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# EXTREMAL FIRST DIRICHLET EIGENVALUE OF DOUBLY CONNECTED PLANE DOMAINS AND DIHEDRAL SYMMETRY

AHMAD EL SOUFI AND ROLA KIWAN

ABSTRACT. We deal with the following eigenvalue optimization problem: Given a bounded domain  $D \subset \mathbb{R}^2$ , how to place an obstacle  $B$  of fixed shape within  $D$  so as to maximize or minimize the fundamental eigenvalue  $\lambda_1$  of the Dirichlet Laplacian on  $D \setminus B$ . This means that we want to extremize the function  $\rho \mapsto \lambda_1(D \setminus \rho(B))$ , where  $\rho$  runs over the set of rigid motions such that  $\rho(B) \subset D$ . We answer this problem in the case where both  $D$  and  $B$  are invariant under the action of a dihedral group  $\mathbb{D}_n$ ,  $n \geq 2$ , and where the distance from the origin to the boundary is monotonous as a function of the argument between two axes of symmetry. The extremal configurations correspond to the cases where the axes of symmetry of  $B$  coincide with those of  $D$ .

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The relations between the shape of a domain and the eigenvalues of its Dirichlet or Neumann Laplacian, have been intensively investigated since the 1920's when Faber [5] and Krahn [12] have proved independently the famous eigenvalue isoperimetric inequality first conjectured by Rayleigh (1877): the first Dirichlet eigenvalue  $\lambda_1(\Omega)$  of any bounded domain  $\Omega \subset \mathbb{R}^n$  satisfies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*),$$

where  $\Omega^*$  is a ball having the same volume as  $\Omega$ . We refer to the review papers of Ashbaugh [1, 2] and Henrot [9] for a survey of recent results on optimization problems involving eigenvalues.

The present work deals with the following eigenvalue optimization problem: Given a bounded domain  $D$ , we want to place an obstacle (or a hole)  $B$ , of fixed shape, inside  $D$  so as to maximize or minimize the fundamental eigenvalue  $\lambda_1$  of the Laplacian or Schrödinger operator on  $D \setminus B$  with Zero Dirichlet conditions on the boundary.

In other words, the problem is to optimize the principal eigenvalue function  $\rho \mapsto \lambda_1(D \setminus \rho(B))$ , where  $\rho$  runs over the set of rigid motions such that  $\rho(B) \subset D$ .

The first result obtained in this direction concerned the case where both  $D$  and  $B$  are disks of given radii. Indeed, it follows from Hersch's work [10] that the maximum of  $\lambda_1$  is achieved when the disks are concentric (see also [14]). This result has been extended to any dimension by several authors (Harrell, Kröger and Kurata [8], Kesavan [11], ...). Actually, Harrell, Kröger and Kurata [8] gave a more general result showing that, if the domain  $D$  satisfies an interior symmetry property with respect to a hyperplane  $P$  passing through the center of the spherical obstacle  $B$

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(which means that the image by the reflection with respect to  $P$  of one component of  $D \setminus P$  is contained in  $D$ ), then the Dirichlet fundamental eigenvalue  $\lambda_1(D \setminus B)$  decreases when the center of  $B$  moves perpendicularly to  $P$  in the direction of the boundary of  $D$ . In the particular case where both the domain  $D$  and the obstacle  $B$  are balls, this implies that the minimum of  $\lambda_1(D \setminus B)$  corresponds to the limit case where  $B$  touches the boundary of  $D$ .

Notice that when the obstacle  $B$  is a disk, only translations of  $B$  may affect the  $\lambda_1$  of  $D \setminus B$  and the optimal placement problem reduces to the choice of the center of  $B$  inside  $D$ .

In the present work we investigate a kind of dual problem in the sense that we consider a *nonspherical* obstacle  $B$  whose center of mass is fixed inside  $D$ , and seek the optimal positions while turning  $B$  around its center.

It is of course hopeless to expect a universal solution to this problem. In fact, we will restrict our investigation to a class of domains satisfying a dihedral symmetry and a monotonicity conditions.

Thus, let  $D$  be a simply-connected plane domain and assume that the following conditions are satisfied:

(i) ( $\mathbb{D}_n$ -symmetry) for an integer  $n \geq 2$ ,  $D$  is invariant under the action of the dihedral group  $\mathbb{D}_n$  of order  $2n$  generated by the rotation  $\rho_{\frac{2\pi}{n}}$  of angle  $\frac{2\pi}{n}$  and a reflection  $S$ . Such a domain admits  $n$  axes of symmetry passing through the origin and such that the angle between 2 consecutive axes is  $\frac{\pi}{n}$ .

(ii) (monotonicity of the boundary) the distance  $d(O, x)$  from the origin to a point  $x$  of the boundary of  $D$  is monotonous as a function of the argument of  $x$ , in a sector delimited by two consecutive symmetry axes.

Notice that assumption (i) guarantees that the center of mass of  $D$  is at the origin. Regular  $n$ -gons centered at the origin are the simplest examples of domains satisfying these assumptions. More generally, if  $g$  is any positive even  $\frac{2\pi}{n}$ -periodic continuous function that is monotonous on the interval  $(0, \frac{\pi}{n})$ , then the domain

$$D = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\},$$

satisfies assumptions (i) and (ii). Actually, up to a rigid motion, any domain satisfying assumptions (i) and (ii) can be parametrized in such a manner.

It is worth noticing that, due to the monotonicity condition, the “distance to the origin” function on the boundary of  $D$  achieves its maximum and its minimum alternatively at the intersection points of  $\partial D$  with the  $2n$  half-axes of symmetry. The  $n$  points of  $\partial D$  at maximal (resp. minimal) distance from the origin will be called “outer vertices” (resp. “inner vertices”) of  $D$ .

Our main result is the following

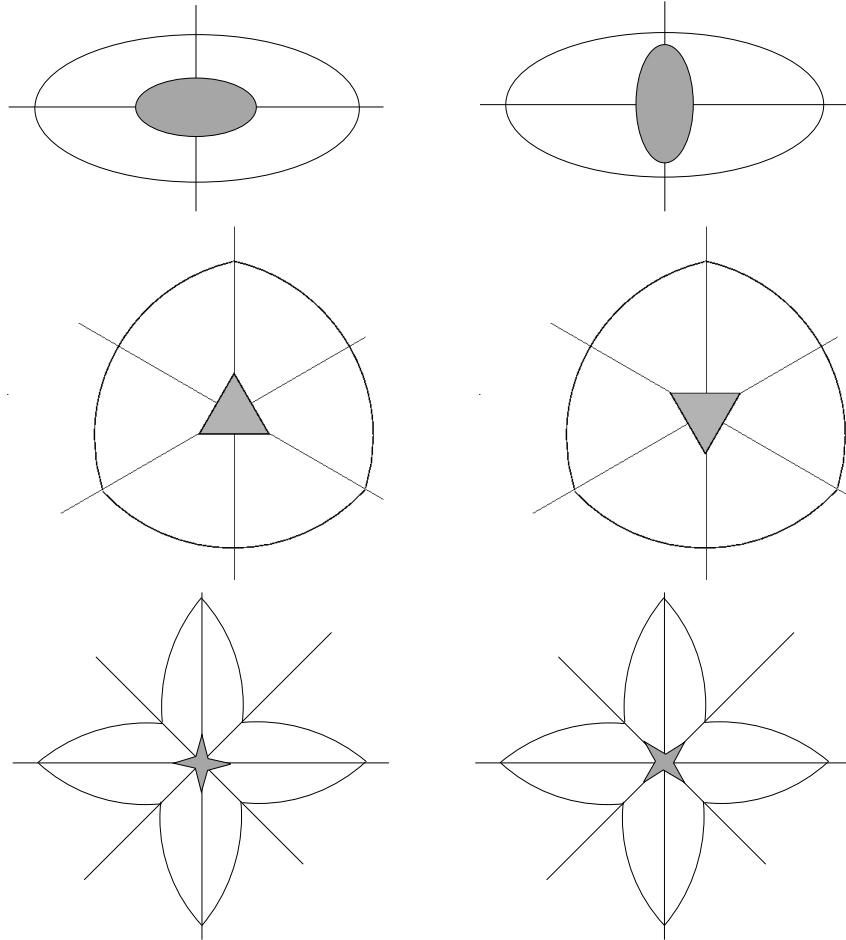
**Theorem 1.** *Let  $D$  and  $B$  be two plane domains satisfying the assumptions of  $\mathbb{D}_n$ -symmetry and monotonicity (i) and (ii) above for an integer  $n \geq 2$ . Assume furthermore that  $B$  has  $C^2$  boundary and that  $\rho(B) \subset D$  for all  $\rho \in SO(2)$ . Then, the fundamental Dirichlet eigenvalue  $\lambda_1(D \setminus B)$  of  $D \setminus B$  is optimized exactly when the axes of symmetry of  $B$  coincide with those of  $D$ .*

*The maximizing configuration corresponds to the case where the outer vertices of  $B$  and  $D$  lie on the same half-axes of symmetry (we will then say that  $B$  occupies the “ON” position in  $D$ ).*

The minimizing configuration corresponds to the case where the outer vertices of  $B$  lie on the half-axes of symmetry passing through the inner vertices of  $D$  (this is what will be called the “OFF” position).

Actually, we will prove that, except for the trivial case where  $D$  or  $B$  is a disk, the fundamental Dirichlet eigenvalue of  $D \setminus B$  decreases gradually when  $B$  switches from “ON” to “OFF”.

The main ingredients of the proof of Theorem 1 are Hadamard’s variation formula for  $\lambda_1$  and the technique of domain reflection initiated by Serrin [17] in PDE’s setting.



Examples of maximal (left) and minimal (right) configurations with  $n = 2, 3$  and  $4$  respectively

Extensions of Theorem 1 to the following situations can be obtained up to slight changes in the proof (indeed, only the Hadamard formula should be replaced by the variation formula corresponding to the new functional):

- (1) Soft obstacles: instead considering the Dirichlet Laplacian on  $D \setminus B$ , we consider the Schrödinger type operator

$$H(\alpha, B) := \Delta - \alpha\chi_B$$

acting on  $H_0^1(D)$ , where  $\alpha > 0$  and  $\chi_B$  is the indicator function of  $B$ . Optimization problems related to the fundamental eigenvalue of operators of this kind have been investigated in particular in [8] and [3]. Under the assumptions of Theorem 1 on  $D$  and  $B$ ,  $\forall \alpha > 0$ , the fundamental eigenvalue of  $H(\alpha, B)$  achieves its maximum at the ‘‘ON’’ position and its minimum at the ‘‘OFF’’ position.

- (2) Wells: this case corresponds to the operator  $H(\alpha, B)$  with  $\alpha < 0$ . Under the circumstances of Theorem 1,  $\forall \alpha < 0$ , the first eigenvalue of  $H(\alpha, B)$  achieves its maximum at the ‘‘OFF’’ position and its minimum at the ‘‘ON’’ position.
- (3) Stationary problem : the problem now is to optimize the Dirichlet energy  $J(D \setminus B) := \int_{D \setminus B} |\nabla u|^2 dx$  of the unique solution  $u$  of the problem

$$\begin{cases} \Delta u = -1 & \text{in } D \setminus B \\ u = 0 & \text{on } \partial(D \setminus B), \end{cases}$$

This problem was treated in [11, Section 2] in the case where both  $D$  and  $B$  are balls. Under the assumptions of Theorem 1 on  $D$  and  $B$ , one can prove that  $J(D \setminus B)$  achieves its maximum when  $B$  is at the ‘‘ON’’ position and its minimum when  $B$  is at the ‘‘OFF’’ position.

## 2. PROOF OF THE MAIN RESULT

Without loss of generality, we may assume that the domain  $D$  and the obstacle  $B$  are centered at the origin and are both symmetric with respect to the  $x_1$ -axis so that they can be parametrized in polar coordinates by

$$\begin{aligned} D &= \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\}, \\ B &= \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < f(\theta)\}, \end{aligned}$$

where  $f$  and  $g$  are two positive even  $\frac{2\pi}{n}$ -periodic functions which are *nondecreasing* on  $(0, \frac{\pi}{n})$ . To avoid technicalities, we suppose throughout that  $g$  is continuous and  $f$  is  $C^2$ . Extensions of our result to a wider class of domains would certainly be possible up to some additional technical difficulties.

The condition that the obstacle  $B$  can freely rotate around his center inside  $D$ , that is  $\rho(\bar{B}) \subset D$  for all  $\rho \in SO(2)$ , amounts to the following:

$$f\left(\frac{\pi}{n}\right) = \max_{0 \leq \theta \leq 2\pi} f(\theta) < \min_{0 \leq \theta \leq 2\pi} g(\theta) = g(0).$$

Let us denote, for all  $t \in \mathbb{R}$ , by  $\rho_t$  the rotation of angle  $t$ , that is,  $\forall \zeta \in \mathbb{R}^2 \cong \mathbb{C}$ ,  $\rho_t(\zeta) = e^{it}\zeta$ , and set

$$B_t := \rho_t(B) \text{ and } \Omega(t) := D \setminus B_t.$$

Let  $\lambda(t)$  be the fundamental eigenvalue of the Dirichlet Laplacian on  $\Omega(t)$ . It is well known that, since it is simple, the first Dirichlet eigenvalue  $\lambda(t)$  is a differentiable function of  $t$  (see [6, 15]). We denote by  $u(t)$  the one parameter family of nonnegative first eigenfunctions satisfying,  $\forall t \in \mathbb{R}$ ,

$$\begin{cases} \Delta u(t) = -\lambda(t)u(t) & \text{in } \Omega(t) \\ u(t) = 0 & \text{on } \partial\Omega(t) \\ \int_{\Omega(t)} u^2(t) = 1. \end{cases}$$

The derivative of  $\lambda(t)$  is then given by the following so-called Hadamard formula (see [4, 6, 7, 16]):

$$(1) \quad \lambda'(t) = \int_{\partial B_t} \left| \frac{\partial u(t)}{\partial \eta_t} \right|^2 \eta_t \cdot v \, d\sigma,$$

where  $\eta_t$  is the inward unit normal vector field of  $\partial\Omega(t)$  (hence, along  $\partial B_t$  the vector  $\eta_t$  is outward with respect to  $B_t$ ) and  $v$  denotes the restriction to  $\partial\Omega(t) = \partial D \cup \partial B_t$  of the deformation vector field. In our case, the vector  $v$  vanishes on  $\partial D$  and is given by  $v(\zeta) = \mathbf{i}\zeta$  for all  $\zeta \in \partial B_t$ .

Since both  $\Omega$  and  $B$  are invariant by the dihedral group  $\mathbb{D}_n$ , it follows that,  $\forall t \in \mathbb{R}$ ,  $\Omega(t + \frac{2\pi}{n}) = \Omega_t$ . Moreover, if we denote by  $S_0$  the reflection with respect to the  $x_1$ -axis, then we clearly have  $\rho_{-t} = S_0 \circ \rho_t \circ S_0$  which gives  $B_{-t} = S_0(B_t)$  and  $\Omega_{-t} = S_0(\Omega_t)$ . Hence, as a function of  $t$ , the first Dirichlet eigenvalue of  $\Omega_t$  is even and periodic of period  $\frac{2\pi}{n}$ , that is,  $\forall t \in \mathbb{R}$ ,

$$\lambda(t + \frac{2\pi}{n}) = \lambda(t) \text{ and } \lambda(-t) = \lambda(t).$$

Therefore, it suffices to investigate the variations of  $\lambda(t)$  on the interval  $[0, \frac{\pi}{n}]$  and Theorem 1 is a consequence of the following:

**Theorem 2.** *Assume that neither  $D$  nor  $B$  is a disk.*

- (i)  $\forall t \in (0, \frac{\pi}{n})$ ,  $\lambda'(t) < 0$ . Hence,  $\lambda(t)$  is strictly decreasing on  $(0, \frac{\pi}{n})$ .
- (ii)  $\forall k \in \mathbb{Z}$ ,  $\lambda'(k\frac{\pi}{n}) = 0$  and  $k\frac{\pi}{n}$ ,  $k \in \mathbb{Z}$ , are the only critical points of  $\lambda$  on  $\mathbb{R}$ .

Hence,  $\lambda(t)$  achieves its maximum for  $t = 0 \pmod{\frac{2\pi}{n}}$  which corresponds to the ‘‘ON’’ position, and its minimum for  $t = \frac{\pi}{n} \pmod{\frac{2\pi}{n}}$  which corresponds to the ‘‘OFF’’ position. Of course, if  $D$  or  $B$  is a disk, then the function  $\lambda(t)$  is constant.

In what follows we will denote, for any  $\alpha \in \mathbb{R}$ , by  $z_\alpha$  the  $\theta = \alpha$  axis, that is  $z_\alpha := \{re^{i\alpha}; r \in \mathbb{R}\}$ , and by  $z_\alpha^+$  the half-axis  $\{re^{i\alpha}; r \geq 0\}$ .

We start the proof with the following elementary lemma.

**Lemma 1.** *Let  $K$  be a plane domain defined in polar coordinates by  $K = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < h(\theta)\}$ , where  $h$  is a positive  $2\pi$ -periodic function of classe  $C^1$ , and let  $v$  be a vector field whose restriction to  $\partial K$  is given by*

$$v(\theta) := v(h(\theta)e^{i\theta}) = \mathbf{i}h(\theta)e^{i\theta} = h(\theta)e^{i(\theta + \frac{\pi}{2})}.$$

*We denote by  $\eta$  the unit outward normal vector field of  $\partial K$ . One has, at any point  $h(\theta)e^{i\theta}$  of  $\partial K$  where  $\eta$  is defined,*

- (i)  $\eta(\theta) := \eta(h(\theta)e^{i\theta}) = \frac{h(\theta)e^{i\theta} - \mathbf{i}h'(\theta)e^{i\theta}}{\sqrt{h^2(\theta) + h'^2(\theta)}}$
- (ii)  $\eta \cdot v(\theta) = \frac{-h(\theta)h'(\theta)}{\sqrt{h^2(\theta) + h'^2(\theta)}}$ . Hence,  $\eta \cdot v(\theta)$  has constant sign on an interval  $I$  if and only if  $h$  is monotonous in  $I$ .
- (iii) if for some  $\alpha > 0$ , the domain  $K$  is symmetric with respect to the axis  $z_\alpha$ , then the function  $\eta \cdot v$  is antisymmetric w.r.t this axis, that is

$$\eta \cdot v(\alpha + \theta) = -\eta \cdot v(\alpha - \theta).$$

*Proof.* Assertions (i) and (ii) are direct consequences from the definition of  $K$ . The fact that  $K$  is symmetric with respect to the axis  $z_\alpha$  implies that the function  $h$  satisfies  $h(\alpha + \theta) = h(\alpha - \theta)$ . Therefore, (iii) follows immediately from (ii).  $\square$

We will denote by  $S_\alpha$  the symmetry with respect to the axis  $z_\alpha$ . We will also denote, for  $\alpha < \beta$ , by  $\sigma(\alpha, \beta)$  the sector delimited by  $z_\alpha^+$  and  $z_\beta^+$ , that is

$$\sigma(\alpha, \beta) = \{re^{i\theta}; r > 0 \text{ and } \alpha < \theta < \beta\}.$$

**Lemma 2.** *Let  $D$  be as above. For all  $t \in (0, \frac{\pi}{n})$ , we have:*

$$S_{\frac{\pi}{n}+t} \left( D \cap \sigma \left( \frac{\pi}{n} + t, \frac{2\pi}{n} + t \right) \right) \subseteq D \cap \sigma \left( t, \frac{\pi}{n} + t \right).$$

Moreover, if  $D$  is not a disk, then

$$S_{\frac{\pi}{n}+t} \left( \partial D \cap \sigma \left( \frac{\pi}{n} + t, \frac{2\pi}{n} + t \right) \right) \cap D \neq \emptyset.$$

*Proof.* The action of the symmetry  $S_{\frac{\pi}{n}+t}$  is given in polar coordinates by  $S_{\frac{\pi}{n}+t}(re^{i\theta}) = re^{i(2(\frac{\pi}{n}+t)-\theta)}$ . Hence,

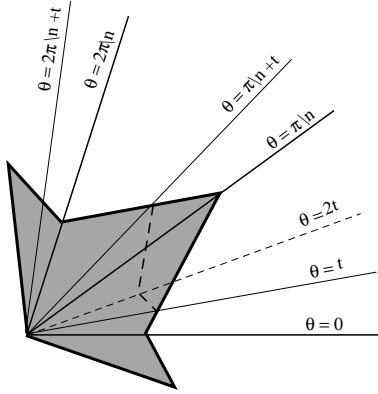
$$S_{\frac{\pi}{n}+t} \left( D \cap \sigma \left( \frac{\pi}{n} + t, \frac{2\pi}{n} + t \right) \right) = S_{\frac{\pi}{n}+t}(D) \cap \sigma \left( t, \frac{\pi}{n} + t \right).$$

Moreover, the domain  $D$  being parametrized by a positive even  $\frac{2\pi}{n}$ -periodic function  $g(\theta)$ , that is  $D = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < g(\theta)\}$ , its image  $S_{\frac{\pi}{n}+t}(D)$  can be parametrized in the same manner by the function  $g^*(\theta) = g(\theta - 2t)$ . Thus

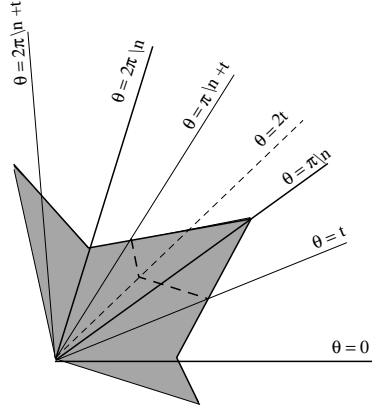
$$S_{\frac{\pi}{n}+t}(D) \cap \sigma \left( t, \frac{\pi}{n} + t \right) = \{re^{i\theta}; \theta \in \left( t, \frac{\pi}{n} + t \right), 0 \leq r < g(\theta - 2t)\}.$$

Therefore, we need to prove that  $F(\theta) = g(\theta) - g^*(\theta)$  is nonnegative for every  $\theta$  in the interval  $(t, \frac{\pi}{n} + t)$ . This will be possible thanks to the assumptions of symmetry (that is  $g$  is even and  $\frac{2\pi}{n}$ -periodic) and monotonicity (that is  $g$  is nondecreasing on  $[0, \frac{\pi}{n}]$ ). Indeed, these properties imply that on the interval  $(t, \frac{\pi}{n} + t)$ ,

- $g$  achieves its maximum at  $\theta = \frac{\pi}{n}$ ,
- $g^*$  achieves its minimum at  $\theta = 2t$ .



case  $2t < \frac{\pi}{n}$



case  $2t > \frac{\pi}{n}$

Four cases must be considered separately:

- If  $t < \theta \leq \min\{2t, \frac{\pi}{n}\}$ , we may write, since  $g$  is even,  $F(\theta) = g(\theta) - g(2t - \theta)$ , with  $0 \leq 2t - \theta < \theta \leq \frac{\pi}{n}$ . Since  $g$  is nondecreasing on  $[0, \frac{\pi}{n}]$ , we get  $F(\theta) \geq 0$ .
- If  $\max\{2t, \frac{\pi}{n}\} \leq \theta < \frac{\pi}{n} + t$ , we may write, since  $g$  is even and  $\frac{2\pi}{n}$ -periodic,  $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(\theta - 2t)$  with  $0 \leq \theta - 2t < 2\frac{\pi}{n} - \theta \leq \frac{\pi}{n}$ . Hence,  $F(\theta) \geq 0$ .

- If  $2t < \frac{\pi}{n}$  and  $2t \leq \theta \leq \frac{\pi}{n}$ , then  $0 \leq \theta - 2t < \theta \leq \frac{\pi}{n}$  and, then,  $F(\theta) = g(\theta) - g(\theta - 2t) \geq 0$ .
- If  $2t > \frac{\pi}{n}$  and  $\frac{\pi}{n} \leq \theta \leq 2t$ , then  $0 \leq 2t - \theta < 2\frac{\pi}{n} - \theta \leq \frac{\pi}{n}$  and, then,  $F(\theta) = g(2\frac{\pi}{n} - \theta) - g(2t - \theta) \geq 0$ .

Hence,  $F(\theta)$  is nonnegative for all  $\theta$  in  $(t, \frac{\pi}{n} + t)$ .

Now, if  $D$  is not a disk, then  $g$  is nonconstant on  $[0, \frac{\pi}{n}]$ . Following the arguments above, we deduce that the function  $F(\theta)$  is positive somewhere on  $(t, \frac{\pi}{n} + t)$  which means that  $S_{\frac{\pi}{n}+t}(\partial D \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t))$  meets the interior of  $D$ .  $\square$

*Proof of Theorem 2.* Notice first that, since  $\lambda$  is an even and  $\frac{2\pi}{n}$ -periodic function of  $t$ , one immediately gets,  $\forall k \in \mathbb{Z}$ ,  $\lambda(k\frac{\pi}{n} - t) = \lambda(k\frac{\pi}{n} + t)$  and, then,

$$\lambda' \left( k\frac{\pi}{n} \right) = 0.$$

Alternatively, one can deduce that  $\lambda' \left( k\frac{\pi}{n} \right) = 0$  from Hadamard's variation formula (1) after noticing that the domain  $\Omega(k\frac{\pi}{n})$  is symmetric with respect to the  $x_1$ -axis and that the first Dirichlet eigenfunction  $u(k\frac{\pi}{n})$  satisfies  $u \circ S_0 = u$ , where  $S_0$  is the symmetry with respect to the  $x_1$ -axis.

Let us fix a  $t$  in  $(0, \frac{\pi}{n})$  and denote by  $u$  the nonnegative first Dirichlet eigenfunction of  $\Omega(t)$  satisfying  $\int_{\Omega(t)} u^2 = 1$ . The domain  $\Omega(t)$  is clearly invariant by the rotation  $\rho_{\frac{2\pi}{n}}$  of angle  $\frac{2\pi}{n}$ , hence  $u \circ \rho_{\frac{2\pi}{n}} = u$ . On the other hand, the domain  $B$  being parametrized by a positive even  $\frac{2\pi}{n}$ -periodic function  $f(\theta)$ , that is  $B = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < f(\theta)\}$ , one has

$$B_t = \{re^{i\theta}; \theta \in [0, 2\pi), 0 \leq r < h(\theta)\},$$

with  $h(\theta) = f(\theta - t)$ . Hence, the function  $\eta_t \cdot v$  is invariant by  $\rho_{\frac{2\pi}{n}}$  (Lemma 1) and we have (Hadamard formula (1))

$$\lambda'(t) = \int_{\partial B_t} \left| \frac{\partial u}{\partial \eta_t} \right|^2 \eta_t \cdot v \, d\sigma = n \int_{\partial B_t \cap \sigma(t, \frac{2\pi}{n} + t)} \left| \frac{\partial u}{\partial \eta_t} \right|^2 \eta_t \cdot v \, d\sigma.$$

Since  $B_t$  is symmetric with respect to the axis  $z_{\frac{\pi}{n}+t}$ , we have (Lemma 1),  $\eta_t \cdot v(\frac{\pi}{n} + t + \theta) = -\eta_t \cdot v(\frac{\pi}{n} + t - \theta)$  or, equivalently,  $\eta_t \cdot v(x) = -\eta_t \cdot v(x^*)$ , where  $x^*$  denotes the symmetric of  $x$  with respect to  $z_{\frac{\pi}{n}+t}$ . This yields

$$\lambda'(t) = n \int_{\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)} \left( \left| \frac{\partial u}{\partial \eta_t}(x) \right|^2 - \left| \frac{\partial u}{\partial \eta_t}(x^*) \right|^2 \right) \eta_t \cdot v(x) \, d\sigma$$

Notice that the function  $h(\theta)$  is decreasing between  $\frac{\pi}{n} + t$  and  $\frac{2\pi}{n} + t$  and, then,  $\eta_t \cdot v$  is nonnegative on  $\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)$  (Lemma 1).

Let  $H(t) := \Omega(t) \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)$ . Applying Lemma 2, and since  $B_t$  is symmetric with respect to the axis  $z_{\frac{\pi}{n}+t}$ , one gets

$$S_{\frac{\pi}{n}+t}(H(t)) \subset \Omega(t) \cap \sigma(t, \frac{\pi}{n} + t).$$

Hence, the function  $w(x) = u(x) - u(x^*)$  is well defined on  $H(t)$  and satisfies  $w(x) = 0$  for all  $x$  in  $\partial H(t) \cap (\partial B_t \cup z_{\frac{\pi}{n}+t} \cup z_{\frac{2\pi}{n}+t})$ . Moreover, since  $u$  vanishes on  $\partial D$  and is positive inside  $\Omega(t)$ ,  $w(x) \leq 0$  for all  $x$  in  $\partial H(t) \cap \partial D$  and  $w(x) < 0$  for certain  $x$  in  $\partial H(t) \cap \partial D$  (recall that  $D$  is not a disk and apply the second part of Lemma 2).



Therefore, the nonconstant function  $w$  satisfies the following:

$$\begin{cases} \Delta w = -\lambda(t)w & \text{in } H(t) \\ w \leq 0 & \text{on } \partial H(t). \end{cases}$$

Hence,  $w$  must be nonpositive on the whole of  $H(t)$ . Otherwise, a nodal domain  $V \subset H(t)$  of  $w$  would have the same first Dirichlet eigenvalue as  $\Omega(t)$ . But, due to the invariance of  $\Omega(t)$  by  $\rho_{\frac{2\pi}{n}}$ , the domain  $\Omega(t)$  would contain  $n$  copies of  $V$  leading to a strong contradiction with the domain monotonicity theorem for eigenvalues. Therefore,  $\Delta w \geq 0$  in  $H(t)$  and  $w$  achieves its maximal value (i.e. zero) on  $\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t) \subset \partial H(t)$ . The Hopf maximum principle (see [13, Theorem 7, ch.2]) then implies that, at any regular point  $x$  of  $\partial B_t \cap \sigma(\frac{\pi}{n} + t, \frac{2\pi}{n} + t)$ , one has

$$\frac{\partial w}{\partial \eta_t}(x) = \frac{\partial u}{\partial \eta_t}(x) - \frac{\partial u}{\partial \eta_t}(x^*) < 0.$$

It follows that  $\lambda'(t) \leq 0$  and that the equality holds if and only if  $\eta_t \cdot v \equiv 0$ . By Lemma 1, this last equality occurs if and only if  $f$  is constant which means that  $B$  is a disk. □

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