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Lie antialgebras

V. Ovsienko

Натке

Abstract

We introduce a new class of algebras that we call Lie antialgebras. The structure of a Lie antialgebra is defined by a changing parity operation on a \mathbb{Z}_2 -graded space satisfying a set of identities amazingly “opposite” to those of a Lie superalgebra. We classify Lie antialgebras of rank 1 and consider several examples of simple Lie antialgebras. We also define an analog of the Lie-Poisson structure on the space dual to a Lie antialgebra and study the notion of central extensions.

1 Introduction

We call *antialgebra* a \mathbb{Z}_2 -graded algebra $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ with an *odd* product, that is a bilinear map from $\mathfrak{a} \otimes \mathfrak{a}$ to \mathfrak{a} changing the parity: if $a \in \mathfrak{a}_i$ and $b \in \mathfrak{a}_j$ with $i, j \in \{0, 1 \mid \text{mod} 2\}$, then $a \cdot b \in \mathfrak{a}_{i+j+1}$. Such structures are known quite well. One example is the Buttin bracket (or the “Poisson antibracket”), see, e.g., [2, 8, 9]. Another one is the usual Schouten bracket on the space of skew-symmetric contravariant tensors in the Batalin-Vilkovisky formalism, see [7] for a survey. All these notions are related to graded Lie superalgebras.

We introduce a class of antialgebras that seems to be new and cannot be reduced to any known structure.

Definition 1.1. An antialgebra \mathfrak{a} is called a *Lie antialgebra* if its product, that we denote by \lrcorner , $\lrcorner : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}$, satisfies the following properties:

1. the operation \lrcorner , $\lrcorner : \mathfrak{a}_1 \otimes \mathfrak{a}_1 \rightarrow \mathfrak{a}_1$ equips the odd part \mathfrak{a}_1 with a structure of a *commutative associative* algebra, i.e., it satisfies two conditions: $\lrcorner \alpha, \beta \lrcorner = \lrcorner \beta, \alpha \lrcorner$ for all $\alpha, \beta \in \mathfrak{a}_1$ and

$$\lrcorner \alpha, \lrcorner \beta, \gamma \lrcorner = \lrcorner \alpha, \beta \lrcorner, \gamma \lrcorner \quad (1.1)$$

for all $\alpha, \beta, \gamma \in \mathfrak{a}_1$;

2. the map \lrcorner , $\lrcorner : \mathfrak{a}_1 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_0$ is *skew-symmetric*: $\lrcorner \alpha, a \lrcorner = -\lrcorner a, \alpha \lrcorner$ for $\alpha \in \mathfrak{a}_1$ and $a \in \mathfrak{a}_0$ and one has

$$\lrcorner \alpha, \lrcorner \beta, a \lrcorner + \lrcorner \beta, \lrcorner \alpha, a \lrcorner = \lrcorner \alpha, \beta \lrcorner, a \lrcorner \quad (1.2)$$

for all $\alpha, \beta \in \mathfrak{a}_1$ and $a \in \mathfrak{a}_0$, so that the map $\text{ad}^0 : \mathfrak{a}_1 \rightarrow \text{End}(\mathfrak{a}_0)$ given by $\alpha \mapsto \lrcorner \alpha, \cdot \lrcorner$, defines an *action* of the commutative algebra \mathfrak{a}_1 on the space \mathfrak{a}_0 in the following (unusual for commutative associative algebras) sense:

$$\text{ad}_\alpha^0 \circ \text{ad}_\beta^0 + \text{ad}_\beta^0 \circ \text{ad}_\alpha^0 = \text{ad}_{\lrcorner \alpha, \beta \lrcorner}^0;$$

3. the map $], [: \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$ is *skew-symmetric*: $]a, b[= -]b, a[$ for all $a, b \in \mathfrak{a}_0$ and satisfies the Leibniz identity

$$] \alpha,]b, c[[=]]\alpha, b[, c[+]b,]\alpha, c[[\quad (1.3)$$

for all $\alpha \in \mathfrak{a}_1$ and $b, c \in \mathfrak{a}_0$;

4. The *Jacobi identity* is satisfied:

$$]a,]b, c[[+]b,]c, a[[+]c,]a, b[[= 0 \quad (1.4)$$

for all $a, b, c \in \mathfrak{a}_0$.

All the vector spaces considered in this paper are defined over $\mathbb{K} = \mathbb{C}$ or \mathbb{R} .

Remark 1.2. The identity (1.3) has precisely the form of the condition of \mathfrak{a}_1 -*invariance* of the bilinear map $], [: \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$. However, to speak of invariance, one needs to define an action of the commutative algebra \mathfrak{a}_1 on $\mathfrak{a}_0 \otimes \mathfrak{a}_0$ which implicitly means one fixes a coproduct $\Delta : \mathfrak{a}_1 \rightarrow \mathfrak{a}_1 \otimes \mathfrak{a}_1$. This is, actually, the case for the main examples of Lie antialgebras we consider in this paper, but one does not need a-priori to impose the existence of a coproduct.

Definition 1.1 should be compared with the usual notion of a Lie superalgebra. Recall that a \mathbb{Z}_2 -graded space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra if it is equipped with an *even* bilinear operation $], [: \mathfrak{g} \rightarrow \mathfrak{g}$ such that:

1. the operation $], [: \mathfrak{g}_0 \otimes \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ defines a Lie algebra structure on \mathfrak{g}_0 ;
2. the map $], [: \mathfrak{g}_0 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ is a Lie algebra action of \mathfrak{g}_0 on \mathfrak{g}_1 ;
3. the map $], [: \mathfrak{g}_1 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ is symmetric and \mathfrak{g}_0 -invariant;
4. the Jacobi identity $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$ is satisfied for all $\xi, \eta, \zeta \in \mathfrak{g}_1$.

Note that the above definition of a Lie superalgebra is more conceptual than the usual (and much shorter) one saying that the bracket $], [$ is skew-symmetric in the \mathbb{Z}_2 -graded sense and satisfies the Jacobi identity with the usual \mathbb{Z}_2 -graded sign rule. Definition 1.1 is, of course, very similar to the definition of a Lie superalgebra. It is even almost identical if one inverts the parity and replaces the Lie algebra \mathfrak{g}_0 by a commutative associative algebra \mathfrak{a}_1 .

We associate with any Lie antialgebra \mathfrak{a} the Lie superalgebra of *symmetry* denoted by $\mathfrak{g}_\mathfrak{a}$. This notion is important for us since it often allows to characterize the initial Lie antialgebra \mathfrak{a} as the unique Lie antialgebra preserved by the $\mathfrak{g}_\mathfrak{a}$ -action. Let us stress that the $\mathfrak{g}_\mathfrak{a}$ -invariance condition should be understood in the graded sense:

$$\varrho_x]a, b[= -(1)^{p(x)}]\varrho_x a, b[+ (1)^{p(x)(1+p(a))}]a, \varrho_x b[, \quad (1.5)$$

where $x \in \mathfrak{g}_\mathfrak{a}$, $a, b \in \mathfrak{a}$, $\varrho : \mathfrak{g}_\mathfrak{a} \rightarrow \text{End}(\mathfrak{a})$ is the action and p is the parity function. The sign rule in (1.5) is due to the fact that $], \cdot[$ is an odd operation.

Remark 1.3. The relation (1.5) shows, in particular, that the usual functor $\Pi : \mathfrak{a}_0 \oplus \mathfrak{a}_1 \rightarrow \mathfrak{a}_1 \oplus \mathfrak{a}_0$ inverting the parity does not make much sense here since it has no invariant meaning.

The main purpose of this paper is to show that the notion of Lie antialgebra is by no means artificial and this algebraic structure appears very naturally, for instance, in the context of symplectic and contact geometry.

The main results of this paper are as follows.

A. We construct several examples of Lie antialgebras. The most interesting examples are *simple* Lie antialgebras. Two of them are finite-dimensional (of dimension $2|1$ and $4|2$) and one is infinite-dimensional. We do not know other finite-dimensional simple Lie antialgebras. We also classify Lie antialgebras of rank 1. This, in particular, provides a number of examples of Lie antialgebras other than simple ones.

B. We introduce the notion of *canonical bivector* on the dual space \mathfrak{a}^* of a Lie antialgebra \mathfrak{a} . This bivector is an analog of the classical Poisson-Lie-Berezin-Kirillov-Kostant-Souriau structure, and this is the way Lie antialgebras are related to symplectic and contact geometry.

C. We introduce the notions of central extension of Lie antialgebras. This notion should be included into the general theory of cohomology of Lie antialgebras, a theory yet to be developed.

This paper is organized as follows.

We start in Section 2 with an example of a simple Lie antialgebra of dimension $2|1$, that we call $\mathfrak{asl}(2)$. We show how this structure naturally appears directly from the standard $2|1$ -dimensional symplectic space, as the only odd bivector preserved by the supergroup $\mathrm{OSp}(1|2)$ of linear symplectic transformations. We realize the Lie superalgebra $\mathfrak{osp}(1|2)$ as the algebra of symmetry of $\mathfrak{asl}(2)$. We also define the odd Lie-Poisson type bivector on the dual space $\mathfrak{asl}(2)^*$ that should be considered as analog of the usual Lie-Poisson structure. This first example illustrates the nature and the origins of the very notion of Lie antialgebra.

In Section 3 we develop our general framework and introduce the basic notations and notions, such as the rank of a Lie antialgebra. We explain the relations to symplectic geometry and to an old problem of linear algebra: study of pencils of skew-symmetric forms.

In Section 4 we define the notion of (central) extension of a Lie antialgebra. We prove that there are no non-trivial central extensions in the case where \mathfrak{a}_0 contains the unit. The notion of central extension will be crucial for the classification result in rank 1.

In Section 5 we obtain a complete classification of Lie antialgebras of rank 1 (i.e., with one-dimensional odd part). We show that $\mathfrak{asl}(2)$ is the unique simple Lie antialgebra of rank 1, we also find a number of different examples of Lie antialgebras.

In Section 6 we define the odd Lie-Poisson type bivector on the space dual to a Lie antialgebra. We also define the notion of symmetry superalgebra and conformal symmetry superalgebra and relate the context to multi-Hamiltonian formalism.

In Section 7 we introduce our second example of simple Lie antialgebra. This Lie antialgebra is of dimension $4|2$ and we denote it $\mathfrak{ao}(4)$. We calculate the symmetry Lie superalgebra of $\mathfrak{ao}(4)$ acting on the dual space $\mathfrak{ao}(4)^*$ by bi-Hamiltonian vector fields.

In Section 8 we give an example of a simple infinite-dimensional Lie antialgebra $\widehat{\mathfrak{asl}(2)}$. This antialgebra has the superconformal Lie superalgebra $\mathcal{K}(1)$ as algebra of symmetry. A geometric way to define $\widehat{\mathfrak{asl}(2)}$ is to say that this is the algebra of vector fields on $S^{1|1}$ tangent to the contact structure.

In the end of this paper, we formulate a number of open problems and suggest some further development.

2 The simple Lie antialgebra $\text{asl}(2)$

We introduce our most elementary and perhaps the most important example of a Lie antialgebra. The algebra $\text{asl}(2)$ is a simple Lie antialgebra of dimension $2|1$.

2.1 The definition

The basis of $\text{asl}(2)$ is denoted by $\{a, b; \varepsilon\}$ and the relations between the elements of the basis are as follows:

$$]a, b[= \frac{1}{2} \varepsilon, \quad]\varepsilon, a[= \frac{1}{2} a, \quad]\varepsilon, b[= \frac{1}{2} b, \quad]\varepsilon, \varepsilon[= \varepsilon. \quad (2.1)$$

It is easy to check that the conditions 1–4 of Definition 1.1 are satisfied. The Lie antialgebra (2.1) is obviously *simple*, i.e., it contains no ideal except for the trivial ideal $\{0\}$ and $\text{asl}(2)$ itself.

The reason for the notation $\text{asl}(2)$ is the similitude of this antialgebra with the classical Lie algebra $\text{sl}(2)$. The next property shows, however, that $\text{asl}(2)$ is quite different as nothing similar occurs in the classical case.

2.2 The Lie superalgebra $\text{osp}(1|2)$ as algebra of symmetry

The Lie antialgebra $\text{asl}(2)$ is characterized by its invariance with respect to the action of the Lie superalgebra $\text{osp}(1|2)$.

Let us recall that $\text{osp}(1|2)$ is a $3|2$ -dimensional Lie superalgebra with the basis

$$\{x_{-1}, x_0, x_1; \xi_{-\frac{1}{2}}, \xi_{\frac{1}{2}}\}$$

and the following relations

$$\begin{aligned} [x_i, x_j] &= (j - i) x_{i+j}, \\ [x_i, \xi_j] &= (j - \frac{i}{2}) \xi_{i+j}, \\ [\xi_i, \xi_j] &= x_{i+j}. \end{aligned} \quad (2.2)$$

Of course, the even part $\text{osp}(1|2)_0$ is nothing but $\text{sl}(2)$.

We define the vector space $\mathbb{K}^{2|1}$ in terms of the supercommutative algebra of functions $C^\infty(\mathbb{K}^{2|1})$. The elements of this algebra are of the form:

$$F(p, q; \tau) = F_0(p, q) + \tau F_1(p, q),$$

where $F_0, F_1 \in C^\infty(\mathbb{K}^2)$. The standard symplectic form on $\mathbb{K}^{2|1}$ is give by

$$\omega = dp \wedge dq + d\tau \wedge d\tau.$$

The Lie superalgebra $\text{osp}(1|2)$ is the the algebra of linear transformations of the space $\mathbb{K}^{2|1}$ preserving ω .

Considering the standard Poisson bracket $\{F, G\} = \frac{1}{2} \langle \mathcal{P}, dF \wedge dG \rangle$, where \mathcal{P} is the Poisson bivector:

$$\mathcal{P} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \frac{\partial}{\partial \tau} \wedge \frac{\partial}{\partial \tau}. \quad (2.3)$$

The Lie superalgebra $\text{osp}(1|2)$ is identified with the space of quadratic polynomials:

$$\{x_{-1}, x_0, x_1; \xi_{-\frac{1}{2}}, \xi_{\frac{1}{2}}\} \longleftrightarrow \{p^2, pq, q^2; \frac{1}{2} p\tau, \frac{1}{2} q\tau\}, \quad (2.4)$$

so that the Poisson bracket corresponds precisely to the Lie bracket (2.2).

It turns out that there exists another, *odd*, bivector on $\mathbb{K}^{2|1}$, invariant with respect to the $\mathfrak{osp}(1|2)$ -action, namely,

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \mathcal{E} + \tau \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}, \quad (2.5)$$

where

$$\mathcal{E} = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} + \tau \frac{\partial}{\partial \tau}$$

is the Euler field.

Proposition 2.1. *The bivector (2.5) is the unique (up to a multiplicative constant) odd bivector invariant with respect to the action of $\mathfrak{osp}(1|2)$.*

Proof. The $\mathfrak{osp}(1|2)$ -invariance of Λ is a very easy check. Note that we will prove a much stronger statement in Section 8, see Proposition 8.7.

Let us prove the uniqueness. An arbitrary odd bivector on $\mathbb{K}^{2|1}$ is given by

$$\Lambda = \frac{\partial}{\partial \tau} \wedge A + \tau F \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q},$$

where A is an even vector field and F is an even function. Let X be an even vector field, one has

$$L_X \Lambda = \frac{\partial}{\partial \tau} \wedge [X, A] + \tau X(F) \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \tau F L_X \left(\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} \right).$$

If, furthermore, $X \in \mathfrak{osp}(1|2)$, then it preserves the even part $\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$ of the Poisson bivector. The condition $L_X \Lambda = 0$ then implies:

$$[X, A] = 0 \quad \text{and} \quad X(F) = 0.$$

It is easy to see that an even vector field A commuting with any even element of $\mathfrak{osp}(1|2)$ is proportional to the Euler field and an even function killed by all even elements of $\mathfrak{osp}(1|2)$ is constant. One has $A = c_1 \mathcal{E}$ and $F = c_2$.

Finally, applying to Λ an odd element of $\mathfrak{osp}(1|2)$, one immediately gets $c_1 = c_2$. \square

Define the following antialgebra structure on the space of functions:

$$]F, G[= \frac{1}{2} \langle \Lambda, dF \wedge dG \rangle.$$

The explicit formula is as follows:

$$]F, G[= \frac{1}{2} \left(\frac{\partial F}{\partial \tau} \mathcal{E}(G) - (-1)^{p(F)} \mathcal{E}(F) \frac{\partial G}{\partial \tau} + \tau \left(\frac{\partial F}{\partial p} \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} \right) \right). \quad (2.6)$$

This is a *linear antibracket* in the sense that the space of linear functions on $\mathbb{K}^{2|1}$ is obviously closed with respect to the antibracket (2.6).

Lemma 2.2. *The space of linear functions on $\mathbb{K}^{2|1}$ equipped with the bracket (2.6) is a Lie antialgebra isomorphic to $\mathfrak{asl}(2)$.*

Proof. One checks that after the identification

$$\{a, b; \varepsilon\} \longleftrightarrow \{q, -p; \tau\}, \quad (2.7)$$

the brackets (2.1) and (6.3) on linear functions coincide. \square

Lie superalgebra $\mathfrak{osp}(1|2)$ acts on the space of linear functions by the Poisson bracket with the quadratic polynomials (2.4) and, thanks to the identification (2.7), on the Lie antialgebra $\mathfrak{asl}(2)$. Proposition 2.1 implies that this action preserves the Lie antialgebra structure.

Remark 2.3. The bivector (2.5) (with the antibracket (2.6)) should be considered as analog of the canonical *Lie-Poisson* structure on the dual space of $\mathfrak{asl}(2)$, see Section 6 below. Proposition 2.1 shows how this bivector (and thus the structure of $\mathfrak{asl}(2)$ itself) can be defined intrinsically through the condition of $\mathfrak{osp}(1|2)$ -invariance.

The bivector (2.5) and the antibracket (2.6) were found in [3].

2.3 The highest weight module structure

The Lie antialgebra $\mathfrak{asl}(2)$ can be viewed as (the smallest) highest weight module of $\mathfrak{osp}(1|2)$. This module is of dimension $2|1$, it is generated by the highest weight vector v and characterized by the relations

$$x_1 v = \xi_{\frac{1}{2}} v = 0, \quad x_0 v = \frac{1}{2} v.$$

The Lie antialgebra $\mathfrak{asl}(2)$ is identified with this $\mathfrak{osp}(1|2)$ -module explicitly by

$$\{a, b; \varepsilon\} \longleftrightarrow \{v, -x_{-1} v, \xi_{-\frac{1}{2}} v\}.$$

The above highest weight module is now equipped with an $\mathfrak{osp}(1|2)$ -invariant structure of Lie antialgebra induced from (2.1).

Note also that formula (8.2) below provides another (explicit) expression for the $\mathfrak{osp}(1|2)$ -action on $\mathfrak{asl}(2)$.

3 Basic notions

In this section we fix the notations and introduce some basic notions that we consider throughout this paper.

3.1 Rank of a Lie antialgebra

Given a Lie antialgebra \mathfrak{a} , we consider the commutative subalgebra \mathfrak{a}_1 as analog of the notion of “Cartan subalgebra” of \mathfrak{a} . This justifies the following terminology.

Definition 3.1. We call the dimension of the odd part \mathfrak{a}_1 of a Lie antialgebra \mathfrak{a} the *rank* of \mathfrak{a} :

$$\mathrm{rk} \mathfrak{a} := \dim \mathfrak{a}_1.$$

Rank is an important characteristic of a Lie antialgebra that simplifies classification problems, cf. Section 5.

3.2 The pencil of presymplectic forms

Let \mathfrak{a} be a Lie antialgebra of rank r . Fix an arbitrary basis $\{\alpha_1, \dots, \alpha_r\}$ of the odd part \mathfrak{a}_1 . One obtains a set of r bilinear skew-symmetric (or presymplectic) forms: $\{\omega_1, \dots, \omega_r\}$ on \mathfrak{a}_0 by projecting the antibracket to each basic element:

$$]a, b[= \sum_{i=1}^r \omega_i(a, b) \alpha_i. \quad (3.1)$$

Changing the basis, one obtains linear changes of the corresponding set of skew-symmetric forms. Therefore, the pencil $\langle \omega_1, \dots, \omega_r \rangle$ (e.g., the vector space spanned by the above skew-symmetric forms) is well-defined.

Remark 3.2. Classification of pencils of r linear skew-symmetric forms on a vector space is a classical problem of linear algebra already considered by Kronecker. This problem turned out to be closely related to integrable systems, see, [6] and references therein.

3.3 Adjoint representation

We will use the standard notation $x \mapsto \text{ad}_x$ for the adjoint action $\text{ad} : \mathfrak{a} \rightarrow \text{End}(\mathfrak{a})$ given by

$$\text{ad}_x y =]x, y[, \quad (3.2)$$

for all $x, y \in \mathfrak{a}$. We will also use the notations

$$\text{ad}_x^0 := \text{ad}_x |_{\mathfrak{a}_0}, \quad \text{ad}_x^1 := \text{ad}_x |_{\mathfrak{a}_1}$$

for the restrictions of ad_x to the even and odd parts of \mathfrak{a} , respectively. The following statement is straightforward.

Proposition 3.3. *The identities (1.1–1.4) are equivalent to the following three relations:*

$$\text{ad}_\alpha \text{ad}_\beta + \text{ad}_\beta \text{ad}_\alpha = \text{ad}_{] \alpha, \beta [}^0 + 2 \text{ad}_{] \alpha, \beta [}^1, \quad (3.3)$$

$$\text{ad}_\alpha \text{ad}_a - \text{ad}_a \text{ad}_\alpha = \text{ad}_{] \alpha, a [}^0 - \text{ad}_{] \alpha, a [}^1, \quad (3.4)$$

$$\text{ad}_a \text{ad}_b - \text{ad}_b \text{ad}_a = \text{ad}_{] a, b [}^0 - \text{ad}_{] a, b [}^1, \quad (3.5)$$

where $\alpha, \beta \in \mathfrak{a}_1$ and $a, b \in \mathfrak{a}_0$.

Example 3.4. The adjoint representation of $\text{asl}(2)$ is given by

$$\text{ad}_\varepsilon = \left(\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad \text{ad}_a = \left(\begin{array}{cc|c} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ \hline 0 & \frac{1}{2} & 0 \end{array} \right), \quad \text{ad}_b = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ \hline -\frac{1}{2} & 0 & 0 \end{array} \right).$$

One easily checks that the relations (2.1) together with (3.3–3.5) are satisfied.

3.4 Representations: a general definition

The expressions in the left-hand-side of formulæ (3.3–3.5) can be written in a “unified manner” with the help of the following antibracket on $\text{End}(\mathfrak{a})$:

$$\llbracket A, B \rrbracket := AB - (-1)^{(p(A)+1)(p(B)+1)} BA. \quad (3.6)$$

Indeed, the operator ad_x is *even* if $x \in \mathfrak{a}_1$ and *odd* if $x \in \mathfrak{a}_0$.

Given a Lie antialgebra \mathfrak{a} and a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$, a linear map $\rho : \mathfrak{a} \rightarrow \text{End}(V)$ satisfying the relation

$$\llbracket \rho_x, \rho_y \rrbracket := \rho_{[x,y]}^0 - (-2)^{p(x)p(y)} \rho_{]x,y[}^1, \quad (3.7)$$

where ρ^0 and ρ^1 are restrictions of ρ to V_0 and V_1 , respectively. is called a *representation* of \mathfrak{a} . The space V is then called an \mathfrak{a} -*module*.

Remark 3.5. Given any \mathbb{Z}_2 -graded vector space V , the antibracket (3.6) defines a structure of an antialgebra on $\text{End}(V)$. This is, however, *not* a Lie antialgebra structure, since the identities (1.1–1.4) are not satisfied.

3.5 Adjoint action of the unit

Assume that the associative commutative algebra \mathfrak{a}_1 contains the unit element ε . By definition,

$$\text{ad}_\varepsilon^1 = \text{Id}.$$

Let us consider the action of ε on the even part of \mathfrak{a} . The identity (1.2) implies the “half-projector” relation $\text{ad}_\varepsilon^0 \circ \text{ad}_\varepsilon^0 = \frac{1}{2} \text{ad}_\varepsilon^0$. Therefore, the even part of the Lie antialgebra \mathfrak{a} is split to a sum of $\frac{1}{2}$ - and 0-eigenspaces of ad_ε^0 :

$$\mathfrak{a}_0 = \mathfrak{a}_{0,\frac{1}{2}} \oplus \mathfrak{a}_{0,0}, \quad \text{ad}_\varepsilon^0 \big|_{\mathfrak{a}_{0,\frac{1}{2}}} = \frac{1}{2} \text{Id}, \quad \text{ad}_\varepsilon^0 \big|_{\mathfrak{a}_{0,0}} = 0.$$

Theorem 1. (i) *The space $\mathfrak{a}_{0,0}$ coincides with the center, $Z(\mathfrak{a})$, of the Lie antialgebra \mathfrak{a} .*

(ii) *The Lie antialgebra \mathfrak{a} is a direct sum:*

$$\mathfrak{a} = \bar{\mathfrak{a}} \oplus Z(\mathfrak{a}) \quad (3.8)$$

of a subalgebra $\bar{\mathfrak{a}}$ and the abelian Lie antialgebra $Z(\mathfrak{a})$.

Proof. Part (i). Let $a \in \mathfrak{a}_{0,0}$, that is, $]\varepsilon, a[= 0$. One has to show that $]x, a[= 0$ for all $x \in \mathfrak{a}$.

Let first $x = \alpha$ be an element of \mathfrak{a}_1 . The identity (1.2) implies

$$]\varepsilon,]\alpha, a[[=]\varepsilon,]\alpha, a[[+]\alpha,]\varepsilon, a[[=]]\varepsilon, \alpha[, a[=]\alpha, a[.$$

But then $\text{ad}_\varepsilon^0 \circ \text{ad}_\varepsilon^0 = \frac{1}{2} \text{ad}_\varepsilon^0$ implies $]\alpha, a[= 0$.

Let now $x = b$ be an element of \mathfrak{a}_0 . One has

$$]b, a[=]\varepsilon,]b, a[[=]]\varepsilon, b[, a[+]b,]\varepsilon, a[[=]]\varepsilon, b[, a[.$$

If, furthermore, $b \in \mathfrak{a}_0^0$, then the last term is zero. Finally, if $b \in \mathfrak{a}_0^{\frac{1}{2}}$, then the last term is equal to $\frac{1}{2}]b, a[$, which again implies $]b, a[= 0$. \square

We postpone the proof of part (ii) of the theorem to Section 4.4.

4 Central extensions

In this section we define the notion of extension of a Lie antialgebra \mathfrak{a} with coefficients in any \mathfrak{a} -module. It will be useful for the classification result of Section 5. The notion of extension should be a part of a general cohomology theory that we will not develop here.

4.1 Abelian extensions

Let \mathfrak{a} be a Lie antialgebra and V an \mathfrak{a} -module. We will consider V as a trivial (or abelian) Lie antialgebra.

Definition 4.1. An exact sequence of Lie antialgebras

$$0 \longrightarrow V \longrightarrow \tilde{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow 0 \quad (4.1)$$

is called an *abelian extension* of the Lie antialgebra \mathfrak{a} with coefficients in V . As a vector space, $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus V$, and the subspace V is obviously an \mathfrak{a} -module. If the subspace V belongs to the center of $\tilde{\mathfrak{a}}$, then the extension (4.1) is called a *central extension*.

A central extension is called *non-trivial* if the Lie antialgebra $\tilde{\mathfrak{a}}$ is not isomorphic to the direct sum $\mathfrak{a} \oplus V$.

Since any central extension can be obtained by iteration of one-dimensional central extensions, it suffice to consider only the case of one-dimensional central extensions. One then has two possibilities:

$$\dim V = 1|0, \quad \text{or} \quad \dim V = 0|1.$$

We then say that the one-dimensional central extension is of *type I* or of *type II*, respectively.

4.2 Central extensions of type I

The general form of central extensions of type I is as follows.

Proposition 4.2. (i) A central extension of type I is defined by an odd linear map

$$C : \mathfrak{a} \wedge \mathfrak{a} \rightarrow \mathbb{K} \quad (4.2)$$

satisfying the following two identities:

$$\begin{aligned} C(\alpha,]\beta, a]) + C(\beta,]\alpha, a]) - C(] \alpha, \beta[, a) &= 0 \\ C(] a, b[, c) + C(] b, c[, a) + C(] c, a[, b) &= 0, \end{aligned} \quad (4.3)$$

for all $\alpha, \beta \in \mathfrak{a}_1$ and $a, b, c \in \mathfrak{a}_0$.

(ii) A central extension is trivial if and only if there exists an even linear function $f : \mathfrak{a}_0 \rightarrow \mathbb{K}$ such that

$$C(\alpha, a) = f(] \alpha, a]) \quad (4.4)$$

for all $\alpha \in \mathfrak{a}_1$ and $a \in \mathfrak{a}_0$.

Proof. Part (i). Given a map C as in (4.2), let us define an antibracket on $\mathfrak{a} \oplus \mathbb{K}$. We fix an element $z \in \mathbb{K}$ and set:

$$\begin{aligned}]x, y[&=]x, y[_{\mathfrak{a}} + C(x, y)z, \\]x, z[&= 0, \end{aligned} \tag{4.5}$$

for all $x, y \in \mathfrak{a}$. Note that, since the map C is skew-symmetric and odd, one has

$$C(x, y) = C(x_1, y_0) - C(y_1, x_0). \tag{4.6}$$

One then easily checks that formula (4.5) defines a structure of a Lie antialgebra if and only if the relations (4.3) are satisfied.

Conversely, a Lie antialgebra structure on $\mathfrak{a} \oplus \mathbb{K}$ such that the subspace \mathbb{K} belongs to the center is obviously of the form (4.5).

Part (ii). In the case where C is as in (4.4), the linear map $\mathfrak{a} \oplus \mathbb{K} \rightarrow \mathfrak{a} \oplus \mathbb{K}$ given by $(x, z) \mapsto (x, z + f(x))$ intertwines the structure (4.5) with the trivial direct sum structure. This means that the central extension is trivial.

Conversely, if the extension is trivial, then there exists an intertwining map $\mathfrak{a} \oplus \mathbb{K} \rightarrow \mathfrak{a} \oplus \mathbb{K}$ sending the structure (4.5) to the trivial one. This map can, again, be chosen in the form $(x, z) \mapsto (x, z + f(x))$, since a different choice of the embedding of \mathbb{K} does not change the structure. \square

We will call a map C satisfying (4.3) a *2-cocycle of type I*. A 2-cocycle of the form (4.4) will be called a *coboundary*.

4.3 Central extensions of type II

Let us describe the central extensions of type II.

Proposition 4.3. (i) *A central extension of type II is defined by an even linear map*

$$C : \mathfrak{a} \wedge \mathfrak{a} \rightarrow \mathbb{K}^{0|1} \tag{4.7}$$

satisfying the following two identities:

$$\begin{aligned} C(\alpha,]a, b[) &= C(] \alpha, a[, b) + C(a,] \alpha, b[) \\ C(] \alpha, \beta[, \gamma) &= C(\alpha,] \beta, \gamma[), \end{aligned} \tag{4.8}$$

for all $\alpha, \beta, \gamma \in \mathfrak{a}_1$ and $a, b \in \mathfrak{a}_0$.

(ii) *The extension is trivial if and only if there exists an even linear functional $f : \mathfrak{a}_1 \rightarrow \mathbb{K}^{0|1}$ such that*

$$C(x, x) = f(]x, y[), \tag{4.9}$$

for all $x, y \in \mathfrak{a}$.

Proof. The proof is similar to that of Proposition 4.2. \square

We will call an even map (4.7) satisfying (4.8) a *2-cocycle of type II*. In the case where it is given by (4.9), the map C is called a coboundary.

4.4 The role of the unit: proof of Theorem 1

Let us show that the presence of the unit element in \mathfrak{a}_1 prevents \mathfrak{a} of existence of central extensions.

Proposition 4.4. *If the commutative algebra \mathfrak{a}_1 contains the unit element, then the Lie antialgebra \mathfrak{a} has no non-trivial central extensions.*

Proof. Let C be a 2-cocycle of type I on \mathfrak{a} . Apply the first identity (4.3) to $\beta = \varepsilon$, where ε is the unit. One has

$$C(\alpha, a) - C(\alpha,]\varepsilon, a[) = C(\varepsilon,]\alpha, a[).$$

If $a \in Z(\mathfrak{a})$, this formula implies $C(\alpha, a) = 0$. If a is an element of the $\frac{1}{2}$ -eigenspace $\mathfrak{a}_{0, \frac{1}{2}}$ of the unit element ε , then one obtains

$$C(\alpha, a) = 2C(\varepsilon,]\alpha, a[).$$

Therefore, the cocycle C is a coboundary.

Let now C be a 2-cocycle of type II. It can be decomposed into a pair (C_0, C_1) of maps

$$C_0 : \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathbb{K}^{0|1}, \quad C_1 : \mathfrak{a}_1 \otimes \mathfrak{a}_1 \rightarrow \mathbb{K}^{0|1},$$

where C_0 is skew-symmetric and C_1 is symmetric. The first condition (4.8) gives

$$C_1(\varepsilon,]a, b[) = C_0(] \varepsilon, a[, b) + C_0(a,] \varepsilon, b[),$$

so that $C_0(a, b) = C_1(\varepsilon,]a, b[)$. The second condition (4.8) implies

$$C_1(\alpha, \beta) = C_1(\varepsilon,]\alpha, \beta[).$$

Therefore, the cocycle C is, again, a coboundary. □

Let us complete the proof of Theorem 1. The Lie antialgebra \mathfrak{a} is a central extension of its quotient $\bar{\mathfrak{a}} = \mathfrak{a}/Z(\mathfrak{a})$. We just proved that this center extension is trivial. The decomposition (3.8) follows.

Theorem 1 is proved.

4.5 Kernel of the presymplectic pencil

Let us now describe a “natural source” of extensions of type I.

Given an arbitrary Lie antialgebra \mathfrak{a} , consider the following space $\mathcal{I} \subset \mathfrak{a}_0$:

$$\mathcal{I} = \bigcap_{i=1}^r \ker \omega_i,$$

where $r = \text{rk} \mathfrak{a}$, the forms ω_i are defined by (3.1), and where $\ker \omega \subset \mathfrak{a}_0$ consists of elements a of \mathfrak{a}_0 such that $\omega(a, b) = 0$ for every $b \in \mathfrak{a}_0$. In other words,

$$\mathcal{I} = \{a \in \mathfrak{a}_0 \mid]a, b[= 0, \text{ for all } b \in \mathfrak{a}_0\}.$$

Proposition 4.5. *The subspace \mathcal{I} is an abelian ideal of \mathfrak{a} .*

Proof. By definition, \mathcal{I} is an abelian subalgebra and the bracket of $a \in \mathcal{I}$ with any element $b \in \mathfrak{a}_0$ vanishes. One has to show that $]\alpha, a[\in \mathcal{I}$, for arbitrary $\alpha \in \mathfrak{a}_1$ and $a \in \mathcal{I}$. Indeed, using the identity (1.3), one obtains

$$])\alpha, a[, b[=]\alpha,]a, b[[-]a,]\alpha, b[[= 0$$

since for $a \in \mathcal{I}$ and every $b \in \mathfrak{a}_0$, one has $]\alpha, b[= 0$. \square

It follows that the Lie antialgebra \mathfrak{a} is an abelian extension of type I of the quotient-algebra \mathfrak{a}/\mathcal{I} .

Proposition 4.6. *If the map $], [: \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$ is surjective, then the ideal \mathcal{I} belongs to the center of \mathfrak{a} .*

Proof. The ideal \mathcal{I} belongs to the center of \mathfrak{a} if and only if the action of \mathfrak{a}_1 on \mathcal{I} is trivial. Surjectivity means that for every $\alpha \in \mathfrak{a}_1$ there are $a, b \in \mathfrak{a}_0$ such that $\alpha =]a, b[$. Using the Jacobi identity (1.4), one obtains for every $c \in \mathcal{I}$:

$$]\alpha, c[=])a, b[, c[= -])b, c[, a[-])c, a[, b[= 0,$$

since both summands in the right-hand-side vanish. \square

The Lie antialgebra \mathfrak{a} is therefore a central extension of \mathfrak{a}/\mathcal{I} (of type I).

5 Lie antialgebras of rank 1

In this section we systematically study the Lie antialgebras of rank 1: we obtain a number of examples and finally give a complete classification of these algebras.

The commutative algebra \mathfrak{a}_1 is one-dimensional and therefore one has a choice between two algebras (as shows the classification result below, both cases may occur):

1. $\mathfrak{a}_1 = \mathbb{K}$, with the unit element $]\varepsilon, \varepsilon[= \varepsilon$;
2. \mathfrak{a}_1 is nilpotent such that $]\alpha, \alpha[\equiv 0$.

Since \mathfrak{a}_1 is one-dimensional, the map $], [: \mathfrak{a}_0 \otimes \mathfrak{a}_0 \rightarrow \mathfrak{a}_1$ is characterized by one bilinear skew-symmetric form, ω , on \mathfrak{a}_0 (see Section 3.2):

$$]a, b[= \omega(a, b) \alpha,$$

where α is a (unique up to a constant) non-zero element of \mathfrak{a}_1 .

5.1 Uniqueness of $\mathfrak{asl}(2)$

Let us start with investigation of the case of simple Lie antialgebras.

Proposition 5.1. *The Lie antialgebra $\mathfrak{asl}(2)$ is the unique simple Lie antialgebra of rank 1.*

Proof. Let \mathfrak{a} be a simple $n|1$ -dimensional Lie antialgebra. By Proposition 4.5, the 2-form ω is non-degenerate, i.e., $\ker \omega = 0$.

Choose a canonical basis $\{a_1, \dots, a_n, b_1, \dots, b_n\}$, of \mathfrak{a}_0 , so that one has

$$\omega(a_i, b_j) = 1, \quad \omega(a_i, a_j) = \omega(b_i, b_j) = 0,$$

where $i, j = 1, \dots, n$.

(a) Case $n > 1$. Let us show that, in this case, the odd part \mathfrak{a}_1 is the center of \mathfrak{a} ; in particular, \mathfrak{a} is nilpotent and cannot be simple. Indeed, the identity (1.4) implies

$$] \alpha, a_j [=] a_i, b_i [, a_j [= -] b_i, a_j [, a_i [-] a_j, a_i [, b_i [= 0$$

for any $i \neq j$. Similarly, one shows that $] \alpha, b_i [= 0$, for any $i = 1, \dots, n$. It remains to prove that $] \alpha, \alpha [= 0$. Indeed, one has

$$] \alpha, \alpha [=] \alpha, a_i, b_i [[=] \alpha, a_i [, b_i [+] a_i] \alpha, b_i [[= 0,$$

for any $i = 1, \dots, n$.

(b1) Consider the case $n = 1$, and where the one-dimensional algebra \mathfrak{a}_1 is nilpotent, i.e., $] \alpha, \alpha [= 0$, the relation (1.2) implies $\text{ad}_\alpha^2 = 0$ and one can choose the basis $\{a, b\}$ in such a way that $] \alpha, a [= \kappa b$ and $] \alpha, b [= 0$. The Lie antialgebra \mathfrak{a} is reducible for $\kappa \neq 0$ and nilpotent for $\kappa = 0$, in both cases, this antialgebra cannot be simple.

(b2) Consider finally the case $n = 1$, where the algebra \mathfrak{a}_1 contains the unit element ε . Theorem (1) implies $\text{ad}_\varepsilon^0 = \frac{1}{2} \text{Id}$, and one obtains precisely the algebra $\text{asl}(2)$ in this case.

Proposition 5.1 is proved. \square

5.2 The (twisted) Heisenberg antialgebra and the affine antialgebra

Let us now give more examples of Lie antialgebras of rank 1.

A. Consider the case where the form ω is non-degenerate.

(A1) The $2n|1$ -dimensional nilpotent Lie antialgebra with the basis $\{a_1, b_1, \dots, a_n, b_n; \alpha\}$ that appeared in the case a) of the above proof is characterized by the relations

$$] a_i, b_j [= \delta_{ij} \alpha, \quad] a_i, a_j [=] b_i, b_j [= 0, \quad] \alpha, a_i [=] \alpha, b_i [= 0, \quad] \alpha, \alpha [= 0, \quad (5.1)$$

where $i, j = 1, \dots, n$. We call this Lie anti algebra the *Heisenberg antialgebra* and denote it by ah_n .

Remark 5.2. Notice, that the relations (5.1) are exactly as those of the standard Heisenberg Lie algebra, but the central element α is odd. As in the usual Lie case, the Heisenberg antialgebra ah_n is a central extension of type II of the abelian Lie antialgebra \mathbb{K}^{2n} .

(A2) Another interesting example is that of a reducible $2|1$ -dimensional Lie antialgebras appeared in the case (b1) of the above proof. This is a family of algebras with basis $\{a, b, \alpha\}$, characterized by the relations

$$] a, b [= \alpha, \quad] \alpha, a [= \kappa b, \quad] \alpha, b [= 0, \quad] \alpha, \alpha [= 0, \quad (5.2)$$

where κ is a constant. If $\kappa = 0$, then this is just the Heisenberg antialgebra ah_1 , if $\kappa \neq 0$, then the defined Lie antialgebra is not isomorphic to ah_1 , and therefore one obtains a non-trivial deformation of ah_1 .

It is easy to see that, in the complex case $\mathbb{K} = \mathbb{C}$, all of the Lie antialgebras (5.2) with $\kappa \neq 0$ are isomorphic to each other. We call this algebra *twisted Heisenberg antialgebra* and denote it by $\widetilde{\mathfrak{ah}}_1$.

If $\mathbb{K} = \mathbb{R}$, however, the sign of κ is an invariant and one gets two different algebras $\widetilde{\mathfrak{ah}}_1^+$ and $\widetilde{\mathfrak{ah}}_1^-$. We thus assume:

$$\kappa = 1, \quad \text{if } \mathbb{K} = \mathbb{C}, \quad \kappa = \pm 1, \quad \text{if } \mathbb{K} = \mathbb{R}. \quad (5.3)$$

Let us summarize the above considerations and the proof of Proposition 5.1

Proposition 5.3. *The Lie algebras $\mathfrak{asl}(2)$, \mathfrak{ah}_n together with $\widetilde{\mathfrak{ah}}_1$ in the case $\mathbb{K} = \mathbb{C}$ (or $\widetilde{\mathfrak{ah}}_1^+$ and $\widetilde{\mathfrak{ah}}_1^-$ in the case $\mathbb{K} = \mathbb{R}$) form the complete list of the Lie antialgebras of rank 1 with a non-degenerate 2-form ω .*

B. Consider another “extremal case”: $\omega \equiv 0$. The Lie antialgebra \mathfrak{a} is then determined by the operator ad_α .

(B1) If \mathfrak{a}_1 contains the unit element ε , then \mathfrak{a} is split into a direct sum (3.8). The centerless Lie antialgebra $\bar{\mathfrak{a}}$ has the basis $\{a_1, \dots, a_n; \varepsilon\}$ with the following set of relations:

$$]a_i, a_j[= 0, \quad]\varepsilon, a_i[= \frac{1}{2} a_i, \quad]\varepsilon, \varepsilon[= \varepsilon. \quad (5.4)$$

We call this Lie antialgebra the *affine antialgebra* and denote by $\text{aaf}(n)$. One then has $\mathfrak{a} = \text{aaf}(n) \oplus Z$, where Z is the (even) center of \mathfrak{a} .

(B2) If $] \alpha, \alpha [= 0$ for $\alpha \in \mathfrak{a}_1$, then $\text{ad}_\alpha^0 \circ \text{ad}_\alpha^0 = 0$. This is a very degenerated Lie antialgebra. The classification in this case is the classification of nilpotent (of order 2) linear operators that we do not discuss.

5.3 The classification result

We are now ready to formulate the main statement of this section.

Theorem 2. *A Lie antialgebra of rank 1 is of the form $\bar{\mathfrak{a}} \oplus Z$ where $\bar{\mathfrak{a}}$ belongs to one of the following three classes:*

a) $\bar{\mathfrak{a}}$ is one of the algebras $\mathfrak{asl}(2)$, $\widetilde{\mathfrak{ah}}_1$ over \mathbb{C} (resp. $\widetilde{\mathfrak{ah}}_1^+$, $\widetilde{\mathfrak{ah}}_1^-$ over \mathbb{R}), \mathfrak{ah}_n with $n \geq 2$, or $\text{aaf}(n)$;

b) $\bar{\mathfrak{a}}$ is either the 3|1-dimensional algebra with the basis $\{a, b, z; \alpha\}$ and the relations

$$]a, b[= \alpha, \quad]\alpha, a[= z, \quad]\alpha, \alpha[= 0 \quad (5.5)$$

while z is in the center, or $\bar{\mathfrak{a}}$ is the 4|1-dimensional algebra with the basis $\{a, b, z_1, z_2; \alpha\}$ and the relations

$$]a, b[= \alpha, \quad]\alpha, a[= z_1, \quad]\alpha, b[= z_2, \quad]\alpha, \alpha[= 0 \quad (5.6)$$

while z_1, z_2 are central elements.

c) $\bar{\mathfrak{a}}$ is a degenerate algebra of type (B2) above.

Proof. We already proved the theorem in the following two cases: the form ω is non-degenerate, or $\omega \equiv 0$, see Sections 5.1 and 5.2. It remains to consider the intermediate case where the 2-form ω is not identically zero but with a non-trivial kernel:

$$\mathcal{I} = \ker \omega \neq \{0\}.$$

The space \mathcal{I} is then an abelian ideal (see Proposition 4.5) and, furthermore, belongs to the center (see Proposition 4.6). We summarize this in a form of a

Lemma 5.4. *The Lie antialgebra \mathfrak{a} is a central extension of \mathfrak{a}/\mathcal{I} .*

To complete the classification, one now has to classify the central extensions of type I of the antialgebras with non-degenerate form ω , that is, of (5.1) and (5.2).

Lemma 5.5. *The Lie antialgebras $\mathfrak{asl}(2), \mathfrak{ah}_n$ with $n \geq 2$ and $\widetilde{\mathfrak{ah}}_1$ (resp. $\widetilde{\mathfrak{ah}}_1^+, \widetilde{\mathfrak{ah}}_1^-$) have no non-trivial central extensions of type I.*

Proof. In the case of $\mathfrak{asl}(2)$ this follows from Theorem 1.

For the Lie antialgebra \mathfrak{ah}_n with $n \geq 2$, let C be an arbitrary 2-cocycle of type I. One has

$$C(\alpha, a_i) = C([a_j, b_j], a_i) = -C([b_j, a_i], a_j) - C([a_i, a_j], b_j) = 0$$

(from the second identity (4.3)) for all $i \neq j$. Similarly, $C(\alpha, b_i) = 0$ for all i . Therefore, C is identically zero.

For the Lie antialgebra $\widetilde{\mathfrak{ah}}_1$ and an arbitrary 2-cocycle C of type I, one has

$$C(\alpha, b) = \frac{1}{\kappa} C(\alpha, [a, a]) = \frac{1}{2\kappa} C([a, a], a) = 0.$$

The cocycle C is then defined by its values on α and a . Let us show that the corresponding extension is trivial. Let $C(\alpha, a) = cz$, where z is an arbitrary generator of the center and c arbitrary constant. Set $b' = b + \frac{c}{\kappa} z$. Then in the new basis $(a, b', z; \alpha)$, the cocycle C' vanishes. \square

Lemma 5.6. *The algebra \mathfrak{ah}_1 has a unique non-trivial central extension of type I; the result of this central extension is the Lie antialgebra (5.5).*

Proof. Indeed, let C be a 2-cocycle of type I on \mathfrak{ah}_1 . It is given by the formula

$$C(\alpha, a) = c_1 z, \quad C(\alpha, b) = c_2 z,$$

where z is an element of the center and c_1 and c_2 are arbitrary constants. Assume (without loss of generality) that $c_1 \neq 0$, then choose another element of the basis: $b' = b + \frac{c_2}{c_1} a$. One obtains $C(\alpha, b) = 0$. Furthermore, taking $z' = c_1 z$, one gets precisely the Lie antialgebra (5.5). This Lie antialgebra is not isomorphic to \mathfrak{ah}_1 so that the extension is, indeed, non-trivial. \square

Lemma 5.7. *The algebra (5.6) has a unique non-trivial central extension of type I; the result of this central extension is the Lie antialgebra (5.6).*

Proof. It is similar to the proof of Lemma 5.6: one shows that the Lie antialgebra (5.6) is precisely the unique (up to isomorphism) central extension of (5.5). \square

In the same way, one proves the following

Lemma 5.8. *The Lie antialgebra (5.6) has no non-trivial central extensions of type I.*

We classified all the non-trivial central extensions of the Lie antialgebras of rank 1 with non-degenerate form ω . This completes the proof of Theorem 2. \square

6 Odd Lie-Poisson structure

In this section we introduce the analog of the Lie-Poisson (Berezin-Kirillov-Kostant-Souriau) structure on the dual of a Lie antialgebra. This notion is important for several reasons: it relates Lie antialgebras to symplectic geometry and (multi-)Hamiltonian formalism; it allows to define infinite-dimensional analogs of Lie antialgebras and also a conformal symmetry algebra associated with a Lie antialgebra.

6.1 Canonical bivector on the dual of a Lie antialgebra

For the sake of simplicity, we will again fix a basis $\{\alpha_1, \dots, \alpha_r\}$ of \mathfrak{a}_1 , but the construction will not depend on the choice.

We denote by (τ_1, \dots, τ_r) the coordinates on \mathfrak{a}_1^* dual to the chosen basis and by $\omega_1, \dots, \omega_r$ the bilinear forms on \mathfrak{a}_0 defined by (3.1). Using a vector space structure, one can understand these 2-forms as *constant Poisson bivector fields* on $\mathfrak{a}_0^* \cong \mathbb{K}^n$ that we denote by π_1, \dots, π_r .

We associate a linear operator on $A_\alpha : \mathfrak{a} \rightarrow \mathfrak{a}$ with each element $\alpha \in \mathfrak{a}_1$, by

$$A_\alpha |_{\mathfrak{a}_0} = 2 \operatorname{ad}_\alpha, \quad A_\alpha |_{\mathfrak{a}_1} = \operatorname{ad}_\alpha, \quad (6.1)$$

and denote A_1, \dots, A_r the operators corresponding to the elements of the basis $\{\alpha_1, \dots, \alpha_r\}$. These linear operators can be, of course, viewed as *linear vector fields* on \mathfrak{a}^* .

Definition 6.1. We call the Lie-Poisson structure of the Lie antialgebra \mathfrak{a} the following odd bivector on \mathfrak{a}^* :

$$\Lambda = \sum_{i=1}^r \left(\frac{\partial}{\partial \tau_i} \wedge A_i + \tau_i \pi_i \right). \quad (6.2)$$

The corresponding antibracket is of the form:

$$]F, G[= \frac{1}{2} \langle \Lambda, dF \wedge dG \rangle \quad (6.3)$$

is defined on the space of (polynomial, smooth, etc.) functions on \mathfrak{a}^* . This antibracket is obviously *linear*, i.e., the space of linear functions $\mathfrak{a}^{**} \cong \mathfrak{a}$ is stable.

Proposition 6.2. *The space of linear functions on \mathfrak{a}^* is a Lie antialgebra with respect to the bracket (6.3) which is isomorphic to \mathfrak{a} .*

Proof. The antibracket of two even linear functions is precisely given by (3.1). The odd linear functions on \mathfrak{a}^* are linear combinations of τ_1, \dots, τ_r . The antibracket of an odd and an even linear functions is given by

$$]\tau_i, \ell[= A_i(\ell),$$

where $\ell \in \mathfrak{a}_0$. This corresponds precisely to the adjoint action of α_i on ℓ . For the antibracket of two odd linear functions one has

$$] \tau_i, \tau_j [= A_i(\tau_j) + A_j(\tau_i)$$

that corresponds to

$$\text{ad}_{\alpha_i} \alpha_j + \text{ad}_{\alpha_j} \alpha_i =] \alpha_i, \alpha_j [.$$

□

Corollary 6.3. *The bivector (6.2) and the antibracket (6.3) are independent of the choice of the basis.*

Proof. Any bivector is uniquely defined by its values on linear functions F, G . □

Example 6.4. The bivector (2.5) is precisely the canonical bivector (6.2) of the Lie antialgebra $\text{asl}(2)$.

6.2 Algebra of symmetry

The space of linear operators $\text{End}(\mathfrak{a}^*)$ form a Lie superalgebra isomorphic to $\mathfrak{gl}(n|r)$.

Definition 6.5. The Lie superalgebra $\mathfrak{g}_{\mathfrak{a}} \subset \text{End}(\mathfrak{a}^*)$ of linear operators on \mathfrak{a}^* preserving the bivector (6.2) is called the *algebra of symmetry* of \mathfrak{a} .

Clearly, the Lie superalgebra $\mathfrak{g}_{\mathfrak{a}}$ acts on the space of linear functions on \mathfrak{a}^* and preserves the structure of Lie antialgebra \mathfrak{a} .

Example 6.6. In the case of $\text{asl}(2)$, the Lie superalgebra $\text{osp}(1|2)$ is the algebra of symmetry, see Section 2.

6.3 Conformal algebra of symmetry and multi-Hamiltonian vector fields

Consider the Lie superalgebra $\text{Vect}(\mathfrak{a}^*) (\cong \text{Vect}(\mathbb{K}^{n|r}))$ of all (smooth, or polynomial, etc.) vector fields on the dual space to a Lie antialgebra \mathfrak{a} .

Definition 6.7. The Lie superalgebra $\mathfrak{G}_{\mathfrak{a}} \subset \text{Vect}(\mathfrak{a}^*)$ of vector fields on \mathfrak{a}^* preserving the bivector (6.2) is called the *conformal algebra of symmetry* of \mathfrak{a} .

The Lie superalgebra $\mathfrak{G}_{\mathfrak{a}}$ is, in general, infinite-dimensional; it does not act on the Lie antialgebra \mathfrak{a} since it does not preserve the space of linear functions on \mathfrak{a}^* . The most interesting feature of $\mathfrak{G}_{\mathfrak{a}}$ is its relation to the multi-Hamiltonian formalism.

Consider the case where \mathfrak{a}_1 contains the unit element. We introduce the following *r constant* Poisson structures on \mathfrak{a}^* , namely the bivector

$$\mathcal{P}_{\varepsilon} = \pi_{\varepsilon} + \sum_{i=1}^r \frac{\partial}{\partial \tau_i} \wedge \frac{\partial}{\partial \tau_i} \quad (6.4)$$

together with its Lie derivatives:

$$\mathcal{P}_i = L_{A_i}(\mathcal{P}_{\varepsilon}). \quad (6.5)$$

Note that these Poisson structures are *compatible*, i.e., any their linear combination is, again, a Poisson bivector, since these are constant bivectors.

Let X be a vector field on \mathfrak{a}^* which is Hamiltonian with respect to the structure \mathcal{P}_ε and commuting with all the vector fields (6.1) of the basis A_1, \dots, A_r :

$$[X, A_i] = 0.$$

Then X is a *multi-Hamiltonian* vector field with respect to all the structures (6.5). If, furthermore, X is purely even, i.e., it is independent of odd coordinates:

$$[X, \frac{\partial}{\partial \tau_i}] = 0, \quad L_X(\tau_i) = 0,$$

then is clear that X belongs to the Lie superalgebra $\mathfrak{G}_\mathfrak{a}$, since it preserves the bivector Λ . The Lie superalgebra $\mathfrak{G}_\mathfrak{a}$ is thus related to the Lie superalgebra of *multi-Hamiltonian* vector fields.

We do not know whether the above property holds for an arbitrary, not necessarily even Hamiltonian vector field X .

6.4 The unit element and homotheties

In the important case where \mathfrak{a}_1 contains the unit element ε and $Z(\mathfrak{a}) = 0$ (see Theorem 1), one has

$$A_\varepsilon = \mathcal{E},$$

where \mathcal{E} is the Euler vector field on the vector space \mathfrak{a}^* , i.e., the generator of the \mathbb{K}^* -action. Choosing arbitrary linear coordinates (x_1, \dots, x_n) on \mathfrak{a}_0^* , one obtains:

$$A_\varepsilon = \sum_{i=1}^r \tau_i \frac{\partial}{\partial \tau_i} + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}. \quad (6.6)$$

The quotient with respect to the vector field A_ε defines a natural projection to the $(n-1)|r$ -dimensional projective space:

$$\mathfrak{a}^* \setminus \{0\} \rightarrow \mathbb{P}^{n-1|r}.$$

This is the way contact geometry enters the picture, since the Poisson structure (6.4) on $\mathbb{K}^{n|r}$ corresponds to a contact structure after the projectivization, cf. [10].

7 Real simple Lie antialgebra of rank 2 over \mathbb{R}

We introduce a simple Lie antialgebra over \mathbb{R} . It is of dimension $4|2$ and is denoted by $\mathfrak{ao}(4)$. This is the second example after $\mathfrak{asl}(2)$ of a simple Lie antialgebra. Let us mention that the complexification of $\mathfrak{ao}(4)$ splits: $\mathfrak{ao}(4)_\mathbb{C} = \mathfrak{asl}(2) \oplus \mathfrak{asl}(2)$.

We describe the structure of $\mathfrak{ao}(4)$ in the details, we also consider the corresponding odd Lie-Poisson structure and the associated symmetry Lie superalgebra.

7.1 Introducing $\mathfrak{ao}(4)$

The definition goes in three steps.

1. Consider the commutative with the basis $\{\varepsilon, \sigma\}$ and the relations

$$]\varepsilon, \varepsilon[= \varepsilon, \quad]\varepsilon, \sigma[= \sigma, \quad]\sigma, \sigma[= -\varepsilon. \quad (7.1)$$

2. Define an action of the above algebra on the even 4-dimensional space \mathbb{R}^4 with basis $\{a_0, b_0, a_1, b_1\}$ as follows:

$$\text{ad}_\varepsilon^0 = \frac{1}{2} \text{Id}$$

and

$$\begin{aligned}]\sigma, a_0[&= \frac{1}{2} b_1, &]\sigma, a_1[&= -\frac{1}{2} b_0, \\]\sigma, b_0[&= \frac{1}{2} a_1, &]\sigma, b_1[&= -\frac{1}{2} a_0. \end{aligned} \quad (7.2)$$

As in Section 6, we will use the notations A_ε and A_σ for the operators defined by (6.1) corresponding to the action of ε and σ , respectively.

3. Define the skew-symmetric bilinear map $], [: \mathbb{K}^4 \rightarrow \mathbb{K}^2$ by

$$]a, b[= \omega_\varepsilon(a, b) \varepsilon + \omega_\sigma(a, b) \sigma. \quad (7.3)$$

where ω_ε and ω_σ are the following two symplectic forms on \mathbb{K}^4 :

$$\omega_\varepsilon = a_0^* \wedge b_0^* + a_1^* \wedge b_1^*, \quad \omega_\sigma = a_0^* \wedge a_1^* - b_0^* \wedge b_1^*. \quad (7.4)$$

Proposition 7.1. *Formulæ (7.1)–(7.3) define a simple Lie antialgebra.*

Proof. Let us show that the defined antialgebra satisfies all the conditions of Definition 1.1. The identities (1.1) and (1.2) are evident.

The identity (1.3) follows from the relations

$$\omega_\sigma(\cdot, \cdot) = \frac{1}{2} (\omega_\varepsilon(A_\sigma \cdot, \cdot) + \omega_\varepsilon(\cdot, A_\sigma \cdot)) \quad (7.5)$$

and

$$\omega_\varepsilon(\cdot, \cdot) = \frac{1}{2} (\omega_\sigma(A_\sigma \cdot, \cdot) + \omega_\sigma(\cdot, A_\sigma \cdot)). \quad (7.6)$$

Let us check the Jacobi identity (1.4). Indeed, using formulæ (7.3) and (7.2), one has

$$]]a_0, b_0[, a_1[+]]b_0, a_1[, a_0[+]]a_1, a_0[, b_0[=]\varepsilon, a_1[-]\sigma, b_0[= \frac{1}{2} a_1 - \frac{1}{2} a_1 = 0.$$

Other terms entering the Jacobi identity are similar.

Let us show that the defined Lie antialgebra is simple. Indeed, if $I = I_0 \oplus I_1$ is an ideal, then the odd part I_1 has to be zero since ad_α^0 is a non-degenerate operator for any $\alpha = \lambda\varepsilon + \mu\sigma$. But, then I_0 has to belong to the space $\ker \omega_\varepsilon \cap \ker \omega_\sigma = \{0\}$. \square

This Lie antialgebra will be denoted by $\mathfrak{ao}(4)$.

Remark 7.2. The matrix A_σ exchanges the symplectic blocks of ω_ε . Note that the pairs of the form of the type $(a_0^* \wedge b_0^*, a_0^* \wedge a_1^*)$ are sometimes called the Kronecker blocks, see [6].

7.2 The canonical bivector on $\mathfrak{ao}(4)^*$

This section is similar to Section 2.2. We will calculate the canonical odd bivector on the dual space $\mathfrak{ao}(4)$ and determine the Lie superalgebra of symmetry.

Consider the 4|2-dimensional space $\mathbb{K}^{4|2}$ with coordinates $(p_1, q_1, p_2, q_2; \tau_1, \tau_2)$. The canonical odd bivector corresponding to the Lie antialgebra $\mathfrak{ao}(4)$ is as follows

$$\Lambda = \frac{\partial}{\partial \tau_1} \wedge \mathcal{E} + \frac{\partial}{\partial \tau_2} \wedge A_\sigma + \tau_1 \pi_\varepsilon + \tau_2 \pi_\sigma, \quad (7.7)$$

where \mathcal{E} is the Euler field (6.6) and

$$A_\sigma = q_2 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial p_2} - p_1 \frac{\partial}{\partial q_2} + \tau_2 \frac{\partial}{\partial \tau_1} - \tau_1 \frac{\partial}{\partial \tau_2}. \quad (7.8)$$

and where

$$\pi_\varepsilon = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}, \quad \pi_\sigma = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2}.$$

Indeed, the Euler field \mathcal{E} corresponds to the action of the unit element ε (cf. Example 6.4), while the vector field A_σ corresponds to the action of σ . The standard Poisson bivector π_ε and the bivector π_σ are dual to the forms (7.4) on the even part \mathbb{K}^4 .

As in Section 6, define the antibracket $]F, G[= \frac{1}{2} \langle \Lambda, dF \wedge dG \rangle$. It follows now from Proposition 6.2 that the space of linear coordinates $\text{Span}(p_1, q_1, p_2, q_2, \tau_1, \tau_2)$ is stable and form a Lie antialgebra isomorphic to $\mathfrak{ao}(4)$.

7.3 The symmetry algebra of $\mathfrak{ao}(4)$

The symmetry algebra of the Lie antialgebra $\mathfrak{ao}(4)$ is, as in the $\mathfrak{asl}(2)$ -case, much bigger than the antialgebra itself.

Proposition 7.3. *The symmetry algebra of $\mathfrak{ao}(4)$ is the simple 6|4-dimensional Lie superalgebra $\mathfrak{o}(4|2)$.*

Proof. The symmetry algebra is the Lie superalgebra of linear vector fields Hamiltonian with respect to the standard Poisson structure on $\mathbb{K}^{4|2}$:

$$\mathcal{P}_\varepsilon = \sum_{1 \leq i \leq 2} \left(\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \frac{\partial}{\partial \tau_i} \wedge \frac{\partial}{\partial \tau_i} \right) \quad (7.9)$$

and commuting with the vector field A_σ . One checks that there are 6 even generators corresponding to the quadratic Hamiltonians

$$\{p_1^2 - q_2^2, \quad p_2^2 - q_1^2, \quad p_1 p_2 + q_1 q_2, \quad p_1 q_1 - p_2 q_2, \quad p_1 q_2, \quad q_1 p_2\}$$

that span the well-known action of $\mathfrak{o}(4)$ on the 4-dimensional symplectic space. The odd generators are

$$\{p_1 \tau_1 + q_2 \tau_2, \quad p_2 \tau_1 - q_1 \tau_2, \quad q_1 \tau_1 + p_2 \tau_2, \quad q_2 \tau_1 - p_1 \tau_2\}.$$

Recall that the Lie superalgebra $\mathfrak{o}(4|2)$ is simple (unlike $\mathfrak{o}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$). \square

Remark 7.4. Note that another way to characterize the symmetry algebra of $\mathfrak{ao}(4)$ is to say that this is the Lie superalgebra of linear bi-Hamiltonian vector fields with respect to the standard structure (7.9) together with the second structure $\mathcal{P}_\sigma = \mathcal{P}_\varepsilon \circ A_\sigma$:

$$\mathcal{P}_\sigma = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2} - \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial q_2} + 2 \frac{\partial}{\partial \tau_1} \wedge \frac{\partial}{\partial \tau_2}. \quad (7.10)$$

8 Conformal Lie antialgebra $\widehat{\mathfrak{asl}(2)}$

In this section we introduce an example of a simple infinite-dimensional Lie antialgebra. This Lie antialgebra is a conformal version of $\mathfrak{asl}(2)$.

The Lie antialgebra $\widehat{\mathfrak{asl}(2)}$ is closely related to the famous conformal Lie superalgebra, $\mathcal{K}(1)$, also known as the (centerless) Neveu-Schwarz algebra. More precisely, we show that $\mathcal{K}(1)$ is, on the one hand the algebra of conformal symmetry of $\mathfrak{asl}(2)$ and, on the other hand, the algebra of symmetry of $\widehat{\mathfrak{asl}(2)}$.

We will also show how $\widehat{\mathfrak{asl}(2)}$ is related to contact geometry. It is given by an amazing “antibracket” on the space of vector fields *tangent* to the contact structure.

8.1 The definition

Consider the infinite-dimensional vector space with the basis

$$\{a_i, i \in \mathbb{Z} + \frac{1}{2}; \quad \alpha_n, n \in \mathbb{Z}\}$$

Let us stress that the even generators a_i are labeled by semi-integer indices, while the odd generators α_n are labeled by integer ones.

Define the following operation:

$$\begin{aligned}]a_i, a_j[&= \frac{1}{2} (j - i) \alpha_{i+j}, \\]\alpha_n, a_i[&= -]a_i, \alpha_n[= \frac{1}{2} a_{n+i}, \\]\alpha_n, \alpha_m[&= \alpha_{n+m}. \end{aligned} \quad (8.1)$$

Proposition 8.1. *The bracket (8.1) defines a structure of a simple Lie antialgebra.*

Proof. The properties (1.1) and (1.2) are evident. The invariance condition (1.3) is as follows:

$$] \alpha_n,]a_i, a_j[[=] \alpha_n, a_i[, a_j[+]a_i, \alpha_n, a_j[[$$

One obtains in the left-hand-side $\frac{1}{2} (j - i) \alpha_{i+j+n}$ and in the right-hand-side the sum of two terms: $\frac{1}{4} (j - (i + n)) \alpha_{i+j+n}$ and $\frac{1}{4} (j + n - i) \alpha_{i+j+n}$, so that the identity (1.3) is satisfied. Finally, the Jacobi identity reads:

$$]]a_i, a_j[, a_k[+]]a_j, a_k[, a_i[+]]a_k, a_i[, a_j[= 0.$$

One has the sum of $\frac{1}{4} (j - i) \alpha_{i+j+n}$ with $\frac{1}{4} (k - j) \alpha_{i+j+n}$ and $\frac{1}{4} (i - k) \alpha_{i+j+n}$ which is zero. \square

We call the defined Lie antialgebra the *conformal antialgebra* and denote it by $\widehat{\mathfrak{asl}(2)}$. This example of a Lie antialgebra was defined in [3]. It is related to the notion of supertransvectants, see [5].

Remark 8.2. It is interesting to notice that the commutative algebra $\widehat{\mathfrak{asl}(2)}_1$ is a group algebra, namely

$$\widehat{\mathfrak{asl}(2)}_1 = \mathbb{K}[\mathbb{Z}].$$

In particular, the element $\varepsilon := \alpha_0$ is the unit of $\widehat{\mathfrak{asl}(2)}_1$. Note also that the antialgebra $\widehat{\mathfrak{asl}(2)}$ contains infinitely many copies of $\mathfrak{asl}(2)$ generated by $\{a_{-i}, a_i; \alpha_0\}$.

8.2 The action of the conformal Lie superalgebra

The conformal Lie superalgebra $\mathcal{K}(1)$ is spanned by

$$\{x_n, n \in \mathbb{Z}; \quad \xi_i, i \in \mathbb{Z} + \frac{1}{2}\}$$

with the defining relations given by formula (2.2). One easily checks that this is, indeed, a Lie superalgebra containing infinitely many copies of $\mathfrak{osp}(1|2)$ with the generators $\{x_{-n}, x_0, x_n; \xi_{-\frac{n}{2}}, \xi_{\frac{n}{2}}\}$.

Define the following action of the algebra $\mathcal{K}(1)$ on $\widehat{\mathfrak{asl}(2)}$:

$$\begin{aligned} x_n(a_i) &= \left(i + \frac{n}{2}\right) a_{n+i}, \\ x_n(\alpha_m) &= m \alpha_{n+m}, \\ \xi_i(a_j) &= \frac{1}{2} (j - i) \alpha_{i+j}, \\ \xi_i(\alpha_n) &= a_{i+n}. \end{aligned} \tag{8.2}$$

Note that this formula is well-known in the literature and is nothing but the action of the superconformal algebra $\mathcal{K}(1)$ on the space of weighted densities of weight $-\frac{1}{2}$ (cf., e.g., [4]).

Proposition 8.3. *The action (8.2) preserves the Lie antialgebra structure (8.1).*

Proof. The proof is, again, straightforward. Let us however give here some details in order to better illustrate the sign rule. For instance, one has

$$\xi_i (]a_j, \alpha_k[) = -\frac{1}{2} \xi_i(a_{j+k}) = -\frac{1}{4} (j + k - i) \alpha_{i+j+k},$$

together with

$$] \xi_i(a_j), \alpha_k[= \frac{1}{2} (j - i)] \alpha_{i+j}, \alpha_k[= \frac{1}{2} (j - i) \alpha_{i+j+k}$$

and

$$] a_j, \xi_i(\alpha_k)[= \frac{1}{2}] a_j, a_{i+k}[= \frac{1}{4} (i + k - j) \alpha_{i+j+k}.$$

One finally gets:

$$\xi_i (]a_j, \alpha_k[) = -] \xi_i(a_j), \alpha_k[-] a_j, \xi_i(\alpha_k)[$$

which is precisely the invariance condition.

The other terms are similar. □

Remark 8.4. Note the sign rule in the above relation that is due to the fact that the antibracket $] , [$ is odd, as well as the element ξ_i , see formula (1.5).

8.3 $\widehat{\text{asl}(2)}$ as conformal version of $\text{asl}(2)$

Let us now clarify the relations of the Lie antialgebra $\widehat{\text{asl}(2)}$ with symplectic geometry. This algebra is naturally related to the odd Poisson $\text{asl}(2)$ -bivector (2.5) and the corresponding antibracket (2.6). Instead of linear functions on $\mathbb{K}^{2|1}$, we consider the *homogeneous* functions of degree 1 on $\mathbb{K}^{2|1} \setminus \{0\}$. Recall that a function F is homogeneous of degree 1 if

$$\mathcal{E}(F) = F.$$

Proposition 8.5. *The space of homogeneous of degree 1 functions on $\mathbb{K}^{2|1} \setminus \{0\}$ is a Lie antialgebra with respect to the antibracket (2.6).*

Proof. The statement can be checked directly, but let us instead relate the situation with the Lie antialgebra (8.1). A homogeneous of degree 1 function can be written (at least locally) in the form

$$F(p, q; \tau) = p f_0\left(\frac{q}{p}\right) + \tau f_1\left(\frac{q}{p}\right),$$

where f_0 and f_1 are arbitrary functions in one even variable. Choose the following Taylor basis (dense in any reasonable topology):

$$a_i = p \left(\frac{q}{p}\right)^{i+\frac{1}{2}}, \quad \alpha_n = \tau \left(\frac{q}{p}\right)^n$$

and substitute it into the antibracket (2.6). One obtains the commutation relations (8.1). \square

8.4 $\mathcal{K}(1)$ as the algebra of conformal symmetry of $\text{asl}(2)$

The conformal Lie superalgebra $\mathcal{K}(1)$ also has a nice symplectic realization.

Proposition 8.6. *The space of homogeneous of degree 2 functions on $\mathbb{K}^{2|1} \setminus \{0\}$ is a Lie superalgebra with respect to the Poisson bracket (2.3).*

Proof. A homogeneous of degree 2 function can be written (at least locally) in the form

$$H(p, q; \tau) = p^2 h_0\left(\frac{q}{p}\right) + \tau p h_1\left(\frac{q}{p}\right),$$

Choose the basis

$$x_n = p^2 \left(\frac{q}{p}\right)^{n+1}, \quad \xi_i = \tau \left(\frac{q}{p}\right)^{i+\frac{1}{2}}$$

and substitute it into (2.3). One obtains the commutation relations (2.2). \square

Note that homogeneous of degree 2 functions correspond precisely to Hamiltonian vector fields homogeneous of degree 0, namely,

$$[\mathcal{E}, X_H] = 0, \tag{8.3}$$

where X_H is the Hamiltonian vector field with Hamiltonian H . The Lie superalgebra of Hamiltonian vector fields satisfying (8.3) is a geometric version of the superconformal algebra. Abusing the notations, we will also denote this Lie superalgebra by $\mathcal{K}(1)$.

Furthermore, the action (8.2) is, again, given by the Poisson bracket (2.3). Let us formulate a geometric version of Proposition 8.3.

Proposition 8.7. *The Lie superalgebra $\mathcal{K}(1)$ is the algebra of conformal symmetry of $\text{asl}(2)$.*

Proof. Let us first show that $\mathcal{K}(1)$ preserves the bivector (2.5).

The Lie derivative of Λ along a Hamiltonian vector field X_H satisfying the condition (8.3) is as follows:

$$L_{X_H}\Lambda = L_{X_H}\left(\frac{\partial}{\partial\tau}\right) \wedge \mathcal{E} + X_H(\tau)P + \tau L_{X_H}(P).$$

If H is even, the above expression obviously vanishes. Consider now an odd function $H = \tau H_1$, then one gets from (2.3)

$$X_{\tau H_1} = \tau X_{H_1} + H_1 \frac{\partial}{\partial\tau}$$

and so $L_{X_{\tau H_1}}\left(\frac{\partial}{\partial\tau}\right) = X_{H_1}$ and $L_{X_{\tau H_1}}(P) = -\frac{\partial}{\partial\tau} \wedge X_{H_1}$. One obtains:

$$L_{X_{\tau H_1}}(\Lambda) = X_{H_1} \wedge \mathcal{E} + H_1 P + \tau \frac{\partial}{\partial\tau} \wedge X_{H_1} = X_{H_1} \wedge E + H_1 P,$$

where $E = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$ is the even part of the Euler field. Finally, using the fact that $E(H_1) = H_1$, one obtains the following nice expression

$$L_{X_{\tau H_1}}(\Lambda) = \langle P \wedge E, dH_1 \rangle \tag{8.4}$$

where d is the de Rham differential. However, the even tri-vector $P \wedge E$ vanishes since the even dimension of $\mathbb{R}^{2|1}$ is 2.

Conversely, it is clear that any vector field preserving the bivector (2.5) is a Hamiltonian vector field commuting with \mathcal{E} , and therefore belonging to $\mathcal{K}(1)$. \square

Note that the above statement also implies Proposition 8.3.

8.5 Vector fields tangent to the contact distribution

“Homogeneous symplectic” means “contact” and the converse is also true, see [1] (and also [10] for a more recent discussion). The quotient

$$\mathbb{P}^{1|1} = (\mathbb{K}^{2|1} \setminus \{0\})/\mathcal{E}$$

is a 1|1-dimensional contact manifold (cf., e.g., [8] and [4]), and the Lie superalgebra $\mathcal{K}(1)$ of Hamiltonian vector fields satisfying (8.3) is precisely the Lie superalgebra of contact vector fields on $\mathbb{P}^{1|1}$. But what is the relation of the Lie antialgebra $\widehat{\text{asl}(2)}$ to contact geometry?

The contact structure on $\mathbb{P}^{1|1}$ can be defined in terms of a 1-form $\theta = dz + \zeta d\zeta$, where $z = \frac{q}{p}$ and $\zeta = \tau \frac{1}{p}$ are the affine coordinates. The 0|1-dimensional contact distribution is defined by the odd vector field

$$D = \frac{\partial}{\partial\zeta} - \zeta \frac{\partial}{\partial z}$$

which is the generator of $\ker \theta$. (This vector field is also known in physical literature as “SUSY-structure”.) A *tangent* vector field is a vector field proportional to D . The space of tangent vector fields is not a Lie superalgebra since the Lie bracket of two tangent vector fields is not a tangent vector field.

Definition 8.8. Define the following *antibracket* of tangent vector fields:

$$]f D, g D[= f D \circ g D + (-1)^{(p(f)+1)(p(g)+1)} g D \circ f D, \quad (8.5)$$

where we understand the vector fields as first-order differential operators.

Notice that the operation (8.5) is similar to the anticommutator (3.6).

Proposition 8.9. *The space of vector fields tangent to the contact distribution is closed with respect to the operation (8.5) and form a Lie antialgebra isomorphic to $\widehat{\text{asl}(2)}$.*

Proof. One checks that $]f D, g D[= \Lambda(df \wedge dg) D$, where Λ is the bivector (2.5). \square

Discussion

Let us formulate here some open problems and outline a few ideas for the further developments.

It seems that the notion of Lie antialgebra cannot be deduced from known algebraic structures. For instance, the idea to define a Lie antialgebra with the help of the anticommutator (3.6) fails, since the space $\text{End}(V)$ equipped with this structure does not satisfy the required conditions. Another idea would be to characterize Lie antialgebra in terms the corresponding symmetry algebra. Such an attempt also fails since there cannot be any one-to-one correspondence (for instance, the weaker is the Lie antialgebra structure, the richer is its symmetry algebra). One needs more examples of Lie antialgebras for a better understanding of this notion and its relations to the other, known, structures.

Most of the examples of Lie antialgebras we considered in this paper are related with some Lie superalgebras. More precisely, we introduced the notions of symmetry algebra and of conformal symmetry algebra associated to a Lie antialgebra. Some of the Lie antialgebras we introduced are uniquely characterized by the corresponding symmetry algebra. We think that relations between Lie antialgebras and Lie superalgebras should be better studied. For instance, it would be natural to formulate an inverse problem: given a (simple) Lie superalgebra \mathfrak{g} and a highest weight module V over \mathfrak{g} , *under what condition there exists a \mathfrak{g} -invariant structure of Lie antialgebra on V ?* Perhaps, such an algebraic structure could be useful for representation theory since for any two singular vectors $v, w \in V$, the vector $]v, w[$ is also singular.

Keeping in mind multiple application of the classical Lie-Poisson structures (in representation theory, integrable systems, quantization, etc.), we hope that the canonical odd bivector on the dual of a Lie antialgebra defined in this paper also can be useful. As one of possible applications, one can consider analogs of Euler equations on the dual space \mathfrak{a}^* of a Lie antialgebra \mathfrak{a} . Such equations would not necessarily be Hamiltonian in the usual sense, but there exists a great number of interesting dynamical systems which cannot be realized as Hamiltonian vector fields.

Another intriguing problem is the relation of Lie antialgebras to contact geometry; an example is given in Section 8.5. It would be interesting to investigate the general situation. Given a contact (super)manifold M , consider the space $\text{TVect}(M)$ of vector fields tangent to the contact distribution. Of course, this space is not a Lie (super)algebra since the very definition of a contact structure means that the Lie bracket of two tangent vector fields is *almost never* a tangent vector field. Nevertheless, there are several natural algebraic structures on the space $\text{TVect}(M)$. To give an example, the simplest algebraic structure uses the decomposition of the full space of vector fields on M into a direct sum of contact and tangent vector fields:

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M).$$

If X and Y are two tangent fields, then there is an *invariant* projection of the Lie bracket $[X, Y]$ to $\text{TVect}(M)$. This structure has never been studied and the properties of the resulting algebra are unknown. This is however not the structure that we considered in Section 8.5. We believe that the antibracket of tangent fields on $S^{1|1}$ has multi-dimensional generalizations and should be related to a multi-Hamiltonian approach. The simple Lie antialgebra $\mathfrak{ao}(4)$ can provide another example. In this case, there is a natural projection (quotient by homotheties) on the contact supermanifold $\mathbb{P}^{3|2}$ with an additional structure given by the projection of the vector field A_σ . Perhaps one should consider the tangent fields commuting with A_σ .

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