Regions of Feasible Point-to-Point Trajectories in the Cartesian Workspace of Fully-Parallel Manipulators

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ABSTRACT

The goal of this paper is to define the n-connected regions in the Cartesian workspace of fully-parallel manipulators, i.e. the maximal regions where it is possible to execute point-to-point motions. The manipulators considered in this study may have multiple direct and inverse kinematic solutions. The N-connected regions are characterized by projection, onto the Cartesian workspace, of the connected components of the reachable configuration space defined in the Cartesian product of the Cartesian space by the joint space. Generalized octree models are used for the construction of all spaces. This study is illustrated with a simple planar fully-parallel manipulator.

INTRODUCTION

The Cartesian workspace of fully-parallel manipulators is generally defined as the set of all reachable configurations of the moving platform. However, this definition is misleading since the manipulator may not be able to move its platform between two prescribed configurations in the Cartesian workspace. This feature is well known in serial manipulators when the environment includes obstacles (Wenger, 91). For fully-parallel manipulators, point-to-point motions may be infeasible even in obstacle-free environments. For manipulators with one unique solution to their inverse kinematics (like Gough-platforms), one configuration of the moving platform is associated with one unique joint configuration and the connected-components of the singularity-free regions of the Cartesian workspace are the maximal regions of point-to-point motions (Chablat, 98a). Unfortunately, this result does not hold for manipulators which have multiple solutions to both their direct and inverse kinematics. For such manipulators which are the subject of this study, the singularity locus in the Cartesian workspace depends on the choice of the inverse kinematic solution (Chablat, 97) and the actual reachable space must be firstly defined in the Cartesian product of the Cartesian space by the joint space. The goal of this paper is to define the N-connected regions in the Cartesian workspace of fully-parallel manipulators, i.e., the maximal regions where it is possible to execute any point-to-point motion. The N-connected regions are characterized by projection, onto the Cartesian space, of the connected components of the manipulator configuration space defined in the Cartesian product of the Cartesian space by the joint space. Generalized Octree models are used for the construction of all spaces. This study is illustrated with a simple planar fully-parallel manipulator.

1 Preliminaries

Some useful definitions are recalled in this section.

1.1 Fully-parallel manipulators

Definition 1. *A fully-parallel manipulator is a mechanism that includes as many elementary kinematic chains as the moving platform does admit degrees of freedom. In addition, every elementary kinematic chain possesses only one actuated joint (prismatic, pivot or kneecap). Besides, no segment of an elementary kinematic chain can be linked to more than two bodies (Merlet, 97).*

In this study, kinematic chains, also called "leg" (Angeles, 97), will be always independent.

1.2 Kinematics

The input vector $\mathbf{q}$ (the vector of actuated joint values) is related to the output vector $\mathbf{X}$ (the vector of configuration of the moving platform) through the following general equation :

$$F(\mathbf{X},\mathbf{q}) = 0 \tag{1}$$

Vector ($\mathbf{X}$, $\mathbf{q}$) will be called *manipulator configuration* and $\mathbf{X}$ is the platform configuration and will be more simply termed *configuration*. Differentiating equation (1) with respect to time leads to the velocity model

$$\mathbf{A}\mathbf{t} + \mathbf{B}\dot{\mathbf{q}} = 0 \tag{2}$$

With $\mathbf{t} = [w, \dot{\mathbf{c}}]^T$, for planar manipulators (w is the scalar angular-velocity and $\dot{\mathbf{c}}$ is the two-dimensional velocity vector of the operational point of the moving platform), $\mathbf{t} = [\mathbf{w}]^T$, for spherical manipulators and $\mathbf{t} = [\mathbf{w}, \dot{\mathbf{c}}]^T$, for spatial manipulators ($\dot{\mathbf{c}}$ is the three-dimensional velocity vector and $\dot{\mathbf{w}}$ is the three-dimensional angular velocity-vector of the operational point of moving platform).

Moreover, $\mathbf{A}$ and $\mathbf{B}$ are respectively the direct-kinematics and the inverse-kinematics matrices of the manipulator. A singularity occurs whenever $\mathbf{A}$ or $\mathbf{B}$, (or both) can no longer be inverted. Three types of singularities exist (Gosselin, 90):

$$\begin{aligned} det(\mathbf{A}) &= 0 \\ det(\mathbf{B}) &= 0 \\ det(\mathbf{A}) &= 0 \quad and \quad det(\mathbf{B}) = 0 \end{aligned}$$

1.3 Parallel singularities

Parallel singularities occur when the determinant of the direct kinematics matrix $\mathbf{A}$ vanishes. The corresponding singular configurations are located inside the Cartesian workspace. They are particularly undesirable because the manipulator can not resist any effort and control is lost.

1.4 Serial singularities

Serial singularities occur when the determinant of the inverse kinematics matrix $\mathbf{B}$ vanishes. By definition, the inverse-kinematic matrix is always diagonal: for a manipulator with n degrees of freedom, the inverse kinematic matrix $\mathbf{B}$ can be written like in equation (3). Each term $\mathbf{B}_{jj}$ is associated with one leg. A serial singularity occurs whenever at least one of these terms vanishes.

$$\mathbf{B} = Diag\,[\mathbf{B}_{11}, ..., \mathbf{B}_{jj}, ..., \mathbf{B}_{nn}] \tag{3}$$

When the manipulator is in serial singularity, there is a direction along which no Cartesian velocity can be produced.

1.5 Postures

The *postures* are defined for fully-parallel manipulators with multiple inverse kinematic solutions (Chablat, 97). Let W be the reachable Cartesian workspace, that is, the set of all reachable configurations of the moving platform ((Kumar, 92) and (Pennock, 93)). Let Q be the reachable joint space, that is, the set of all joint vectors reachable by the actuated joints.

Definition 2. *For a given configuration* $\mathbf{X}$ *in* W*, a* posture *is defined as a solution to the inverse kinematics of the manipulator.*

According to the joint limit values, all postures do not necessarily exist. Changing posture is equivalent to changing the posture of one or several legs.

1.6 Point-to-point trajectories

There are two major types of tasks to consider : point-to-point motions and continuous path tracking. Only point-to-point motions will be considered in this study.

Definition 3. *A point-to-point trajectory* T *is defined by a set of* p *configurations in the Cartesian workspace :* $T = \{\mathbf{X}_1, ..., \mathbf{X}_i, ..., \mathbf{X}_p\}$.

By definition, no path is prescribed between any two configurations $\mathbf{X}_i$ and $\mathbf{X}_j$.

Hypothesis : In a point-to-point trajectory, the moving platform can not move through a parallel singularity.

Although it was shown recently that in some particular cases a parallel singularity could be crossed (Nenchev, 97), hypothesis 1 is set for the most general cases.

A point-to-point trajectory T will be feasible if there exists a continuous path in the Cartesian product of the Cartesian space by the joint space which does not meet a parallel singularity and which makes the moving platform pass through all prescribed configurations $\mathbf{X}_i$ of the trajectory T.

Remark : A fully-parallel manipulator with several inverse kinematic solutions can change its posture between two prescribed configurations. Such a manoeuver may enable the manipulator to avoid a parallel singularity (Figure 1). More generally, the choice of the posture for each configuration $\mathbf{X}_i$ of the trajectory T can be established by any other criteria like stiffness or cycle time (Chablat, 98b). Note that a change of posture makes the manipulator run into a serial singularity, which is not redhibitory for the feasibility of point-to-point trajectories.

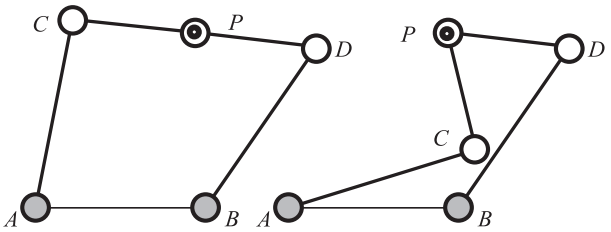

Figure 1. Singular (left) and a regular (right) configurations (the actuated joints are A and B)

1.7 The generalized octree model

The quadtree and octree models are hierachical data structures based on a recursive subdivision of the plane and the space, respectively (Meagher, 81). There are useful for representing complex 2-D and 3-D shapes. In this paper, we use a generalization of this model to dimension k, with $k > 3$, the 2^k-tree (Chablat, 98a). This model is suitable for Boolean operations like union, difference and intersection. Since this structure has an implicit adjacency graph, path-connectivity analyses and trajectory planning can be naturally achieved.

When $k > 3$, it is not possible to represent graphically the 2^k-tree. It is necessary to project this structure onto a lower dimensional space (quadtree or octree). For a n-dof fully-parallel manipulator, the Cartesian product of the Cartesian space by the joint space defines generalized octree with dimension $2n$. When $n = 3$ (respectively $n = 2$), the projection onto the Cartesian space and the joint space yields octree models (respectively quadtree models).

2 The moveability in the Cartesian workspace

Definition 4. *The* N-connected regions *of the Cartesian workspace are the maximal regions where any point-to-point trajectory is feasible.*

For manipulators with multiple inverse and direct kinematic solutions, it is not possible to study the joint space and the Cartesian space separately. First, we need to define the *regions of manipulator reachable configurations* in the Cartesian product of the Cartesian space by the joint space $W.Q$.

Definition 5. *The regions of manipulator reachable configurations R_j are defined as the maximal sets in $W.Q$ such that*

$R_j \in W.Q$,
R_j is connected,
$R_j = \{\mathbf{X}, \mathbf{q}\}$ such that $det(\mathbf{A}) \neq 0$

In other words, the regions R_j are the sets of all configurations ($\mathbf{X}$, $\mathbf{q}$) that the manipulator can reach without meeting a parallel singularity and which can be linked by a continuous path in $W.Q$.

Proposition : A trajectory $T = \{\mathbf{X}_1, ..., \mathbf{X}_p\}$ defined in the Cartesian workspace W is feasible if and only if :

$$\begin{cases} \forall \mathbf{X} \in \{\mathbf{X}_1, ..., \mathbf{X}_p\} \\ \exists \mathbf{q}_i \in Q, \exists R_j \end{cases} \textit{such that } (\mathbf{X}_i, \mathbf{q}_i) \in R_j$$

In other words, for each configuration $\mathbf{X}_i$ in T, there exists at least one posture $\mathbf{q}_i$ and one region of manipulator reachable configurations R_j such that the manipulator configuration $(\mathbf{X}_i, \mathbf{q})$ is in R_j.

Proof : *Indeed, if for all configurations $\mathbf{X}_i$, there is one joint configuration $\mathbf{q}_i$ such that $(\mathbf{X}_i, \mathbf{q}_i) \in R_j$ then the trajectory is feasible because, by definition, a region of manipulator reachable configurations is connected and free of parallel singularity. Conversely, if for a given configuration $\mathbf{X}_i$, it is not possible to find a posture $\mathbf{q}_i$ such that $(\mathbf{X}_i, \mathbf{q}_i) \in R_j$, then no continuous, parallel singularity-free path exists in $W.Q$ which can link the other prescribed configurations.*

Theorem : The N-connected regions W_{Nj} are the projection Π_W of the region of manipulator reachable configurations R_j onto the Cartesian space :

$$W_{Nj} = \Pi_W R_j$$

Proof : *This results is a straightforward consequence of the above proposition.*

The N-connected regions cannot be used directly for planning trajectories in the Cartesian workspace since it is necessary to choose one joint configuration $\mathbf{q}$ for each configuration $\mathbf{X}$ of the moving platform such that $(\mathbf{X}, \mathbf{q})$ is included in the same region of manipulator reachable configurations R_j. However, the N-connected regions provide interesting global information with regard to the performances of a fully-parallel manipulators because they define the maximal regions of the Cartesian workspace where it is possible to execute any point-to-point trajectory.

A consequence of the above theorem is that the Cartesian workspace W is N-connected if and only if there exists a N-connected region W_{Nj} which is coincident with the Cartesian workspace :

$$W_{Nj} = W$$

3 Example: A Two-DOF fully-parallel manipulator

For more legibility, a planar manipulator is used as illustrative example in this paper. This is a five-bar, revolute (R)-closed-loop linkage, as displayed in figure 2. The actuated joint variables are θ_1 and θ_2, while the Output values are the (x, y) coordinates of the revolute center P. The passive joints will always

be assumed unlimited in this study. Lengths L_0, L_1, L_2, L_3, and L_4 define the geometry of this manipulator entirely. The dimensions are defined in table 1 in certain units of length that we need not specify.

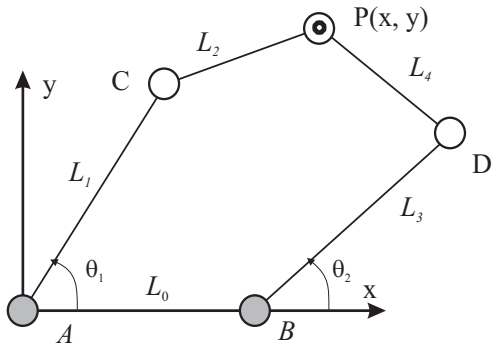

Figure 2. A two-dof fully-parallel manipulator

L_0	L_1	L_2	L_3	L_4	θ_{1min}	θ_{1max}	θ_{2min}	θ_{2max}
7	8	5	8	5	0	π	0	π

Table 1. The dimensions of the RR-RRR studied

As shown in table 1, the actuated joints are limited. The Cartesian workspace is shown in figure 3. We want to know whether this manipulator can execute any point-to-point motion in the Cartesian workspace. To answer this question, we need to determine the the N-connected regions.

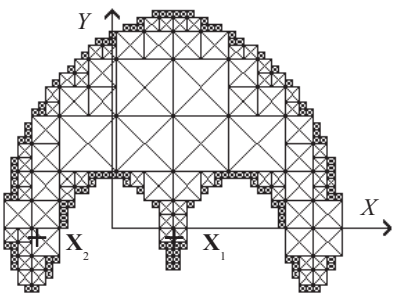

Figure 3. The Cartesian workspace

3.1 Singularities

For the manipulator studied, the parallel singularities occur whenever the points C, D, and P are aligned (Figure 4). Manipulator postures whereby $\theta_3 - \theta_4 = k\pi$ denote a singular matrix $\mathbf{A}$, and hence, define the boundary of the joint space of the manipulator. For the manipulator at hand, the serial singularities occur

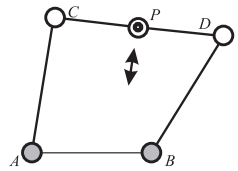

Figure 4. Example of parallel singularity

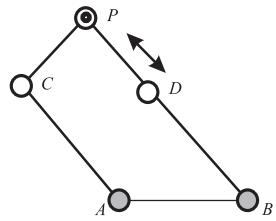

Figure 5. Example of serial singularity

whenever the points A, C, and P or the points B, D, and P are aligned (Figure 5). Manipulator postures whereby $\theta_3 - \theta_1 = k\pi$ or $\theta_4 - \theta_2 = k\pi$ denote a singular matrix $\mathbf{B}$, and hence, define the boundary of the Cartesian workspace of the manipulator.

3.2 Postures

The manipulator under study has four postures, as depicted in figure 6. According to the posture, the parallel singularity locus changes in the Cartesian workspace, as already shown in figure 1.

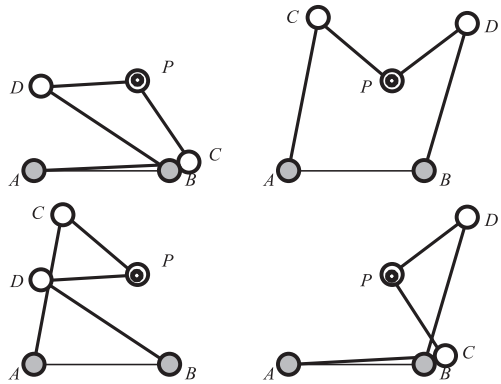

Figure 6. The four postures

3.3 The N-connected regions

It turns out that the Cartesian workspace of the manipulator at hand is not N-connected, e.g. the manipulator cannot move its platform between any set of configurations in the Cartesian workspace. In effect, due to the existence of limits on the actuated joints, not all postures are accessible for any configuration in the Cartesian workspace. Thus, the manipulator may loose its ability to avoid a parallel singularity when moving from one configuration to another. This is what happens between points $\mathbf{X}_1$ and $\mathbf{X}_2$ (Figure 3). These two points cannot be linked by the manipulator although they lie in the Cartesian workspace which is connected in the mathematical sense (path-connected) but not N-connected. In fact, there are two separate N-connected regions which do not coincide with the Cartesian workspace and the two points do not belong to the same N-connected region (Figures 7 and 8).

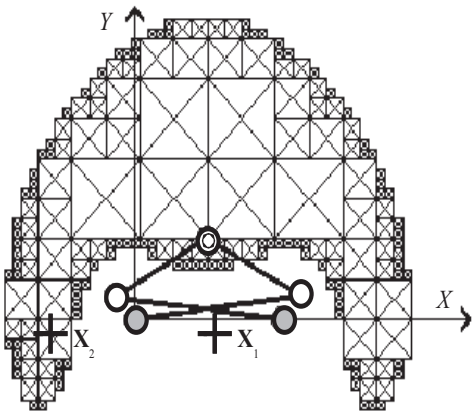

Figure 7. The first N-connected region of the Cartesian workspace when $0.0 \leq \theta_1, \theta_2 \leq \pi$

Physically, any attempt in moving the point P from $\mathbf{X}_1$ to $\mathbf{X}_2$ will cause the manipulator either cross a parallel singularity or reach a joint limit.

In effect, point $\mathbf{X}_1$ is accessible only in the manipulator configuration shown in figure 9a because of the joint limits. When moving towards point $\mathbf{X}_4$, the manipulator cannot remain in its initial posture because it would meet a parallel singularity (Figure 9b). Thus, it must change its posture, let say at $\mathbf{X}_3$ (Figure 9c). The only new posture which can be chosen is the one depicted in figure 9d because any other posture would make the manipulator meet a parallel singularity (Figure 9e). Then, it is apparent that the manipulator cannot reach $\mathbf{X}_1$ from $\mathbf{X}_4$ since joint A attains its limits (figure 9f).

If we change the values of the joint limits ($\theta_{1min} = \theta_{2min} = -\pi$), the Cartesian workspace is now N-connected since the

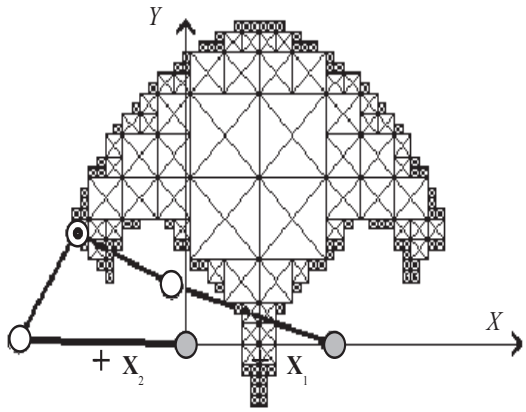

Figure 8. The second N-connected region of the Cartesian workspace when $0.0 \leq \theta_1, \theta_2 \leq \pi$

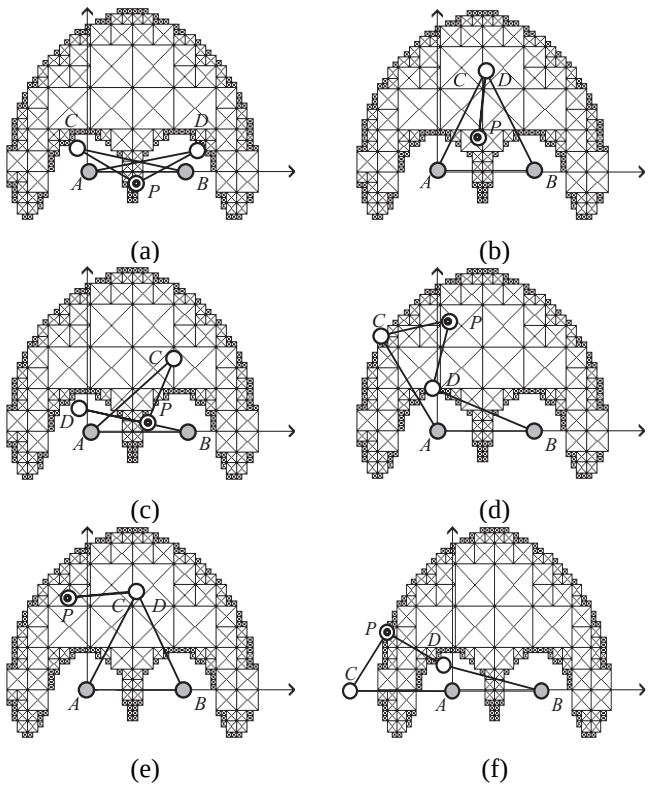

Figure 9. Moving from $\mathbf{X}_1$ to $\mathbf{X}_4$

computed N-connected regions are coincident with the Cartesian workspace (Figure 10). In effect, it can be verified in this case that for every configuration of the moving platform, there are four postures which define two regions of accessible configurations whose projection onto the Cartesian space yields the full

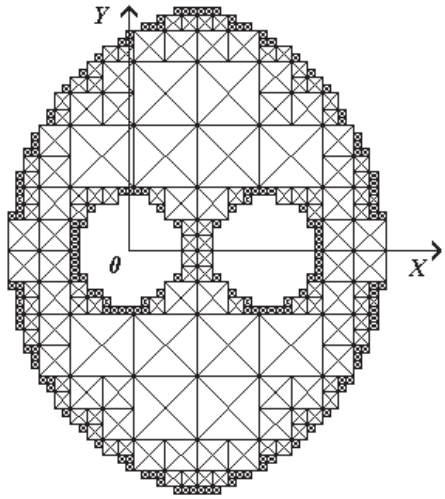

Figure 10. The N-connected regions of the Cartesian workspace when $-\pi \leq \theta_1, \theta_2 \leq \pi$

Cartesian workspace.

4 Conclusions

The aim of this paper was the characterization of the N-connected regions in the Cartesian workspace of fully-parallel manipulators, i.e. the regions of feasible point-to-point trajectories. The word feasible means that the manipulator should be able to move between all prescribed configurations while never meeting a parallel singularity. The manipulators considered in this study have multiple solutions to their direct and inverse kinematics. The N-connected regions were defined by first determining the maximum path-connected, parallel singularity-free regions in the Cartesian product of the Cartesian workspace by the joint space. The projection of these regions onto the Cartesian workspace were shown to define the N-connected regions.

The N-connectivity analysis of the Cartesian workspace is of high interest for the evaluation of manipulator global performances as well as for off-line task programming.

Further research work is being conducted by the authors to take into account the collisions and to characterize the maximum regions of the Cartesian workspace where the manipulator can track any continuous trajectory.

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