\Lambda$  which is preserved by the Lie supergroup of linear orthosymplectic transformations<sup>1</sup>. In this way we obtain our first example of a  $2|1$ -dimensional Lie antialgebra that we denote by  $\text{asl}(2, \mathbb{K})$ . The Lie antialgebra  $\text{asl}(2, \mathbb{K})$  simple, that is, it contains no non-trivial ideal.

Already this first example provides as many examples of Lie antialgebras as of commutative ones. Since the tensor product of a Lie antialgebra with an arbitrary commutative algebra  $\mathfrak{C}$  is again a Lie antialgebra, one defines

$$\text{asl}(2, \mathfrak{C}) = \mathfrak{C} \otimes_{\mathbb{K}} \text{asl}(2, \mathbb{K}).$$

The odd part of this Lie antialgebra is  $\mathfrak{C}$  and the even part is  $\mathfrak{C} \oplus \mathfrak{C}$ . If, furthermore,  $\mathfrak{C}$  is a field (e.g., of rational functions on an algebraic manifold, or Laurent series, etc.) then  $\text{asl}(2, \mathfrak{C})$  is a simple infinite-dimensional Lie antialgebra.

The same odd bivector  $\Lambda$  on  $\mathbb{R}^{2|1}$  defines an antibracket on the space of homogeneous functions and leads to a very interesting simple infinite-dimensional Lie antialgebra that we call the conformal Lie antialgebra and denote  $\mathcal{AK}(1)$ . This is a conformal version of  $\text{asl}(2, \mathbb{R})$ . This Lie antialgebra is closely related to the well-known conformal Lie superalgebra  $\mathcal{K}(1)$ , also known as the (centerless) Neveu-Schwarz algebra; we show that  $\mathcal{K}(1)$  is the algebra of symmetry:

$$\mathcal{K}(1) = \text{Der}(\mathcal{AK}(1)).$$

---

<sup>1</sup>The bivector  $\Lambda$  was introduced in [4])

The odd part of the conformal Lie antialgebra is the algebra of Laurent polynomials in one variable:

$$\mathcal{K}(1)_1 = \mathbb{K}[t, t^{-1}],$$

or Laurent series according to the choice of topology.

It is natural to consider a similar problem for the symplectic space of any dimension  $n|m$ . However, in this case, the situation is more complicated: there is no odd bivector preserved by the supergroup of linear symplectic transformations. We consider the space  $\mathbb{R}^{4|2}$  equipped with a pair of linear symplectic structures. There exists an odd bivector preserved by the supergroup of linear “bi-symplectic” transformations. This 4|2-dimensional odd invariant bivector is related to  $\mathfrak{asl}(2, \mathbb{C})$ , the conformal version is  $\mathcal{AK}(1)_{\mathbb{C}}$ . The algebra of conformal symmetry is the Lie superalgebra  $\mathcal{K}(1)_{\mathbb{C}}$ . We conjecture that, in the multi-dimensional case, Lie superalgebras are always related to multi-Hamiltonian structures.

Let us mention that Lie antialgebras are related not only to symplectic geometry but also to contact geometry. In the natural case where the commutative subalgebra  $\mathfrak{a}_1$  contains the unit element  $\varepsilon$ , its action is given by the Euler vector field, the generator of homotheties. This defines a natural projection

$$\mathbb{R}^{2n|m} \setminus \{0\} \longrightarrow \mathbb{RP}^{2n-1|m}$$

from the symplectic vector space to the contact projective space. Amazingly, Lie antialgebras are represented by vector fields tangent to the contact distribution, whereas the corresponding symmetry algebras are represented by contact vector fields.

The results of this paper are as follows.

**A.** We classify finite-dimensional simple Lie antialgebras. It turns out that  $\mathfrak{asl}(2, \mathbb{C})$  is the only complex simple finite-dimensional Lie antialgebra; in the real case, there are two simple finite-dimensional Lie antialgebras:  $\mathfrak{asl}(2, \mathbb{R})$  and  $\mathfrak{asl}(2, \mathbb{C})$  viewed as a 4|2-dimensional real Lie antialgebra.

We also classify Lie antialgebras of rank 1. This, in particular, provides a number of examples of Lie antialgebras other than simple ones.

**B.** We introduce the conformal Lie antialgebra  $\mathcal{AK}(1)$  and define its representation by vector fields on  $\mathbb{R}^{1|1}$  tangent to the contact distribution. The complexified version  $\mathcal{AK}(1)_{\mathbb{C}}$  (and  $\mathfrak{asl}(2, \mathbb{C})$ ) is described in terms of bi-Hamiltonian formalism.

**C.** We also consider a few very simple ingredients of the general theory. We introduce the notions of representation and module over a Lie antialgebra. We define the canonical odd bivector on the dual space  $\mathfrak{a}^*$  of an arbitrary Lie antialgebra  $\mathfrak{a}$ . This bivector is an analog of the classical Poisson-Lie-Berezin-Kirillov-Kostant-Souriau structure on the dual of a Lie (super)algebra, and this is the way Lie antialgebras are related to symplectic and contact geometry. We define the notion of symmetry superalgebra and conformal symmetry superalgebra and relate the context to multi-Hamiltonian formalism. We also introduce the notions of central extension of a Lie antialgebra. We prove that existence of the unit element  $\varepsilon \in \mathfrak{a}_1$  implies that the Lie antialgebra  $\mathfrak{a}$  has no non-trivial central extension.

In the end of this paper, we formulate a number of open problems and suggest some further developments.

## 2 Simple Lie antialgebras

We call a Lie antialgebra  $\mathfrak{a}$  *simple* if it contains no ideal except for the trivial one and  $\mathfrak{a}$  itself. It turns out that the classification of finite-dimensional simple Lie antialgebras is similar to that of commutative algebras.

**Theorem 1.** (i) *There exists a unique finite-dimensional complex simple Lie antialgebra.*  
(ii) *There are two finite-dimensional simple Lie antialgebras over  $\mathbb{R}$ .*

Recall that the only simple finite-dimensional commutative algebras over  $\mathbb{R}$  are  $\mathbb{R}$  and  $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ . In the complex case there is only  $\mathbb{C}$  itself.

In this section, we construct the simple Lie antialgebras build on  $\mathbb{R}$  and  $\mathbb{C}$  as the odd parts. We also calculate the corresponding Lie superalgebras of symmetry. The proof of the uniqueness part of Theorem 1 is postponed to Section 5.1.

### 2.1 Introducing $\mathfrak{asl}(2)$ and the odd bivector

Our first example of a Lie antialgebra is a simple Lie antialgebra of dimension  $2|1$ . The ground field in this section is  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ .

**Definition 2.1.** The basis of  $\mathfrak{asl}(2, \mathbb{K})$  is denoted by  $\{a, b; \varepsilon\}$  and the relations between the elements of the basis are as follows:

$$]a, b[ = \frac{1}{2} \varepsilon, \quad ]\varepsilon, a[ = \frac{1}{2} a, \quad ]\varepsilon, b[ = \frac{1}{2} b, \quad ]\varepsilon, \varepsilon[ = \varepsilon. \quad (2.1)$$

It is easy to check that the conditions 1–4 of Definition 1.1 are satisfied. The Lie antialgebra  $\mathfrak{asl}(2, \mathbb{K})$  is obviously simple.

The origin of the defined structure is related to  $2|1$ -dimensional linear symplectic geometry. Consider the vector space  $\mathbb{K}^{2|1}$  equipped with the standard symplectic form

$$\omega = dp \wedge dq + \frac{1}{2} d\tau \wedge d\tau, \quad (2.2)$$

where  $p$  and  $q$  are the usual even coordinates on  $\mathbb{K}^2$  and  $\tau$  is the formal Grassmann variable so that  $\tau^2 = 0$ . An equivalent way to define the above symplectic structure is to introduce the Poisson bivector on  $\mathbb{K}^{2|1}$ :

$$\mathcal{P} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \frac{1}{2} \frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau}. \quad (2.3)$$

which is inverse to the symplectic form:  $\mathcal{P} = \omega^{-1}$ .

The Lie superalgebra  $\mathfrak{osp}(1|2)$  is defined as the of linear transformations preserving the symplectic structure. The bivector (2.3) is the unique (up to a multiplicative constant) even bivector invariant with respect to the action of  $\mathfrak{osp}(1|2)$ .

It turns out that there exists another, *odd*,  $\mathfrak{osp}(1|2)$ -invariant bivector on  $\mathbb{K}^{2|1}$ .

**Proposition 2.2.** *There exists a unique (up to a multiplicative constant) odd bivector invariant with respect to the action of  $\mathfrak{osp}(1|2)$ :*

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \varepsilon + \tau \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}, \quad (2.4)$$

where

$$\mathcal{E} = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} + \tau \frac{\partial}{\partial \tau}$$

is the Euler field.

*Proof.* The  $\mathfrak{osp}(1|2)$ -invariance of  $\Lambda$  is a very easy check. Note that in the next section we will prove a much stronger statement see Theorem 2.

Let us prove the uniqueness. An arbitrary odd bivector on  $\mathbb{K}^{2|1}$  is given by

$$\Lambda = \frac{\partial}{\partial \tau} \wedge A + \tau F \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q},$$

where  $A$  is an even vector field and  $F$  is an even function. Let  $X$  be an even vector field, one has

$$L_X \Lambda = \frac{\partial}{\partial \tau} \wedge [X, A] + \tau X(F) \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \tau F L_X \left( \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} \right).$$

If, furthermore,  $X \in \mathfrak{osp}(1|2)$ , then it preserves the even part  $\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$  of the Poisson bivector. The condition  $L_X \Lambda = 0$  then implies:

$$[X, A] = 0 \quad \text{and} \quad X(F) = 0.$$

The even part of  $\mathfrak{osp}(1|2)$  is just the Lie algebra  $\mathfrak{sl}(2, \mathbb{K})$  generated by the Hamiltonian vector fields with quadratic Hamiltonians  $\langle p^2, pq, q^2 \rangle$ . It is easy to see that an even vector field  $A$  commuting with any even element of  $\mathfrak{osp}(1|2)$  is of the form

$$A = c_1 \tau \frac{\partial}{\partial \tau} + c_2 E,$$

where  $c_1$  and  $c_2$  are arbitrary constants and

$$E = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}.$$

An even function killed by all even elements of  $\mathfrak{osp}(1|2)$  is constant:  $F = c_3$ .

The odd part of  $\mathfrak{osp}(1|2)$  is spanned by the following two vector fields:

$$X_{\tau p} = \tau \frac{\partial}{\partial q} + p \frac{\partial}{\partial \tau}, \quad X_{\tau q} = -\tau \frac{\partial}{\partial p} + q \frac{\partial}{\partial \tau}.$$

Applying to  $\Lambda$  any of the above odd elements of  $\mathfrak{osp}(1|2)$ , one immediately gets  $c_1 = c_2 = c_3$ .  $\square$

The relation of the bivector  $\Lambda$  to the Lie antialgebra  $\mathfrak{asl}(2, \mathbb{K})$  is as follows. Any bivector defines an algebraic structure on the space of functions. Consider the bracket associated with the bivector (2.4):

$$]F, G[ = \frac{1}{2} \langle \Lambda, dF \wedge dG \rangle, \tag{2.5}$$

where  $F$  and  $G$  are arbitrary functions on  $\mathbb{K}^{2|1}$ , that is,  $F = F_0(p, q) + \tau F_1(p, q)$ .

**Lemma 2.3.** *The space of linear functions on  $\mathbb{K}^{2|1}$  equipped with the bracket (2.5) is a Lie antialgebra isomorphic to  $\mathfrak{asl}(2, \mathbb{K})$ .*

*Proof.* One checks that after the identification

$$\{a, b; \varepsilon\} \longleftrightarrow \{p, q; \tau\},$$

the bracket (2.1) coincides with the bracket (2.5) restricted to linear functions.  $\square$

We proved that the Lie superalgebra  $\mathfrak{osp}(1|2)$  preserves the bivector  $\Lambda$ . Since  $\mathfrak{osp}(1|2)$  acts on  $\mathbb{K}^{2|1}$  by linear vector fields, it also preserves the space of linear functions. It follows that

$$\mathfrak{osp}(1|2) = \text{Der}(\mathfrak{asl}(2, \mathbb{K})),$$

or, in other words,  $\mathfrak{osp}(1|2)$  is the algebra of symmetry of  $\mathfrak{asl}(2, \mathbb{K})$ .

**Remark 2.4.** (i) The bivector  $\Lambda$  given by (2.4), and the Lie antialgebra  $\mathfrak{asl}(2, \mathbb{K})$  are equivalent structures, they contain the same information. The above lemma provides the identification of the dual space:

$$\mathfrak{asl}(2, \mathbb{K})^* \cong (\mathbb{K}^{2|1}, \Lambda).$$

The bivector  $\Lambda$  is the therefore analog of the Lie-Poisson structure of  $\mathfrak{asl}(2, \mathbb{K})$ , cf. Section 4.2 for a general setting.

(ii) The similar question in the case of orthosymplectic Lie superalgebra in higher dimension, has a *negative* answer: the only  $\mathfrak{osp}(m|2n)$ -invariant bivector is the standard, even, Poisson bivector. There is no analog of the odd bivector (2.4) in this case.

## 2.2 Lie antialgebra $\mathfrak{asl}(2, \mathbb{C})$ and a pair of symplectic structures

Consider now the complex Lie antialgebra  $\mathfrak{asl}(2, \mathbb{C})$  viewed as a simple 4|2-dimensional Lie antialgebra over  $\mathbb{R}$ . Our goal is to introduce the corresponding odd bivector, which is an analog of the bivector (2.4), and calculate the algebra of symmetry.

Consider the vector space  $\mathbb{R}^{4|2}$  with linear coordinates  $(p_1, q_1, p_2, q_2; \tau_1, \tau_2)$ . Introduce the following odd bivector:

$$\Lambda^C = \frac{\partial}{\partial \tau_1} \wedge \mathcal{E} + \frac{\partial}{\partial \tau_2} \wedge \mathcal{J} + \tau_1 \pi_\varepsilon + \tau_2 \pi_\sigma, \quad (2.6)$$

where  $\mathcal{E}$  is the Euler vector field

$$\mathcal{E} = \sum_{i=1,2} \left( p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} + \tau_i \frac{\partial}{\partial \tau_i} \right)$$

and

$$\mathcal{J} = q_2 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_2} - p_1 \frac{\partial}{\partial q_2} + \tau_2 \frac{\partial}{\partial \tau_1} - \tau_1 \frac{\partial}{\partial \tau_2}. \quad (2.7)$$

and where

$$\pi_\varepsilon = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}, \quad \pi_\sigma = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2}.$$

Define the antibracket corresponding to the bivector (2.6), as in formula (2.5). One easily checks the following

**Proposition 2.5.** *The space of linear functions  $\text{Span}(p_1, q_1, p_2, q_2; \tau_1, \tau_2)$  form a Lie antialgebra isomorphic to  $\mathfrak{asl}(2, \mathbb{C})$ .*



The space  $(\mathbb{R}^{4|2}, \Lambda^C)$  is thus identified with the space dual to the Lie antialgebra  $\mathfrak{asl}(2, \mathbb{C})$ :

$$\mathfrak{asl}(2, \mathbb{C})^* \cong (\mathbb{R}^{4|2}, \Lambda^C).$$

The symmetry algebra associated to  $\mathfrak{asl}(2, \mathbb{C})$  is, of course,  $\mathfrak{osp}(1|2, \mathbb{C})$ . We realize it as the Lie superalgebra of linear vector preserving the bivector  $\Lambda^C$ .

**Proposition 2.6.** *The subalgebra  $\mathfrak{osp}(1|2, \mathbb{C}) \subset \mathfrak{gl}(4|2, \mathbb{R})$  preserving the bivector (2.6) is the Lie superalgebra of linear vector fields that can be equivalently characterized in the following two ways:*

(i) *it preserves the symplectic form*

$$\omega_\varepsilon = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + \frac{1}{2}(d\tau_1 \wedge d\tau_1 - d\tau_2 \wedge d\tau_2)$$

*and commuting with the vector field (2.7);*

(ii) *this is the Lie superalgebra of linear vector fields preserving two symplectic forms: the form  $\omega_\varepsilon$  together with*

$$\omega_\sigma = dp_1 \wedge dp_2 - dq_1 \wedge dq_2 + d\tau_1 \wedge d\tau_2.$$

*Proof.* The forms  $\omega_\varepsilon$  and  $\omega_\sigma$  are the even and the odd part of the complex 2-form (2.2) with the complex coordinates

$$p = p_1 + iq_2, \quad q = q_1 + ip_2, \quad \tau = \tau_1 + i\tau_2. \quad (2.8)$$

The complex Lie superalgebra preserving this 2-form is  $\mathfrak{osp}(1|2, \mathbb{C})$ .

Since the Poisson bivector  $\mathcal{P}_\varepsilon$  is inverse to the standard symplectic form  $\omega$  on  $\mathbb{R}^{4|2}$ , every symplectic vector field is  $\mathcal{P}_\varepsilon$ -Hamiltonian. Furthermore, the two symplectic structures are related by

$$\omega_\sigma = \frac{1}{2} L_{\mathcal{J}}(\omega_\varepsilon), \quad \omega_\varepsilon = -\frac{1}{2} L_{\mathcal{J}}(\omega_\sigma).$$

Therefore, a  $\omega_\varepsilon$ -Hamiltonian vector field commutes with the vector field  $\mathcal{J}$  if and only if is also Hamiltonian with respect to the symplectic structure  $\omega_\sigma$ .  $\square$

**Remark 2.7.** The symmetry algebra  $\mathfrak{osp}(1|2, \mathbb{C})$  is thus obtained as intersection of two copies of the real symplectic Lie superalgebra:

$$\mathfrak{osp}(1|2, \mathbb{C}) = \mathfrak{osp}_\varepsilon(1, 1|4) \cap \mathfrak{osp}_\sigma(1, 1|4)$$

corresponding to the symplectic forms  $\omega_\varepsilon$  and  $\omega_\sigma$ , respectively.

**Proposition 2.8.** *The bivector  $\Lambda^C$  is the unique (up to a multiplicative constant)  $\mathfrak{osp}(1|2, \mathbb{C})$ -invariant bivector.*

*Proof.* The  $\mathfrak{osp}(1|2, \mathbb{C})$ -action on  $\mathbb{R}^{4|2}$  is spanned by 6 even bi-Hamiltonian vector fields corresponding to the quadratic Hamiltonians

$$\{p_1^2 - q_2^2, \quad p_2^2 - q_1^2, \quad p_1 p_2 + q_1 q_2, \quad p_1 q_1 - p_2 q_2, \quad p_1 q_2, \quad q_1 p_2\}$$

and 4 odd bi-Hamiltonian vector fields with the Hamiltonians

$$\{p_1 \tau_1 - q_2 \tau_2, \quad p_2 \tau_1 + q_1 \tau_2, \quad q_1 \tau_1 - p_2 \tau_2, \quad q_2 \tau_1 + p_1 \tau_2\}.$$

The uniqueness can then be proved in a similar way that in Proposition 2.2.  $\square$

**Remark 2.9.** The above 6|4-dimensional subspace of quadratic polynomials is characterized by the property  $\mathcal{J}(\mathcal{J}(F)) = -4F$ . the corresponding bi-Hamiltonian vector fields satisfy

$$X_F^\varepsilon = -\frac{1}{2} X_{\mathcal{J}(F)}^\sigma, \quad X_F^\sigma = \frac{1}{2} X_{\mathcal{J}(F)}^\varepsilon,$$

where  $X_F^\varepsilon$  and  $X_F^\sigma$  are the Hamiltonian vector fields with the Hamiltonian  $F$ , with respect to  $\omega_\varepsilon$  and  $\omega_\sigma$ .

### 3 Conformal Lie antialgebra $\mathcal{AK}(1)$

In this section we determine the maximal space of functions on  $\mathbb{R}^{2|1}$  that form a Lie antialgebra with respect to the bracket (2.5). We introduce a simple infinite-dimensional Lie antialgebra, that we denote  $\mathcal{AK}(1)$ ; this is a *conformal version* of  $\text{asl}(2, \mathbb{R})$ . The Lie antialgebra  $\mathcal{AK}(1)$  first appeared in [4] in the context of supertransvectants, see [6].

We will show that the famous conformal Lie superalgebra  $\mathcal{K}(1)$ , also known as the (centerless) Neveu-Schwarz algebra is the algebra of symmetry of  $\mathcal{AK}(1)$ ; this is the maximal subalgebra of the Poisson Lie algebra on  $\mathbb{R}^{2|1}$  that preserves the bivector (2.4).

**Definition 3.1.**  $\mathcal{AK}(1)$  is the infinite-dimensional Lie antialgebra with the basis

$$\{a_i, i \in \mathbb{Z} + \frac{1}{2}; \quad \alpha_n, n \in \mathbb{Z}\}$$

and the following relations

$$\begin{aligned} ]\alpha_n, \alpha_m[ &= \alpha_{n+m}, \\ ]\alpha_n, a_i[ &= \frac{1}{2} a_{n+i}, \\ ]a_i, a_j[ &= \frac{1}{2} (j - i) \alpha_{i+j}. \end{aligned} \tag{3.1}$$

We call  $\mathcal{AK}(1)$  the *conformal antialgebra*.

**Proposition 3.2.** *The bracket (3.1) defines a structure of a simple Lie antialgebra.*

*Proof.* Let us first check that the defined structure is, indeed, a Lie antialgebra. The properties (1.2) and (1.3) are evident. The invariance condition (1.4) is as follows:

$$]\alpha_n, ]a_i, a_j[[ = ]]\alpha_n, a_i[, a_j[ + ]a_i, ]\alpha_n, a_j[[$$

One obtains in the left-hand-side  $\frac{1}{2} (j - i) \alpha_{i+j+n}$  and in the right-hand-side the sum of two terms:  $\frac{1}{4} (j - (i + n)) \alpha_{i+j+n}$  and  $\frac{1}{4} (j + n - i) \alpha_{i+j+n}$ , so that the identity (1.4) is satisfied. Finally, the Jacobi identity reads:

$$]]a_i, a_j[, a_k[ + ]]a_j, a_k[, a_i[ + ]]a_k, a_i[, a_j[ = 0.$$

One has the sum of  $\frac{1}{4} (j - i) \alpha_{i+j+n}$  with  $\frac{1}{4} (k - j) \alpha_{i+j+n}$  and  $\frac{1}{4} (i - k) \alpha_{i+j+n}$  which is zero.

Let us now proof that the defined Lie antialgebra is simple. The commutative subalgebra  $\mathcal{AK}(1)_1$  is a group algebra, namely

$$\mathcal{AK}(1)_1 = \mathbb{K}[\mathbb{Z}],$$

(in other words, the algebra of Laurent polynomials in one variable). In particular, the element  $\varepsilon = \alpha_0$  is the unit of  $\mathcal{AK}(1)_1$ . Obviously,  $\mathcal{AK}(1)_1$  has no ideal. Assume that the Lie antialgebra  $\mathcal{AK}(1)$  has an ideal. It follows that the odd part of the ideal is either zero or coincides with  $\mathcal{AK}(1)_1$ . But, in the first case the whole ideal is obviously zero, while, in the second case, it contains the unit  $\varepsilon$  and thus coincides with  $\mathcal{AK}(1)$ .  $\square$

### 3.1 Relation to symplectic geometry

In this section we show that the Lie antialgebra  $\mathcal{AK}(1)$  can be obtained, as  $\mathfrak{asl}(2, \mathbb{R})$ , from the odd Poisson bivector (2.4).

Consider first the space of all smooth functions  $C^\infty(\mathbb{R}^{2|1})$ :

$$F(p, q; \tau) = F_0(p, q) + \tau F_1(p, q),$$

where  $F_0(p, q)$  and  $F_1(p, q)$  are smooth functions of two real variables and  $\tau$  is a formal Grassmannian variable, i.e.,  $\tau^2 = 0$ . The antibracket (2.5) defines a structure of antialgebra on  $C^\infty(\mathbb{R}^{2|1})$ ; the explicit formula is as follows:

$$]F, G[ = \frac{1}{2} \left( \frac{\partial F}{\partial \tau} \mathcal{E}(G) - (-1)^{p(F)} \mathcal{E}(F) \frac{\partial G}{\partial \tau} + \tau \left( \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} \right) \right). \quad (3.2)$$

However, the full antialgebra of functions  $C^\infty(\mathbb{R}^{2|1})$  equipped with this bracket is *not* a Lie antialgebra.

Consider the space, denoted by  $\mathcal{F}_\lambda$ , of *homogeneous functions of degree  $\lambda$*  on  $\mathbb{R}^{2|1} \setminus \mathbb{R}^{1|1}$ , consisting of the functions (with singularities at  $p = 0$ ) satisfying the condition

$$\mathcal{E}(F) = \lambda F,$$

where  $\mathcal{E}$  is the Euler field. Homogeneous functions on  $\mathbb{R}^{2|1} \setminus \mathbb{R}^{1|1}$  correspond to functions on  $\mathbb{R}^{1|1}$ . Given a function

$$f(x, \xi) = f_0(x) + \xi f_1(x)$$

in one even and one Grassmann variable, one defines a homogeneous of degree  $\lambda$  function

$$F_f^\lambda(p, q; \tau) = p^\lambda f\left(\frac{q}{p}, \frac{\tau}{p}\right). \quad (3.3)$$

This is of course a one-to-one correspondence:

$$\mathcal{F}_\lambda \cong C^\infty(\mathbb{R}^{1|1}).$$

**Proposition 3.3.** *The space,  $\mathcal{F}_1$ , of homogeneous of degree 1 functions on  $\mathbb{R}^{2|1} \setminus \mathbb{R}^{1|1}$  is a Lie antialgebra with respect to the antibracket (3.2) that contains  $\mathcal{AK}(1)$ .*

*Proof.* The space of homogeneous of degree 1 functions is obviously stable with respect to the antibracket (3.2):

$$F_{[f, g]}^1 := \frac{1}{2} \langle \Lambda, dF_f^1 \wedge dF_g^1 \rangle. \quad (3.4)$$

One then easily checks the Lie antialgebra conditions.

Choose the following Taylor basis (dense in any reasonable topology):

$$a_i = p \left( \frac{q}{p} \right)^{i+\frac{1}{2}}, \quad \alpha_n = \tau \left( \frac{q}{p} \right)^n$$

and substitute it into the antibracket (2.5). One obtains the commutation relations (3.1), so that the Lie antialgebra  $\mathcal{AK}(1)$  is a subalgebra of  $\mathcal{F}_1$ .  $\square$

**Remark 3.4.** Geometrically speaking, the map (3.3) is an isomorphism between the space of homogeneous of degree  $\lambda$  functions on  $\mathbb{R}^{2|1} \setminus \{0\}$  and tensor densities of degree  $-\frac{1}{2}\lambda$  on  $S^{1|1}$ , see, e.g., [5]. We do not dwell on this compactification problem and rather develop a purely algebraic viewpoint.

### 3.2 Conformal Lie superalgebra $\mathcal{K}(1)$ as the algebra of symmetry

The conformal Neveu-Schwarz superalgebra  $\mathcal{K}(1)$  is spanned by the basis

$$\{x_n, n \in \mathbb{Z}; \quad \xi_i, i \in \mathbb{Z} + \frac{1}{2}\}$$

with the following commutation relations

$$\begin{aligned} [x_i, x_j] &= (j - i) x_{i+j}, \\ [x_i, \xi_j] &= (j - \frac{i}{2}) \xi_{i+j}, \\ [\xi_i, \xi_j] &= 2 x_{i+j}. \end{aligned} \tag{3.5}$$

This is a Lie superalgebra that contains infinitely many copies of  $\mathfrak{osp}(1|2)$  with the generators  $\{x_{-n}, x_0, x_n; \xi_{-\frac{n}{2}}, \xi_{\frac{n}{2}}\}$ . Define the following action of  $\mathcal{K}(1)$  on  $\mathcal{AK}(1)$ :

$$\begin{aligned} x_n(a_i) &= (i - \frac{n}{2}) a_{n+i}, \\ x_n(\alpha_m) &= m \alpha_{n+m}, \\ \xi_i(a_j) &= (j - i) \alpha_{i+j}, \\ \xi_i(\alpha_n) &= a_{i+n}. \end{aligned} \tag{3.6}$$

**Remark 3.5.** This formula is well-known and represents the action of the superconformal algebra  $\mathcal{K}(1)$  on the space of tensor densities of weight  $-\frac{1}{2}$  (cf., e.g., [5]).

**Proposition 3.6.** *The action (3.6) preserves the Lie antialgebra structure (3.1).*

*Proof.* Let us check the most non-trivial relations; the other terms are similar. One has

$$\xi_i([a_j, \alpha_k]) = -\frac{1}{2} \xi_i(a_{j+k}) = -\frac{1}{2} (j + k - i) \alpha_{i+j+k},$$

together with

$$]\xi_i(a_j), \alpha_k[ = (j - i) ]\alpha_{i+j}, \alpha_k[ = (j - i) \alpha_{i+j+k}$$

and

$$]a_j, \xi_i(\alpha_k)[ = ]a_j, a_{i+k}[ = \frac{1}{2} (i + k - j) \alpha_{i+j+k}.$$

One finally gets:

$$\xi_i([a_j, \alpha_k]) = -]\xi_i(a_j), \alpha_k[ - ]a_j, \xi_i(\alpha_k)[$$

which is precisely the invariance condition. Note that the sign rule in the above relation is due to the fact that the antibracket  $], [$  is odd, as well as the element  $\xi_i$ , cf. formula (1.6).  $\square$

The conformal Lie superalgebra  $\mathcal{K}(1)$  also has a symplectic realization.

**Proposition 3.7.** *The space  $\mathcal{F}_2$  of homogeneous of degree 2 functions on  $\mathbb{R}^{2|1} \setminus \mathbb{R}^{1|1}$  is a Lie superalgebra with respect to the Poisson bracket (2.3) that contains  $\mathcal{K}(1)$ .*

*Proof.* The Poisson bracket of two homogeneous of degree 2 functions is, again, a homogeneous of degree 2 function. Therefore,  $\mathcal{F}_2$  is, indeed, a Lie superalgebra.

A homogeneous of degree 2 function can be written in the form (3.3) with  $\lambda = 2$ . Choosing the basis of the space of all such functions:

$$x_n = \frac{p^2}{2} \left( \frac{q}{p} \right)^{n+1}, \quad \xi_i = \tau p \left( \frac{q}{p} \right)^{i+\frac{1}{2}}$$

and substituting it into the Poisson bracket (2.3), one immediately obtains the commutation relations (3.5). Therefore,  $\mathcal{K}(1)$  is a subalgebra of  $\mathcal{F}_2$ .  $\square$

**Remark 3.8.** (a) The Lie superalgebra  $\mathcal{F}_2$  is a “geometric version” of the superconformal algebra  $\mathcal{K}(1)$ , which is a polynomial part of  $\mathcal{F}_2$ . Similarly,  $\mathcal{AK}(1)$  is the polynomial part of the Lie antialgebra  $\mathcal{F}_1$ .

(b) The action (3.6) written in terms of homogeneous functions is, again, given by the standard Poisson bracket (2.3).

**Theorem 2.** *The Lie superalgebra  $\mathcal{F}_2$  is the maximal Lie superalgebra of vector fields that preserves the bivector (2.4).*

*Proof. Part 1.* Let us first show that  $\mathcal{K}(1)$  preserves the bivector (2.4). Given a function  $H \in \mathcal{F}_2$ , the corresponding Hamiltonian vector field is homogeneous of degree 0:

$$[\mathcal{E}, X_H] = 0. \quad (3.7)$$

Consider a more general case, where  $P$  is an purely even (independent of  $\tau$ ) Poisson bivector homogeneous of degree  $-2$  with respect to a vector field  $E$ , that is

$$L_E(P) = -2P.$$

Let  $\Lambda$  be the odd bivector field

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \mathcal{E} + \tau \wedge P,$$

where

$$\mathcal{E} = E + \tau \frac{\partial}{\partial \tau}.$$

(Note that in our case  $E = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$ ,  $P = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$ .)

Let  $X_H$  be a Hamiltonian (with respect to the Poisson structure  $P$ ) vector field satisfying the homogeneity condition (3.7). The Lie derivative of  $\Lambda$  along  $X_H$  is as follows:

$$L_{X_H} \Lambda = L_{X_H} \left( \frac{\partial}{\partial \tau} \right) \wedge \mathcal{E} + X_H(\tau) P + \tau L_{X_H}(P).$$

If  $H$  is even, the above expression obviously vanishes. Consider now an odd function  $H = \tau H_1$ , then one gets from (2.3)

$$X_{\tau H_1} = \tau X_{H_1} + H_1 \frac{\partial}{\partial \tau}.$$

**Lemma 3.9.** *One has*

$$L_{X_{\tau H_1}}(\Lambda) = \langle P \wedge E, dH_1 \rangle \quad (3.8)$$

where  $d$  is the de Rham differential.

*Proof.* Using the obvious expressions

$$[X_{\tau H_1}, \frac{\partial}{\partial \tau}] = X_{H_1}, \quad L_{X_{\tau H_1}}(P) = -\frac{\partial}{\partial \tau} \wedge X_{H_1},$$

one obtains:

$$L_{X_{\tau H_1}}(\Lambda) = X_{H_1} \wedge \mathcal{E} + H_1 P + \tau \frac{\partial}{\partial \tau} \wedge X_{H_1} = X_{H_1} \wedge E + H_1 P.$$

Finally, using the fact that  $E(H_1) = H_1$ , one obtains the expression (3.8).  $\square$

The even tri-vector  $P \wedge E$  obviously vanishes on  $\mathbb{R}^{2|1}$ , and so we proved that  $X_H$ , indeed, preserves the bivector (2.4).

*Part 2.* Conversely, one has to show that any vector field preserving the bivector (2.4) is a Hamiltonian vector field commuting with  $\mathcal{E}$ .

If  $X$  is a purely even vector field, i.e.,

$$[X, \frac{\partial}{\partial \tau}] = 0 \quad \text{and} \quad X(\tau) = 0,$$

then  $L_X(\Lambda) = 0$  implies that  $X$  commutes with  $\mathcal{E}$  and preserves the even bivector  $P = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$ , so that  $X$  is Hamiltonian.

If  $X$  is an odd vector field:

$$X = F_0 \frac{\partial}{\partial \tau} + \tau X_0,$$

where  $F_0$  is an even function and  $X_0$  an even vector field, then one obtains explicitly

$$L_X \Lambda = (E(F_0) - 1) \frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \tau} \wedge (X_{F_0} + [E, X_0]) + (F_0 P + X_0 \wedge E).$$

The assumption  $L_X(\Lambda) = 0$  implies that each of the three summands in this expression vanishes. It follows from

$$E(F_0) - 1 = 0,$$

that  $F_0$  is a homogeneous of degree 1 function. The condition

$$X_{F_0} + [E, X_0] = 0$$

then implies that  $X_0$  is a vector field homogeneous of degree  $-1$ , since so is  $X_{F_0}$ , and thus  $X_0 = X_{F_0}$ . We proved that the vector field  $X$  is Hamiltonian and  $[\mathcal{E}, X] = 0$ .

Theorem 2 is proved.  $\square$

One then readily has

**Corollary 3.10.**  $\mathcal{K}(1)$  is the algebra of symmetry of  $\mathcal{AK}(1)$ , that is,  $\mathcal{K}(1) = \text{Der}(\mathcal{AK}(1))$ .

Indeed, the subalgebra  $\mathcal{K}(1) \subset \mathcal{F}_2$  corresponds precisely to the space of vector fields preserving the basis of  $\mathcal{AK}(1)$ .

### 3.3 Representation of $\mathcal{AK}(1)$ by tangent vector fields

In this section we investigate the relation of the Lie antialgebra  $\mathcal{AK}(1)$  to contact geometry. It turns out that, in some sense,  $\mathcal{AK}(1)$  provides a way to “integrate” the contact structure.

The natural projection

$$\mathbb{R}^{2|1} \setminus \mathbb{R}^{1|1} \longrightarrow \mathbb{R}^{1|1},$$

obtained as the quotient by homotheties, equips  $\mathbb{R}^{1|1}$  with a structure of 1|1-dimensional contact manifold. This contact structure can be defined in terms of a contact 1-form

$$\alpha = dx + \xi d\xi,$$

or, equivalently, in terms of an odd vector field<sup>2</sup>

$$D = \frac{1}{2} \left( \xi \frac{\partial}{\partial x} - \frac{\partial}{\partial \xi} \right),$$

since  $D$  spans the kernel of  $\alpha$ . A vector field *tangent* to the contact distribution is a vector field proportional to  $D$ , that is,  $X = f D$  for some function  $f(x, \xi)$ .

**Definition 3.11.** We introduce the following *anticommutator* of tangent vector fields:

$$]f D, g D[ := f D \circ g D + (-1)^{(p(f)+1)(p(g)+1)} g D \circ f D. \quad (3.9)$$

Note that the sign in this operation is inverse to that of usual commutator of vector fields.

The space of tangent vector fields is not a Lie superalgebra since the Lie bracket of two tangent vector fields is not a tangent vector field (this is equivalent to non-integrability of the contact distribution). It turns out that the anticommutator of two tangent vector fields is again a tangent vector field.

Define a map from  $\mathcal{AK}(1)$  to the space of tangent vector fields as follows. To each homogeneous of degree 1 function  $F_f^1(p, q, \tau)$ , cf. formula (3.3), we associate a tangent vector field by

$$\chi_f = \bar{f}(x, \xi) D, \quad (3.10)$$

where  $\bar{f}(x, \xi) = f_0(x) - \xi f_1(x)$ .

**Proposition 3.12.** *The map (3.10) intertwines the antibracket (2.5) on the Lie antialgebra  $\mathcal{AK}(1)$  and the anticommutator (3.9):*

$$] \chi_f, \chi_g [ = \chi_{]f, g[},$$

where  $] , [$  stands for the anticommutator (3.9) in the left-hand-side and for the antibracket (2.5) in the right-hand-side.

*Proof.* Let us calculate the explicit formula of the anticommutator (3.9).

Let first both of the functions  $f = \xi f_1$  and  $g = \xi g_1$  be odd, then one has

$$f D \circ g D + g D \circ f D = \frac{1}{2} f g \frac{\partial}{\partial x} + (f D(g) + g D(f)) D.$$

---

<sup>2</sup>This vector field is also known in physical literature as “SUSY-structure”.

The first summand is zero since it contains  $\xi^2 = 0$ , while the second one is equal to  $\xi (f_1 g_1) D$ , so that the anticommutator (3.9) on the odd functions reads

$$]f_1, g_1[ = f_1 g_1.$$

This corresponds to the product of two elements of the odd part  $\mathcal{AK}(1)_1$ , see formula (3.1).

If  $f = \xi f_1$  is odd and  $g = g_0$  is even, then the Leibniz rule  $D \circ f = D(f) - f D$  implies

$$f D \circ g D + g D \circ f D = g D(f) D = \frac{1}{2} f_1 g_0 D,$$

so that one gets

$$]f_1, g_0[ = \frac{1}{2} f_1 g_0,$$

accordingly to the  $\mathcal{AK}(1)_1$ -action on  $\mathcal{AK}(1)_0$ , cf. (3.1).

If, finally, the both functions  $f = f_0$  and  $g = g_0$  are even, then

$$f D \circ g D - g D \circ f D = (f D(g) - g D(f)) D = \frac{1}{2} \xi (f_0 g'_0 - g_0 f'_0) D,$$

gives the skew-symmetric product

$$]f_0, g_0[ = \frac{1}{2} (f_0 g'_0 - g_0 f'_0),$$

on  $\mathcal{AK}(1)_0$  with values in  $\mathcal{AK}(1)_1$ , see formula (3.1). □

Let us also stress that the map (3.10) is nothing but the bivector (2.4) contracted with the elements of  $\mathcal{AK}(1)$ . One checks that

$$\chi_f(g) = \frac{1}{2} \langle \Lambda, dF_f^1 \wedge dF_g^0 \rangle, \quad (3.11)$$

where  $F_f^1$  and  $F_g^0$  are functions on  $\mathbb{R}^{2|1}$  homogeneous of degree 1 and 0, respectively, obtained as the lift of  $f$  and  $g$  according to (3.3). It is interesting to compare the above formula with (3.4).

**Remark 3.13.** The relation of the conformal Lie superalgebra  $\mathcal{K}(1)$  to contact geometry is well-known (see [13]). More precisely,  $\mathcal{K}(1)$  can be realized as the Lie superalgebra of contact vector fields on  $S^{1|1}$ . Every contact vector field on  $S^{1|1}$  is of the form

$$X_h = h(x, \xi) \frac{\partial}{\partial x} + 2 D(h(x, \xi)) D,$$

where  $h(x, \xi) = h_0(x) + \xi h_1(x)$  is an arbitrary function. The map  $h \mapsto F_h^2$ , see (3.3), provides then a Lie superalgebra isomorphism.

### 3.4 $\mathcal{AK}(1)_{\mathbb{C}}$ and $\mathcal{K}(1)_{\mathbb{C}}$ : the bi-Hamiltonian formalism

In this section we construct a realization of  $\mathcal{AK}(1)_{\mathbb{C}}$  and  $\mathcal{K}(1)_{\mathbb{C}}$  in terms of real rational harmonic functions on  $\mathbb{R}^{4|2}$ . This is related to the bi-Hamiltonian formalism defined by the pair of symplectic structures  $\omega_{\varepsilon}$  and  $\omega_{\sigma}$  from Section 2.2.



The Lie antialgebra  $\mathcal{AK}(1)_{\mathbb{C}}$  is represented by the homogeneous of degree 1 harmonic functions on  $\mathbb{R}^{4|2}$  with the complex structure (2.7). The odd generators of  $\mathcal{AK}(1)_{\mathbb{C}}$  are as follows

$$\alpha_n = \frac{1}{2^n} \operatorname{Re} \tau \left( \frac{q}{p} \right)^n, \quad \beta_n = \frac{1}{2^n} \operatorname{Im} \tau \left( \frac{q}{p} \right)^n,$$

and the even ones are

$$a_n = \frac{1}{2^n} \operatorname{Re} p \left( \frac{q}{p} \right)^n, \quad b_n = \frac{1}{2^n} \operatorname{Im} p \left( \frac{q}{p} \right)^n,$$

where  $p, q, \tau$  are the complex coordinates (2.8). One then checks the relations in  $\mathcal{AK}(1)_{\mathbb{C}}$ .

The conformal Lie superalgebra  $\mathcal{K}(1)_{\mathbb{C}}$  is realized by homogeneous bi-Hamiltonian vector fields on  $\mathbb{R}^{4|2}$ . For each function  $F \in C^\infty(\mathbb{R}^{4|2})$ , denote by  $X_F^\varepsilon$  and  $X_F^\sigma$  the Hamiltonian vector fields on  $\mathbb{R}^{4|2}$  with respect to the symplectic form  $\omega_\varepsilon$  and  $\omega_\sigma$ , respectively.

One checks that the following three conditions are equivalent.

1.  $F$  is a homogeneous of degree 2 harmonic function:

$$F = \operatorname{Re} p^2 f \left( \frac{q}{p}, \frac{\tau}{p} \right),$$

where  $f$  is an arbitrary function.

2. The function  $F$  satisfies the relations:

$$\mathcal{E}(F) = 2F, \quad \mathcal{J}(\mathcal{J}(F)) = -4F.$$

3. The Hamiltonian vector fields with the Hamiltonian  $F$  commute with  $\mathcal{E}$  and  $\mathcal{J}$ :

$$[\mathcal{E}, X_F^\varepsilon] = [\mathcal{E}, X_F^\sigma] = [\mathcal{J}, X_F^\varepsilon] = [\mathcal{J}, X_F^\sigma] = 0$$

and are bi-Hamiltonian such that

$$X_F^\varepsilon = -\frac{1}{2} X_{\mathcal{J}(F)}^\sigma, \quad X_F^\sigma = \frac{1}{2} X_{\mathcal{J}(F)}^\varepsilon.$$

This space of homogeneous harmonic bi-Hamiltonian vector fields is a Lie superalgebra isomorphic to  $\mathcal{K}(1)_{\mathbb{C}}$ .

**Proposition 3.14.** *The Lie superalgebra  $\mathcal{K}(1)_{\mathbb{C}}$  is the maximal algebra of vector fields on  $\mathbb{R}^{4|2}$  preserving the bivector (2.6).*

*Proof.* Theorem 2 implies that  $\mathcal{K}(1)_{\mathbb{C}}$ , indeed, preserves the bivector  $\Lambda^C$ . The proof of maximality is similar to that of Theorem 2, Part 2. We omit here the corresponding straightforward computations.  $\square$

## 4 Elements of the general theory

In this section we some basic notions and discuss some very general properties of Lie antialgebras. The most important definitions are those of representations and of the analog of the Lie-Poisson (Berezin-Kirillov-Kostant-Souriau) structure on the dual of a Lie antialgebra. This notion is important for several reasons: it relates Lie antialgebras to symplectic geometry and (multi-)Hamiltonian formalism; it allows to define infinite-dimensional (conformal) analog of Lie antialgebras and of the corresponding symmetry algebras.

## 4.1 Representations and modules

In this section we discuss important notions of representation and of module over Lie antialgebras. We believe that these two notions (which are, usually, almost tautologically the same) should be separated in our context.

**A. Representations.** Let  $V = V_0 \oplus V_1$  be a  $\mathbb{Z}_2$ -graded vector space. Introduce the following anticommutator on the space  $\text{End}(V)$ :

$$]A, B[ := AB + (-1)^{p(A)p(B)} BA. \quad (4.1)$$

Note that this operation the sign rule inverse to that of the usual  $\mathbb{Z}_2$ -graded commutator (the sign is “−” if and only if both  $A$  and  $B$  are odd). This is, of course, *not* a Lie antialgebra structure on the space  $\text{End}(V)$ .

We already considered a particular case of the anticommutator (4.1), namely the operation (3.9) on the space of tangent vector fields.

**Definition 4.1.** Let  $\mathfrak{a}$  be a Lie antialgebra, an odd linear maps

$$\chi : \mathfrak{a} \rightarrow \text{End}(V)$$

is called a *representation* of  $\mathfrak{a}$ , if

$$]\chi_x, \chi_y[ = \chi_{[x, y]}, \quad (4.2)$$

for all  $x, y \in \mathfrak{a}$ .

**Example 4.2.** The map (3.10) defines a representation of the conformal Lie antialgebra  $\mathcal{AK}(1)$  in the space of differential operators on  $\mathbb{R}^{1|1}$ .

**B. Modules.** Let, again,  $V = V_0 \oplus V_1$  be a  $\mathbb{Z}_2$ -graded vector space.

**Definition 4.3.** The space  $V$  is called an  $\mathfrak{a}$ -*module* if there is an odd linear maps

$$\rho : \mathfrak{a} \rightarrow \text{End}(V),$$

such that the direct sum  $\mathfrak{a} \oplus V$  equipped with the antibracket

$$](x, v), (y, w)[ = (]\chi_x, \chi_y[, \rho_x w - (-1)^{p(y)p(v)} \rho_y w) \quad (4.3)$$

is again a Lie antialgebra.

We call the Lie antialgebra structure (4.3) a *semi-direct* product and denote it by  $\mathfrak{a} \ltimes V$ .

**Example 4.4.** The “adjoint representation”  $\text{ad} : \mathfrak{a} \rightarrow \text{End}(\mathfrak{a})$  given by  $\text{ad}_x y = ]x, y[$ , for all  $x, y \in \mathfrak{a}$  is *not* a representation of  $\mathfrak{a}$  since it does not satisfy (4.2). However,  $\mathfrak{a}$  is of course a module over itself. This follows, for instance, from the fact that the tensor product  $\mathfrak{C} \otimes \mathfrak{a}$  of a Lie antialgebra  $\mathfrak{a}$  with a commutative algebra  $\mathfrak{C}$  is again a Lie antialgebra. Indeed, consider  $\mathfrak{C} = \mathbb{K}[t]/(t^2)$ , then one has

$$\mathfrak{C} \otimes \mathfrak{a} = \mathfrak{a} \ltimes \mathfrak{a}.$$

## 4.2 The odd Lie-Poisson bivector

In this section we introduce the notion of canonical odd bivector field on the dual space  $\mathfrak{a}^*$  of an arbitrary Lie antialgebra  $\mathfrak{a}$ . This is the most important ingredient of our theory that relates Lie antialgebras to differential geometry. Our construction uses a choice of coordinates on  $\mathfrak{a}^*$ , but we will prove that the defined bivector is independent of this choice.

Given a Lie antialgebra  $\mathfrak{a}$ , one can consider the commutative subalgebra  $\mathfrak{a}_1$  as analog of the notion of “Cartan subalgebra” of  $\mathfrak{a}$ . This justifies the following terminology. We call the dimension of the odd part  $\mathfrak{a}_1$  of a Lie antialgebra  $\mathfrak{a}$  the *rank* of  $\mathfrak{a}$ :

$$\mathrm{rk} \mathfrak{a} := \dim \mathfrak{a}_1.$$

Let  $\mathfrak{a}$  be a Lie antialgebra of rank  $r$ . Fix an arbitrary basis  $\{\alpha_1, \dots, \alpha_r\}$  of the odd part  $\mathfrak{a}_1$ . One obtains a set of  $r$  bilinear skew-symmetric (or presymplectic) forms:  $\{\omega_1, \dots, \omega_r\}$  on  $\mathfrak{a}_0$  by projecting the antibracket to each basic element:

$$]a, b[ = \sum_{i=1}^r \omega_i(a, b) \alpha_i. \quad (4.4)$$

Changing the basis, one obtains linear changes of the corresponding set of skew-symmetric forms. Therefore, the *pencil of presymplectic forms*

$$\langle \omega_1, \dots, \omega_r \rangle$$

is well-defined.

**Remark 4.5.** The classification of pencils of  $r$  linear skew-symmetric forms on a vector space is a classical problem of linear algebra already considered by Kronecker. This problem turned out to be closely related to integrable systems, see, [7] and references therein.

We denote by  $(\tau_1, \dots, \tau_r)$  the coordinates on  $\mathfrak{a}_1^*$  dual to the chosen basis. To each 2-form  $\omega_i$ , one associates a bivector

$$\pi_i \in \wedge^2 \mathfrak{a}_0^*,$$

that we can understand as a constant bivector field on  $\mathfrak{a}_0^*$ . Furthermore, to each element  $\alpha \in \mathfrak{a}_1$ , one associates a linear operator  $A_\alpha : \mathfrak{a} \rightarrow \mathfrak{a}$  defined by

$$A_\alpha|_{\mathfrak{a}_0} = 2 \mathrm{ad}_\alpha, \quad A_\alpha|_{\mathfrak{a}_1} = \mathrm{ad}_\alpha. \quad (4.5)$$

We denote  $A_1, \dots, A_r$  the operators corresponding to the elements of the basis  $\{\alpha_1, \dots, \alpha_r\}$ . These linear operators can be, of course, viewed as *linear vector fields* on  $\mathfrak{a}^*$ .

**Example 4.6.** In the important case where  $\mathfrak{a}_1$  contains the *unit element*  $\varepsilon$  and the center of  $\mathfrak{a}$  is trivial (see Theorem 3 below), one has

$$A_\varepsilon = \mathcal{E},$$

where  $\mathcal{E}$  is the Euler vector field on the vector space  $\mathfrak{a}^*$ , i.e., the generator of the  $\mathbb{K}^*$ -action by homotheties. Choosing arbitrary linear coordinates  $(x_1, \dots, x_n)$  on  $\mathfrak{a}_0^*$ , one obtains:

$$A_\varepsilon = \sum_{i=1}^r \tau_i \frac{\partial}{\partial \tau_i} + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}.$$

The quotient with respect to the vector field  $A_\varepsilon$  defines a natural projection to the  $(n-1)|r$ -dimensional projective space:

$$\mathfrak{a}^* \setminus \{0\} \rightarrow \mathbb{P}^{n-1|r}.$$

This is the way contact geometry enters the picture.

**Definition 4.7.** We call the Lie-Poisson structure of the Lie antialgebra  $\mathfrak{a}$  the following odd bivector on  $\mathfrak{a}^*$ :

$$\Lambda_{\mathfrak{a}} = \sum_{i=1}^r \left( \frac{\partial}{\partial \tau_i} \wedge A_i + \tau_i \pi_i \right). \quad (4.6)$$

The corresponding antibracket is of the form:

$$]F, G[ = \frac{1}{2} \langle \Lambda_{\mathfrak{a}}, dF \wedge dG \rangle \quad (4.7)$$

is defined on the space of (polynomial, smooth, etc.) functions on  $\mathfrak{a}^*$ . This antibracket is obviously *linear*, i.e., the space of linear functions  $\mathfrak{a}^{**} \cong \mathfrak{a}$  is stable.

**Proposition 4.8.** *The space of linear functions on  $\mathfrak{a}^*$  is a Lie antialgebra with respect to the bracket (4.7) which is isomorphic to  $\mathfrak{a}$ .*

*Proof.* The antibracket of two even linear functions is precisely given by (4.4).

The odd linear functions on  $\mathfrak{a}^*$  are linear combinations of  $\tau_1, \dots, \tau_r$ . The antibracket of an odd and an even linear functions is given by

$$]\tau_i, \ell[ = \frac{1}{2} A_i(\ell),$$

where  $\ell \in \mathfrak{a}_0$ . This corresponds precisely to the adjoint action of  $\alpha_i$  on  $\ell$ .

Finally, the antibracket of two odd linear functions is given by

$$]\tau_i, \tau_j[ = \frac{1}{2} (A_i(\tau_j) + A_j(\tau_i)) = \frac{1}{2} (\text{ad}_{\alpha_i} \alpha_j + \text{ad}_{\alpha_j} \alpha_i) = ]\alpha_i, \alpha_j[.$$

Hence the result. □

**Corollary 4.9.** *The bivector (4.6) and the antibracket (4.7) are independent of the choice of the basis.*

*Proof.* Any bivector is uniquely defined by its values on linear functions. □

**Example 4.10.** The bivectors (2.4) and (2.6) that played so important role in Sections 2 and 3 are precisely the canonical bivectors associated to the Lie antialgebras  $\text{asl}(2, \mathbb{R})$  and  $\text{asl}(2, \mathbb{C})$ , respectively.

### 4.3 Algebras of symmetry

In this section, we associate with any Lie antialgebra two different Lie superalgebras. The first one is the standard derivation algebra that we already discussed in Introduction, see formula (1.6); the second one is the (infinite-dimensional) algebra of “conformal symmetry”. Both of these Lie superalgebras can be defined in terms of the canonical bivector (4.6).

**Definition 4.11.** The Lie superalgebra  $\mathfrak{g}_{\mathfrak{a}} \subset \text{End}(\mathfrak{a})$  of linear operators on  $\mathfrak{a}$  preserving the Lie antialgebra structure is called the *algebra of symmetry* of  $\mathfrak{a}$ .

**Example 4.12.** The algebra of symmetry of  $\text{asl}(2, \mathbb{K})$  is the Lie superalgebra  $\text{osp}(1|2, \mathbb{K})$ . The algebra of symmetry of  $\mathcal{AK}(1)$  is the conformal Lie superalgebra  $\mathcal{K}(1)$ .

The standard notation for the Lie superalgebra of symmetry is, of course,

$$\mathfrak{g}_{\mathfrak{a}} = \text{Der}(\mathfrak{a}).$$

The Lie superalgebra  $\mathfrak{g}_{\mathfrak{a}}$  acts on the dual space  $\mathfrak{a}^*$  and preserves the canonical bivector (4.6). This action is given by vector fields with *linear coefficients* and therefore preserves the space of linear functions on  $\mathfrak{a}^*$ , which is nothing but the Lie antialgebra  $\mathfrak{a}$  itself, cf. Proposition 4.8.

**Definition 4.13.** Consider the Lie superalgebra of all (smooth, or polynomial, or analytic, etc.) vector fields on  $\mathfrak{a}^*$ ; we call the subalgebra consisting of vector fields preserving the bivector (4.6), the *conformal algebra of symmetry* of  $\mathfrak{a}$ . This Lie superalgebra will be denoted by  $\mathfrak{G}_{\mathfrak{a}}$ .

The Lie superalgebra  $\mathfrak{G}_{\mathfrak{a}}$  is, in general, infinite-dimensional; it does not act on  $\mathfrak{a}$  since it does not preserve the space of linear functions on  $\mathfrak{a}^*$ .

**Example 4.14.** In the case of  $\text{asl}(2, \mathbb{R})$  and  $\text{asl}(2, \mathbb{C})$ , the conformal algebra of symmetry is the conformal Lie superalgebra  $\mathcal{K}(1)$  and  $\mathcal{K}(1)_{\mathbb{C}}$ , respectively, see Section 3.

The most interesting feature of  $\mathfrak{G}_{\mathfrak{a}}$  is its relation to the multi-Hamiltonian formalism. Consider, for simplicity, the case where  $\mathfrak{a}_1$  contains the unit element  $\varepsilon$  and introduce the following Poisson bivector on  $\mathfrak{a}^*$ :

$$\mathcal{P}_{\varepsilon} = \pi_{\varepsilon} + \frac{1}{2} \sum_{i=1}^r \frac{\partial}{\partial \tau_i} \wedge \frac{\partial}{\partial \tau_i}. \quad (4.8)$$

Consider also the Poisson bivectors given by the Lie derivatives

$$\mathcal{P}_i = L_{A_i}(\mathcal{P}_{\varepsilon}). \quad (4.9)$$

Note that all of these Poisson structures are obviously *compatible*, i.e., any their linear combination is, again, a Poisson bivector. Indeed, these bivectors are with constant coefficients.

**Conjecture 4.15.** *The Lie superalgebra of conformal symmetry  $\mathfrak{G}_{\mathfrak{a}}$  can be characterized in the following two equivalent ways:*

(i)  $\mathfrak{G}_{\mathfrak{a}}$  consists of vector fields on  $\mathfrak{a}^*$  Hamiltonian with respect to the structure  $\mathcal{P}_{\varepsilon}$  and commuting with all of the vector fields (4.5) of the basis  $A_1, \dots, A_r$ :

$$[X, A_i] = 0.$$

(ii)  $\mathfrak{G}_{\mathfrak{a}}$  consists of vector fields on  $\mathfrak{a}^*$  Hamiltonian with respect to all of the structures (4.8) and (4.9) and commuting with the Euler vector field  $\mathcal{E}$ .

The statement is evident if  $X$  is a purely even vector field, i.e.,

$$\left[ X, \frac{\partial}{\partial \tau_i} \right] = 0, \quad X(\tau_i) = 0.$$

Note also that Theorem 2 and Proposition 3.14 are particular cases.

One also easily shows that the spaces of vector fields (i) and (ii) coincide since the Poisson structures are related by (4.9). Furthermore, a vector field  $X_F^\varepsilon$  Hamiltonian with respect to  $\mathcal{P}_\varepsilon$  with the Hamiltonian function  $F$  is Hamiltonian with respect to  $\mathcal{P}_i$  with the Hamiltonian function  $A_i(F)$ , namely

$$X_F^\varepsilon = -X_{A_i(F)}^i.$$

Indeed, one has

$$[A_i, X_F^\varepsilon] = \langle L_{A_i}(\mathcal{P}_\varepsilon), dF \rangle + \langle \mathcal{P}_\varepsilon, d(A_i(F)) \rangle = 0.$$

However, we failed to prove Conjecture 4.15 in full generality.

#### 4.4 Central extensions

In this section we define the notion of extension of a Lie antialgebra  $\mathfrak{a}$  with coefficients in any  $\mathfrak{a}$ -module. It will be useful for the classification result of Section 5.2. The notion of extension should be a part of a general cohomology theory that we will not develop here.

Let  $\mathfrak{a}$  be a Lie antialgebra and  $V$  an  $\mathfrak{a}$ -module. We will consider  $V$  as a trivial (or abelian) Lie antialgebra.

**Definition 4.16.** (a) An exact sequence of Lie antialgebras

$$0 \longrightarrow V \longrightarrow \tilde{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow 0 \quad (4.10)$$

is called an *abelian extension* of the Lie antialgebra  $\mathfrak{a}$  with coefficients in  $V$ . As a vector space,  $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus V$ , and the subspace  $V$  is obviously an  $\mathfrak{a}$ -module.

(b) An extension (4.10) is called *non-trivial* if the Lie antialgebra  $\tilde{\mathfrak{a}}$  is not isomorphic to the semi-direct sum  $\mathfrak{a} \ltimes V$ .

(c) If the subspace  $V$  belongs to the center of  $\tilde{\mathfrak{a}}$ , then the extension (4.10) is called a *central extension*.

Since any central extension can be obtained by iteration of one-dimensional central extensions, it suffice to consider only the case of one-dimensional central extensions. One then has two possibilities:

$$\dim V = 1|0, \quad \text{or} \quad \dim V = 0|1.$$

We then say that the one-dimensional central extension is of *type I* or of *type II*, respectively.

The general form of central extensions of type I is as follows.

**Proposition 4.17.** (i) A central extension of type I is defined by an odd linear map

$$C : \mathfrak{a} \wedge \mathfrak{a} \rightarrow \mathbb{K} \quad (4.11)$$

satisfying the following two identities:

$$\begin{aligned} C(\alpha, ]\beta, a[) &= C(\beta, ]\alpha, a[) = \frac{1}{2} C(]\alpha, \beta[, a) \\ C(]a, b[, c) + C(]b, c[, a) + C(]c, a[, b) &= 0, \end{aligned} \quad (4.12)$$

for all  $\alpha, \beta \in \mathfrak{a}_1$  and  $a, b, c \in \mathfrak{a}_0$ .

(ii) A central extension is trivial if and only if there exists an even linear function  $f : \mathfrak{a}_0 \rightarrow \mathbb{K}$  such that

$$C(\alpha, a) = f([\alpha, a]) \quad (4.13)$$

for all  $\alpha \in \mathfrak{a}_1$  and  $a \in \mathfrak{a}_0$ .

*Proof.* Part (i). Given a map  $C$  as in (4.11), let us define an antibracket on  $\mathfrak{a} \oplus \mathbb{K}$ . We fix an element  $z \in \mathbb{K}$  and set:

$$\begin{aligned} ]x, y[ &= ]x, y[_{\mathfrak{a}} + C(x, y) z, \\ ]x, z[ &= 0, \end{aligned} \quad (4.14)$$

for all  $x, y \in \mathfrak{a}$ . Note that, since the map  $C$  is skew-symmetric and odd, one has

$$C(x, y) = C(x_1, y_0) - C(y_1, x_0). \quad (4.15)$$

One then easily checks that formula (4.14) defines a structure of a Lie antialgebra if and only if the relations (4.12) are satisfied.

Conversely, a Lie antialgebra structure on  $\mathfrak{a} \oplus \mathbb{K}$  such that the subspace  $\mathbb{K}$  belongs to the center is obviously of the form (4.14).

Part (ii). In the case where  $C$  is as in (4.13), the linear map  $\mathfrak{a} \oplus \mathbb{K} \rightarrow \mathfrak{a} \oplus \mathbb{K}$  given by  $(x, z) \mapsto (x, z + f(x))$  intertwines the structure (4.14) with the trivial direct sum structure. This means that the central extension is trivial.

Conversely, if the extension is trivial, then there exists an intertwining map  $\mathfrak{a} \oplus \mathbb{K} \rightarrow \mathfrak{a} \oplus \mathbb{K}$  sending the structure (4.14) to the trivial one. This map can, again, be chosen in the form  $(x, z) \mapsto (x, z + f(x))$ , since a different choice of the embedding of  $\mathbb{K}$  does not change the structure.  $\square$

We will call a map  $C$  satisfying (4.12) a *2-cocycle of type I*. A 2-cocycle of the form (4.13) will be called a *coboundary*.

Let us describe the central extensions of type II.

**Proposition 4.18.** (i) A central extension of type II is defined by an even linear map

$$C : \mathfrak{a} \wedge \mathfrak{a} \rightarrow \mathbb{K}^{0|1} \quad (4.16)$$

satisfying the following two identities:

$$\begin{aligned} C(\alpha, ]a, b[) &= C([\alpha, a], b) + C(a, ]\alpha, b[) \\ C([\alpha, \beta], \gamma) &= C(\alpha, ]\beta, \gamma[), \end{aligned} \quad (4.17)$$

for all  $\alpha, \beta, \gamma \in \mathfrak{a}_1$  and  $a, b \in \mathfrak{a}_0$ .

(ii) The extension is trivial if and only if there exists an even linear functional  $f : \mathfrak{a}_1 \rightarrow \mathbb{K}^{0|1}$  such that

$$C(x, y) = f([\alpha, y]), \quad (4.18)$$

for all  $x, y \in \mathfrak{a}$ .

*Proof.* The proof is similar to that of Proposition 4.17.  $\square$

We will call an even map (4.16) satisfying (4.17) a *2-cocycle of type II*. In the case where it is given by (4.18), the map  $C$  is called a coboundary.

## 4.5 Kernel of the presymplectic pencil

Let us now describe a “natural source” of extensions of type I.

Given an arbitrary Lie antialgebra  $\mathfrak{a}$ , consider the following space  $\mathcal{I} \subset \mathfrak{a}_0$ :

$$\mathcal{I} = \bigcap_{i=1}^r \ker \omega_i,$$

where  $r = \text{rk} \mathfrak{a}$ , the forms  $\omega_i$  are defined by (4.4), and where  $\ker \omega \subset \mathfrak{a}_0$  consists of elements  $a$  of  $\mathfrak{a}_0$  such that  $\omega(a, b) = 0$  for every  $b \in \mathfrak{a}_0$ . In other words,

$$\mathcal{I} = \{a \in \mathfrak{a}_0 \mid ]a, b[ = 0, \text{ for all } b \in \mathfrak{a}_0\}.$$

**Proposition 4.19.** *The subspace  $\mathcal{I}$  is an abelian ideal of  $\mathfrak{a}$ .*

*Proof.* By definition,  $\mathcal{I}$  is an abelian subalgebra and the bracket of  $a \in \mathcal{I}$  with any element  $b \in \mathfrak{a}_0$  vanishes. One has to show that  $] \alpha, a[ \in \mathcal{I}$ , for arbitrary  $\alpha \in \mathfrak{a}_1$  and  $a \in \mathcal{I}$ . Indeed, using the identity (1.4), one obtains

$$]] \alpha, a[, b[ = ] \alpha, ]a, b[[ - ]a, ] \alpha, b[[ = 0$$

since for  $a \in \mathcal{I}$  and every  $b \in \mathfrak{a}_0$ , one has  $]a, b[ = 0$ .  $\square$

It follows that the Lie antialgebra  $\mathfrak{a}$  is an abelian extension of type I of the quotient-algebra  $\mathfrak{a}/\mathcal{I}$ .

**Proposition 4.20.** *If the map  $] , [ : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$  is surjective, then the ideal  $\mathcal{I}$  belongs to the center of  $\mathfrak{a}$ .*

*Proof.* The ideal  $\mathcal{I}$  belongs to the center of  $\mathfrak{a}$  if and only if the action of  $\mathfrak{a}_1$  on  $\mathcal{I}$  is trivial. Surjectivity means that for every  $\alpha \in \mathfrak{a}_1$  there are  $a, b \in \mathfrak{a}_0$  such that  $\alpha = ]a, b[$ . Using the Jacobi identity (1.5), one obtains for every  $c \in \mathcal{I}$ :

$$] \alpha, c[ = ]]a, b[, c[ = - ]]b, c[, a[ - ]]c, a[, b[ = 0,$$

since both summands in the right-hand-side vanish.  $\square$

The Lie antialgebra  $\mathfrak{a}$  is therefore a central extension of  $\mathfrak{a}/\mathcal{I}$  (of type I).

## 4.6 Restrictive role of the unit

In this section we show that, in the case where the associative commutative algebra  $\mathfrak{a}_1$  contains the unit element  $\varepsilon$ , the Lie antialgebra  $\mathfrak{a}$  has no non-trivial central extensions.

By definition, the action of  $\varepsilon$  on the odd part  $\mathfrak{a}_1$  is given by the identity map:

$$\text{ad}_\varepsilon^1 = \text{ad}_e|_{\mathfrak{a}_1} = \text{Id}.$$

Let us consider the action of  $\varepsilon$  on the even part  $\mathfrak{a}_0$ . The identity (1.3) implies the “half-projector” relation for the operator  $\text{ad}_\varepsilon^0 = \text{ad}_e|_{\mathfrak{a}_0}$ :

$$\text{ad}_\varepsilon^0 \circ \text{ad}_\varepsilon^0 = \frac{1}{2} \text{ad}_\varepsilon^0.$$



Therefore, the even part  $\mathfrak{a}_0$  is split

$$\mathfrak{a}_0 = \mathfrak{a}_{0, \frac{1}{2}} \oplus \mathfrak{a}_{0,0}$$

to a sum of  $\frac{1}{2}$ - and 0-eigenspaces of  $\text{ad}_\varepsilon^0$ , respectively. That is,  $\text{ad}_\varepsilon|_{\mathfrak{a}_{0, \frac{1}{2}}} = \frac{1}{2}\text{Id}$  and  $\text{ad}_\varepsilon|_{\mathfrak{a}_{0,0}} = 0$ .

**Theorem 3.** (i) The space  $\mathfrak{a}_{0,0}$  coincides with the center of  $\mathfrak{a}$ :

$$\mathfrak{a}_{0,0} = Z(\mathfrak{a}).$$

(ii) The Lie antialgebra  $\mathfrak{a}$  is a direct sum:

$$\mathfrak{a} = \bar{\mathfrak{a}} \oplus Z(\mathfrak{a}), \quad (4.19)$$

where  $\bar{\mathfrak{a}} = \mathfrak{a}/Z(\mathfrak{a})$ .

*Proof.* Part (i). Let  $a \in \mathfrak{a}_{0,0}$ , that is,  $]\varepsilon, a[ = 0$ . One has to show that  $]x, a[ = 0$  for all  $x \in \mathfrak{a}$ .

Let first  $x = \alpha$  be an element of  $\mathfrak{a}_1$ . The identity (1.3) implies

$$]\varepsilon, ]\alpha, a[[ = ]\varepsilon, ]\alpha, a[[ + ]\alpha, ]\varepsilon, a[[ = ]]\varepsilon, \alpha[, a[ = ]\alpha, a[.$$

But then  $\text{ad}_\varepsilon^0 \circ \text{ad}_\varepsilon^0 = \frac{1}{2} \text{ad}_\varepsilon^0$  implies  $]\alpha, a[ = 0$ .

Let now  $x = b$  be an element of  $\mathfrak{a}_0$ . One has

$$]b, a[ = ]\varepsilon, ]b, a[[ = ]]\varepsilon, b[, a[ + ]b, ]\varepsilon, a[[ = ]]\varepsilon, b[, a[.$$

If, furthermore,  $b \in \mathfrak{a}_0^0$ , then the last term is zero. Finally, if  $b \in \mathfrak{a}_0^{\frac{1}{2}}$ , then the last term is equal to  $\frac{1}{2}]b, a[$ , which again implies  $]b, a[ = 0$ .

Part (ii). Let us show that if  $\mathfrak{a}_1$  contains the unit element  $\varepsilon$ , then  $\mathfrak{a}$  has no non-trivial central extensions.

Let  $C$  be a 2-cocycle of type I on  $\mathfrak{a}$ . Apply the first identity (4.12) to  $\beta = \varepsilon$ , where  $\varepsilon$  is the unit. One has

$$C(\alpha, a) - C(\alpha, ]\varepsilon, a[) = C(\varepsilon, ]\alpha, a[).$$

If  $a \in Z(\mathfrak{a})$ , this formula implies  $C(\alpha, a) = 0$ . If  $a$  is an element of the  $\frac{1}{2}$ -eigenspace  $\mathfrak{a}_{0, \frac{1}{2}}$  of the unit element  $\varepsilon$ , then one obtains

$$C(\alpha, a) = 2C(\varepsilon, ]\alpha, a[).$$

Therefore, the cocycle  $C$  is a coboundary.

Let now  $C$  be a 2-cocycle of type II. It can be decomposed into a pair  $(C_0, C_1)$  of maps

$$C_0 : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathbb{K}^{0|1}, \quad C_1 : \mathfrak{a}_1 \otimes \mathfrak{a}_1 \rightarrow \mathbb{K}^{0|1},$$

where  $C_0$  is skew-symmetric and  $C_1$  is symmetric. The first condition (4.17) gives

$$C_1(\varepsilon, ]a, b[) = C_0(]\varepsilon, a[, b) + C_0(a, ]\varepsilon, b[),$$

so that  $C_0(a, b) = C_1(\varepsilon, ]a, b[)$ . The second condition (4.17) implies

$$C_1(\alpha, \beta) = C_1(\varepsilon, ]\alpha, \beta[).$$

Therefore, the cocycle  $C$  is, again, a coboundary.

Let us complete the proof of Theorem 3. The Lie antialgebra  $\mathfrak{a}$  is a central extension of its quotient  $\bar{\mathfrak{a}} = \mathfrak{a}/Z(\mathfrak{a})$ . We just proved that this center extension is trivial. The decomposition (4.19) follows.  $\square$

## 5 Classification results

In this section we prove that the only finite-dimensional simple Lie antialgebras are  $\mathfrak{asl}(2, \mathbb{R})$  and  $\mathfrak{asl}(2, \mathbb{C})$ . More precisely, we prove the uniqueness of  $\mathfrak{asl}(2, \mathbb{R})$  and  $\mathfrak{asl}(2, \mathbb{C})$  in the real case, while in the complex case there nothing else than  $\mathfrak{asl}(2, \mathbb{R})$ . This means that the situation is similar to the case of commutative algebras.

We also obtain a complete classification of finite-dimensional Lie antialgebras of rank 1, i.e., with  $\dim \mathfrak{a}_1 = 1$ . This, in particular, provides a number of examples of Lie antialgebras, other than we already considered.

### 5.1 Proof of Theorem 1

We will now give a proof of Theorem 1.

Let us start with the complex case. Let  $\mathfrak{a}$  be a simple finite-dimensional Lie antialgebra. We will first assume that the commutative algebra  $\mathfrak{a}_1$  has no nilpotent elements. As it is very well known, see, e.g., [2],  $\mathfrak{a}_1$  is of the form

$$\mathfrak{a}_1 = \mathbb{C} \oplus \cdots \oplus \mathbb{C}.$$

We will prove that if  $\mathfrak{a}$  is simple, then  $r = 1$ .

Choose a basis  $\{\alpha_1, \dots, \alpha_r\}$  in  $\mathfrak{a}_1$  such that  $[\alpha_i, \alpha_j] = \delta_{ij}$ . As in Section 4.2, one associates with each element  $\alpha_i$  a presymplectic form  $\omega_i$  on  $\mathfrak{a}_0$  (see formula 4.4).

**Lemma 5.1.** *Each form  $\omega_i$  is of rank 2.*

*Proof.* Choose a canonical (Darboux) basis  $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ , of  $\omega_i$ , so that one has

$$\omega_i(a_k, b_\ell) = \delta_{k\ell}, \quad \omega(a_k, a_\ell) = \omega(b_k, b_\ell) = 0,$$

where  $k, \ell = 1, \dots, n$ . Let us show that  $n = 1$ .

Assume that  $n > 1$ . The identity (1.5) implies

$$[\alpha_i, a_\ell] = [[a_k, b_k], a_\ell] = -[[b_k, a_\ell], a_k] - [a_\ell, a_k][b_k] = 0$$

for any  $k \neq \ell$ . It follows that  $[\alpha_i, a] = 0$ , for any  $a \in \mathfrak{a}_0$ . Furthermore,

$$[\alpha, \alpha] = [\alpha, [a_i, b_i]] = [[\alpha, a_i], b_i] + [a_i, [\alpha, b_i]] = 0,$$

for any  $i = 1, \dots, n$ . Therefore,  $\alpha_i$  belongs to the center of  $\mathfrak{a}$ ; in particular,  $\mathfrak{a}$  cannot be simple. This is a contradiction.  $\square$

The following statement shows that, if  $r > 1$ , then the algebra  $\mathfrak{a}$  contains a non-trivial ideal. Consider the subspace  $\ker \omega_1 \subset \mathfrak{a}_0$  consisting of the elements  $a \in \mathfrak{a}_0$  such that, for all  $b$  one has:  $[a, b]$  is a combination of  $\alpha_i$  with  $i \geq 2$ . Consider the subspace of  $\mathfrak{a}$ :

$$\mathcal{I} = \ker \omega_1 \oplus \langle \alpha_2, \dots, \alpha_r \rangle.$$

**Lemma 5.2.** *The space  $\mathcal{I}$  is an ideal of  $\mathfrak{a}$ .*

*Proof.* One has to prove that for arbitrary  $\alpha \in \mathfrak{a}_1$  and  $a \in \mathfrak{a}_0$  one has: (a)  $\alpha\mathcal{I} \subset \mathcal{I}$  and (b)  $a\mathcal{I} \subset \mathcal{I}$ .

Part (a). Let  $a \in \ker \omega_1$ , by identity (1.4), one has for an arbitrary  $b \in \mathfrak{a}_0$ :

$$]\alpha, a[, b[ = ]\alpha, ]a, b[[ - ]a, ]\alpha, b[[.$$

The both terms in the right-hand-side are combinations of  $\alpha_i$  with  $i \geq 2$ . Therefore,  $]\alpha, a[ \in \ker \omega_1$ .

Part (b). Let  $\alpha \in \langle \alpha_2, \dots, \alpha_r \rangle$ , then  $]\alpha_1, \alpha[ = 0$ . Let  $a \in \mathfrak{a}_0$  be arbitrary, one has to prove that, again,  $]\alpha, a[ \in \ker \omega_1$ . Choose an arbitrary  $b \in \mathfrak{a}_0$  and consider again the expression  $]\alpha, a[, b[$ . Since this is an odd element of  $\mathfrak{a}$ , it can be written in the form:

$$]\alpha, a[, b[ = \sum_{i=1}^r c_i \alpha_i.$$

One has to prove that  $c_1 = 0$ . One has

$$]\alpha_1, ]\alpha, a[, b[[ = ]\alpha_1, ]\alpha, a[[, b[ + ]\alpha, a[, ]\alpha_1, b[[.$$

But the first summand in the right-hand-side vanishes. Indeed, by (1.3), one has

$$]\alpha_1, ]\alpha, a[[ = \frac{1}{2} ]\alpha_1, \alpha[, a[ = 0,$$

since  $]\alpha_1, \alpha[ = 0$ . In the same way, one obtains

$$]\alpha_1, ]\alpha_1, ]\alpha, a[, b[[[ = ]\alpha, a[, ]\alpha_1, ]\alpha_1, b[[[.$$

However, for the left-hand-side, one obtains using (1.2):

$$]\alpha_1, ]\alpha_1, ]\alpha, a[, b[[[ = ]\alpha_1, \alpha_1[, ]\alpha, a[, b[[ = ]\alpha_1, ]\alpha, a[, b[[ = c_1$$

since  $]\alpha_1, \alpha_1[ = \alpha_1$ ; while, for the right-hand-side, one gets using (1.3):

$$]\alpha, a[, ]\alpha_1, ]\alpha_1, b[[[ = \frac{1}{2} ]\alpha, a[, ]\alpha_1, \alpha_1[, b[[ = \frac{1}{2} ]\alpha, a[, ]\alpha_1, b[[ = ]\alpha_1, ]\alpha, a[, b[[ = \frac{1}{2} c_1.$$

Therefore,  $c_1 = 0$  and so  $]\alpha, a[ \in \ker \omega_1$ . The result follows.  $\square$

Lemma 5.1 and Lemma 5.2 imply Theorem 1 in the complex case, where the commutative algebra  $\mathfrak{a}_1$  has no nilpotent elements. If now  $\mathfrak{a}_1 = \mathbb{C}^n \ltimes \mathcal{N}$ , where  $\mathcal{N}$  is a nilpotent ideal, then the same arguments prove that  $n \leq 1$  and  $\ker \omega_1 \oplus \mathcal{N}$  is an ideal. Theorem 1 is proved in the complex case.

The real case immediately follows from the complex one. Indeed, let  $\mathfrak{a}$  be a real simple Lie antialgebra, the standard arguments show that the complexification  $\mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$  is either simple or the direct sum of two isomorphic simple ideals.

Theorem 1 is proved.

## 5.2 Lie antialgebras of rank 1

Our next task is to classify the Lie antialgebras of rank 1. We thus assume that the commutative algebra  $\mathfrak{a}_1$  is one-dimensional.

There exist two different one-dimensional (commutative) algebras:

1.  $\mathfrak{a}_1 = \mathbb{K}$ , with the unit element  $]\varepsilon, \varepsilon[ = \varepsilon$ ;
2.  $\mathfrak{a}_1$  is nilpotent such that  $]\alpha, \alpha[ \equiv 0$ .

Since  $\mathfrak{a}_1$  is one-dimensional, the map  $], [ : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$  is characterized by one bilinear skew-symmetric form,  $\omega$ , on  $\mathfrak{a}_0$  (see Section 4.2):

$$]a, b[ = \omega(a, b) \alpha,$$

where  $\alpha$  is a (unique up to a constant) non-zero element of  $\mathfrak{a}_1$ .

Let us first construct several examples of Lie antialgebras of rank 1.

**A.** Consider the case where the form  $\omega$  is non-degenerate.

(A1) The  $2n|1$ -dimensional nilpotent Lie antialgebra with the basis  $\{a_1, b_1, \dots, a_n, b_n; \alpha\}$  that appeared in the case a) of the above proof is characterized by the relations

$$\begin{aligned} ]a_i, b_j[ &= \delta_{ij} \alpha, \\ ]a_i, a_j[ &= 0, \quad ]b_i, b_j[ = 0, \\ ]\alpha, a_i[ &= 0, \quad ]\alpha, b_i[ = 0, \\ ]\alpha, \alpha[ &= 0, \end{aligned} \tag{5.1}$$

where  $i, j = 1, \dots, n$ . We call this Lie antialgebra the *Heisenberg antialgebra* and denote it by  $\text{ah}_n$ .

**Remark 5.3.** Notice, that the relations (5.1) are exactly as those of the standard Heisenberg Lie algebra, but the central element  $\alpha$  is odd. As in the usual Lie case, the Heisenberg antialgebra  $\text{ah}_n$  is a central extension of type II of the abelian Lie antialgebra  $\mathbb{K}^{2n}$ .

(A2) Another interesting example is a family of Lie antialgebras of dimension  $2|1$ . The basis of these Lie antialgebras will be denoted by  $\{a, b, \alpha\}$ ; the commutation relations are

$$\begin{aligned} ]a, b[ &= \alpha, \\ ]\alpha, a[ &= \kappa b, \quad ]\alpha, b[ = 0, \\ ]\alpha, \alpha[ &= 0, \end{aligned} \tag{5.2}$$

where  $\kappa$  is a constant. If  $\kappa = 0$ , then this is just the Heisenberg antialgebra  $\text{ah}_1$ , if  $\kappa \neq 0$ , then the defined Lie antialgebra is not isomorphic to  $\text{ah}_1$ , and therefore one obtains a non-trivial deformation of  $\text{ah}_1$ .

It is easy to see that, in the complex case  $\mathbb{K} = \mathbb{C}$ , all of the Lie antialgebras (5.2) with  $\kappa \neq 0$  are isomorphic to each other. We call this algebra *twisted Heisenberg antialgebra* and denote it by  $\widetilde{\text{ah}}_1$ .

If  $\mathbb{K} = \mathbb{R}$ , however, the sign of  $\kappa$  is an invariant and one gets two different algebras  $\widetilde{\mathfrak{ah}}_1^+$  and  $\widetilde{\mathfrak{ah}}_1^-$ . We thus assume:

$$\kappa = 1, \quad \text{if } \mathbb{K} = \mathbb{C}, \quad \kappa = \pm 1, \quad \text{if } \mathbb{K} = \mathbb{R}.$$

We are ready to formulate a partial result.

**Proposition 5.4.** *The complete list of the real Lie antialgebras of rank 1 with a non-degenerate 2-form  $\omega$  is as follows:*

$$\mathfrak{asl}(2, \mathbb{R}), \quad \mathfrak{ah}_n, \quad \widetilde{\mathfrak{ah}}_1^+, \quad \widetilde{\mathfrak{ah}}_1^-; \quad (5.3)$$

*in the complex case, the two last algebras are isomorphic.*

*Proof.* Consider first the case where  $\mathfrak{a}_1$  is not nilpotent, i.e.,  $[\alpha, \alpha] \neq 0$ . We already proved that, in this case,  $\mathfrak{a}$  is of dimension  $2|1$ , see Lemma 5.1. Therefore,  $\mathfrak{a} = \mathfrak{asl}(2, \mathbb{K})$ .

Assume that  $[\alpha, \alpha] = 0$ .

In the case where the form  $\omega$  is of rank  $n > 1$ , one proves, in the same way as in Lemma 5.1, that  $[\alpha, a] = 0$  for all  $a$ , so that  $\mathfrak{a} = \mathfrak{ah}_n$ .

If, finally,  $\omega$  is of rank 1, then  $\text{ad}_\alpha^0$  does not necessarily vanish but the identity (1.3) implies that this operator has to be nilpotent:  $(\text{ad}_\alpha^0)^2 = 0$ . One then easily shows that any such operator on  $\mathfrak{a}$  is equivalent to  $\text{ad}_\alpha^0$  in (5.2) up to the area preserving changes of the basis. It follows that  $\mathfrak{a} = \widetilde{\mathfrak{ah}}_1^+$  or  $\widetilde{\mathfrak{ah}}_1^-$ .

Hence the result.  $\square$

**B.** Consider the case  $\omega = 0$ . The Lie antialgebra  $\mathfrak{a}$  is then determined by the operator  $\text{ad}_\alpha$ .

(B1) If  $\mathfrak{a}_1$  contains the unit element  $\varepsilon$ , then  $\mathfrak{a}$  is split into a direct sum (4.19). The centerless Lie antialgebra  $\overline{\mathfrak{a}}$  has the basis  $\{a_1, \dots, a_n; \varepsilon\}$  with the following set of relations:

$$[a_i, a_j] = 0, \quad [\varepsilon, a_i] = \frac{1}{2} a_i, \quad [\varepsilon, \varepsilon] = \varepsilon.$$

We call this Lie antialgebra the *affine antialgebra* and denote by  $\text{aaf}(n)$ . One then has

$$\mathfrak{a} = \text{aaf}(n) \oplus Z,$$

where  $Z$  is the (even) center of  $\mathfrak{a}$ .

(B2) If  $[\alpha, \alpha] = 0$  for  $\alpha \in \mathfrak{a}_1$ , then  $\text{ad}_\alpha^0 \circ \text{ad}_\alpha^0 = 0$ . This is a very degenerated Lie antialgebra. The classification in this case is the classification of nilpotent (of order 2) linear operators that we do not discuss.

Let us summarize the above considerations.

**Proposition 5.5.** *A Lie antialgebras of rank 1 with  $\omega = 0$  is one of two classes:*

$$\mathfrak{a} = \text{aaf}(n) \oplus Z, \quad \mathfrak{a} \text{ is of type (B2)}. \quad (5.4)$$

**C.** Consider finally the “mixed case” where  $2 < \text{rk } \omega < \dim \mathfrak{a}_0$ .

(C1) Define a  $3|1$ -dimensional Lie antialgebra with the basis  $\{a, b, z; \alpha\}$  and the relations

$$\begin{aligned} [a, b] &= \alpha, & [a, z] &= 0, & [b, z] &= 0, \\ [\alpha, a] &= z, & [\alpha, b] &= 0, & [\alpha, z] &= 0, \\ [\alpha, \alpha] &= 0. \end{aligned} \quad (5.5)$$

We denote this Lie antialgebra  $\widehat{\mathfrak{ah}}_1$ . The element  $z$  spans the center, so that this is a central extension (of type I) of the Heisenberg antialgebra  $\mathfrak{ah}_1$ .

(C2) Define a 4|1-dimensional Lie antialgebra with the basis  $\{a, b, z_1, z_2; \alpha\}$  and the relations

$$\begin{aligned} ]a, b[ &= \alpha, & ]a, z_i[ &= 0, & ]b, z_i[ &= 0, \\ ]\alpha, a[ &= z_1, & ]\alpha, b[ &= z_2, & ]\alpha, z_i[ &= 0, \\ ]\alpha, \alpha[ &= 0. \end{aligned} \tag{5.6}$$

We denote this Lie antialgebra  $\widehat{\widehat{\mathfrak{ah}}}_1$ . The center of  $\widehat{\widehat{\mathfrak{ah}}}_1$  is spanned by  $z_1$  and  $z_2$ , so that this is a central extension (of type I) of the above antialgebra  $\widehat{\mathfrak{ah}}_1$ .

We are now ready to formulate the main statement of this section.

**Theorem 4.** *A Lie antialgebra of rank 1 is of the form*

$$\mathfrak{a} = \bar{\mathfrak{a}} \oplus Z,$$

where  $\bar{\mathfrak{a}}$  belongs either to the list (5.3), or to the list (5.4), or one of the antialgebras  $\widehat{\mathfrak{ah}}_1, \widehat{\widehat{\mathfrak{ah}}}_1$ .

*Proof.* We already proved the theorem in the following two cases: the form  $\omega$  is non-degenerate, or  $\omega \equiv 0$ . It remains to consider the intermediate case where the 2-form  $\omega$  is not identically zero but with a non-trivial kernel:

$$\mathcal{I} = \ker \omega \neq \{0\}.$$

The space  $\mathcal{I}$  is then an abelian ideal (see Proposition 4.19) and, furthermore, belongs to the center (see Proposition 4.20). We summarize this in a form of a

**Lemma 5.6.** *The Lie antialgebra  $\mathfrak{a}$  is a central extension of  $\mathfrak{a}/\mathcal{I}$ .*

To complete the classification, one now has to classify the central extensions of type I of the antialgebras with non-degenerate form  $\omega$ , that is, of (5.1) and (5.2).

**Lemma 5.7.** *The Lie antialgebras  $\mathfrak{asl}(2, \mathbb{R}), \mathfrak{ah}_n$  with  $n \geq 2$  and  $\widetilde{\mathfrak{ah}}_1$  (resp.  $\widetilde{\mathfrak{ah}}_1^+, \widetilde{\mathfrak{ah}}_1^-$ ) have no non-trivial central extensions of type I.*

*Proof.* In the case of  $\mathfrak{asl}(2, \mathbb{R})$  this follows from Theorem 3.

For the Lie antialgebra  $\mathfrak{ah}_n$  with  $n \geq 2$ , let  $C$  be an arbitrary 2-cocycle of type I. One has

$$C(\alpha, a_i) = C([a_j, b_j], a_i) = -C([b_j, a_i], a_j) - C([a_i, a_j], b_j) = 0$$

(from the second identity (4.12)) for all  $i \neq j$ . Similarly,  $C(\alpha, b_i) = 0$  for all  $i$ . Therefore,  $C$  is identically zero.

For the Lie antialgebra  $\widetilde{\mathfrak{ah}}_1$  and an arbitrary 2-cocycle  $C$  of type I, one has

$$C(\alpha, b) = \frac{1}{\kappa} C(\alpha, [\alpha, a]) = \frac{1}{2\kappa} C([\alpha, \alpha], a) = 0.$$

The cocycle  $C$  is then defined by its values on  $\alpha$  and  $a$ . Let us show that the corresponding extension is trivial. Let  $C(\alpha, a) = cz$ , where  $z$  is an arbitrary generator of the center and  $c$  arbitrary constant. Set  $b' = b + \frac{c}{\kappa} z$ . Then in the new basis  $(a, b', z; \alpha)$ , the cocycle  $C'$  vanishes.  $\square$

**Lemma 5.8.** *The algebra  $\mathfrak{ah}_1$  has a unique non-trivial central extension of type I; the result of this central extension is the Lie antialgebra (5.5).*

*Proof.* Indeed, let  $C$  be a 2-cocycle of type I on  $\mathfrak{ah}_1$ . It is given by the formula

$$C(\alpha, a) = c_1 z, \quad C(\alpha, b) = c_2 z,$$

where  $z$  is an element of the center and  $c_1$  and  $c_2$  are arbitrary constants. Assume (without loss of generality) that  $c_1 \neq 0$ , then choose another element of the basis:  $b' = b + \frac{c_2}{c_1} a$ . One obtains  $C(\alpha, b) = 0$ . Furthermore, taking  $z' = c_1 z$ , one gets precisely the Lie antialgebra (5.5). This Lie antialgebra is not isomorphic to  $\mathfrak{ah}_1$  so that the extension is, indeed, non-trivial.  $\square$

**Lemma 5.9.** *The algebra (5.6) has a unique non-trivial central extension of type I; the result of this central extension is the Lie antialgebra (5.6).*

*Proof.* It is similar to the proof of Lemma 5.8: one shows that the Lie antialgebra (5.6) is precisely the unique (up to isomorphism) central extension of (5.5).  $\square$

In the same way, one proves the following

**Lemma 5.10.** *The Lie antialgebra (5.6) has no non-trivial central extensions of type I.*

We classified all the non-trivial central extensions of the Lie antialgebras of rank 1 with non-degenerate form  $\omega$ . This completes the proof of Theorem 4.  $\square$

## Discussion

Let us formulate some open problems and outline a few ideas for the further development.

**A.** The subject of this paper is closely related to so-called *invariant differential operators*, see [3]. Indeed, the simplest example of a Lie antialgebra,  $\mathfrak{asl}(2)$ , is related to the bivector (2.4) and the corresponding antibracket (2.5) that can be characterized as the unique odd bilinear  $\mathcal{K}(1)$ -invariant operator on a 1|1-dimensional contact manifold. This is a very classical subject and the classification of bilinear differential operators on (super)manifolds invariant with respect to the full diffeomorphism (super)group is well known, see [8, 10]. The operator (2.5) is invariant only with respect to the supergroup of contact diffeomorphisms and therefore does not appear in the known classifications. Let us mention that the operator (2.5) coincides with the first supertransvectant, see [6].

**B.** The most important geometric object related to a Lie antialgebra  $\mathfrak{a}$  is the canonical odd bivector  $\Lambda_{\mathfrak{a}}$  on the dual space  $\mathfrak{a}^*$ , see formula (4.7). The relation between  $\mathfrak{a}$  and  $\Lambda_{\mathfrak{a}}$  is exactly the same as that between a Lie (super)algebra and the corresponding classical (even) Lie-Poisson bivector. However, unlike a Poisson bivector,  $\Lambda_{\mathfrak{a}}$  does not define a Lie antialgebra structure on the space of all functions on  $\mathfrak{a}^*$ . More precisely, the space  $C^\infty(\mathfrak{a}^*)$  is not a Lie antialgebra with respect to the antibracket (4.7). It is then natural to ask the same question that in Section 3: *what is the maximal subspace of functions on  $\mathfrak{a}^*$  which this is a Lie antialgebra with respect to the antibracket (4.7)?* In other words, we wish to define a “conformal version” of an arbitrary Lie antialgebra  $\mathfrak{a}$ . Similarly, the full space of linear operators equipped with the anticommutator (4.1) is not a Lie antialgebra. What is the maximal subspace that satisfies this property? These questions are closely related to Conjecture 4.15 which is a challenging problem.

**C.** The present paper makes only the very first steps towards a complete general theory of Lie antialgebras. Among the most important problems is the problem of construction of enveloping algebras and cohomology theory. Such a theory should, of course, include the notion of central extension, see Section 4.4. We hope to treat this subject in a subsequent paper.

**D.** We think that relations between Lie antialgebras and Lie superalgebras should be better studied. We associated with any Lie antialgebra  $\mathfrak{a}$  a Lie superalgebra of symmetry:  $\mathfrak{g}_{\mathfrak{a}} = \text{Der}(\mathfrak{a})$ . It is natural to formulate an inverse problem: given a (simple) Lie superalgebra  $\mathfrak{g}$  and a  $\mathfrak{g}$ -module  $V$ , *under what condition there exists a  $\mathfrak{g}$ -invariant structure of Lie antialgebra on  $V$ ?* More particularly, it would be interesting to find Lie antialgebras related to other superconformal algebras than the simplest Neveu-Schwarz algebra, see, e.g., [11, 9].

**E.** It is not at all clear whether there is any notion of “group object” related to a Lie antialgebra, in the same way as the notion of Lie (super)group is related with that of Lie (super)algebras. Since Lie antialgebras generalize commutative algebras, such an object should generalize the notion of spectrum.

**F.** Keeping in mind multiple application of the classical Lie-Poisson structures (in representation theory, integrable systems, quantization, etc.), we hope that the canonical odd bivector (4.7) also can be useful for the same purposes. As one of possible applications, one can consider analog of Euler equations on the dual space  $\mathfrak{a}^*$  of a Lie antialgebra  $\mathfrak{a}$ . Such equations would not necessarily be Hamiltonian in the usual sense, but there exists a great number of interesting dynamical systems which cannot be realized as Hamiltonian vector fields.

**G.** Another intriguing problem is the relation of Lie antialgebras to contact geometry; an example is given in Section 3.3. It would be interesting to investigate the general situation. Given a contact (super)manifold  $M$ , consider the space  $\text{TVect}(M)$  of vector fields tangent to the contact distribution. Of course, this space is not a Lie (super)algebra since the very definition of a contact structure means that the Lie bracket of two tangent vector fields is *almost never* a tangent vector field. Nevertheless, there are several natural algebraic structures on the space  $\text{TVect}(M)$ . To give an example, the simplest algebraic structure uses the decomposition of the full space of vector fields on  $M$  into a direct sum of contact and tangent vector fields:

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M),$$

see [15]. If  $X$  and  $Y$  are two tangent fields, then there is an *invariant* projection of the Lie bracket  $[X, Y]$  to  $\text{TVect}(M)$ . This structure has never been studied and the properties of the resulting algebra are unknown. This is however not the anticommutator (4.1) that we considered in Section 3.3. We believe that the antibracket of tangent fields on  $S^{1|1}$  has multi-dimensional generalizations and should be related to a multi-Hamiltonian approach. The simple Lie antialgebra  $\text{asl}(2, \mathbb{C})$  provide an example. In this case, the a natural projection (quotient by homotheties) defines a structure of bi-contact supermanifold on  $\mathbb{P}^{3|2}$  that corresponds to the symplectic structures  $\omega_{\varepsilon}$  and  $\omega_{\sigma}$ .

**Acknowledgments.** I am pleased to thank C. Duval, D. Fuchs, J. Germoni and S. Tabachnikov for a careful reading of the first version of this paper, corrections, and extremely valuable discussions. I am also grateful to F. Chapoton, C. Conley, Y. Fregier, H. Gargoubi, D. Leites, O. Kravchenko, S. Morier-Genoud, S. Parmentier, N. Poncin, C. Roger and A. Tchoudjem for enlightening discussions at different stages of preparation of this paper.



## References

- [1] C. Buttin, *Théorie des opérateurs différentiels gradués sur les formes différentielles*, Bull. Soc. Math. France, **102** (1974), 49–73.
- [2] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, **150**. Springer-Verlag, N.Y., 1995.
- [3] D. Fuks, *Cohomology of infinite-dimensional Lie algebras*. Consultants Bureau, New York, 1986.
- [4] H. Gargoubi, V. Ovsienko, *Supertransvectants and symplectic geometry*, hal-00145402.
- [5] H. Gargoubi, N. Mellouli, V. Ovsienko, *Differential operators on supercircle: conformally equivariant quantization and symbol calculus*, Lett. Math. Phys. **79** (2007) 51–65.
- [6] F. Gieres, *Conformally covariant operators on Riemann surfaces (with applications to conformal and integrable models)*, Int. J. Mod. Phys. A, **8** (1993), 1–58.
- [7] I. Gelfand, I. Zakharevich, *Webs, Lenard schemes, and the local geometry of bi-Hamiltonian Toda and Lax structures*, Selecta Math. (NS) **6** (2000), 131–183.
- [8] P. Grozman, *Classification of bilinear invariant operators on tensor fields*, Func. Anal. Appl. **14** (1980) 58–59.
- [9] P. Grozman, D. Leites, I. Shchepochkina, *Lie superalgebras of string theories*, Acta Math.Vietnam, **26** (2001), 27–63.
- [10] P. Grozman, D. Leites, I. Shchepochkina, *Invariant operators on supermanifolds and standard models*, in: Multiple facets of quantization and supersymmetry, 508–555, M. Olshanetski, A. Vainstein (eds.), World Scientific, 2002.
- [11] V. Kac, J. van de Leur, *On classification of superconformal algebras*, in *Strings-88* World Scientific, 1989, 77–106.
- [12] Y. Kosmann-Schwarzbach, *Derived brackets*, Lett. Math. Phys. **69** (2004), 61–87.
- [13] D. Leites, *Supermanifold theory*, Petrozavodsk 1983 (Russian).
- [14] D. Leites, I. Shchepochkina, *How should an antibracket be quantized?* Theoret. and Math. Phys. **126** (2001), 281–306.
- [15] V. Ovsienko, *Vector fields in the presence of a contact structure*, Enseignement Mathématique, **52** (2006), 215–229.

CNRS,  
Institut Camille Jordan,  
Université Lyon 1,  
Villeurbanne Cedex, F-69622,  
France;  
ovsienko@math.univ-lyon1.fr