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Letter to the Editor

Merger as Intermittent Accretion

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Abstract

Aims. The Self-Similar Secondary Infall Model (SSIM) is modified to simulate a merger event.
Methods. The model encompasses spherical versions of tidal stripping and dynamical friction that agrees with the Syer & White merger paradigm's behaviour.
Results. The SSIM shows robustness in absorbing even comparable mass perturbations and returning to its original state.
Conclusions. It suggests the approach to be invertible and allows to consider accretion as smooth mass inflow merging and mergers as intermittent mass inflow accretion.

Key words. Cosmology:theory – Dark Matter – Galaxies:formation – galaxies:halos – gravitation

1. Introduction

Structure formation in the Cold Dark Matter (CDM, or more simply DM) paradigm is dominated by the hierarchical picture of repeated mergers. This picture was emphasised by Syer & White (1998), explaining the dynamical formation of halo density profile with a feedback mechanism provided by repeated mergers. Whereas it is now believed that isotropisation of the velocity dispersion (angular momentum; see Le Delliou & Henriksen 2003; Barnes et al. 2005; MacMillan et al. 2006) via the radial-orbit instability (also viewed as adiabatic variability of self-similarity, Henriksen 2007) is responsible for the density profile formation, their picture remains a widely accepted description of the merger digestion mechanism. Despite its simple spherical symmetry and apparent lack of compliance with the merger paradigm, some studies have shown that the Secondary Infall Model (SIM) is a viable model to predict the structure and density profile evolutions of DM haloes as compared to N-body simulations (Ascasibar et al. 2007; Salvador-Solé et al. 2007).

This letter proposes to understand this paradox by examining the merger paradigm within the SIM and studying how merger events impact on the relaxation and structure of a CDM halo.

The SIM stems from the seminal work of Gunn & Gott (1972), and the SSIM (Self-similar SIM) started when Fillmore & Goldreich (1984) and Bertschinger (1984) independently found self-similar solutions to the SIM. It was later shown that those solutions can be reached from non-self-similar initial conditions (e.g. in Hoffman & Shaham 1985; White & Zaritsky 1992; Ryden 1993; Henriksen & Widrow 1995, 1997; Avila-Reese et al. 1999; Henriksen & Widrow 1999; del Popolo et al. 2000; Henriksen & Le Delliou 2002; Le Delliou & Henriksen 2003) and a systematic approach to the SSIM was used in Henriksen & Widrow (1995, 1997, 1999); Henriksen & Le Delliou (2002); Le Delliou & Henriksen (2003), derived from the Carter-Henriksen formalism (Carter & Henriksen 1991, hereafter CH). Some extensions to the SIM were proposed that included the effects of angular momentum to explain flat halo cusps (Hiotelis 2002; Le Delliou & Henriksen 2003; Ascasibar et al. 2004; Williams et al. 2004; Lu et al. 2006), but no fundamental attempt was made before Le Delliou (2002) to confront the SIM with the merger paradigm.

The following section (Sec. 2) will describe how and why the SSIM can be extended to model a merger event. Then Sec. 3 will discuss how the symmetry of the SSIM still allows for a form of tidal stripping and dynamical friction, before presenting the consequences of such a merger in the SSIM in Sec. 4, and to make some concluding remarks in Sec. 5.

2. Merger in an Infall

Modelling a merger event in a spherical geometry may appear contradictory but it is possible to a certain extent. To understand this it is important to realise the following: a very small amount of substructures are seen in N-body simulations; Diemand et al. (2007) find that only 5.3% of the total mass fraction of haloes lie in subhaloes. In the Syer & White (1998) picture, incoming satellite haloes merge with their parent, fall in the centre and contribute to the density profile and to the parent's relaxation and virialisation. However, in simulations, subobjects swing back and forth several times in their parents before being digested. That process can be modelled in a simpler way: on average, spherical symmetry is not bad (Ascasibar et al. 2007) as it reproduces the correct time scales and density profiles. Shell codes are much **simpler** than N-body codes and therefore provide with robust tests of certain aspects of their results. Other simplifying approaches have been used to understand halo formation, such as phase-space coarse graining (Le Delliou & Henriksen 2003; Henriksen 2004, 2006) or in the one dimensional slab model used in Binney (2004), where it was shown to explain the formation of cosmic web sheets through the interplay of phase mixing and violent relaxation, also present in spherical models. Henriksen & Widrow (1999) have shown that relaxation is moderately violent (in their figure 9) and induced by a phase space instability (Henriksen & Widrow 1997). Section 3 will detail how another perspective of phase mixing and moderately violent relaxation through phase space instability can be interpreted as some sort of tidal stripping and dynamical friction.

In this paper the SSIM is implemented in a shell code (see details in Le Delliou 2002, and references therein) with fully dy-

ϵ, panel	M_{ratio}	D_{ratio}	M_{OD}/M_{BG}
3/2, upper panel	0.751	0.282	1.173
3/2, middle panel	4.25×10^{-2}	7.10×10^{-2}	9.38×10^{-2}
3/2, lower panel	6.92×10^{-2}	0.168	1.453
5/2, upper panel	0.889	5.51×10^{-2}	0.319
5/2, middle panel	0.439	5.54×10^{-2}	0.290
5/2, lower panel	0.178	0.454	1.133
2.9, upper panel	0.753	9.19×10^{-2}	0.416
2.9, middle panel	0.407	0.641	1.118
2.9, lower panel	0.301	9.71×10^{-2}	0.344

Table 1. Density, mass and mass perturbation ratios defining the satellite initial OD for the mergers in the SSIM. The mass perturbation measures the perturbation of the OD compared to the background halo region it spans, just before entering the core. First column gives parent initial power law seed and panel order in reference to figures 1, 2 and 3.

namical Lagrangian treatment of infall using the CH (Carter & Henriksen 1991) self-similar variables that reveals when the system reaches naturally a self-similar regime. A halo is modelled from a radial power law perturbation $\delta\rho/\rho \propto r^{-\epsilon}$ on an Einstein-de Sitter homogeneous background, that is evolved to reach its quasi-stationary self-similar regime in its core[1] (Henriksen & Widrow 1999). The SIM is known to establish a self-similar infall phase (Henriksen & Widrow 1997), which then leads to a semi-universal power law density profile (Fillmore & Goldreich 1984; Bertschinger 1984): for initial power index $\epsilon \leq 2$, the isothermal sphere ($\rho \propto r^{-\mu}$ with $\mu = 2$) is the semi-universal attractor, whereas with $\epsilon > 2$, there is a continuum of attractors with $\mu = 3\epsilon/(1+\epsilon)$. Positive overdensity and the requirement of a finite initial core mass in the centre limit the range to $0 \leq \epsilon < 3$. The cores explored here were chosen, as presented in Table 1, according to their SSIM behaviour defined by their initial power index: typical shallow ($\epsilon = 3/2$) and steep ($\epsilon = 5/2$) profiles, with the addition of an extreme steep case ($\epsilon = 2.9$) to test the behaviour of a highly concentrated parent halo. The steep and shallow denominations refer to the comparison relative to the isothermal sphere.

In this geometry, an overdensity (hereafter OD, or satellite), representing a spherically averaged satellite halo, is a region of overdense shells close to the edge of the core, the parent halo (hereafter core, or parent).

The OD is evolved dynamically from an initial gaußian density profile added on top of the background halo profile over a finite region. That evolution runs long enough to observe the signature of the OD's own stationary regime in phase space. This is manifested in the mixing of its Liouville sheet during the OD's dynamical mass accretion of halo shells from its environment. The OD's definition as a set of particles (shells) is frozen when the core swallows it.

At that point are recorded the ratios of OD-over-core masses, M_{ratio}, of their densities, D_{ratio}, and the measure of the perturbation provided by the OD on its background surroundings, in mass, M_{OD}/M_{BG}. For each case, three different satellites were chosen, trying to obtain various types of mass and density ratios between satellites and parents.

Since they were allowed to accrete mass dynamically from their environment, ODs were laid close to the edge of the core to maintain some control over the final frozen mass and density ratios. Some configurations of those ratios were too difficult to obtain: in the shallow case, with high M_{ratio}, lower values for D_{ratio} were prevented by the high density background the OD accretes from, while for the steep cases, also with high M_{ratio}, higher D_{ratio} couldn't be obtained because of their cores' poor density backgrounds which tended to spread the ODs (see Sec. 4's tidal effect).

[1] The core, or self gravitating system, is defined as the set of shells in the multiple flow region. Its edge's radius is that of the outermost shell that has passed only once through the centre, as seen in phase space.

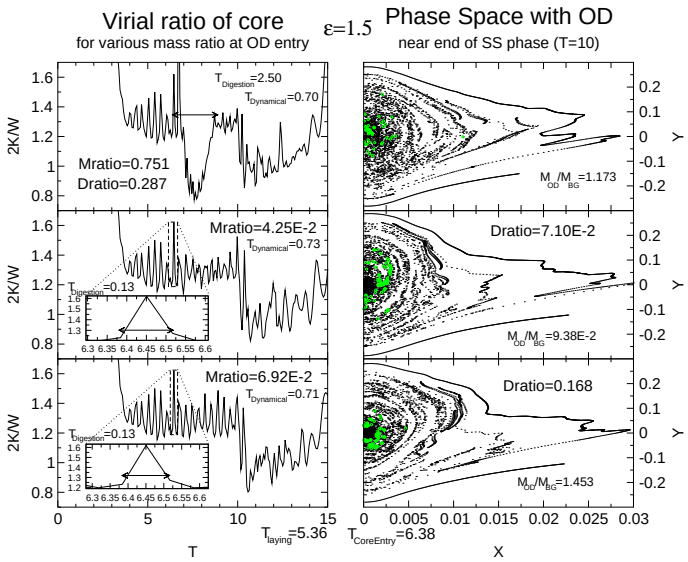

Figure 1. Shallow case: Virial ratio and phase space diagrams at the end of the self-similar phase for three sets of ODs in the $\epsilon = \frac{3}{2} = 1.5$ case. Overdensity shells are emphasised in green in phase space. Digestion time is defined from OD core entry (pre-spike, see Sec. 4) to virial recovery (measured on left panels). Zoomed encapsulation of those spikes in middle and lower left panels show their measure in troughless cases. T, X and Y are respectively the self-similar time, radius and radial velocity, which units are set by $G = M(\infty) = 1$ (Henriksen & Widrow 1997).

The ratios indicated are measured at the time of core entry. The explored values are presented in Table 1.

It is crucial to point out that the numerical implementation of the SSIM entails a shell code where finite size shells model the continuous system. That will play a role in the discussion of the results.

3. Merger paradigm and SSIM

Syer & White (1998) have attempted to define the singularity of mergers in an effort, at the time, to explain the universality of the density profile found in N-body simulation by Navarro et al. (1996, hereafter NFW): their key feature is the feedback mechanism between dynamical friction from the parent halo and tidal stripping of the satellite. Even though this is not anymore considered to hold the key to the formation of the density profile, their merger digestion mechanisms is still widely accepted to describe the behaviour of satellites. I argue that both mechanisms can be modelled within the SSIM despite its one-dimensional nature.

Tidal acceleration on an infinitesimal shell of mass $dm = 4\pi\rho r^2 dr$ – located at radius r, containing the system mass M and with thickness dr – can be defined as the differential gravity between its boundaries. Defining the cumulative average density profile

$$\langle\rho\rangle_r = \frac{M(r)}{4\pi r^3/3}, \tag{1}$$

the inward oriented elementary tidal acceleration reads, to leading order,

$$dT = 4\pi G dr\left(\rho - \frac{2}{3}\langle\rho\rangle_r\right). \tag{2}$$

It is thus clear that regions of peak density below the cumulative average ($\rho < \frac{2}{3}\langle\rho\rangle_r$) will experience a net disruptive tidal acceleration spreading apart shells in those regions, in the radial direction. In this spherically averaged study of a merger, this models tidal stripping.

ϵ, panel	$T_{digestion}$	$T_{dynamical}$	$\frac{T_{digestion}}{T_{dynamical}}$	$M_{ratio}.D_{ratio}$
$\frac{3}{2}$, upper p.	2.50	0.70	3.57	0.212
$\frac{3}{2}$, middle p.	0.13	0.73	0.178	3.017×10^{-3}
$\frac{3}{2}$, lower p.	0.13	0.71	0.183	1.163×10^{-2}
$\frac{5}{2}$, upper p.	4.21	1.21	3.48	4.989×10^{-2}
$\frac{5}{2}$, middle p.	3.07	1.12	2.74	2.432×10^{-2}
$\frac{5}{2}$, lower p.	2.11	0.98	2.15	8.081×10^{-2}
2.9, upper p.	4.83	1.17	4.13	6.920×10^{-2}
2.9, middle p.	4.94	1.10	4.49	2.609×10^{-1}
2.9, lower p.	3.07	1.11	2.77	2.923×10^{-2}

Table 2. Digestion and dynamical times and strength parameter of the OD for the mergers in the SSIM. Again, first column gives parent initial power law seed and panel order in figures 1, 2 and 3.

Dynamical friction classically is defined as the creation of a wake by a moving mass in a gravitating medium which back reaction entails a net drag force upon the moving mass. In the SSIM, a massive shell is crossing the core's shell in its travelling inwards or outwards. We will see that a radial drag force, with the correct orientation, is experienced as a result of this motion in the spherically averaged model.

This crossing of shells by the OD's results in shells just outside of it feeling more or less mass pulling inwards, depending on the direction of the motion of the massive OD shells. That leads to a differential tightening or spreading of core's shell behind the moving mass, in the fashion of a wake. However in *spherical symmetry*, an outer wake **does not** contribute to the pull on the OD. Nevertheless, its mass corresponds to shells that *defected* from inside because of OD motion, and their effect can be seen in the dynamics (see Appendix A).

In a similar fashion, the dynamical effect on the OD from its motion can be *described* in terms of a drag force: the crossing of core shells by massive OD shell lead to a decrease, or increase, of the resulting inner mass of the moving OD, depending on the direction of motion. Thus, with inner mass goes the inner pull, which can be interpreted a dragging force that adds to the total force, that should be experienced in the opposite direction of the motion.

Therefore, the SSIM with an outer overdensity can be interpreted to model the main features of the merger paradigm.

4. Digestions

Indeed, it is possible to keep track, in the Lagrangian shell model, of the defined satellite's (OD's) components once they have been absorbed by the parent (core). The core can be considered isolated at the end of the accretion phase (Henriksen & Widrow 1997). The phase space configurations of simulated merged haloes are displayed on Figs. 1, 2, and 3's right panels, distinguishing between the core and OD's accreted shells. This reveals how the different ODs, in their various (shallow or steep) environments, either retain some degree of coherence after being ingested by the core or have been digested and scattered over the core's phase space.

The left panels of Figs. 1, 2, and 3 examine the Virial ratios of the corresponding cores, and show a remarkable robustness in the SSIM: the quasi-stable self-similar phase[2] is shown to be either marginally or strongly disturbed by the OD absorption, but to return to the **original** undisturbed level of the parent after a digestion time $T_{digestion}$, provided a mass flow still fuels the self-similar equilibrium. Digestion is manifested by a more or less pronounced initial decrease (entry of extra mass in core increases W), followed by a spike (first crossing of centre gives m_{OD} high velocities, thus peaks K) and then, for stronger disturbance, a trough (energy exchanges from phase space instability, shells spend on average more time at low velocities, thus

[2] with Virial markedly different from usual value of 1!

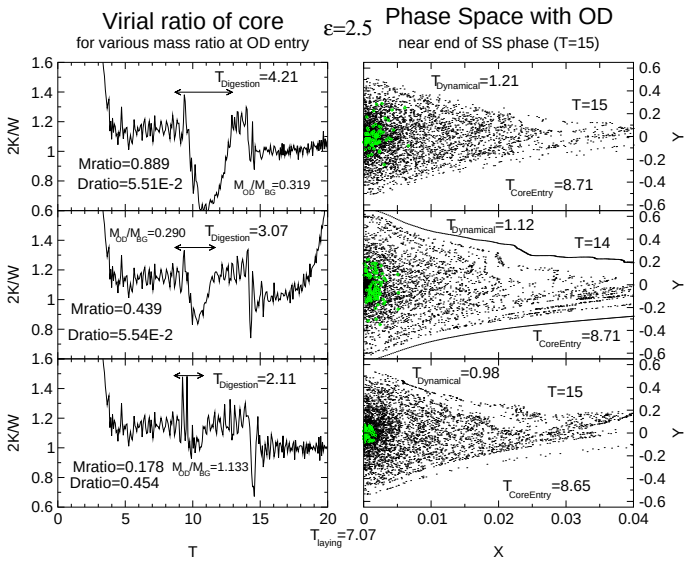

Figure 2. Steep case: Virial ratio and phase space diagrams at the end of the self-similar phase, for three sets of ODs in the $\epsilon = \frac{5}{2} = 2.5$ case, including an emphasis on digested overdensity shells in phases space and a measure of digestion time. Same units as in Fig. 1.

lower Virial, Henriksen & Widrow 1999). Its deepness depends primarily on M_{ratio}. Digestion time measurements are shown on Figs. 1, 2, and 3's left panels (double horizontal arrows), and are summarised in Table 2. There, they are compared with the OD's free fall dynamical time through the core, $T_{dynamical}$, also indicated on the figures. $T_{dynamical}$ is defined as the free fall time to the centre of a test shell across a constant density distribution, equivalent to the core, in self-similar variables. From Table 2, without Fig. 1's two lowest panels, where the definition of $T_{digestion}$ is problematic, the average $\left\langle T_{digestion}/T_{dynamical}\right\rangle = 3.33$, with a standard deviation of 0.77, can be computed. It shows the core digests the OD in 2 to 4 passages in the central relaxation region of phase space. This is comparable to the number of distinguishable Lagrange-Liouville streams present in the core's outer phase space regions, as seen from Figs. 1, 2, and 3's right panels.

From the OD's point of view, the mergers display their effects in phase spaces, represented on Figs. 1, 2, and 3's right panels, on which two features are crucial: the spread (or compactness) of the OD over the core at the end of the infall phase and the presence of some, or all, of its shells in the centre of the core's phase space. This reflects the digestion mechanisms adopted by Syer & White (1998). Their proposal aimed at a dynamical explanation of the NFW profile. Although this explanation is not anymore considered (see Sec. 1), it is interesting to note that the presently discussed single merger model in the SSIM shows signs of inflections (central flattening and edge steepening) from its semi-universal, almost isothermal, density profile. However this is not the focus of this paper.

The OD's compactness resists to tidal stripping while its final presence in the centre is driven by dynamical friction. The fate of a model satellite in the SSIM displays behaviour well in agreement with the merger digestion mechanisms proposed by Syer & White: in the SSIM a combination of density and mass ratios leads to emphasise each effect. High D_{ratio}s seem to be the dominant factor for OD's compactness, while high M_{ratio}s promote the sinking of the OD to the centre of the core's phase space.

All possible qualitative types of behaviour are present: if both ratios, M_{ratio} and D_{ratio}, are strong enough, the OD survives almost intact to the centre of phase space (Figs. 2's lower and 3's middle right panels). If only M_{ratio} is high while D_{ratio} is low, the OD is scattered at the centre (Figs. 1, 2 and 3's upper right panels). Conversely, a high D_{ratio} and low M_{ratio} lead to a compact

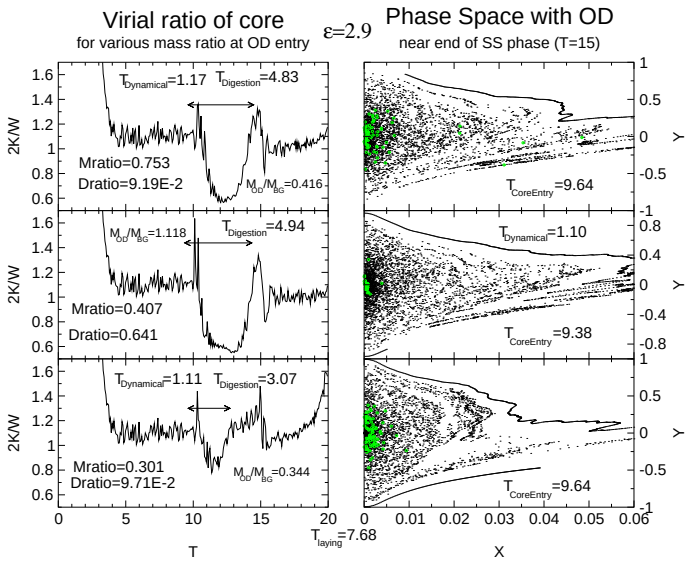

Figure 3. Extreme steep case: Virial ratio and phase space diagrams at the end of the self-similar phase, for three sets of ODs in the $\epsilon = 2.9$ case, including an emphasis on digested overdensity shells in phase space and a measure of digestion time. Same units as in Fig. 1.

OD around, but not reaching, the centre of phase space (Fig. 1's lower right panel). Finally if both ratios are too low, the OD is scattered without reaching the centre of phase space (Figs. 1 and 2's middle and 3's lower right panels).

A step further in this phenomenology would be to note that a combination of both ratios should be taken ($M_{ratio}.D_{ratio}$, see Table 2), for which a threshold can be defined for reaching the centre and another for compactness of the OD. However this classification seems to require an additional dependency with the steepness of the initial profile. Indeed the available data offer different ranges for each initial profile case. The shallow case calls for higher values for the $M_{ratio}.D_{ratio}$ thresholds than the steep cases. This reflects the shallow case's wider spread of material, compared with the steep cases, that the OD has to cross on its journey towards the centre of phase space.

As an illustration of our model, we can assume the Milky Way (hereafter MW) to have a shallow profile and use the corresponding reliable digestion time model, that is with $\epsilon = 1.5$, $M_{ratio} = 0.751$ and $T_{digestion} = 2.50$. The corresponding satellite S would have a mass $M_S \simeq 44 M_{LMC}$ compared to the Large Magellanic Cloud (hereafter LMC), which is huge. The model then yields a very short digestion time, also compared with the age of the oldest stars in the MW $T_{MW} = 13.2$Gyr, as

$$T_{digestion} \simeq 584\text{Myr} \simeq \frac{T_{MW}}{22.6}. \tag{3}$$

Its dynamical time $T_{dynamical} \simeq 234$Myr indicates that at the end of digestion, this satellite's shells would be lined between the second incoming and second outgoing inner streams of the core and the model suggests it to then sink to the centre by the end of the MW formation as seen on Fig. 1's upper right panel.

5. Discussion and conclusions

The SSIM has proven its capacity to model a merger event. Its simplicity allows one to probe the dynamics of the merger and the most remarkable result of this work shows that the self-similar quasi-stable regime of quasi-Virial equilibrium is extremely **robust** to perturbations that can be of comparable size to the core (equal mass mergers): the Virial ratio, after a more or less long period of digestion returns to its stabilised **original** undisturbed level, after only 2 to 4 passages in the centre, and continues its usual evolution. The spreading and sinking of the satellite's particles across the parents and towards its centre agree with the tidal stripping and dynamical friction picture from Syer & White (1998), provided some adaptation to the language of the SSIM's symmetry. Finally, and this is the claim of this paper, the numerical implementation of the model requiring discretisation, the rapid oscillations of the Virial ratio in the accretion phase offer a novel interpretation in the light of the SSIM merger model: instead of a continuous stream of mass, the model presents a repeated bombardment of finite mass shells that can be understood as small overdensities; Fig. 1's zoomed two lowest right panels show a spike to manifest the weakest mergers digestion; thus the wiggles in the Virial ratio can be interpreted as manifestation of repeated mergers that are at this level indistinguishable from accretion. Therefore there is *no fundamental difference* between mergers and accretion, the latter being a series of repeated merger with vanishing mass, while the latter is just intermittent accretion. This reconciles approaches such as Salvador-Solé et al. (2007) where accretion was presented as a memory loss mechanism, eliminating the need to refer to mergers.

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Appendix A: Spherical model of dynamical friction

A thin OD shell in the inward/outward direction, adding mass $\pm dm_{OD}$, crossing shells at r creates a differential acceleration w.r.t. the state without OD which induces an infinitesimal displacement, thus a wake,

$$dr = \mp \frac{G(dt)^2 dm_{OD}}{2r^2}. \tag{A.1}$$

This wake of mass $dm_W = \rho r^2 dr$ induces on the OD an acceleration (backreaction)

$$a_{drag} = -\frac{G dm_W}{r^2} = -G\rho dr = \pm \frac{(Gdt)^2 dm_{OD}}{2r^2}\rho, \tag{A.2}$$

opposite to the direction of motion. In addition, the amplitude of the drag force is shown proportional to $dm_{OD}\rho$, related to $M_{ratio}.D_{ratio}$.