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Supertransvectants and symplectic geometry

H. Gargoubi [‡] V. Ovsienko [§]

Abstract

We consider the $\mathfrak{osp}(1|2)$ -invariant bilinear operations on weighted densities on the supercircle $S^{1|1}$ called the supertransvectants. These operations are analogues of the famous Gordan transvectants (or Rankin-Cohen brackets). We prove that these operations coincide with the iterated Poisson and ghost Poisson brackets on $\mathbb{R}^{2|1}$ and apply this result to construct star-products involving the supertransvectants.

1 Introduction

The *supertransvectants* are 1|1-dimensional generalization of the famous Gordan's transvectants. These are $\mathfrak{OSp}(1|2)$ -invariant bilinear differential operators on spaces of weighted densities on the supercircle $S^{1|1}$:

$$J_k^{\lambda,\mu} : \mathcal{F}_\lambda(S^{1|1}) \otimes \mathcal{F}_\mu(S^{1|1}) \rightarrow \mathcal{F}_{\lambda+\mu+k}(S^{1|1}),$$

where $k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ and (λ, μ) are (complex) parameters. The supertransvectants were introduced by Gieres and Theisen in [6] and [5], see also [9]. It is an interesting feature of the supersymmetric case, that there are twice more operators than in the S^1 -case. The operators labeled by integer k are *even*, and by semi-integer k are *odd*.

The two first examples of the supertransvectant, namely for $k = \frac{1}{2}$ and $k = 1$, play a particular role. These operations are not only $\mathfrak{OSp}(1|2)$ -invariant, but also invariant with respect to the full infinite-dimensional conformal Lie superalgebra $\mathcal{K}(1)$ (also known as the centerless Neveu-Schwarz algebra). The first-order supertransvectant J_1 is nothing but the well-known Poisson (contact, Schouten, etc.) bracket on $S^{1|1}$. The odd supertransvectant $J_{\frac{1}{2}}$ is much less known; we will show that this is a very interesting operation of "antibracket" on the $\mathcal{K}(1)$ -modules of densities.

The main purpose of this paper is to give an interpretation of the supertransvectants in terms of the linear symplectic superspace $\mathbb{R}^{2|1}$ with coordinates (p, q, τ) and the standard symplectic form $\omega = dp \wedge dq + d\tau \wedge d\tau$. This interpretation simplifies considerably the explicit expression of the supertransvectants and their definition. It also allows one to apply some algebraic constructions of Poisson geometry, as star-products and suggests multi-dimensional generalizations of the supertransvectants.

The standard Poisson bracket on $\mathbb{R}^{2|1}$ is given by

$$\{F, G\} = F_p G_q - F_q G_p + F_\tau G_\tau. \quad (1.1)$$

The space of homogeneous functions of degree 2:

$$\mathcal{E}(F) = 2F \quad \iff \quad [\mathcal{E}, X_F] = 0,$$

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where

$$\mathcal{E} = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} + \xi \frac{\partial}{\partial \xi}$$

is the Euler field and X_F is the Hamiltonian field with Hamiltonian F , is stable with respect to the bracket (1.1). Therefore, this space is a Lie (but not Poisson) superalgebra. This is nothing but the conformal superalgebra $\mathcal{K}(1)$.

We introduce one more, odd, operation on $C^\infty(\mathbb{R}^{2|1})$:

$$\{F, G\}^\wp = F_\tau \mathcal{E}(G) - (-1)^{p(F)} \mathcal{E}(F) G_\tau + \tau (F_p G_q - F_q G_p), \quad (1.2)$$

where p is the parity function. We call it the *ghost Poisson bracket*.

We will study the geometric and algebraic meaning of operation (1.2). Its crucial property is $\mathcal{K}(1)$ -invariantness.

Theorem 1. *The ghost bracket (1.2) is invariant with respect to the action of the conformal algebra $\mathcal{K}(1)$.*

It turns out that the Poisson bracket restricted to the homogeneous functions coincides with the supertransvectant J_1 , while the ghost Poisson bracket coincides with $J_{\frac{1}{2}}$. In the framework of deformation quantization, we will consider “iterated” Poisson brackets (1.1) and (1.2).

Theorem 2. *The supertransvectants J_k with integer k coincide with the iterated Poisson bracket (1.1), while those with semi-integer k are obtained by the iteration of (1.1) with (1.2).*

To the best of our knowledge, operations of type (1.2) have not been studied (see [11] for a survey of algebraic structures in Poisson geometry and [8] for that in supergeometry). Note that the invariance with the full Poisson superalgebra $(C^\infty(\mathbb{R}^{2|1}), \{, \})$ fails which is probably the reason why the operation (1.2) was omitted.

2 The transvectants and linear Poisson bracket

Consider the space, denoted by \mathcal{F}_λ , of smooth (complex valued) functions on S^1 equipped with the following $\mathrm{SL}(2, \mathbb{R})$ -action:

$$f(x) \mapsto f\left(\frac{ax+b}{cx+d}\right) (cx+d)^{-2\lambda},$$

where x is the affine coordinate and $\lambda \in \mathbb{R}$ (or \mathbb{C}) is a parameter.

Classification of $\mathrm{SL}(2, \mathbb{R})$ -invariant bilinear differential operators on S^1 is a famous classical result of the invariant theory. For every $k = 1, 2, 3, \dots$ there exists the $\mathrm{SL}(2, \mathbb{R})$ -invariant bilinear differential operator

$$J_k^{\lambda, \mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+k}$$

given by the following explicit formula

$$J_k^{\lambda, \mu}(f, g) = \sum_{i+j=k} (-1)^i \binom{2\lambda+k-1}{j} \binom{2\mu+k-1}{i} f^{(i)} g^{(j)}, \quad (2.1)$$

where $f^{(i)}(x) = \frac{d^i f(x)}{dx^i}$ and where $\binom{a}{b}$ is the standard binomial coefficient: $\binom{a}{i} = \frac{a(a-1)\dots(a-i+1)}{i!}$ that makes sense for arbitrary (non-integer) a . The operators (2.1) are called *transvectants* and

were found in 1885 by Gordan [7]; for almost all (λ, μ) these operators are unique $\mathrm{SL}(2, \mathbb{R})$ -invariant bilinear differential operators on S^1 . Note that one can also consider holomorphic functions on the upper half-plane \mathcal{H} and assume λ (half)integer. However, the C^∞ -case is more general.

Transvectants have been rediscovered by Rankin [18] and Cohen [1] in the theory of modular forms and by Janson and Peetre [10] in differential projective geometry. Coincidence between the Rankin-Cohen brackets and Gordan's transvectants is noticed by Zagier [19] (see also [14]). It was shown in [4] that the transvectants are in one-to-one correspondence with singular vectors in the tensor product of two $\mathfrak{sl}(2, \mathbb{C})$ Verma modules.

Perhaps the best way to understand the operators (2.1) is to rewrite them in terms of symplectic geometry (see [16] and [17]). Consider the plane \mathbb{R}^2 with linear coordinates (p, q) and the standard symplectic form $\omega = dp \wedge dq$ and the Poisson bracket

$$\{F, G\} = F_p G_q - F_q G_p.$$

The symmetry group in this case is the group $\mathrm{Sp}(2, \mathbb{R})$ isomorphic to $\mathrm{SL}(2, \mathbb{R})$. It is easy to describe all the $\mathrm{Sp}(2, \mathbb{R})$ -invariant bilinear differential operators on $C^\infty(\mathbb{R}^2)$. For every positive integer k , there exists a bilinear differential operator of order $2k$ given by the differential binomial of the form

$$B_k(F, G) := \sum_{i+j=k} (-1)^i \binom{k}{i} F \underbrace{p \dots p}_i \underbrace{q \dots q}_j G \underbrace{p \dots p}_j \underbrace{q \dots q}_i. \quad (2.2)$$

The operators B_k are, in fact, iterations of the Poisson bracket in the following sense. Consider the operator B on $C^\infty(\mathbb{R}^2) \otimes C^\infty(\mathbb{R}^2)$ given by

$$B(F \otimes G) = F_p \otimes G_q - F_q \otimes G_p$$

and the natural projection $\mathrm{Tr}(F \otimes G) = FG$. Then obviously

$$B_k = \mathrm{Tr} \circ B^k.$$

The expression (2.2) is of course much simpler than (2.1); in particular, it is independent of λ and μ . Nevertheless, these operators coincide up to a multiple. Identify the space $\mathcal{F}_\lambda(S^1)$ and the space of functions on $\mathbb{R}^2 \setminus \{0\}$ homogeneous of degree -2λ by

$$f(x) \mapsto F_f(p, q) = p^{-2\lambda} f\left(\frac{q}{p}\right), \quad (2.3)$$

so that the affine coordinate is chosen as $x = q/p$. The following observation was done in [16].

Proposition 2.1. *One has: $B_k(F_f, F_g) = k! F_{J_k^{\lambda, \mu}(f, g)}$.*

A simple corollary of Proposition 2.1 is the fact that the operators (2.1) can be used to construct $\mathrm{SL}(2, \mathbb{R})$ -invariant star-products on T^*S^1 (see [2], [16], [15] and [17]). Another application of the same idea leads to a multi-dimensional generalization of the transvectants as $\mathrm{Sp}(2n, \mathbb{R})$ -invariant bilinear differential operators on the sphere S^{2n-1} , see [17]. Simple expression (2.2) allows one to avoid any non-trivial combinatorics.

Remark 2.2. Formula (2.3) is somewhat mysterious, but it has a geometric sense. Every vector field on S^1 admits a unique ‘‘symplectic lifting’’ to a homogeneous Hamiltonian vector field on $\mathbb{R}^2 \setminus \{0\}$ and (2.3) is the unique lifting of weighted densities commuting with the vector fields lifting (cf. [17]).

3 The supertransvectants

We define the supercircle $S^{1|1}$ in terms of its superalgebra of functions, denoted by $C_{\mathbb{C}}^{\infty}(S^{1|1})$ and consisting of elements of the form

$$f(x, \xi) = f_0(x) + \xi f_1(x)$$

where f_0, f_1 are smooth functions on S^1 and ξ is an (formal Grassmann) coordinate such that $\xi^2 = 0$. The parity function p is defined by $p(f(x)) = 0$ and $p(\xi) = 1$.

3.1 Fractional-linear transformations

The action of the supergroup $\text{OSp}(1|2)$ on $S^{1|1}$ is given by the fractional-linear transformations

$$(x, \xi) \mapsto \left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi} \right),$$

where $ad - bc - \alpha\beta = 1$, $e^2 + 2\gamma\delta = 1$, $\alpha e = a\delta - c\gamma$ and $\beta e = b\delta - d\gamma$ (cf. [2]).

From now on, denote by \mathcal{F}_{λ} the space of functions $C_{\mathbb{C}}^{\infty}(S^{1|1})$ equipped with the following $\text{OSp}(1|2)$ -action

$$f(x, \xi) \mapsto f\left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi}\right) (cx + d + \delta\xi)^{-2\lambda}, \quad (3.1)$$

where $\lambda \in \mathbb{C}$ is a parameter.

As usual, in view of any explicit calculation of $\text{OSp}(1|2)$ -invariants, it is much easier to deal with the infinitesimal version of this action. The action of the orthosymplectic Lie superalgebra $\text{osp}(1|2)$ on $S^{1|1}$ corresponding to the $\text{OSp}(1|2)$ -action is spanned by three even and two odd vector fields:

$$\begin{aligned} \text{osp}(1|2)_0 &= \text{Span} \left(\frac{\partial}{\partial x}, \quad x \frac{\partial}{\partial x} + \frac{1}{2} \xi \frac{\partial}{\partial \xi}, \quad x^2 \frac{\partial}{\partial x} + x \xi \frac{\partial}{\partial \xi} \right), \\ \text{osp}(1|2)_1 &= \text{Span} (D, \quad x D), \end{aligned}$$

where

$$D = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}$$

is an odd vector field satisfying $\frac{1}{2} [D, D] = \frac{\partial}{\partial x}$.

The action of $\text{osp}(1|2)$ on \mathcal{F}_{λ} corresponding to the group action (3.1) is easy to calculate:

$$\begin{aligned} L_{\frac{\partial}{\partial x}}^{\lambda} &= \frac{\partial}{\partial x}, & L_{x \frac{\partial}{\partial x}}^{\lambda} &= x \frac{\partial}{\partial x} + \lambda, & L_{x^2 \frac{\partial}{\partial x} + x \xi \frac{\partial}{\partial \xi}}^{\lambda} &= x^2 \frac{\partial}{\partial x} + x \xi \frac{\partial}{\partial \xi} + 2\lambda x \\ L_D^{\lambda} &= D, & L_{xD}^{\lambda} &= x D + 2\lambda \xi \end{aligned} \quad (3.2)$$

which is nothing but the Lie derivative of λ -densities (see, e.g., [3]).

Remark 3.1. Note that the odd elements D and $x D$ generate the whole $\text{osp}(1|2)$ so that an operator commuting with the action of these two elements commutes with the $\text{OSp}(1|2)$ -action.

We will also use the following odd vector field on $S^{1|1}$

$$\bar{D} = \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial x},$$

sometimes called the canonical SUSY-structure (see [13]) or also the contact structure on $S^{1|1}$ (see [12]). It is characterized by the relations

$$[D, \bar{D}] = 0, \quad \frac{1}{2} [\bar{D}, \bar{D}] = -\frac{\partial}{\partial x}.$$

An important property of \bar{D} is that its direction remains invariant under the $\text{OSp}(1|2)$ -action. For instance, one has $[xD, \bar{D}] = -\xi \bar{D}$.

Remark 3.2. The direction field generated by \bar{D} is the kernel of the contact 1-form $\alpha = dx + \xi d\xi$ on $S^{1|1}$.

Every differential operator on $S^{1|1}$ can be written as an expression in \bar{D} (cf., e.g., [3]).

3.2 Supertransvectants: explicit formula

The supertransvectants are the bilinear $\text{OSp}(1|2)$ -invariant maps $J_k^{\lambda, \mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+k}$ where $k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. These operators were calculated in [6] (see also [9]), let us give their (slightly modified) explicit formula.

One has

$$J_k^{\lambda, \mu}(f, g) = \sum_{i+j=2k} J_{i,j}^k \bar{D}^i(f) \bar{D}^j(g), \quad (3.3)$$

where the numeric coefficients are

$$J_{i,j}^k = (-1)^{\left(\left[\frac{i+1}{2}\right] + j(i+p(f))\right)} \frac{\binom{[k]}{\left[\frac{2j+1+(-1)^{2k}}{4}\right]} \binom{2\lambda + [k - \frac{1}{2}]}{\left[\frac{2j+1-(-1)^{2k}}{4}\right]}}{\binom{2\mu + \left[\frac{j-1}{2}\right]}{\left[\frac{j+1}{2}\right]}}, \quad (3.4)$$

where $[a]$ denotes the integer part of $a \in \mathbb{R}$ and, as above, the binomial coefficients $\binom{a}{b}$ are well-defined if b is integer. It can be checked directly that these operators are, indeed, $\text{OSp}(1|2)$ -invariant.

4 The Poisson bracket and the ghost bracket

Let us consider the first examples of supertransvectants: $J_{\frac{1}{2}}^{\lambda, \mu}$ and $J_1^{\lambda, \mu}$. To simplify the notations, throughout this section, we denote these operators by $(,)$ and $[,]$, respectively.

4.1 Two operations

The supertransvectant of order $\frac{1}{2}$ is

$$(f, g) = \mu \bar{D}(f) g - (-1)^{p(f)} \lambda f \bar{D}(g). \quad (4.1)$$

This odd operator is extremely interesting. We will show that it is invariant with respect to the full infinite-dimensional superconformal algebra (and therefore has a geometric meaning).

The first-order supertransvectant is

$$[f, g] = \mu f' g - \lambda f g' - (-1)^{p(f)} \frac{1}{2} \overline{D}(f) \overline{D}(g). \quad (4.2)$$

This even operation is nothing but the well-known Poisson bracket on $S^{1|1}$ (see, e.g., [12], [8] and also [3]).

4.2 The Poisson superalgebra \mathcal{F}

Consider the continuous sum (direct integral) of all spaces \mathcal{F}_λ :

$$\mathcal{F} = \cup_{\lambda \in \mathbb{C}} \mathcal{F}_\lambda,$$

the collection of operations $J_1^{\lambda, \mu}$ defines a bilinear map $[\cdot, \cdot] : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$.

Lemma 4.1. *The operation J_1 defines the structure of a Poisson Lie superalgebra on \mathcal{F} .*

Proof. Indeed, one checks that for all (λ, μ)

$$[f, g] = (-1)^{p(f)p(g)} [g, f]$$

that means J_1 is skew-symmetric. Furthermore, the Jacobi identity is satisfied, that we will write in the form

$$[f, [g, h]] = [[f, g], h] + (-1)^{p(f)p(g)} [g, [f, h]] \quad (4.3)$$

for arbitrary $f \in \mathcal{F}_\lambda, g \in \mathcal{F}_\mu$ and $h \in \mathcal{F}_\nu$. Therefore, \mathcal{F} is a Lie superalgebra. Furthermore, the Leibniz identity

$$[f, gh] = [f, g] h + (-1)^{p(f)p(g)} g [f, h] \quad (4.4)$$

completes the Poisson algebra structure. \square

Remark 4.2. Note that the identity (4.4) means precisely that the usual product of functions (i.e., the transvectant of order zero) is an invariant operation on the Poisson algebra \mathcal{F} . The identity (4.3) has the same meaning but for the Poisson bracket itself.

4.3 The conformal superalgebra $\mathcal{K}(1)$ and its modules

The space $\mathcal{F}_{-1} \subset \mathcal{F}$ is a Lie subalgebra (but not a Poisson subalgebra) since it is stable with respect to the bracket (4.2) (but not with respect to the usual product).

Let us write explicitly the relations in the following Taylor (or Fourier) basis of the space of (polynomial) functions \mathcal{F}_{-1} :

$$X_n = x^{n+1}, \quad \Xi_n = \xi x^{n+\frac{1}{2}}.$$

The formula (4.2) readily gives the system of relations

$$\begin{aligned} [X_n, X_m] &= (m - n) X_{n+m}, \\ [X_n, \Xi_m] &= \left(m - \frac{n}{2}\right) \Xi_{n+m}, \\ [\Xi_n, \Xi_m] &= X_{n+m} \end{aligned} \quad (4.5)$$

very well known in the literature. This is precisely the conformal superalgebra on $S^{1|1}$, also known as the Lie superalgebra of contact vector fields (see [12],[8] and also [3]), or the (centerless) Neveu-Schwarz algebra. Let us denote this Lie subalgebra $\mathcal{K}(1)$.

Each space \mathcal{F}_λ is a $\mathcal{K}(1)$ -module since $\mathcal{K}(1) = \mathcal{F}_{-1}$ and $[\cdot, \cdot] : \mathcal{F}_{-1} \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda$. Choose the following basis of \mathcal{F}_λ

$$V_n = x^{n-\lambda}, \quad \Psi_n = \xi x^{n-\lambda-\frac{1}{2}},$$

one has

$$\begin{aligned} [X_n, V_m] &= (m + \lambda n) V_{n+m}, \\ [X_n, \Psi_m] &= (m + (\lambda + \frac{1}{2})n) \Psi_{n+m}, \\ [\Xi_n, V_m] &= (\frac{1}{2}m + \lambda n) \Psi_{n+m}, \\ [\Xi_n, \Psi_m] &= V_{n+m}. \end{aligned} \tag{4.6}$$

This is the standard formula of the $\mathcal{K}(1)$ -action on the module of “primary fields” of weight λ .

Remark 4.3. The above choice of the basis is adapted to the affine coordinate x . In particular, the elements of this basis are well-defined harmonics in the angular parameter $\varphi = \tan \frac{x}{2}$.

4.4 Invariance of the supertransvectant $J_{\frac{1}{2}}$

The operation (4.1) is an additional, odd, structure on the Poisson algebra \mathcal{F} . The crucial property of this ghost bracket is that it is *invariant* with respect to the action of the conformal subalgebra $\mathcal{K}(1) \subset \mathcal{F}$.

Proposition 4.4. *The operation (4.1) on \mathcal{F} is $\mathcal{K}(1)$ -invariant.*

Proof. One has to check that for $f \in \mathcal{F}_{-1}$ and arbitrary $g \in \mathcal{F}_\mu$ and $h \in \mathcal{F}_\nu$ one has

$$[f, (g, h)] = (-1)^{p(f)} ([f, g], h) + (-1)^{p(f)(p(g)+1)} (g, [f, h]). \tag{4.7}$$

It can be done by straightforward calculation. Note however, that the identity (4.7) is a particular case of Theorem 1 whose proof will be given in Section 5.2. \square

Note the sign rule in (4.7) is due to the fact that the bracket (\cdot, \cdot) is odd.

Remark 4.5. The relation (4.7) fails for $f \in \mathcal{F}_\lambda$ with $\lambda \neq -1$ so that the operation (4.1) is not an invariant differential operator in the usual sense. This is probably the reason why this operation was not considered in the literature (cf. [8] for a survey).

4.5 The algebraic structure on $\mathcal{F}_{-\frac{1}{2}}$

The $\mathcal{K}(1)$ -module $\mathcal{F}_{-\frac{1}{2}}$ is a “square root” of $\mathcal{K}(1) \cong \mathcal{F}_{-1}$. This space is stable with respect to the operation (\cdot, \cdot) . Adopting the above basis

$$V_n = x^{n+\frac{1}{2}}, \quad \Psi_n = \xi x^n,$$

one obtains explicitly

$$\begin{aligned} (V_n, V_m) &= (m - n) \Psi_{n+m}, \\ (\Psi_n, V_m) &= V_{n+m} = -(V_m, \Psi_n), \\ (\Psi_n, \Psi_m) &= 2\Psi_{n+m}. \end{aligned} \tag{4.8}$$

This odd algebraic structure has quite interesting properties.

Proposition 4.6. *The algebra $(\mathcal{F}_{-\frac{1}{2}}, (\cdot, \cdot))$ satisfies the following four properties:*

1. *the odd part $(\mathcal{F}_{-\frac{1}{2}})_1$ is a commutative associative subalgebra;*
2. *the odd part $(\mathcal{F}_{-\frac{1}{2}})_1$ acts on the even part $(\mathcal{F}_{-\frac{1}{2}})_0$ by $\rho_\psi v := (\psi, v)$ and one has*

$$\rho_\varphi \circ \rho_\psi + \rho_\psi \circ \rho_\varphi = \rho_{(\varphi, \psi)}$$

for all $\varphi, \psi \in (\mathcal{F}_{-\frac{1}{2}})_1$;

3. *the map $(\cdot, \cdot) : (\mathcal{F}_{-\frac{1}{2}})_0 \otimes (\mathcal{F}_{-\frac{1}{2}})_0 \rightarrow (\mathcal{F}_{-\frac{1}{2}})_1$ is skew-symmetric and $(\mathcal{F}_{-\frac{1}{2}})_1$ -invariant, namely*

$$\rho_\psi(v, w) = (\rho_\psi v, w) + (v, \rho_\psi w)$$

for all $\psi \in (\mathcal{F}_{-\frac{1}{2}})_1$ and $v, w \in (\mathcal{F}_{-\frac{1}{2}})_0$;

4. *the Jacobi identity is satisfied:*

$$(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0,$$

for all $u, v, w \in (\mathcal{F}_{-\frac{1}{2}})_0$.

Proof. The properties 1–4 of can be checked directly. □

Remark 4.7. The properties 1–4 correspond exactly to the definition of a Lie superalgebra with inversed parity and (skew)-symmetricity of the bracket.

5 Symplectic lifting

In this section we show that the supertransvectants (4.2) and (4.1) coincide with the Poisson bracket (1.1) and the ghost bracket (1.2). We prove Theorem 1.

5.1 Homogeneous functions

Let us define a symplectic lifting of the space \mathcal{F} . To a function $f \in \mathcal{F}_\lambda$ we associate a function on $\mathbb{R}^{2|1}$ homogeneous of degree -2λ . The explicit formula is $f(x, \xi) \mapsto F_f(p, q, \tau)$, where

$$F_f(p, q, \tau) = p^{-2\lambda} f\left(\frac{q}{p}, \frac{\tau}{p}\right) \equiv p^{-2\lambda} f_0\left(\frac{q}{p}\right) + \tau p^{-2\lambda-1} f_1\left(\frac{q}{p}\right) \quad (5.1)$$

and where (p, q, τ) are linear coordinates on $\mathbb{R}^{2|1}$. Abusing the notations, from now on, we will also denote \mathcal{F}_λ the space of homogeneous functions on $\mathbb{R}^{2|1}$ of degree -2λ .

This lifting is invariant in the following sense.

Proposition 5.1. (i) *The first transvectant $J_1^{\lambda, \mu}$, see the explicit formula (4.2), corresponds to the Poisson bracket (1.1):*

$$F_{[f, g]} = \frac{1}{2} \{F_f, F_g\},$$

(ii) *The $\frac{1}{2}$ -transvectant (4.1) corresponds to the odd bracket (1.2):*

$$F_{(f, g)} = -\frac{1}{2} \{F_f, F_g\}^\circ.$$

Proof. Just substitute the expression (5.1) to (1.1) and (1.2) and compare the result with (4.2) and (4.1). \square

A nice feature of the lifting (5.1) is that it intertwines the standard embedding of $\mathfrak{osp}(1|2)$ into the Poisson algebra given by the quadratic polynomials

$$\mathfrak{osp}(1|2) = \text{Span}(p^2, pq, q^2; \quad \tau p, \tau q)$$

with the $\mathfrak{osp}(1|2)$ -action (3.2). Again, the odd elements τp , and τq generate the whole algebra.

Remark 5.2. The lifting (5.1) has a similar geometric meaning as that of (2.3). The Lie superalgebra $\mathcal{K}(1) \cong \mathcal{F}_{-1}$ corresponds to the space of functions on $\mathbb{R}^{2|1}$ homogeneous of degree 2 and formula (5.1) is the unique way to identify weighted densities with homogeneous functions that intertwines (4.2) and (1.1).

5.2 Invariance of the ghost Poisson bracket

Let us prove Theorem 1.

To show that the ghost bracket (1.2) is invariant with respect to the action of $\mathcal{K}(1)$, one has to show that

$$\{F, \{G, H\}^\wp\} = (-1)^{p(F)} \{\{F, G\}, H\}^\wp + (-1)^{p(F)(p(G)+1)} \{G, \{F, H\}\}^\wp$$

for every function $F \in \mathcal{F}_{-1}$ on $\mathbb{R}^{2|1} \setminus \{0\}$, i.e., homogeneous of degree 2. To do this, we adopt the technique common in Poisson geometry.

The bracket (1.2) is given by the following ‘‘ghost Poisson’’ bivector

$$\Lambda = \frac{\partial}{\partial \tau} \wedge \mathcal{E} + \tau P, \tag{5.2}$$

where $P = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}$ is the even part of the Poisson bivector. The equivariance condition is equivalent to the fact that the Hamiltonian vector field, X_F , with respect to the Poisson bracket (1.1) preserves the bivector Λ . Since the Poisson bracket is homogeneous of degree -2 , the vector field X_F is homogeneous of degree 0, that is $[X_F, \mathcal{E}] = 0$. One has for the Schouten bracket

$$[X_F, \Lambda] = [X_F, \frac{\partial}{\partial \tau}] \wedge \mathcal{E} + X_F(\tau) P + \tau [X_F, P].$$

If F is even, the above expression obviously vanishes. Consider now an odd function $F = \tau F_1$, then one gets from (1.1)

$$X_{\tau F_1} = \tau X_{F_1} + F_1 \frac{\partial}{\partial \tau}$$

and so $[X_{\tau F_1}, \frac{\partial}{\partial \tau}] = X_{F_1}$ and

$$[X_{\tau F_1}, P] = [F_1 \frac{\partial}{\partial \tau}, P] = -\frac{\partial}{\partial \tau} \wedge X_{F_1}.$$

One obtains:

$$[X_{\tau F_1}, \Lambda] = X_{F_1} \wedge \mathcal{E} + F_1 P + \tau \frac{\partial}{\partial \tau} \wedge X_{F_1} = X_{F_1} \wedge E + F_1 P,$$

where $E = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$ is the even part of the Euler field. Finally, using the fact that $\mathcal{E}(F_1) = F_1$, one obtains the following nice expression

$$[X_{\tau F_1}, \Lambda] = \langle P \wedge E, dF_1 \rangle \tag{5.3}$$

where d is the de Rham differential. However, the even tri-vector $P \wedge E$ vanishes since the even dimension of $\mathbb{R}^{2|1}$ is 2. We proved that the bivector Λ is $\mathcal{K}(1)$ -invariant.

Theorem 1 is proved.

Remark 5.3. There is a uniqueness statement. It follows from the classification of the supertransvectants, that, for generic (λ, μ) , the ghost bracket (1.2) is a unique odd bilinear homogeneous map $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_\nu$ commuting with the $\mathcal{K}(1)$ -action.

6 Supertransvectants from the symplectic viewpoint

In this section we prove Theorem 2. We realize the supertransvectants in terms of the iterated brackets (1.1) and (1.2). As a corollary of this result, we construct a star-product involving the supertransvectants J_k as k -th order terms.

6.1 Even supertransvectants as the iterated Poisson bracket

Consider the linear operator \mathcal{B} acting on the space $C^\infty(\mathbb{R}^{2|1}) \otimes C^\infty(\mathbb{R}^{2|1})$ given by

$$\mathcal{B}(F \otimes G) = F_p \otimes G_q - F_q \otimes G_p + F_\tau \otimes G_\tau \quad (6.1)$$

The Poisson bracket (1.1) is given by the composition: $\{, \} = \text{Tr} \circ \mathcal{B}$ where Tr is the operator of projection $\text{Tr}(F \otimes G) = FG$.

Define the “iterated Poisson brackets” $\mathcal{B}_k = \text{Tr} \circ \mathcal{B}^k$, with $k = 1, 2, \dots$. One readily gets the explicit formula:

$$\mathcal{B}_k(F, G) = B_k(F, G) + k B_{k-1}(F_\tau, G_\tau), \quad (6.2)$$

where B_k is the iterated bracket (2.2) on \mathbb{R}^2 .

Proposition 6.1. *The iterated Poisson bracket (6.2) is $\text{osp}(1|2)$ -invariant for every integer k .*

Proof. The $\text{osp}(1|2)$ -action on $\mathbb{R}^{2|1}$ is generated by two odd elements: τp and τq . Let us check that

$$\{\tau p, \mathcal{B}_k(F, G)\} = \mathcal{B}_k(\{\tau p, F\}, G) + (-1)^{p(F)} \mathcal{B}_k(F, \{\tau p, G\}).$$

If F, G are even then the above relation is evident. For F even and $G = \tau G_1$ odd one has the condition

$$p B_k(F, G_1) = k B_{k-1}(F_q, G_1) + B_k(F, p G_1)$$

that follows from formula (2.2). Finally, for $F = \tau F_1, G = \tau G_1$, one gets the relation:

$$k \tau (B_{k-1}(F_1, G_1))_q = \tau (B_k(p F_1, G_1) - B_k(F_1, p G_1))$$

which is obviously true. □

The bilinear map \mathcal{B}_k restricted to the homogeneous functions defines the map

$$\mathcal{B}_k : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+k}$$

which is $\text{osp}(1|2)$ -invariant. It follows then from the uniqueness of the supertransvectants that the maps $J_k^{\lambda, \mu}$ and $\mathcal{B}_k|_{\mathcal{F}_\lambda \otimes \mathcal{F}_\mu}$ are proportional. Taking particular functions $p^{-2\lambda}$ and $q^{-2\mu}$, one now checks that the proportionality coefficient is $2^k k!$ and finally

$$F_{J_k^{\lambda, \mu}(f, g)} = \frac{1}{2^k k!} \mathcal{B}_k(F_f, F_g). \quad (6.3)$$

for generic and, therefore, for all (λ, μ) .

6.2 Iterated ghost Poisson bracket and odd supertransvectants

Define an analogous linear operator corresponding to the ghost bracket (1.2) by the following formula:

$$\begin{aligned} \mathfrak{D}(F \otimes G) &= F_\tau \otimes \mathcal{E}(G) - (-1)^{p(F)} \mathcal{E}(F) \otimes G_\tau \\ &+ \chi(f, g) ((\tau F_p) \otimes G_q - (\tau F_q) \otimes G_p + F_p \otimes (\tau G_q) - F_q \otimes (\tau G_p)) \end{aligned} \quad (6.4)$$

where $\chi(f, g)$ is a function depending on the parity of f and g :

$$\chi(f, g) = \begin{cases} \frac{1}{2}, & p(f) = 0 \text{ and } p(g) = 0 \\ 1, & \text{otherwise} \end{cases}$$

Clearly,

$$\{, \}^\wp = \text{Tr} \circ \mathfrak{D}.$$

Remark 6.2. One can also write

$$\chi(f, g) = \frac{1}{2} + \frac{(1 + (-1)^{(p(f)+1)(p(g)+1)})}{4}$$

in order to have a closed formula.

Let us define the odd iterated brackets:

$$\mathcal{B}_{k+\frac{1}{2}} = \text{Tr} \circ \mathfrak{D} \circ \mathcal{B}^k \quad (6.5)$$

for $k = 1, 2, \dots$

Proposition 6.3. *The odd brackets $\mathcal{B}_{k+\frac{1}{2}}$ are $\mathfrak{osp}(1|2)$ -invariant.*

Proof. Similar to the proof of Proposition 6.1. □

Again, the coefficient of the proportionality can be calculated:

$$F_{J_{k+\frac{1}{2}}^{\lambda, \mu}}(f, g) = -\frac{1}{2^k k!} \mathcal{B}_{k+\frac{1}{2}}(F_f, F_g). \quad (6.6)$$

Remark 6.4. (i) The definition (6.5) does not depend on the order of composition of the operators \mathfrak{D} and \mathcal{B} since one has

$$\text{Tr} \circ \mathfrak{D} \circ \mathcal{B}^k = \text{Tr} \circ \mathcal{B}^\ell \circ \mathfrak{D} \circ \mathcal{B}^m,$$

for $\ell + m = k$.

(ii) It turns out that the map (6.4) is the “square root” of the map (6.1) in the following sense. Namely, one has

$$\text{Tr} \circ \mathfrak{D}^2 = \frac{1}{2} \left((1 + (-1)^{p(F)(p(G)+1)}) (\mu + 1) - (1 + (-1)^{p(G)(p(F)+1)}) (\lambda + 1) \right) \text{Tr} \circ \mathcal{B},$$

when restricted to the homogeneous functions $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu$.

Proof. Straightforward. □

6.3 An $\mathfrak{osp}(1|2)$ -invariant star-product

The coincidence (6.3) defines a pull-back of the standard Moyal-Weyl star-product on $\mathbb{R}^{2|1}$ to an invariant star-product on the Poisson algebra \mathcal{F} . The explicit formula is very simple:

$$f * g = fg + \sum_{k=1}^{\infty} t^k J_k^{\lambda, \mu}(f, g), \quad (6.7)$$

for all $f \in \mathcal{F}_\lambda$ and $g \in \mathcal{F}_\mu$. The operation (6.7) is an associative product on the space of formal series $\mathcal{F}[[t]]$ which is a deformation of the standard commutative product of functions. The star-product (6.7) is obviously $\mathfrak{osp}(1|2)$ -invariant.

Note that the operation (6.7) involves only even supertransvectants. It would be interesting to understand if there is another deformation that contains the odd terms as well.

Appendix

For the sake of completeness, let us give here a proof of the fact that, for the generic (λ, μ) , the supertransvectants (3.3) with coefficients (3.4) are the unique $\mathfrak{osp}(1|2)$ -invariant bidifferential operators.

An arbitrary bidifferential operator can be written in the form (3.3) with coefficients $J_{i,j} \in C^\infty(S^{1|1})$. The action of a vector field X on the operator (3.3) is then given by

$$\mathcal{L}(B)(f, g) := \sum_{i+j=k} J_{i,j} \left(L_X(\overline{D}^i)(f) \overline{D}^j(g) + (-1)^{i+p(f)} \overline{D}^i(f) L_X(\overline{D}^j)(g) \right)$$

We will use the generators D and xD of $\mathfrak{osp}(1|2)$. The invariance condition with respect to the first generator D proves that each $J_{i,j}$ is an even constant. Consider the vector field xD . First, we calculate the action of xD on the operators $\overline{D}^i : \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$. One has

$$\begin{aligned} L_{xD}(\overline{D}^{2p+1}) &:= (xD + 2\mu\xi) \overline{D}^{2p+1} + \overline{D}^{2p+1} (xD + 2\lambda\xi) \\ &= (2\lambda + p) \overline{D}^{2p} + (2\mu - 2\lambda - 2p - 1) \xi \overline{D}^{2p+1} \end{aligned}$$

for $i = 2p + 1$ and

$$\begin{aligned} L_{xD}(\overline{D}^{2p}) &:= (xD + 2\mu\xi) \overline{D}^{2p} - \overline{D}^{2p} (xD + 2\lambda\xi) \\ &= p \overline{D}^{2p-1} + (2\mu - 2\lambda - 2p) \xi \overline{D}^{2p} \end{aligned}$$

for $i = 2p$. In particular, if $\mu = \lambda + \frac{i}{2}$, one obtains

$$L_{xD}^{\lambda, \lambda + \frac{i}{2}}(\overline{D}^i) = \begin{cases} (2\lambda + \frac{i-1}{2}) \overline{D}^{i-1}, & \text{if } i \text{ is odd,} \\ \frac{i}{2} \overline{D}^{i-1}, & \text{if } i \text{ is even} \end{cases}.$$

The equivariance equation, $L_{xD}(J)(f, g) = 0$, for a bidifferential operator J gives now the following system:

$$\begin{aligned} (2\lambda + l) J_{2l+1, 2m} &= -(-1)^{p(f)} (2\mu + m) J_{2l, 2m+1} \\ l J_{2l, 2m-1} &= (-1)^{p(f)} m J_{2l-1, 2m} \\ (2\lambda + l) J_{2l+1, 2m-1} &= -(-1)^{p(f)} m J_{2l, 2m} \\ l J_{2l, 2m} &= (-1)^{p(f)} (2\mu + m) J_{2l-1, 2m+1} \end{aligned} \quad (6.1)$$

Explicit resolution of the system (6.1) leads to the following critical (or “resonant”) set

$$I_k = \left\{ 0, -\frac{1}{2}, -1, -\frac{3}{2}, \dots, -\frac{1}{2} \left[\frac{k-1}{2} \right] \right\}$$

and one has to separate the following four cases.

1) If $\lambda, \mu \notin I_k$ then the system (6.1) has a unique (up to a multiplicative constant) solution given by (3.4).

2) If one of the weights λ or μ belongs to I_k but not the second one, then the system (6.1) has a unique (up to a multiplicative constant) solution. If, say, $\lambda = \frac{1-m}{4}$ for some odd m , then the corresponding bilinear $\mathfrak{osp}(1|2)$ -invariant operator is given by

$$f \otimes g \mapsto J_{k-m}^{\frac{1+m}{4}, \mu} (\overline{D}^m(f), g).$$

3) If $\lambda = \frac{1-m}{4}$ for some odd m and $\mu = \frac{1-\ell}{4}$ for some odd ℓ , and if $\ell + m > k$ then the solution is still unique and is of the form

$$f \otimes g \mapsto J_{k-m}^{\frac{1+m}{4}, \mu} (\overline{D}^m(f), g) = J_{k-\ell}^{\lambda, \frac{1+\ell}{4}} (f, \overline{D}^\ell(g)).$$

4) If $\lambda = \frac{1-m}{4}$ for some odd m and $\mu = \frac{1-\ell}{4}$ for some odd ℓ , and if $\ell + m > k$ then there are two independent solutions

$$f \otimes g \mapsto J_{k-m}^{\frac{1+m}{4}, \mu} (\overline{D}^m(f), g) = J_{k-\ell}^{\lambda, \frac{1+\ell}{4}} (f, \overline{D}^\ell(g)).$$

and

$$f \otimes g \mapsto J_{k-m-\ell}^{\frac{1+m}{4}, \frac{1+\ell}{4}} (\overline{D}^m(f), \overline{D}^\ell(g)).$$

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