Bifurcation delay and difference equations.

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Abstract : We prove the existence of complex analytic solutions of difference equations of the form $y(x+\varepsilon) = f(x, y(x))$, where $x, y \in \mathbb{C}$ and ε is a small parameter. We also show that differences of two solutions are exponentially small. We apply these results to the problem of delayed bifurcation at a point of period doubling for real discrete dynamical systems. In contrast to previous publications, the results obtained in this article are global.
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1 Introduction

In classical bifurcation theory, the dynamical system contains a *fixed* parameter, the phase portraits are studied for parameter values in some interval and the qualitative differences for values below and above some threshold are described. The theory of *dynamical bifurcations* [3] consists in studying the behavior of a one parameter family of dynamical systems, where the parameter is not fixed but considered as a variable and changes slowly with time. The behavior of this new system sometimes has a bifurcation diagram different from the *static* one obtained for a fixed parameter.

Consider for example a family of vector fields exhibiting a Poincaré-Andronov-Hopf bifurcation: suppose that the family has a stationary point depending continuously of the parameter, that this stationary point is attractive if the parameter is below a certain critical value but repulsive if it is above this threshold and that an attractive invariant cycle exists close to the repulsive fixed point. In certain cases, the behavior of the system with slowly varying parameter has been shown to exhibit a *bifurcation delay*: when the critical value is passed, instead of leaving the now repulsive curve of stationary points and approaching the attractive cycle, the system continues to stay in the neighborhood of the curve of stationary points for some surprisingly long time.

The same phenomenon occurs also for families of discrete dynamical systems. We consider in the present article a real discrete system defined by the recurrence equation

$$y_{n+1} = f(x, y_n) \tag{1}$$

where x is a real parameter, as is the variable y, and where $f : \mathbb{R}^2 \to \mathbb{R}$ is sufficiently smooth. For simplicity, we assume that f is $\mathcal{C}^3$. Consider a bifurcation point $C = (x_c, y_c)$, i.e. a fixed point ($y_c = f(x_c, y_c)$) that is non-hyperbolic ($|a| = 1$, where $a := \frac{\partial f}{\partial y}(x_c, y_c)$). Suppose, moreover, that there exists in a neighborhood of C a curve of fixed points $y = g_0(x)$ depending smoothly upon x (for example $g_0 \in \mathcal{C}^3$ at the point x_c, $g_0(x_c) = y_c$), and that this fixed point is attractive for x below x_c and repulsive for $x > x_c$.

Static bifurcation. In the beginning of section 2, we briefly recall the classical scenarios of possible bifurcations in the most generic cases. More precisely, we show that in the non-oscillatory case ($a = 1$), generically a *transcritical bifurcation* occurs: a second curve of fixed points intersects the first curve $y = g_0(x)$ in the point C. In the oscillatory case $a = -1$, generically a period doubling bifurcation appears: either the repulsive curve of fixed points is accompanied by a double curve of attractive 2-periodic points or the attractive part is accompanied by a double curve of repulsive 2-periodic points, depending on the sign of $\frac{\partial^2 f^2}{\partial x \partial y}(C) \frac{\partial^3 f^2}{\partial y^3}(C)$ where $f^2(x, y) := f(x, f(x, y))$; the distance of the double curve from g_0 is of the order $\sqrt{x - x_c}$ for x close to x_c.

Even though these scenarios are well known, we find it useful to state them precisely and reprove them in appendix A; the main reason for doing so is the completeness of the present article.

Example. In this article, we are particularly interested in the famous family of quadratic mappings given by

$$f(x, y) = xy(1 - y) \; . \tag{2}$$

Usually, these mappings are considered as mappings of the interval $[0, 1]$ into itself, thus x has to be in $[0, 4]$. Here we will consider them as dynamical systems for all $y \in \mathbb{R}$ and $x > 0$.

A first transcritical bifurcation for the curve of fixed points $y = 0$ appears at the critical value $x_c = 1$. The period doubling appears for the curve $y = g_0(x) = 1 - \frac{1}{x}$ of fixed points at the critical value $x_c = 3$.

In the sequel – and not only for the example – we are only interested in this period doubling bifurcation. In particular, for the example, g_0 always denotes the function $x \mapsto 1 - \frac{1}{x}$.

Dynamic bifurcation. The parameter x is replaced by a variable x_n that changes slowly with each iteration. Thus a small parameter $\varepsilon > 0$ is introduced and the following discrete *slow-fast* system is considered.

$$\left\{ \begin{array}{rcl} x_{n+1} & = & x_n + \varepsilon \\ y_{n+1} & = & f(x_n, y_n) \end{array} \right. \tag{3}$$

In the sequel, we call *orbit* a family $(x_n(\varepsilon), y_n(\varepsilon))_{n \in \mathbb{N}}$ of solutions of (3) depending upon ε.

There are two reasons suggesting this kind of investigation. First, the description of this type of bifurcation in classical works often contains “dynamic” terms, for example:

[12] p.87, l.4 “If the fixed point becomes unstable [...]”
[11] p.86, l.27 “[...] the fixed point of P_γ becomes unstable and undergoes a flip bifurcation in which a stable orbit of period $4\pi/\omega$ appears [...]”

The second reason has to do with applications to physics, in particular non-linear optics [10, 14, 13]. Indeed, the very long time necessary for a system in physics to reach a state of equilibrium requires sometimes to slowly increase the parameter during an experiment.

It is therefore interesting to understand the asymptotic behavior of some orbit when the small parameter tends to 0, and in particular to find out whether the *dynamic* bifurcation reflects the static bifurcation.

In figure 1, we have chosen $\varepsilon = 10^{-3}$ and as initial point $x_0 = 1$, $y_0 = \frac{1}{2}$. The figures differ only in the numerical precision used to calculate the orbits.

The first observation is that with sufficiently high precision the dynamic bifurcation corresponding to system (3) is completely different from the static bifurcation of (1). In particular, the orbits follow the curve of repulsive fixed points instead of switching to the curves of 2-periodic points. This is the so-called *bifurcation delay.* The second observation is an exponential sensitivity of this phenomenon explained by results of previous articles restated in section 2.

The best way to study the dynamical bifurcation of the orbits of (3) is to consider *invariant curves*, i.e. the graphs of the solutions of the associated difference equation

$$\varphi(x+\varepsilon) = f(x, \varphi(x)) \ . \tag{4}$$

As is the case for the orbits, what we call here *solution of* (4) is more precisely a family of solutions φ_ε depending upon the parameter ε. The smoothness of these invariant curves has no influence on the dynamics of the discrete solutions of (3). Their closeness to the slow curve, however, is important. This notion of closeness is defined in section 2.

It is easy to construct invariant curves (which are not necessarily close to the slow curve): Define φ_ε arbitrarily on some interval of length ε, e.g. $\varphi_\varepsilon = g_0$ on $[x_0, x_0+\varepsilon[$, and then use (4) to define φ_ε on the intervals $[x_0+n\varepsilon, x_0+(n+1)\varepsilon[$. In this way, it is even possible to construct invariant curves of class $\mathcal{C}^\infty$. On the other hand the existence of *analytic* invariant curves is not clear, but also unnecessary for studying the discrete dynamics. The existence of any invariant curve, even not measurable, is sufficient, provided it is *close* to the slow curve (in the sense of the following section) on some appropriate interval.

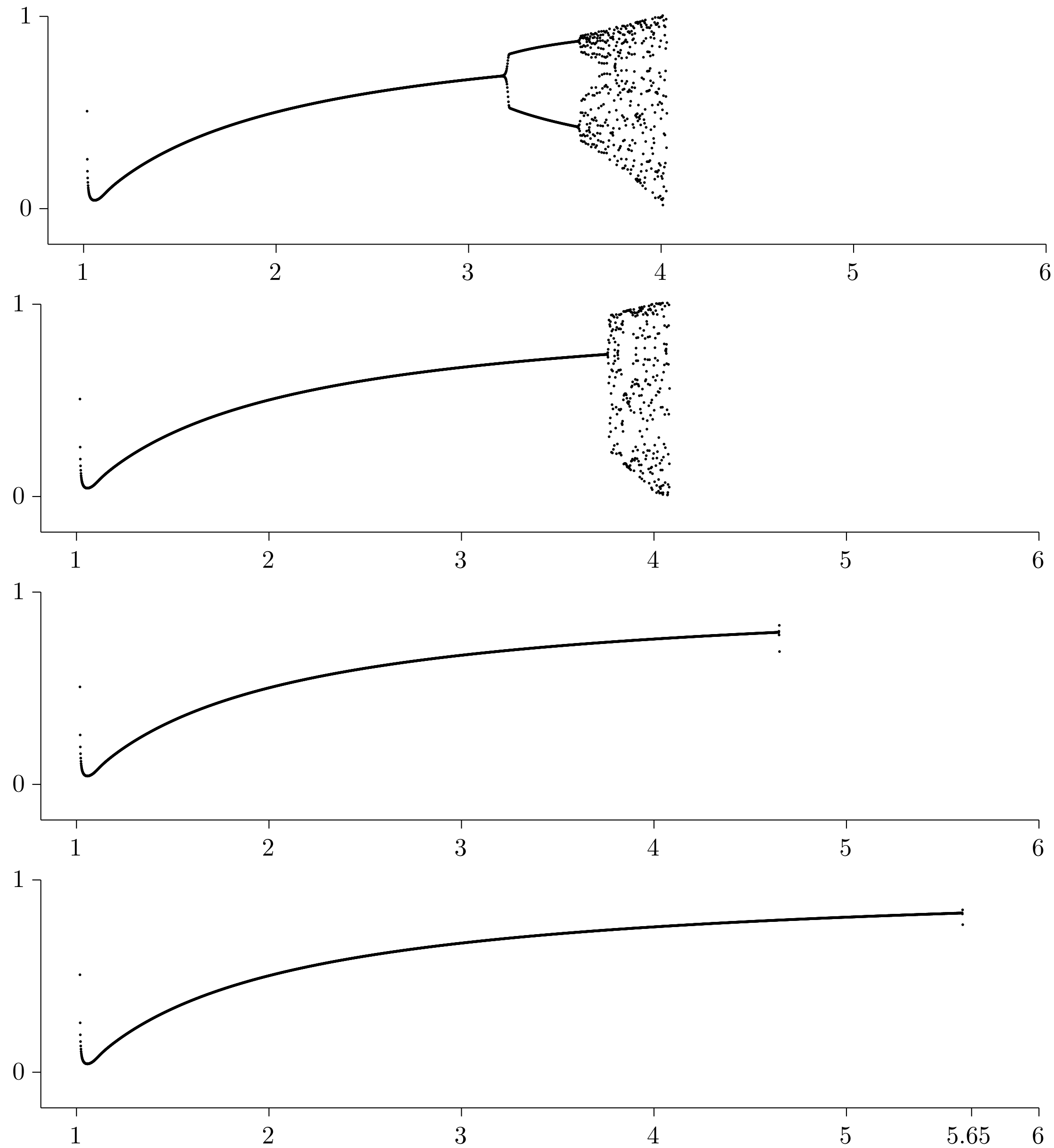

Figure 1: Successively 8, 100, 400 and 1000 digits

Ironically, the invariant curves we will construct are analytic with respect to x. More precisely, we will show in section 3 the existence and exponential closeness of analytic solutions of (4) close to g_0 on some domain Ω satisfying certain geometrical conditions.

Our method of proof relies on the fixed point principle. Therefore we need to construct a linear operator solving equations of the form $z(x+\varepsilon) = a(x)z(x)+\varepsilon g(x)$. This construction is to a large extent analogous to certain operators in [7]. Two types of geometrical conditions are required: the domain Ω has to be "c-ascending" for a certain positive c and "relief-functions" R_0 and R_1 appear as is the case for singularly perturbed differential equations. Indeed, to some extent, equation (4) can be seen as the discrete analog of the differential equation $\varepsilon y' = h(x,y)$, where $h(x,y) = f(x,y) - y$.

In the applications, these geometrical conditions are verified by sketching the level curves (in the complex domain) of the two relief functions; those can be directly calculated from the equation. For our example, the general framework of section 3 allows us to prove in section 4 the following conjecture of Jean-Louis Callot (1987).

Theorem 1.1 *Let x^* be determined by $x^* > 2$ and $\int_1^{x^*} \ln|2-x|dx = 0$. Put $g_0 :]1, x^*[\to \mathbb{R}, x \mapsto 1 - \frac{1}{x}$.*

For every $\delta > 0$, there exist $\varepsilon_0 > 0$ and a real analytic function $\varphi :]1+\delta, x^* - \delta[\times]0, \varepsilon_0[\to \mathbb{R},\ (x,\varepsilon) \mapsto \varphi(x,\varepsilon)$, solution of the difference equation

$$y(x+\varepsilon,\varepsilon) = xy(x,\varepsilon)(1 - y(x,\varepsilon)) \tag{5}$$

such that φ tends to g_0 as ε tend to 0, uniformly for x on $]1+\delta, x^*-\delta[$.

Numerically one finds $x^* \approx 5.65$. Using the preliminary results of the following section, our theorem implies the follwing consequence for the dynamics of the orbits.

Corollary 1.2 *Let (x_0, y_0) an initial condition where $x_0 \in [1,3[$ and $y_0 \in]0,1[$ and let $((x_n, y_n))_{n\in\mathbb{N}^*}$ the sequence defined by*

$$\left\{ \begin{array}{lcc} x_{n+1} & = & x_n + \varepsilon \\ y_{n+1} & = & x_n y_n (1 - y_n) \end{array} \right. . \tag{6}$$

If $x_0 \in [1,2]$ then the orbit of (6) starting at (x_0, y_0) follows the slow curve $y = g_0(x) = 1 - \frac{1}{x}$ from $x = x_0$ up to $x = x_s \in]3, +\infty]$ satisfying $x_s \geq l(x_0)$, where the function $l : [1,3[\to]3, x^*]$ is defined by $\int_x^{l(x)} \ln|2-\xi|d\xi = 0$.

If $x_0 \in]2,3[$ and if $y_0 \neq g_0(x_0)$, then the the exit abscissa $x = x_s$ is equal to $l(x_0)$.

2 Preliminaries

In subsection 2.1, we give precise statements for the scenario of static bifurcation presented in the introduction and justify them. This subsection is not essential for the proofs in the subsequent sections, but might help to better understand static and dynamic bifurcation.

In subsection 2.2, we present the already published definitions and results concerning the dynamic bifurcation of discrete dynamical systems. We reproduce these results not only for the sake of completeness, but also because the framework of the preceding publications had been non-standard analysis, which is not well known. We have therefore translated them into classical terms. We also include some ideas of the proofs; we refer to the cited references for complete proofs. One of the results had not been completely proved; we give a complete proof in appendix B. The idea of the proof is due to Jean-Louis Callot.

2.1 Static bifurcation

Let $f : \mathbb{R}^2 \to \mathbb{R}$ a mapping of class $\mathcal{C}^3$. In this subsection, we are interested in the bifurcations of the fixed points and 2-periodic points of the family of dynamical systems $(y_n)_{n\in\mathbb{N}}$ satisfying (1) which depend upon the parameter x.

$$y_{n+1} = f(x, y_n) \ .$$

By abuse of language, we call *fixed point of f* a point (x, y) satisfying $y = f(x, y)$. We suppose that f admits a bifurcation point $C = (x_c, y_c)$, i.e. a fixed point such that $|a| = 1$, where $a := \frac{\partial f}{\partial y}(x_c, y_c)$. The following well known proposition explains the hypotheses we make in the sequel. For a sketch of the proofs see appendix A.

Proposition 2.1 (a) (transcritical bifurcation) *Suppose $a = 1$.*
If there exists, in some neighborhood of C, a $\mathcal{C}^1$-curve $y = g_0(x)$ of fixed points of f passing through C, then f satisfies

$$\frac{\partial f}{\partial x}(C) = 0 \ , \quad \Delta := \left(\frac{\partial^2 f}{\partial x \partial y}(C)\right)^2 - \frac{\partial^2 f}{\partial x^2}(C)\frac{\partial^2 f}{\partial y^2}(C) \geq 0 \ . \tag{7}$$

Conversely, if f verifies (7) and the "generic" conditions $\Delta \neq 0$ and $\frac{\partial^2 f}{\partial y^2} \neq 0$, then there exists a neighborhood V of C such that the set of fixed points of f in V consists of two $\mathcal{C}^2$-curves $y = g_0(x)$ and $y = \tilde{g}_0(x)$, passing through C. Moreover, the derivatives $g_0'(x_c)$ and $\tilde{g}_0'(x_c)$ are equal to $\left(\frac{\partial^2 f}{\partial x \partial y}(C) \pm \sqrt{\Delta}\right) \Big/ \frac{\partial^2 f}{\partial y^2}(C)$.

(b) (period doubling bifurcation) *Suppose $a = -1$.*
Then there exists a neighborhood V of C such that the set of fixed points of f in V consists of one $\mathcal{C}^3$-curve $y = g_0(x)$, passing through C.

If f also satisfies the condition $\frac{\partial^2 f^2}{\partial x \partial y}(C) \neq 0$ where $f^2(x, y) := f(x, f(x, y))$, then there exists a neighborhood W of C such that the set of 2-periodic points of f consists of some $\mathcal{C}^2$-curve $x = p(y)$ passing through C. Moreover, the functions g_0 and p satisfy $g_0'(x_c) = \frac{1}{2}\frac{\partial f}{\partial x}(C)$, $p'(y_c) = 0$ and $p''(y_c) = -\frac{1}{3}\frac{\partial^3 f^2}{\partial y^3}(C) \Big/ \frac{\partial^2 f^2}{\partial x \partial y}(C)$.

Remark. In the non oscillatory case ($a = 1$), normally the generic bifurcation is the *saddle-node bifurcation*: If f satisfies $\frac{\partial f}{\partial x}(C) \neq 0$, then the implicit function theorem applied to the equation

$$h(x, y) := y - f(x, y) = 0 \tag{8}$$

proves the existence of a $\mathcal{C}^3$-curve $x = x(y)$ of fixed points of f in the neighborhood of C which has a vertical tangent at C (more precisely, the classical saddle-node bifurcation requires also the condition $\frac{\partial^2 f}{\partial y^2}(C) \neq 0$). We consider here a non-generic situation as we suppose the existence of some curve of fixed points that can be parametrized by x; this explains why the saddle-node bifurcation does not appear.

2.2 Dynamic bifurcation

Consider now the system (3):

$$\begin{cases} x_{n+1} &= x_n + \varepsilon \\ y_{n+1} &= f(x_n, y_n)\ . \end{cases}$$

where f is defined and $\mathcal{C}^3$ on $\mathbb{R}^2$, with values in $\mathbb{R}$. This strong global hypothesis is not crucial, but simplifies the presentation. As indicated in the introduction, we suppose that f has a period doubling bifurcation at the point $C = (x_c, y_c)$, i.e. we have

$$(H) f_y(C) = -1,\ f^2_{xy}(C) \neq 0 \text{ and } f^2_{yyy}(C) \neq 0\ ,$$

where the subscripts indicate partial derivatives and where $f^2 : (x, y) \mapsto f(x, f(x, y))$.

We also suppose that the curve of fixed points of f given by proposition 2.1, which will be called *slow curve* in the sequel, is attractive for $x < x_c$ and repulsive for $x > x_c$. In other words, we suppose that the function

$$a : x \mapsto f_y(x, g_0(x))$$

satisfies $a(x) > -1$ if $x < x_c$ and $a(x) < -1$ if $x > x_c$. To avoid technical difficulties, we suppose that $a'(x_c) = f_{xy}(C) > 0$. The partial derivatives of f^2 introduced above can be calculated in terms of f resulting in $f^2_{xy}(C) = -2f_{xy}(C) - f_x(C)f_{yy}(C)$ and $f^2_{yyy}(C) = -2f_{yyy}(C) - 3\,(f_{yy}(C))^2$.

We say that some orbit (i.e. solution) $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ of (3) *follows the slow curve* $y = g_0(x)$ *from the entry point* x_e *to the exit point* x_s if, for every $\delta > 0$ and $\rho > 0$, there exists ε_0 such that:

$$\forall n \in \mathbb{N},\ \forall \varepsilon \in]0, \varepsilon_0[, \quad (x_e + \delta \leq x_n(\varepsilon) \leq x_s - \delta\ \Rightarrow\ |y_n(\varepsilon) - g_0(x_n(\varepsilon))| < \rho)$$

and if the interval $[x_e, x_s]$ is maximal with this property.

The above definition concerns finite entry and exit points; it can easily be adapted to the cases $x_e = -\infty$ or $x_s = +\infty$.

Even though this is not necessary, we have distinguished the two numbers δ and ρ for readability. As for singularly perturbed differential equations, one could say that the orbit $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ has *boundary layers* near $x = x_e$ and $x = x_s$.

If the initial condition (x_0, y_0) satisfies $x_0 < x_c$ and if y_0 is in the domain of attraction of the fixed point $g_0(x_0)$ (for the static system) then, because of the attractiveness of the slow curve $g_0(x)$ for $x < x_c$, we obtain $x_e = x_0$ and $x_s \geq x_c$ (*cf* [4]). Thus, naturally, the orbit follows the slow curve at least on its attractive part.

We say that some orbit $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ *exhibits a bifurcation delay* if $x_s > x_c$; we say that system (3) *exhibits a bifurcation delay* if every orbit with initial point (x_0, y_0), where $x_0 < x_c$ and y_0 is in the domain of attraction of the fixed point $g_0(x_0)$, exhibits a bifurcation delay.

In the references [4, 5, 6] these orbits are called "discrete canards". We avoid the word "canard" which might indicate a certain volatility. In the present context the phenomenon of bifurcation delay is robust in some sense, at least under the hypothesis of analyticity.

We call *invariant curve* of (3) the graph of some solution (depending upon ε) of the associated difference equation (4), i.e.

$$\varphi_\varepsilon(x+\varepsilon) = f(x, \varphi_\varepsilon(x)) \ .$$

We say that the invariant curve $y = \varphi_\varepsilon(x)$ is *close* to the slow curve $y = g_0(x)$ on some compact interval I if $\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = g_0(x)$ uniformly on I. We say that the invariant curve is close to the slow curve on some open interval I if it is close on every compact sub-interval.

To simplify notation, we will not indicate the ε-dependence of an orbit of (3). Thus the notation $((x_n, y_n))_{n\in\mathbb{N}}$ replaces the notation $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ used previously. We will indicate, however, the ε-dependence of the invariant curves.

The following results have already been published. We give a complete proof of the first one in appendix B and ideas of proof for the other ones.

1. There are orbits exhibiting bifurcation delay [4].

2. If some orbit $((x_n, y_n))_{n\in\mathbb{N}}$ exhibits bifurcation delay and if $((x_n, \tilde{y}_n))_{n\in\mathbb{N}}$ is an orbit (having the same first coordinates x_n) with $\tilde{y}_0$ in the basin of attraction of $g_0(x_0)$ then $((x_n, \tilde{y}_n))_{n\in\mathbb{N}}$ also exhibits bifurcation delay.

   Moreover, if n is such that x_n is "properly" between x_e and x_s then the two points (x_n, y_n) and $(x_n, \tilde{y}_n)$ are exponentially close: $\forall\delta > 0\ \exists k, M > 0\ \forall n \in \mathbb{N}\ \forall\varepsilon \in]0, \varepsilon_0],\ (x_e + \delta \le x_n \le x_s - \delta \Rightarrow |y_n - \tilde{y}_n| \le M \exp(-k/\varepsilon))$.

3. System (3) exhibits bifurcation delay if and only if there exists an invariant curve close to the slow curve on some open interval containing x_c [4].

4. If there exists an invariant curve close to the slow curve on some open interval containing x_c then for fixed (i.e. ε-independent) $x_0 < x_c$ sufficiently close to x_c and fixed y_0 in the basin of attraction of $g_0(x_0)$, the orbit with initial point (x_0, y_0) exhibits bifurcation delay. More precisely, if $y_0 \neq g_0(x_0)$ and if the function a does not vanish in I then the orbit with initial point (x_0, y_0) follows the slow curve on the interval $]x_e, x_s[$, where $x_e = x_0$ and where $x_s > x_c$ is determined by the *entry-exit relation* $\int_{x_e}^{x_s} \ln|a(x)|dx = 0$ [4]. (Remark: The hypotheses on a made in the beginning of this section imply that x_s is unique. The point x_0 has to be chosen close enough to x_c to assure that $[x_e, x_s]$ is included in I.)
   Moreover, for n such that x_n is "properly" between x_e and x_s, the point (x_n, y_n) is exponentially close to the invariant curve.

5. Conversely, if $y = \varphi_\varepsilon(x)$ is an invariant curve close to the slow curve on some interval $[a, b]$, $a < x_c < b$ and if $(x_0, y_0(\varepsilon))$ is an initial point (with fixed x_0) exponentially close to the invariant curve (i.e. there exists $k > 0$ such that

$y_0(\varepsilon) - \varphi_\varepsilon(x_0) = \mathcal{O}\left(\exp\left(-k/\varepsilon\right)\right)$ then the orbit with initial point (x_0, y_0) follows the slow curve at least on the interval $[x_0, x]$ for every $x \in]x_0, b]$ satisfying

$$\int_{x_0}^{x} \ln|a(\xi)| d\xi < k \ . \tag{9}$$

Similarly, if $y = \psi_\varepsilon(x)$ is another invariant curve exponentially close to φ_ε on some interval $[x_0, x_0+\varepsilon[\subset [a, b]$ (*i.e.* $\exists k > 0\, \forall x \in [x_0, x_0+\varepsilon[\ ,\ |\psi_\varepsilon(x) - \varphi_\varepsilon(x)| = \mathcal{O}\left(\exp\left(-k/\varepsilon\right)\right)$) then the invariant curve $y = \psi_\varepsilon(x)$ remains close to the slow curve on the interval $[x_0, x]$ for every $x \in]x_0, b]$ satisfying (9).

6. If the function f is real analytic in a neighborhood of C then (3) exhibits bifurcation delay [5, 1].

Ideas of the proofs.

1. First the existence of some orbit with points close to the slow curve at some $c < x_c$ and some $d > x_c$ is proved. Then it is shown that this orbit remains close to the slow curve on the interval $[c, d]$; this is easily shown except near the critical point x_c. For details see appendix B.
2. Using the following change of variables, the exponential closeness of two orbits is expressed differently: $Z_n = \varepsilon \ln|y_n - \tilde{y}_n|$. This yields an equation of the form

$$Z_{n+1} = Z_n + \varepsilon \ln|a(x_n)| + \varepsilon P(x_n, y_n, \tilde{y}_n, \varepsilon)$$

where P is negligible compared to $\ln|a(x_n)|$ if y_n and $\tilde{y}_n$ are close to $g_0(x_n)$. It follows that $Z_n - Z_0$ is close to $\displaystyle\int_{x_0}^{x_n} \ln|a(x)| dx$. This also shows the exponential closeness of y_n and $\tilde{y}_n$.

The proofs of 3.,4. and 5. are analogous using $Z_n = \varepsilon \ln|y_n - \varphi_\varepsilon(x_n)|$ (where $y = \varphi_\varepsilon(x)$ parametrises the slow curve). Without giving any details, we mention that if the function a vanishes between x_e and x_s, the approximation $\displaystyle\int_{x_0}^{x_n} \ln|a(x)| dx$ of Z_n is not always valid and the entry-exit relation might be different. Note that these results remain valid for complex y.

6. Two different methods have been used independently. Both rely on the construction of some quasi-invariant curve, i.e. satisfying equation (4) except for exponentially small error terms. The first method [5] is an adaptation of a technique due to A.I. Neishtadt and consists in a sequence of changes of variable. The second method, due to Claude Baesens [1, 2], is a Gevrey analysis of the formal solution. Both results are local. The only previous global results concerned analytical systems that are linear non homogeneous with respect to y [6].

3 Analytic Solutions of difference equations

Below, we need the logarithm in the complex domain. For simplicity, we consider a simply connected domain Δ of $\mathbb{C}^*$ and the function Log is only defined on Δ. In

section 4, Δ will be chosen as $\mathbb{C} \setminus \mathbb{R}^-$ and $\mathbb{C} \setminus -i\mathbb{R}^+$.
For a subset A of $\mathbb{C}$, we denote by $\mathrm{Cl}(A)$ its closure; the notation $\overline{A}$ is used for the image of A by the complex conjugation.
For simplicity, we only consider *simply connected* domains in the sequel without mentioning this explicitly each time.
A path $\gamma : [0,1] \to \mathbb{C}$ is called *c-ascending*, $c > 0$, if, denoting $x_i = \gamma(\tau_i)$,

$$\forall \tau_1, \tau_2 \in [0,1] \quad (\tau_1 < \tau_2 \Rightarrow \mathrm{Im}\,(x_2 - x_1) \geq c|x_2 - x_1|) \ .$$

A domain B is called *c-ascending* if there exist points x^+, x^- in $\mathrm{Cl}(B)$, called *peeks of B*, where $\mathrm{Im}\,(x)$ is maximal (resp. minimal) and if the boundary of B consists of two c-ascending path from x^- to x^+.
To simplify formulas, we sometimes omit to indicate the ε-dependence.

We consider the following difference equation in the complex domain

$$y_\varepsilon(x + \varepsilon) = f(x, y_\varepsilon(x)) \tag{10}$$

where:

- the variable x varies in a *horizontally convex* domain $D \subset \mathbb{C}$, i.e. a domain satisfying $(x_1, x_2 \in D$ and $\mathrm{Im}\, x_1 = \mathrm{Im}\, x_2) \Rightarrow [x_1, x_2] \subset D$,
- the function $f : D \times \mathbb{C} \to \mathbb{C}$ is holomorphic,
- the letter ε denotes as usual a small positive parameter.

We suppose there exists an analytic function $g_0 : D \to \mathbb{C}$ verifying

$$f(x, g_0(x)) = g_0(x) \tag{11}$$

for all $x \in D$. We define $a(x) = \frac{\partial f}{\partial y}(x, g_0(x))$, and we suppose that, for $x \in D$, the values $a(x)$ are contained in the domain Δ introduced in the beginning of this section.

With some $c \in]0, 1/2]$, consider a c-ascending bounded domain Ω whose closure is contained in D such that a does not have the values 0 or 1 on $\mathrm{Cl}(\Omega)$. Choose another bounded sub-domain U of D such that $\mathrm{Cl}(\Omega) \subset U \subset \mathrm{Cl}(U) \subset D$. We can suppose that U is c-ascending (otherwise reduce the value of c and take an appropriate subdomain) and that a does not have the values 0 or 1 on $\mathrm{Cl}(U)$.

We denote by x^- and x^+ the peeks of Ω, i.e. the points of $\mathrm{Cl}(\Omega)$ such that $\forall x \in \Omega,\ \mathrm{Im}\, x^- < \mathrm{Im}\, x < \mathrm{Im}\, x^+$.

Finally we denote by R_0 and R_1 the *relief functions*, defined on D by

$$R_0 : x \mapsto \mathrm{Re}\,\left(\int_{x_0}^{x} \mathrm{Log}\, a(\xi) d\xi \right) \tag{12}$$

$$R_1 : x \mapsto R_0(x) - \mathrm{Re}\,(2\pi i(x - x_0)) = R_0(x) + 2\pi \mathrm{Im}\,(x - x_0)$$

where x_0 is some arbitrary point of Ω.

These relief functions were already introduced in [6]. Near the end of the present article, other possible choices can be found using more than two relief functions. This modification permits the treat similar problems where two reliefs are not enough, but this is beyond the scope of this article.

Theorem 3.1 *Suppose that for every $x \in \Omega$ there exist two c-ascending paths γ_x^- from x^- to x and γ_x^+ from x to x^+ such that R_0 is decreasing on γ_x^- and R_1 is increasing on γ_x^+.*

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ there exists an analytic solution $y_\varepsilon : \Omega \to \mathbb{C}$ of (10) tending to g_0 as $\varepsilon \to 0$ uniformly on Ω.

The second general result presented here concerns the exponential closeness of solutions of (10). Here, we only present this result in a situation symmetric with respect to the real axis; this is sufficient in our example. To simplify the statement, we suppose that the solutions of (10) are defined on some neighborhood of $\mathrm{Cl}(\Omega)$ and not only on Ω (this, on the other hand, would have allowed to combine both theorems).

Theorem 3.2 *Suppose that the hypotheses of theorem 3.1 are satisfied and that the functions f and g_0 have real values on the real axis. Suppose that y_1 and y_2 are two solutions of (10) defined in U, such that $y_j(x, \varepsilon) = g_0(x) + \mathcal{O}(\varepsilon)$ uniformly on U, $j = 1, 2$.*

Then we have $y_1(x) - y_2(x) = \mathcal{O}(\exp(-r/\varepsilon))$ uniformly on Ω, where

$$r := \min(R_0(x^-) - R_0(x), R_1(x^+) - R_1(x)) \ .$$

The rest of this section is devoted to the proof of the theorems. First we present two preliminary results.

Given two functions $\varphi, \psi : X \times]0, \varepsilon_0] \subset \mathbb{C} \to \mathbb{C}^*$, we say that φ *is of the exact order of ψ on X* if the two quotients φ/ψ and ψ/φ are bounded on $X \times]0, \varepsilon_0]$.

Lemme 3.3 *For any analytic function $A_\varepsilon : U \to \mathbb{C}$ satisfying $A_\varepsilon(x) = a(x) + \mathcal{O}(\varepsilon)$ uniformly on U, there is a function h_ε analytic in some neighborhood V of $\mathrm{Cl}(\Omega)$ of the exact order $\exp(R_0(x)/\varepsilon)$ on V solution of the homogeneous equation*

$$h_\varepsilon(x + \varepsilon) = A_\varepsilon(x) h_\varepsilon(x). \tag{13}$$

Proof. Since the closure of U is compact and contained in D, the closure of its a-image is a compact subset of Δ and hence for sufficiently small ε the function A_ε has values in Δ and $|a|$ is bounded below by some positive constant. Thus the two functions $\mathrm{Log}\, A_\varepsilon$ and $\mathrm{Log}\, a$ are single valued and analytic on U; moreover they satisfy $\mathrm{Log}\, A_\varepsilon = \mathrm{Log}\, a + \mathcal{O}(\varepsilon)$ uniformly on U.
We choose a neighborhood V of $\mathrm{Cl}(\Omega)$ such that its closure is a compact subset of U and which is horizontally convex. Theorem 11 page 82 of [7] yields the existence of a "sum" L_ε of $\mathrm{Log}\, A_\varepsilon$ on V, i.e. an analytic solution $L_\varepsilon : V \to \mathbb{C}$ of the equation

$$L_\varepsilon(x + \varepsilon) - L_\varepsilon(x) = \varepsilon \mathrm{Log}\, A_\varepsilon(x) \tag{14}$$

that is uniformly bounded on V with respect to ε. Moreover, by choosing L_ε such that $L_\varepsilon(x_0) = 0$, the following estimate follows (theorem 3 of [7]):

$$L_\varepsilon(x) = \int_{x_0}^{x} \mathrm{Log}\, a(\xi) d\xi + \mathcal{O}(\varepsilon) \ . \tag{15}$$

uniformly on every compact subset of V.

For the sake of completeness, we indicate briefly how to construct such a "sum". Choose X^+ and X^- in $\mathrm{Cl}(U)$ and $c \in]0,1/2]$ such that there exists a c-ascending domain with peeks X^- and X^+ containing $x-\varepsilon/2$ for every $x \in V$. Denoting the function $\operatorname{Log} A_\varepsilon$ by φ, put

$$\tilde{L}_\varepsilon(x) := \int_{X^-}^{X^+} \frac{\varphi(\xi)d\xi}{e_x(\xi)-1}$$

where the path of integration is ascending (i.e. the imaginary part is increasing along it) and intersects the segment $]x-\varepsilon, x[$ and where e_x is the function given by

$$e_x(\xi) = \exp\left(\frac{2\pi i}{\varepsilon}(\xi - x)\right) .$$

The function $\tilde{L}_\varepsilon$ satisfies (14) because of the residue theorem. To prove the estimate (15), choose a γ-ascending path for some small $\gamma > 0$ passing through $x-\varepsilon/2$ for ε sufficiently small and write $\tilde{L}_\varepsilon$ in the form

$$\tilde{L}_\varepsilon(x) = \int_{X^+}^{x-\varepsilon/2} \varphi(\xi)d\xi + \int_{X^-}^{x-\varepsilon/2} \frac{\varphi(\xi)d\xi}{e_x(\xi)-1} - \int_{x-\varepsilon/2}^{X^+} \frac{\varphi(\xi)d\xi}{e_x(\xi)^{-1}-1} .$$

This leads to the following estimate for x in V (*cf* [9], lemma 3.3)

$$\left| \tilde{L}_\varepsilon(x) - \int_{X^+}^{x-\varepsilon/2} \varphi(\xi)d\xi \right| \leq \frac{\varepsilon}{\pi\gamma^2} \sup_{x\in U} |\varphi(x)| .$$

For $x = x_0$, one hence has $\left|\tilde{L}_\varepsilon(x_0) - \int_{X^+}^{x_0} \varphi\right| = \mathcal{O}(\varepsilon)$. Now put $L_\varepsilon(x) = \tilde{L}_\varepsilon(x) - \tilde{L}_\varepsilon(x_0)$. Equation (14) remains valid for L_ε and the estimate (15) follows immediately. Therefore $\operatorname{Re} L_\varepsilon(x) = R_0(x) + \mathcal{O}(\varepsilon)$ uniformly on V. Finally we choose $h_\varepsilon = \exp(L_\varepsilon/\varepsilon)$. □

Consider now the compact set

$$\Omega_\varepsilon = \{x + \varepsilon t \;;\; x \in \mathrm{Cl}(\Omega),\; t \in [-1/2, 1/2]\} = \mathrm{Cl}(\Omega) + [-\varepsilon/2, \varepsilon/2] .$$

For $\varepsilon_0 > 0$ sufficiently small and for $\varepsilon \in]0, \varepsilon_0[$, Ω_ε is contained in the neighborhood V of the preceding lemma.

Consider some analytic function A_ε on U with $A_\varepsilon = a + \mathcal{O}(\varepsilon)$ uniformly on U and let h_ε be a function given by the preceding lemma.

Denote by $E = \mathcal{H}_b(\Omega_\varepsilon)$ the Banach space of functions holomorphic and bounded on Ω_ε, equipped with the maximum norm and construct a linear operator $T_\varepsilon : E \to E$ in the following way. For $g \in E$, define $T_\varepsilon g$ by

$$T_\varepsilon g(x) = 4 \int_{-\frac{1}{8}}^{\frac{1}{8}} I(g,t,x,\varepsilon)\, dt \tag{16}$$

with

$$I(g,t,x,\varepsilon) = \int_{x^-+\varepsilon t}^{x^++\varepsilon t} \frac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)} \frac{g(\xi)}{A_\varepsilon(\xi)}\, d\xi$$

where again the path of integration is ascending and intersects $]x-\varepsilon, x[$. This operator is constructed using the right inverse V_ε of Δ_ε introduced in [8] and using variation of constants for (17) (see below).

Lemme 3.4 *For $g \in E$, the function $z = T_\varepsilon g$ is a solution of the difference equation*

$$z(x+\varepsilon) = A_\varepsilon(x)z(x) + \varepsilon g(x) \quad (x, x+\varepsilon \in \Omega_\varepsilon)\ , \tag{17}$$

i.e. T_ε is a right inverse of the operator U_ε, $U_\varepsilon z(x) = \frac{1}{\varepsilon}\left(z(x+\varepsilon) - A_\varepsilon(x)z(x)\right)$ defined on a subset of E.
Moreover, $T_\varepsilon : E \to E$ is bounded uniformly with respect to ε.

Proof. The first statement is again an application of the residue theorem: Using $h_\varepsilon(x+\varepsilon) = A_\varepsilon(x)h(x)$, one finds for each $t \in [-1/8, 1/8]$:

$$I(g,t,x+\varepsilon,\varepsilon) - A_\varepsilon(x)I(g,t,x,\varepsilon) = 2\pi i \operatorname{Res}\left(\frac{h_\varepsilon(x+\varepsilon)g(\xi)}{(e_x(\xi)-1)\,h_\varepsilon(\xi+\varepsilon)}\,;\,\xi = x\right) = \varepsilon g(x)\ .$$

The important second statement of the lemma requires a rather technical proof involving appropriate integration paths, in the definition of $T_\varepsilon g$ in (16), such that the quantity $\dfrac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)}$ can be estimated on them. These integration paths are chosen close to the paths γ_x^+ and γ_x^- in the hypothesis of the theorem but modified such that they are not too close to the poles $x - k\varepsilon,\ k \in \mathbb{Z}$, of this quantity.

This is possible except if x is close to the upper or lower boundary of Ω_ε; if x is close to $x^- + \varepsilon t$ or $x^+ + \varepsilon t$ the pole at $\xi = x$ yields a logarithmic singularity for the integral from $x^- + \varepsilon t$ to $x^+ + \varepsilon t$. Precisely, it is proved in appendix C that (uniformly with respect to all t, x, ε under consideration)

- $I(g,t,x,\varepsilon) = \mathcal{O}(\|g\|)$ if $|x - x^- + \varepsilon t| \geq \varepsilon$ and $|x - x^+ + \varepsilon t| \geq \varepsilon$,
- $I(g,t,x,\varepsilon) = \mathcal{O}\left(\|g\| \ln\left(\frac{\varepsilon}{|x-x^-+\varepsilon t|}\right)\right) + \mathcal{O}(\|g\|)$ if $|x - x^- + \varepsilon t| < \varepsilon$
- and similarly $I(g,t,x,\varepsilon) = \mathcal{O}\left(\|g\| \ln\left(\frac{\varepsilon}{|x-x^++\varepsilon t|}\right)\right) + \mathcal{O}(\|g\|)$ if $|x - x^+ + \varepsilon t| < \varepsilon$.

Next, the logarithmic singularity is overcome by "averaging" with respect to t. □

Remark. Since A_ε is close to a which does not have the value 1, any bounded solution of (17) seems to be of order at most ε. This is true, except possibly on the boundary of Ω_ε, and T_ε has no reason to have a norm of order ε. However, if the assumptions of theorem 3.1 are strengthened by requiring that $\gamma_x^\pm$ are C^1, $(R_0 \circ \gamma_x^-)' \leq -\delta$ and $(R_1 \circ \gamma_x^+)' \geq \delta$, for some $\delta > 0$, then a modification of the above proof following the lines of [8], proof of lemma 3, would provide a norm of order ε for T_ε. We chose the weak assumptions in view of our application to theorem 1.1

Proof of theorem 3.1 It is known that (10) has a formal series solution $\hat{y}_\varepsilon = \sum_{n\geq 0} \varepsilon^n g_n$ where g_0 was defined at the beginning of this section, see (11). For the following two coefficients, one finds $g_1 = \frac{-g_0'}{1-a}$ and $g_2 = \frac{1}{1-a}\left(\frac{1}{2}bg_1^2 - \frac{1}{2}g_0'' - g_1'\right)$, with $b(x) = \frac{\partial^2 f}{\partial y^2}(x, g_0(x))$.

Put $y_{2\varepsilon} := g_0 + \varepsilon g_1 + \varepsilon^2 g_2$. One has $f(x, y_{2\varepsilon}(x)) - y_{2\varepsilon}(x+\varepsilon) = \varepsilon^3 c_\varepsilon(x)$ with c_ε analytic in U, uniformly bounded with respect to ε.

Put $A_\varepsilon(x) := \frac{\partial f}{\partial y}(x, y_{2\varepsilon}(x))$. One has $A_\varepsilon = a + \mathcal{O}(\varepsilon)$ uniformly on U. Taylor's formula yields $f(x, y_{2\varepsilon}(x) + u) = f(x, y_{2\varepsilon}(x)) + A_\varepsilon u + u^2 g_\varepsilon(x, u)$ with g_ε analytic in a neighborhood of $\Omega_\varepsilon \times \{0\}$, uniformly bounded with respect to ε. The substitution

$$y_\varepsilon = y_{2\varepsilon} + \varepsilon z_\varepsilon$$

gives the equation

$$z_\varepsilon(x+\varepsilon) = A_\varepsilon(x) z_\varepsilon(x) + \varepsilon^2 c_\varepsilon(x) + \varepsilon z_\varepsilon(x)^2 g_\varepsilon(x, \varepsilon z_\varepsilon(x)) \ . \tag{18}$$

In order to find a solution of (18), we consider a fixed point equation in the Banach space $E = \mathcal{H}_b(\Omega_\varepsilon)$ using the operator T_ε of lemma 3.4:

$$z_\varepsilon = \mathcal{F}_\varepsilon(z_\varepsilon) \qquad \text{with} \qquad \mathcal{F}_\varepsilon(z_\varepsilon) = T_\varepsilon(\varepsilon c_\varepsilon + z_\varepsilon^2 G_\varepsilon(z_\varepsilon)) \tag{19}$$

where $G_\varepsilon(z_\varepsilon)(x) := g_\varepsilon(x, z_\varepsilon(x))$. Obviously, a solution of (19) satisfies (18). Because of lemma 3.4, for sufficiently small ε, the operator $\mathcal{F}_\varepsilon$ is a contraction in some (small) neighborhood of 0 in E. Thus it has a (unique) fixed point z_ε in this neighborhood. This completes the proof of theorem 3.1 □

Proof of theorem 3.2 Put $y_\varepsilon = y_1 - y_2$. The function y_ε is a solution of the homogeneous equation

$$y_\varepsilon(x+\varepsilon) = A_\varepsilon(x) y_\varepsilon(x) \tag{20}$$

where $A_\varepsilon(x) = A(x, y_1(x), y_2(x))$ is given by $f(x, y_1) - f(x, y_2) = A(x, y_1, y_2)(y_1 - y_2)$. By the hypotheses on y_1 and y_2, the function A_ε satisfies $A_\varepsilon(x) = a(x) + \mathcal{O}(\varepsilon)$ uniformly on U. Applying lemma 3.3 (to the function $-a$ instead of a), one obtains that there exists a function h_ε, analytic in Ω, solution of the homogeneous equation

$$h_\varepsilon(x+\varepsilon) = -A_\varepsilon(x) h_\varepsilon(x) \tag{21}$$

of order $\exp\left(\frac{1}{\varepsilon}(R(x) - R(x^-))\right)$, where R denotes the function

$$R(x) = \frac{1}{2}(R_0(x) + R_1(x)) = R_0(x) + \pi \operatorname{Im}(x - x_0) \ .$$

Thus $z_\varepsilon = y_\varepsilon / h_\varepsilon$ is analytic on a neighborhood of Ω and satisfies

$$z_\varepsilon(x+\varepsilon) = -z_\varepsilon(x) \ .$$

Hence z_ε is periodic with period 2ε. Moreover, $z_\varepsilon(x)$ is bounded for $x = x^+ + \mathcal{O}(\varepsilon)$ and for $x = x^- + \mathcal{O}(\varepsilon)$. Thus (*cf* formula (3.12) of [9] and the proof following it) we can conclude that for all x in Ω:

$$z_\varepsilon(x) - z_\varepsilon(\tilde{x}) = \mathcal{O}\left(\exp\left(-\frac{\pi}{\varepsilon}(\operatorname{Im} x^+ - |\operatorname{Im} x|)\right)\right) \ ,$$

where $\tilde{x}$ is some point of $\Omega \cap \mathbb{R}$. Applying this estimate to $x = \tilde{x} + \varepsilon$ and using $z_\varepsilon(\tilde{x} + \varepsilon) = -z_\varepsilon(\tilde{x})$ we obtain $z_\varepsilon(\tilde{x}) = \mathcal{O}(\exp(-\pi \operatorname{Im} x^+ / \varepsilon))$. Thus $y_\varepsilon(x) = \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}(R(x) - R(x^-)) - \frac{\pi}{\varepsilon}(\operatorname{Im} x^+ - |\operatorname{Im} x|)\right)\right)$.

The remark $R(x^-) - R(x) + \pi(\operatorname{Im} x^+ - |\operatorname{Im} x|) = \min(R_0(x^-) - R_0(x), R_1(x^+) - R_1(x)) = r$ completes the proof. □

4 Proof of theorem 1.1

The idea is to construct (using theorem 3.1) two solutions of (5) close to $1-\frac{1}{x}$ on two domains not containing the point 2, then to show using theorem 3.2 that these solutions are exponentially close and finally to deduce the existence of a solution of (10) close to $1-\frac{1}{x}$ on some domain containing the line segment $]1+\delta, x^*-\delta[$ of the theorem. Observe that theorem 3.1 only allows to prove the existence of invariant curves of (4) close to the slow curve on intervals *not* containing 2 and that theorem 3.2 yields the additional necessary ingredients.

In the sequel, the independent variable will be denoted by z and its real and imaginary part will be denoted by x and y.

For the construction of the first solution, we choose as domain D_1 the complex plane cut along the ray $[2,+\infty[$; this allows to choose $\Delta = \mathbb{C} \setminus \mathbb{R}^-$ as domain of the logarithm (here we have $a(z) = 2-z$).

The point z_0 (replacing x_0) is chosen real, e.g. $z_0 = \frac{3}{2}$. For convenience, we add a constant to the function R_0 given by (12) such that it vanishes at the point 1.

Thus the relief function R_0 is given by:

$$R_0(z) = \mathrm{Re}\left(\int_1^z \mathrm{Log}\,(2-\zeta)d\zeta\right) = \mathrm{Re}\,\left((z-2)\mathrm{Log}\,(2-z) - z + 1\right)\ .$$

As this relief is symmetric with respect to the real axis, we only describe it in the

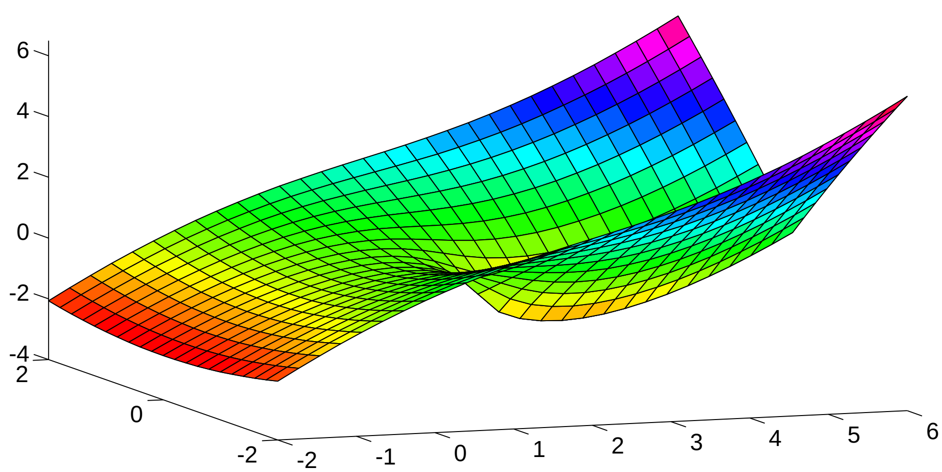

Figure 2: Perspective view of R_0.

lower half plane. The relief has a saddle point at $z=1$.

For $k \in \mathbb{R}$, $k \geq -2$, denote by S_k the level curve k in the quadrant $Q = \{z = x+iy \in \mathbb{C}\ ;\ x \geq 1 \ \text{ and } \ y \leq 0\}$. Thus S_0 is the separatrix of the saddle point

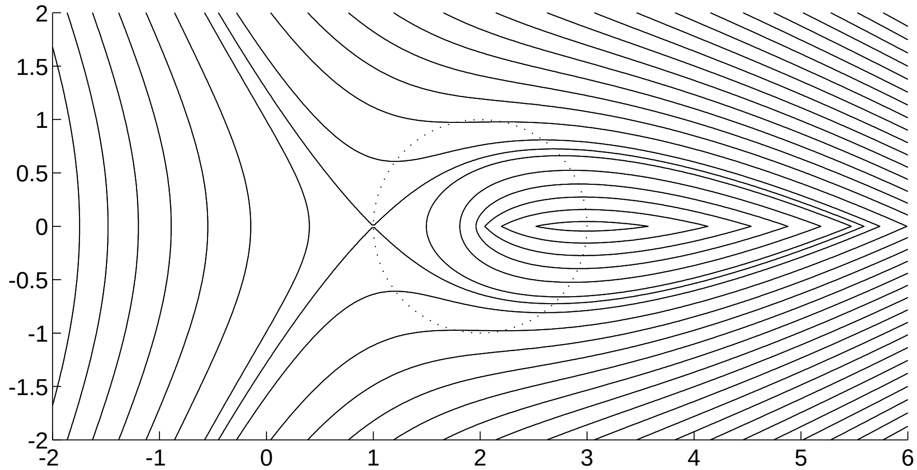

Figure 3: Level curves of R_0.

connecting the points 1 and x^* (in the lower half plane). We denote by B the disk with center 2 and radius 1 and by C its boundary.

On a vertical ray parametrized by $z(t) = x + it,\ t \leq 0$ (where $x \geq 1$ is fixed) one has

$$\frac{d}{dt} R_0(z(t)) = -\mathrm{Im}\,(\ln|2 - z(t)| + i \arg(2 - z(t))) = -\arg(2 - z(t)) \leq 0\ ,$$

hence $R_0(z)$ decreases whenever the imaginary part y increases while the real part x remains constant.

On a horizontal line $z(t) = iy + t$, we find $\frac{d}{dt} R_0(z(t)) = \ln|2 - z(t)|$, hence $R_0(z)$ decreases as x increases if and only if z is in B.

As a consequence, in Q every level curve S_k is the graph $y = f_k(x)$ of some function f_k that is decreasing when the point $x + if_k(x)$ is in the interior of the disk B and increasing otherwise.

We denote by $Z_0 = x_0 + iy_0$ the intersection of S_0 and C other than 1. Let $a_k \in [1, 3]$ and $b_k \in [3, +\infty[$ such that $R_0(a_k) = R_0(b_k) = k$ if $k \in]-2, 0]$, and $a_k = 1,\ R_0(b_k) = k$ otherwise. Finally let $x'_k \in]1, 2[$ and $x_k \in [2, 3]$ be the intersection points of S_k and C (x'_k only exists for $0 < k < R_0(2 - i) = \frac{\pi}{2} - 1$, x_k only for $-2 \leq k < \frac{\pi}{2} - 1$), determined by $|x'_k - 2 + if_k(x'_k)| = |x_k - 2 + if_k(x_k)| = 1$. Then we can give the following description of the relief:

- If $-2 \leq k \leq 0$ then the function f_k is strictly decreasing on the interval $[a_k, x_k]$ and strictly increasing on $[x_k, b_k]$.

- If $0 < k < R_0(2 - i) = \frac{\pi}{2} - 1$ then the function f_k is strictly increasing on $[1, x'_k]$ and $[x_k, b_k]$ but strictly decreasing on $[x'_k, x_k]$.

- If $k \geq \frac{\pi}{2} - 1$ then the function f_k is strictly inceasing on $[1, b_k]$.

We first construct a solution on some domain Ω_1 satisfying the conditions of theorem 3.1 that is close to the symmetric domain whose portion below the real axis is enclosed

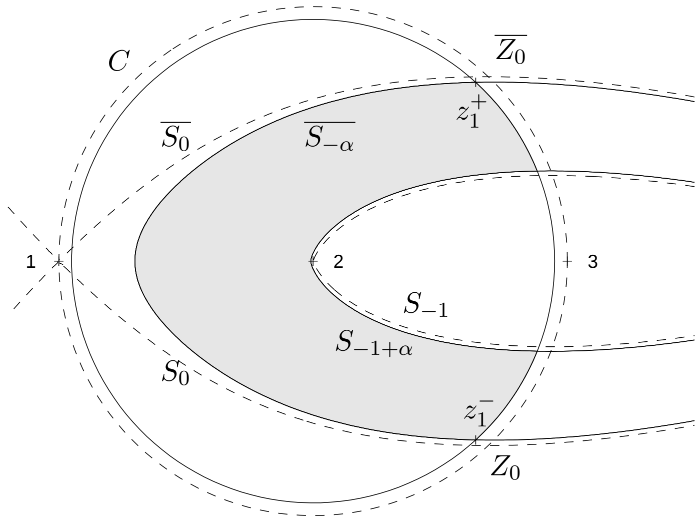

Figure 4: The domain Ω_1 for $\delta = \frac{1}{20}$.

by the two level curves S_0 and S_{-1} (i.e. containing the points 1 and 2) and the portion of C connecting them.

On the one hand, 1 and 2 should not be boundary points of Ω_1. Thus we fix $\alpha > 0$ arbitrarily small and we choose as boundary curves of Ω_1 the level curves $S_{-\alpha}$ and $S_{-1+\alpha}$. On the other hand, it must be possible to choose a path γ_z^- descending the relief R_0 and c-ascending for a certain $c > 0$; hence the level curves may not have horizontal tangents. Therefore we complete the lower boundary of Ω_1 by a circular arc with center 2 and radius $1 - \alpha$ connecting $S_{-\alpha}$ — at its "lowest" point denoted z_1^- — and $S_{-1+\alpha}$. Ω_1 is completed by symmetry in the upper half plane.

Given $z \in \Omega_1$, we choose as γ_z^- the path consisting of the circular arc of radius $1-\alpha$ connecting z_1^- to the level curve of R_0 passing through z (the endpoint of this arc is denoted be $c^-(z)$ and satisfies $\operatorname{Im} c^-(z) < \operatorname{Im} z$), and the portion of the level curve between $c^-(z)$ and z. As γ_z^+, we choose as path the portion of the level curve of $R_0(\zeta) = R_0(z)$ $(\operatorname{Im}\zeta > \operatorname{Im} z)$ connecting z and the point $c^+(z) := \overline{c^-(z)}$, combined with the circular arc connecting $c^+(z)$ and $z_1^+ := \overline{z_1^-}$.

As the functions Im and R_0 are increasing on γ_z^+ and $R_1(\zeta) = R_0(\zeta) + 2\pi \operatorname{Im}\zeta$, the function R_1 is also increasing on γ_z^+. Thus theorem 3.1 can be applied to Ω_1 and thus, we obtain our first solution $y_1 = y_{1,\varepsilon}$.

Moreover because the slow curve $1 - \frac{1}{z}$ is attractive on the disk $|z-2| < 1$, the solution y_1 can be continued (analytically) to the part of the disk

$$B_\alpha := \{z \in \mathbb{C}\ ;\ |z-2| < 1-\alpha\}$$

on the right of Ω_1 and remains close to the slow curve on this region.

Let us now construct a solution y_2 on some domain Ω_2 close to the domain enclosed by the line segment $[Z_0, \overline{Z_0}]$ and the portions of S_0 and $\overline{S_0}$ connecting Z_0 (resp. $\overline{Z_0}$) to x^*.

We choose as domain D_2 the halfplane $\operatorname{Re} z > 2$ and $\Delta = \mathbb{C} \setminus i\mathbb{R}^-$. In other words, the cut used in the definition of R_0 has been turned by $\pi/2$ and is now the ray $2 + i\mathbb{R}^+$. (Equivalently, one could define the two reliefs by $R_0(z) = (z-2)\mathrm{Log}\,(z-2) - z + 1 - \pi \operatorname{Im} z$ and $R_1(z) = (z-2)\mathrm{Log}\,(z-2) - z + 1 + \pi \operatorname{Im} z$ on $\mathbb{C}\setminus]-\infty, 2]$. One could also regard R_0 as the continuation of the preceding relief R_0 "below" 2 and R_1 as the continuation of the same preceding relief R_0 "above" $z = 2$.)

The two reliefs R_1 and R_0 are now symmetric.

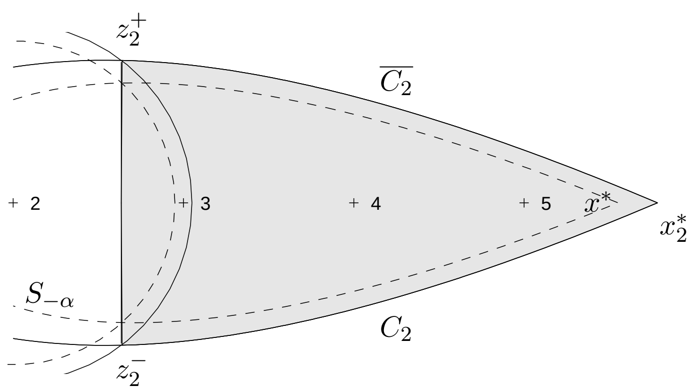

Figure 5: Le domaine Ω_2.

As before, it easy to verify that $R_0(z)$ decreases when $\operatorname{Im} z$ increases while the real part of z remains constant.

Consider the intersection point z_2^- of the circle with center 2 and radius $1 + \alpha$ and the vertical ray $z_1^- - i\mathbb{R}^+$. Let C_2 be the portion of the level curve of R_0 connecting z_2^- to the real axis, meeting it in a point $x_2^* > x^*$, x_2^* close to x^* verifying $R_0(x_2^*) = R_0(z_1)$. For $c > 0$ sufficiently small, C_2 is a c-ascending path.

We choose as Ω_2 the symmetric domain bounded by the line segment $[z_2^-, z_2^+]$, where $z_2^+ = \overline{z_2^-}$, and by the two curves C_2 and $\overline{C_2}$.

For each z in Ω_2, we choose as γ_z^- the union of the vertical line segment below z connecting z to C_2 and of the portion of C_2 connecting z_2^- to this segment . As γ_z^+ we choose the union of the vertical line segment above z connecting z to $\overline{C_2}$ and of the portion of $\overline{C_2}$ connecting z_2^+ to this segment. The preceding discussion shows that these paths are c-ascending and that R_0 is decreasing on the first, R_1 increasing on the latter.

Thus, Ω_2 satisfies the conditions of theorem 3.1 and hence there exists a solution y_2 of (10) on Ω_2.

Now we apply theorem 3.2 to the above solutions y_1 and y_2 where $U = \Omega_1 \cap \Omega_2$ is the portion of B_α right of z_1^- i.e.

$$U = \{z \in \mathbb{C}\ ;\ |z-2| < 1-\alpha \ \text{ and } \ \mathrm{Re}\,(z - z_1^-) > 0\} \quad .$$

For sufficiently small α and for z between $3-\alpha-\varepsilon$ and $3-\alpha$ we obtain:

$$y_1(\varepsilon, z) - y_2(\varepsilon, z) = \mathcal{O}\left(\exp\left(\tfrac{1}{\varepsilon}\left(R_0(3-\alpha) - R_0(z_1^-)\right)\right)\right) \quad .$$

Hence because of $R_0(3-\alpha) = R_0(3) + \mathcal{O}(\alpha^2) = -2 + \mathcal{O}(\alpha^2)$ we find:

$$y_1(\varepsilon, z) - y_2(\varepsilon, z) = \mathcal{O}(\exp((-2+\alpha)/\varepsilon)) \ .$$

Because of the preliminary result 5. of section 2.2, it follows that the solution y_1 remains close to $1 - \frac{1}{z}$ on $\Omega_2 \cap [3, x_\alpha^*]$ where x_α^* is the point on $]3, x^*[$ verifying $R_0(x_\alpha^*) = -2 + \alpha$, hence also on $[1+\delta, x^* - \delta]$ if one chooses α sufficiently small. This completes the proof of theorem 1.1.

5 Remark

The two reliefs are symmetric on Ω_2, i.e. the region where the real dynamics exhibit oscillations, but the asymmetry on Ω_1 is not very nice. It is possible to make the situation more symmetric by considering three reliefs instead of two on Ω_1.

More generally, reconsider the above notation R_0, where Log denotes any branch of the logarithm. For any integers $m \leq 0$ and $n \geq 1$ and for $j = m, ..., n$, put $R_j(x) = R_0(x) + 2j\pi\,\mathrm{Im}\,x$.

Then the conclusion of theorem 3.1 remains valid if we replace its hypothesis by the following:

For each $j = m, ..., n$ there exists $x_j \in \mathrm{Cl}(\Omega)$ such that for all $x \in \Omega$ there is some path $\gamma_{x,j}$ connecting x_j to x on which R_j is decreasing; for $j = m$ (resp. n) the path $\gamma_{x,m}$ (resp. $\gamma_{x,n}$) can be chosen such that it is additionally c-ascending (resp. c-descending).

Indeed, we can construct as in lemma 3.3 the solutions h_j of the homogeneous equation $h_\varepsilon(x+\varepsilon) = a(x) h_\varepsilon(x)$ which are of the order $\exp(R_j/\varepsilon)$, and with $H_j(x,\xi) := \frac{h_j(x)}{h_j(\xi)}$, we choose as operator T_ε solving $y(x+\varepsilon) = a(x)y(x) + \varepsilon g(x)$ the operator $T_\varepsilon = U + I_m + ... + I_{n-1}$ given by

$$I_j g(x) = \int_{x_j}^{x-\frac{\varepsilon}{2}} H_j(x,\xi)\ \frac{g(\xi)}{a(\xi)}\ d\xi \ ,$$

$$Ug(x) = 4\int_{-\frac{1}{8}}^{\frac{1}{8}} \left(\int_{x_m+\varepsilon t}^{x-\frac{\varepsilon}{2}} \frac{H_m(x,\xi)}{e_x(\xi)-1}\ \frac{g(\xi)}{a(\xi)}\ d\xi - \int_{x_n+\varepsilon t}^{x-\frac{\varepsilon}{2}} \frac{H_n(x,\xi)}{e_x(\xi)-1}\ \frac{g(\xi)}{a(\xi)}\ d\xi \right) dt \ .$$

Concerning the exponential closeness of two solutions, it is also possible to replace the hypothesis of theorem 3.2 by the above hypothesis. One obtains the same conclusion as in 3.2, where $r = \min\limits_{j=m,...,n}(R_j(x_j) - R_j(x))$.

A Proof of proposition 2.1

To simplify notation, we indicate partial derivatives as subscripts.

(a) Differentiation of (8) with respect to x where $y = g_0(x)$ yields $g_0'(x) - f_x(x, g_0(x)) - f_y(x, g_0(x))\, g_0'(x) = 0$, hence $f_x(C) = 0$. Therefore the point C is a critical point of the function h, its Hessian in C is $-\Delta$. If Δ were negative, then C would have to be a strict local extremum of h, contradicting the fact that C is not an isolated fixed point.

Conversely, if $\Delta \neq 0$ then C is a non degenerate critical point of h. By conjugating h to its quadratic part using the Morse lemma, we obtain two $\mathcal{C}^2$-curves of fixed points passing through C and the values of the derivatives are easily determined from the quadratic part.

We mention briefly that in the case where one of the curves has a vertical tangent in C (i.e. $f_{yy}(C) = 0$) and if $f_{yyy}(C) \neq 0$, a pitchfork bifurcation appears: the curve under consideration is – in the neighborhood of C – either in the left or in the right halfplane, according to the sign of $f_{xy}(C) f_{yyy}(C)$.

(b) In the oscillatory case $a = -1$, the implicit function theorem applied to equation (8) assures that f has a unique $\mathcal{C}^3$-curve of fixed points $y = g_0(x)$ in some neighborhood of C. The 2-periodic points (together with the fixed points) of f are, of course, the fixed points of $f^2 : (x, y) \mapsto f(x, f(x, y))$.

If $f^2_{xy}(C) \neq 0$ then the function f^2 has a pitchfork-bifurcation in the point C (this means that automatically $f^2_x(C) = f^2_{yy}(C) = 0$). To prove this, one can:

- observe that the curve $y = g_0(x)$ contains all the fixed points of f and hence the other fixed points of f^2 appear in pairs, thus excluding the saddle-node and the transcritical bifurcation for f^2.
- or simply calculate $f^2_x(C)$ and $f^2_{yy}(C)$:
  $f^2_x(x, y) = f_x(x, f(x, y)) + f_y(x, f(x, y))\, f_x(x, y)$ hence $f^2_x(C) = 0$, and
  $f^2_{yy}(x, y) = f_{yy}(x, f(x, y))\, f_y(x, y)^2 + f_y(x, f(x, y))\, f_{yy}(x, y)$, hence $f^2_{yy}(C) = 0$.

The value $g_0'(x_c)$ is found by differentiation of (8) with respect to x using $y = g_0(x)$. In the same way the derivatives of p are calculated by differentiating $f(p(y), f(p(y), y)) = y$. Of course one finds $p'(y_c) = 0$ because f^2 has a pitchfork-bifurcation in C. For the second derivative of p, first differentiate $f^2(p(y), y) = y$: $f^2_x(p(y), y)p'(y) + f^2_y(p(y), y) = 1$, thus

$$f^2_{xx}(p(y), y)p'(y)^2 + 2f^2_{xy}(p(y), y)p'(y) + f^2_{yy}(p(y), y) + f^2_x(p(y), y)p''(y) = 0$$

(this yields no information in the point C) and finally

$$f^2_{xxx}p'^3 + 3f^2_{xxy}p'^2 + 3f^2_{xyy}p' + f^2_{yyy} + 3(f^2_{xx}p' + f^2_{xy})p'' + f^2_x p''' = 0\ ,$$

where the partial derivatives of f^2 are taken at $(p(y), y)$ and the derivatives of p at the point y. At the point C, using $f^2_x = f^2_{yy} = 0$ and $p'(y_c) = 0$, we obtain $3f^2_{xy}(C)p''(y_c) + f^2_{yyy}(C) = 0$, which yields the value of $p''(y_c)$.

In the particular case $g_0 = 0$, more details can be found in [15], 357-374, and in [12] theorem 3.21, p.88.

B Proof of statement 1, subsection 2.2

We choose c, d with $c < x_c < d$ such that $|a(x)| < 1$ if $c \le x < x_c$ and $a(x) < -1$ if $x_c < x \le d$. We first construct a finite orbit $\{(\tilde{x}_n, \tilde{y}_n)\ ;\ 0 \le n \le N(\varepsilon)\}$ of (3) the boundary points of which – having x-coordinates close to c and d – are close to the slow curve. We will then show that this orbit remains always close to the slow curve. Let us first prove:

Statement A: *For sufficiently small $r > 0$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ there exists a finite orbit $(\tilde{x}_n, \tilde{y}_n)_{0 \le n \le N = N(\varepsilon)}$ of (3) having the following properties:*

(22) $\quad c \le \tilde{x}_0 < c + \varepsilon,\ d - \varepsilon < \tilde{x}_N \le d,\ |\tilde{y}_0 - g_0(\tilde{x}_0)| \le r,\ |\tilde{y}_N - g_0(\tilde{x}_N)| \le r\ .$

To show this, we consider the compact tubular neighborhood

$$K_r := \{(x, y) \in \mathbb{R}^2\ ;\ a \le x \le b,\ |y - g_0(x)| \le r\}$$

of the slow curve. For $r, \delta > 0$ sufficiently small, one has $\frac{\partial f}{\partial y}(x, y) < 0$ for all (x, y) in K_r with $x \ge x_c - \delta$. Consider the mapping

$$F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x + \varepsilon, f(x, y))\ .$$

First, we need to discuss whether some point (x, y) and its image $F_\varepsilon(x, y)$ are on the same side of the slow curve $y = g_0(x)$ or not. Thus, we compare the differences $y - g_0(x)$ and $f(x, y) - g_0(x + \varepsilon)$ using the formula

$$f(x, y) - g_0(x + \varepsilon) = \frac{\partial f}{\partial y}(x, \eta)\,(y - g_0(x)) - \varepsilon g_0'(\xi)$$

where η is between y and $g_0(x)$, ξ between x and $x + \varepsilon$.

It follows that there exist $\rho > 0$, $\varepsilon_0 > 0$ such that for all $(x, y) \in K_r$ with $x \ge x_c - \delta$ and all $\varepsilon \in]\,0, \varepsilon_0[$ the following property holds: if $(x, y) \in K_\rho$ then $F_\varepsilon(x, y) \in K_r$; otherwise the signs of $f(x, y) - g_0(x + \varepsilon)$ and $y - g_0(x)$ are different.

This means that if (x, y) is some point of K_r with $x > x_c - \delta$ then its F_ε-image $(x + \varepsilon, f(x, y))$ is also in K_r or on the other side of the slow curve (Note that the F_ε-image of a point of K_r is not necessarily again in K_r).

Now we choose $x_0 = c$ and $y_0 = g_0(x_0)$. Denote $m_0 = (x_0, y_0)$ and $m_n = m_n(\varepsilon) = (x_n, y_n)$ ($n \in \mathbb{N}$ such that $x_0 + n\varepsilon \le d + \varepsilon$) the finite orbit of (3) with initial point m_0 – these are the iterates of m_0 by F_ε. We construct an invariant curve by iterating F_ε on all points of the segment $[m_0, m_1[$. This curve is close to the slow curve on $[x_0, x_c[$; we do not know anything, however, on its behavior for $x \ge x_c$.

We will show by induction on n that for all $n \ge 0$ with $x_0 + n\varepsilon \le d$ there exists a point $p_n(\varepsilon)$ in K_r on the invariant curve whose x-coordinate is between that of m_n and m_{n+1}. This is already true for n such that $x_0 + n\varepsilon < x_c - \delta$, where $\delta > 0$ sufficiently small is fixed and independent of ε.

Suppose the statement is true for $n - 1$. If p_{n-1} is even in K_ρ then its iterate by F_ε is in K_r proving the statement for n . Otherwise this iterate is on the other side of the slow curve.

If p_{n-1} and m_n are on different sides then the invariant curve intersects the slow curve in some point q_{n-1} between p_{n-1} and m_n and thus its image $p_n := F_\varepsilon(q_{n-1})$ proves the statement for n.

If p_{n-1} and m_n are on the same side of the slow curve then m_n and $F_\varepsilon(p_{n-1})$ are not on the same side and hence the invariant curve contains at least one point $p_n \in K_r$ between m_n and $F_\varepsilon(p_{n-1})$. Thus the above statement is proved.

We can now use the above statement for $N = N(\varepsilon)$ such that $d-\varepsilon < x_0+N\varepsilon \leq d$. The orbit $(\tilde{m}_n(\varepsilon))_{n\in\mathbb{N}} = ((\tilde{x}_n, \tilde{y}_n))_{n\in\mathbb{N}}$ — with $\tilde{y}_n = \varphi_\varepsilon(\tilde{x}_n)$ — containing p_N (i.e. $\tilde{m}_N := p_N \in K_r$) satisfies statement A.

It remains to be shown that this orbit exhibits a bifurcation delay, more precisely:

$$(23) \qquad \forall r > 0 \; \exists \varepsilon_0 > 0 \; \forall \varepsilon \in]0, \varepsilon_0] \; \forall n \in \{0, 1, ..., N(\varepsilon)\}, \quad |\tilde{y}_n - g_0(\tilde{x}_n)| \leq r \;.$$

We treat only the case $f^2_{yyy}(C) < 0$ (the case $f^2_{yyy}(C) > 0$ can be treated analogously.) This means that the curve of 2-periodic points is attractive, to the right of C and of nonzero curvature at C; furthermore C is attractive.

As the slow curve is attractive on $[c, x_c[$ and repulsive on $]x_c, d]$, (23) is true for integers n such that $\tilde{x}_n$ is not close to x_c. It remains to be shown for n such that $\tilde{x}_n = x_c + o(1)$.

Consider the subset of K_r defined by

$$L_r := \{(x, y) \in \mathbb{R}^2 \; ; \; |x - x_c| \leq r^3, \; |y - g_0(x)| \leq r\} \;.$$

The term r^3 has been chosen to assure that the curve of the 2-periodic points and the upper and lower boundaries of L_r do not intersect for sufficiently small r; these boundaries are therefore in the attractive region of f.

Statement B : *For sufficiently small r there exists ε_0 such that for every positive $\varepsilon < \varepsilon_0$ the image $(x', y') = F^2_\varepsilon(x, y)$ of some point $(x, y) \in L_r$ satisfies $|y' - g_0(x')| \leq r$.*

Let $M = (x, y)$ a point of L_r, $C_x := (x, g_0(x))$ and $y = g_0(x)+Y$ with $-r \leq Y \leq r$. By definition of L_r and the properties of f^2 at C (pitchfork bifurcation), one finds that $f^2_y(C_x) = 1+\mathcal{O}(r^3)$ and $f^2_{yy}(C_x) = \mathcal{O}(r^3)$. Taylor's formula yields (with a certain point $\tilde{C}_x$ in L_r)

$$\begin{aligned} f^2(M) &= f^2(C_x) + Y f^2_y(C_x) + \tfrac{1}{2} Y^2 f^2_{yy}(C_x) + \tfrac{1}{6} Y^3 f^2_{yyy}(\tilde{C}_x) \\ &= g_0(x) + Y + \tfrac{1}{6} Y^3 f^2_{yyy}(C_x) + \mathcal{O}(r^4) \end{aligned}$$

uniformly on L_r.

Thus for sufficiently small r, the F^2_0-image $(x', y') = F^2_0(x, y)$ of (x, y) satisfies $|y' - g_0(x')| \leq r - \alpha r^3$ with $\alpha = -f^2_{yyy}(C)/12$, say. Statement B then follows from the continuity of F_ε with respect to ε.

Now, choose $r > 0$ such that statements A and B are true. As the slow curve is attractive on $[x_0, x_c - r^3/2]$, we have $|\tilde{y}_n - g_0(\tilde{x}_n)| \leq r$ for ε sufficiently small and for n such that $\tilde{x}_n \in [c, x_c - r^3 + 2\varepsilon]$. Statement B yields the same estimate for $\tilde{x}_n \in]x_c - r^3 + 2\varepsilon, x_c + r^3 + 2\varepsilon]$ (by considering the odd and even indices separately). As the slow curve is repulsive on $[x_c + r^3, d]$ the estimate also follows for $\tilde{x}_n \in]x_c + r^3 + 2\varepsilon, d]$. This proves (23) for sufficiently small r and hence for all r.

C Majorization of the integral in the proof of lemma 3.4

For simplicity, we denote $I(g,t,x,\varepsilon)$ by $I_t(x)$. Recall that $I_t(x)$ is given by

$$I_t(x) := \int_{x^-+\varepsilon t}^{x^++\varepsilon t} \frac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)} \frac{g(\xi)}{A_\varepsilon(\xi)}\, d\xi$$

where $h_\varepsilon(x)$ is of the exact order $\exp(R_0(x)/\varepsilon)$ (*cf.* lemma 3.3). We have to prove (uniformly with respect to all t, x, ε under consideration):

- $I_t(x) = \mathcal{O}(\|g\|)$ if $|x - x^- + \varepsilon t| \geq \varepsilon$ and $|x - x^+ + \varepsilon t| \geq \varepsilon$,
- $I_t(x) = \mathcal{O}\left(\|g\| \ln\left(\frac{\varepsilon}{|x-x^-+\varepsilon t|}\right)\right) + \mathcal{O}(\|g\|)$ if $|x - x^- + \varepsilon t| < \varepsilon$
- and similarly $I_t(x) = \mathcal{O}\left(\|g\| \ln\left(\frac{\varepsilon}{|x-x^++\varepsilon t|}\right)\right) + \mathcal{O}(\|g\|)$ if $|x - x^+ + \varepsilon t| < \varepsilon$.

Since $A_\varepsilon(\xi)^{-1}$ is bounded (uniformly in ε) on the whole domain Ω_ε, this amounts to estimating $\dfrac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)}$.

Without loss in generality we can assume that $x - \frac{\varepsilon}{2} \in \Omega_\varepsilon$. Otherwise, $(x+\varepsilon) - \frac{\varepsilon}{2} = x + \frac{\varepsilon}{2} \in \Omega_\varepsilon$ and the estimate for $I_t(x+\varepsilon)$ together with $I_t(x) = \frac{1}{A_\varepsilon(x)}(I_t(x+\varepsilon) - \varepsilon g(x))$ yields the estimate for $I_t(x)$.

As paths of integration defining $I_t(x)$, we choose paths depending on ε and having at most a distance of order ε to the paths γ_x^+ and γ_x^- of theorem 3.1 and passing through $x - \frac{\varepsilon}{2}$. To simplify notation, we denote these paths again γ_x^+ and γ_x^-. Thus we find uniformly for t on $[0,1]$ and for $x \in \Omega_\varepsilon$ such that also $x - \frac{\varepsilon}{2} \in \Omega_\varepsilon$:

$$R_0(x) \leq R_0(\gamma_x^-(t)) + \mathcal{O}(\varepsilon) \quad \text{and} \quad R_1(x) \leq R_1(\gamma_x^+(t)) + \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0\ .$$

Suppose at first that x is not too close to the lower and upper boundary of Ω_ε. More precisely, suppose that $\operatorname{Im} x^- + \varepsilon c/8 < \operatorname{Im} x < \operatorname{Im} x^+ - \varepsilon c/8$. In this case, the paths of integration can be modified such that they keep a sufficient distance to $x - \varepsilon$ and x, i.e. $|\gamma_x^+(\tau) - x|^{-1} = \mathcal{O}(\varepsilon^{-1})$ for all $\tau \in [0,1]$; similarly for γ_x^- instead of γ_x^+; similarly for $x - \varepsilon$ instead of x. Then, for ξ on γ_x^-, we find $\dfrac{1}{e_x(\xi)-1} = \mathcal{O}(1)$ and for ξ on γ_x^+, we find $\dfrac{1}{e_x(\xi)-1} = \mathcal{O}\left(\exp\left(\dfrac{2\pi}{\varepsilon}\operatorname{Im}(x-\xi)\right)\right)$. Using lemma 3.3 and the properties of the paths $\gamma_x^\pm$ we chose, we obtain:

- for ξ on γ_x^-: $\dfrac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)} = \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}(R_0(x) - R_0(\xi))\right)\right) = \mathcal{O}(1)$,
- and for ξ on γ_x^+: $\dfrac{h_\varepsilon(x)}{(e_x(\xi)-1)h_\varepsilon(\xi)} = \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}(R_1(x) - R_1(\xi))\right)\right) = \mathcal{O}(1)$.

Remark also that, if $|x - x^- + \varepsilon t| \geq \varepsilon$ and $|x - x^+ + \varepsilon t| \geq \varepsilon$, then necessarily $\operatorname{Im} x^- + \varepsilon c/8 < \operatorname{Im} x < \operatorname{Im} x^+ - \varepsilon c/8$. Hence first item is proved.

For points close to the upper or lower boundaries, one of the above estimates remains valid. For example, if $\operatorname{Im} x \in [\operatorname{Im} x^-, \operatorname{Im} x^- + \varepsilon c/8]$, the estimate for γ_x^+ remains valid and it remains to estimate the integral over γ_x^-; this will be done in the sequel.

As γ_x^-, we choose a certain polygonal path. With the notation $x_t = x^- + \varepsilon t$, we distinguish three cases: (a) $\operatorname{Re} x_t \le \operatorname{Re} x - \frac{\varepsilon}{4}$, (b) $\operatorname{Re} x - \frac{\varepsilon}{4} < \operatorname{Re} x_t \le \operatorname{Re} x$ and (c) $\operatorname{Re} x < \operatorname{Re} x_t$.

- In case (a), we choose as γ_x^- the path connecting $x_t, \operatorname{Re} x_t + i \operatorname{Im} x$ and $x - \frac{\varepsilon}{2}$. As Ω is c-ascending, one has $\operatorname{Re} x_t > \operatorname{Re} x - \frac{3\varepsilon}{4}$ and thus the above estimate remains valid on γ_x^-.
- In case (b) the path is chosen to connect $x_t, x^*, x - \frac{\varepsilon}{4}$ and $x - \frac{\varepsilon}{2}$, with

$$x^* := \operatorname{Re}(x - \frac{\varepsilon}{4}) + i \operatorname{Im} x^- \ .$$

Since $h_\varepsilon(x)/h_\varepsilon(\xi)$ and $A_\varepsilon(\xi)^{-1}$ are of order at most 1 on the whole path γ_x^-, it remains to estimate

$$\int_{x_t}^{x-\varepsilon/2} \frac{|d\xi|}{|e_x(\xi) - 1|} \ .$$

On the segments $\left[x^*, x - \frac{\varepsilon}{4}\right]$ and $\left[x - \frac{\varepsilon}{4}, x - \frac{\varepsilon}{2}\right]$ one has $|e_x(\xi) - 1|^{-1} = \mathcal{O}(1)$, whereas on $[x_t, x^*]$ one has for some $C > 0$:

$$|e_x(\xi) - 1|^{-1} \le C|\xi - x|^{-1} \le C\left(|\xi - x_t|^2 + |x_t - x|^2\right)^{-1/2} \ .$$

With $u = x_t - \xi$, and using $|x_t - x| = \mathcal{O}(\varepsilon)$, we get

$$\int_{x_t}^{x^*} \frac{|d\xi|}{|e_x(\xi) - 1|} \le C \int_0^{\varepsilon/4} \frac{du}{\sqrt{u^2 + |x_t - x|^2}}$$

$$= C\ \left(\ln\left(\tfrac{\varepsilon}{4} + \sqrt{(\tfrac{\varepsilon}{4})^2 + |x - x_t|^2}\right) - \ln|x - x_t|\right) = C\ \ln\left(\frac{\varepsilon}{|x - x_t|}\right) + \mathcal{O}(1) \ .$$

Altogether this gives

$$\int_{x^- + \varepsilon t}^{x - \frac{\varepsilon}{2}} \frac{h_\varepsilon(x)}{(e_x(\xi) - 1)h_\varepsilon(\xi)} \frac{g(\xi)}{A_\varepsilon(\xi)}\ d\xi = \mathcal{O}\left(\|g\|\ \ln\left(\frac{\varepsilon}{|x - x_t|}\right)\right) + \mathcal{O}(\|g\|) \ .$$

- In the third case (c) one uses the residue theorem in the following form:

$$\int_{x^- + \varepsilon t}^{x - \frac{\varepsilon}{2}} \frac{h_\varepsilon(x)}{(e_x(\xi) - 1)h_\varepsilon(\xi)} \frac{g(\xi)}{A_\varepsilon(\xi)}\ d\xi = \int_{\tilde{\gamma}_x^-} \frac{h_\varepsilon(x)}{(e_x(\xi) - 1)h_\varepsilon(\xi)} \frac{g(\xi)}{A_\varepsilon(\xi)}\ d\xi + \frac{\varepsilon g(x)}{A_\varepsilon(x)}$$

where the path $\tilde{\gamma}_x^-$ connects $x^- + \varepsilon t$ with $x - \frac{\varepsilon}{2}$ *above* x. By choosing as $\tilde{\gamma}_x^-$ the path connecting $x^- + \varepsilon t$, $\operatorname{Re}(x + \frac{\varepsilon}{8}) + \operatorname{Im} x^-$, $\operatorname{Re}(x + \frac{\varepsilon}{8}) + i(\operatorname{Im} x^- + \frac{\varepsilon c}{4})$, $\operatorname{Re}(x - \frac{\varepsilon}{8}) + i(\operatorname{Im} x^- + \frac{\varepsilon c}{4})$, $x - \frac{\varepsilon}{8}$ and $x - \frac{\varepsilon}{2}$, one obtains the same estimate.

The case of x close to the upper boundary is analogous.

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