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Classification of resonant equations

Augustin Fruchard and Reinhard Schäfke

Abstract: Consider a singularly perturbed equation of the form

$$\varepsilon y'' - f(x, \varepsilon)y' + g(x, \varepsilon)y = 0, \quad (1)$$

where $x, y \in \mathbb{C}$, $\varepsilon > 0$ is a small parameter, and f and g are two analytic functions in a neighborhood of $(0, 0)$, real for real values of x, ε , $f(0, 0) = 0$, $f'(0, 0) > 0$; this means that $x = 0$ is a turning point. Equation (1) is called *resonant* in the sense of Ackerberg-O'Malley, if there is a solution, analytic for x in some neighborhood of 0 and ε in some sector, which tends to a non-trivial solution of the reduced equation $f(x, 0)y' = g(x, 0)y$ as $\varepsilon \rightarrow 0$.

The article presents a classification of such resonant equations with respect to analytic transformations $\tilde{y} = a(x, \varepsilon)y + b(x, \varepsilon)\varepsilon y'$. First of all, $f_0(x) = f(x, 0)$ is a *formal* invariant considered fixed below. Furthermore, to each resonant equation are associated three formal series in ε , which are Gevrey of order 1 and invariant under analytic transformations. It is shown that this correspondence between equivalence classes of resonant equations and triples of Gevrey series is essentially bijective, and that each equivalence class contains an equation of a particular form: $f(x, \varepsilon) = f_0(x)$ and $g(x, \varepsilon) = f_1(x) + \varepsilon f_2(x)$ with $f_1(0) = 0$.

Key-words: Ackerberg-O'Malley resonance, Gevrey asymptotics, singular perturbation.

AMS Classification: 34E.

1 Introduction

In this article we consider the singularly perturbed equation (1) with the above assumptions. Throughout the whole article, the prime will denote differentiation with respect to x .

Let V be a neighborhood of $x = 0$ and S a sector in ε with vertex 0, with a finite radius and containing real positive numbers.

A *resonant solution* of (1) on $V \times S$ is a function $y = y(x, \varepsilon)$ analytic and bounded on $V \times S$ that is a solution of (1) for all $\varepsilon \in S$ and tends to a non trivial solution of the reduced equation

$$f(x, 0)y' = g(x, 0)y \quad (2)$$

uniformly on V as $\varepsilon \rightarrow 0$, $\varepsilon \in S$.

Equation (1) is called *resonant* if it possesses a resonant solution. It is known [3, 4] that a necessary condition for (1) to be resonant is that $n = g(0, 0)/f'(0, 0)$ is a nonnegative integer; this is an important quantity for the formal series (3): the multiplicity of $x = 0$ as a zero of its leading term $y_0(x)$. As shown in section 7, the derivative $y^{(n)}$ satisfies a second order equation analytically equivalent to (1) such that this integer is zero. These equations will be called *0-resonant*; for them, $y_0(0) \neq 0$.

It is well known, too [4, 5], that equation (1) is resonant if and only if it admits a formal solution

$$\hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n \quad (3)$$

with coefficients y_n analytic in a neighborhood of 0. Such a formal solution will also be called *resonant*. The key idea to prove this latter result is to show that the series \hat{y} (except for a factor that is a formal series in ε , i.e. independent of x) is of Gevrey order 1, i.e. its coefficients y_n satisfy

$$\exists A, C, \rho > 0 \forall n \in \mathbb{N} \forall |x| < \rho, \quad |y_n(x)| \leq A C^n n! . \quad (4)$$

Unfortunately the only known explicit examples of resonant equations are particularly simple equations having (non-trivial) convergent formal solutions. In [4], a method is given which allows to construct, using the fixed point theorem, resonant equations with divergent (of Gevrey order 1) formal solutions. It is therefore natural to link resonant equations and Gevrey-1 series.

The purpose of the present article is to characterize equivalence classes of resonant equations by means of some “invariants”. A weak and a strong equivalence relation are considered, depending on whether all transformations analytic near $x = \varepsilon = 0$ are allowed or only those which reduce to the identity for $x = 0$. To each resonant equation (1), we associate as invariants the function $f_0(x) = f(x, 0)$ and a certain triple of Gevrey-1 series. We show that two 0-resonant equations are strongly equivalent if and only if they have the same invariants and that two general resonant equations are weakly equivalent if and only if their invariants satisfy a simple equivalence relation.

We hope that our work will contribute to a better description of resonant equations; it shows that formal solutions of these equations are naturally divergent and that the framework of Gevrey theory reintroduced by Ramis [7] is the natural framework for the local theory of singular perturbations.

Only resonant equations are considered in this article. A classification of all second order linear equations with turning point would be important. We hope that the study of the special case of resonant equations will give ideas for the more general problem.

We would like to mention a different (unpublished) approach to the classification of resonant equations with $f(x, 0) = 2x$ by Sibuya [8]: It involves the Stokes matrices for a collection of transformations defined on ε -sectors reducing (1) to its normal form $\varepsilon y'' - 2xy' + 2ny = 0$; the ε -sectors have to be a good covering (see subsection 3.1) of a neighborhood of $\varepsilon = 0$.

The structure of the article is as follows. In the next section, we present the main results. Section 3 introduces two auxiliary tools. The first one is the construction of a right inverse of the variation operator; this operator was already used in [4] in a special context. The second one is due to Y. Sibuya; roughly speaking, it says that if a (vectorial) formal series solution of a singularly perturbed system of first order differential equations converges for one value of x , then it converges for any x . In the following three sections, we consider the special case of 0-resonant equations (1) with $f(x, \varepsilon) = 2x$. Section 4 contains several preliminary results concerning the invariants and characterizing weak and strong equivalence. The proofs of the main statements in the special case are done in sections 5 and 6. In section 5, we show that in this case the triples of Gevrey-1 series we introduced are indeed invariant with respect to analytic transformations and that

such equations having the same invariants are equivalent. We prove in section 6 that any (admissible) triple of Gevrey-1 series can be attained as invariant of some resonant equation with $f(x, \varepsilon) = 2x$. In section 7 finally, we extend the results to general 0-resonant and resonant equations.

2 Statement of the main results

Before presenting the main results in detail, we make a preliminary reduction and introduce some notation. First of all, substitution of $\varepsilon = c\tilde{\varepsilon}$ allows to reduce to the case $f'(0, 0) = 2$; this will be assumed throughout the article. We denote by \mathcal{R} the set of resonant equations of the form (1), by \mathcal{R}_0 the set of 0-resonant equations of the form (1) (i.e. with $g(0, 0) = 0$). By identifying a resonant equation to its coefficient functions (f, g) , we identify \mathcal{R} to a subset of the space \mathcal{H}_0^2 , $\mathcal{H}_0 := \mathbb{C}\{x, \varepsilon\}$, of pairs of germs of holomorphic functions in a neighborhood of $(0, 0) \in \mathbb{C}^2$.

Using matrix notation, equation (f, g) can be rewritten

$$(R) \quad \varepsilon \mathbf{y}' = R\mathbf{y} ,$$

with $\mathbf{y} = \begin{pmatrix} y \\ \varepsilon y' \end{pmatrix}$ and $R : (x, \varepsilon) \mapsto \begin{pmatrix} 0 & 1 \\ -\varepsilon g(x, \varepsilon) & f(x, \varepsilon) \end{pmatrix}$.

Remark: For simplicity, we will often omit the variables x and ε , and write for instance $R = \begin{pmatrix} 0 & 1 \\ -\varepsilon g & f \end{pmatrix}$. Another way to read this kind of formula is to see an identity between functions, the letters x, ε being the canonical projections and 1 being the constant function.

Definition 1. — We say that (R) and (\tilde{R}) are *weakly equivalent*, and we write $(R) \sim (\tilde{R})$, if there exists a two by two matrix T with entries in \mathcal{H}_0 whose determinant does not vanish identically such that the change of unknowns $\tilde{\mathbf{y}} = T(x, \varepsilon)\mathbf{y}$ transforms equation (R) into equation (\tilde{R}) , i.e. $\tilde{\mathbf{y}}$ is a solution of (\tilde{R}) if \mathbf{y} is a solution of (R) .

We say that (R) and (\tilde{R}) are *strongly equivalent*, and we write $(R) \approx (\tilde{R})$, if moreover T satisfies $T(0, \varepsilon) = \mathbf{1}$ (for all ε in a neighborhood of 0), where $\mathbf{1}$ is the identity matrix of $\mathcal{M}(2, \mathbb{C})$.

If $\tilde{\mathbf{y}} = T(x, \varepsilon)\mathbf{y}$ transforms (R) into (\tilde{R}) , then for some fundamental solution Y of (R) , the matrix function $Z(x, \varepsilon) = T(x, \varepsilon)Y(x, \varepsilon)$ is a solution of (\tilde{R}) . Thus, if \tilde{Y} denotes a fundamental solution of (\tilde{R}) , there exists a matrix $D(\varepsilon)$ of functions analytic at $\varepsilon = 0$ independent of x such that $Z(x, \varepsilon) = \tilde{Y}(x, \varepsilon)D(\varepsilon)$. Therefore, if $e^{\int_0^x f(t, 0)dt/\varepsilon}w(x, \varepsilon)$ and $e^{\int_0^x \tilde{f}(t, 0)dt/\varepsilon}\tilde{w}(x, \varepsilon)$ denote the Wronskian determinants of the equations ($w, \tilde{w} \in \mathcal{H}_0$ and $w(0, 0), \tilde{w}(0, 0) \neq 0$), we must necessarily have $f(x, 0) = \tilde{f}(x, 0)$ and also $\det T(x, \varepsilon) = \frac{\tilde{w}(x, \varepsilon)}{w(x, \varepsilon)} \det D(\varepsilon)$. Hence $S(x, \varepsilon) = \det D(\varepsilon)(T(x, \varepsilon))^{-1}$ has entries in \mathcal{H}_0 and transforms (\tilde{R}) into (R) . This shows that \sim is indeed symmetric and that $f(x, 0)$ is an invariant, i.e. the same for all weakly equivalent equations.

We call a scalar second order equation (f, g) weakly or strongly equivalent to another one (\tilde{f}, \tilde{g}) if the corresponding systems (R) and (\tilde{R}) are weakly or strongly equivalent. Thus an equivalence between two scalars equations is realized by change of unknowns of the form $\tilde{y} = a(x, \varepsilon)y + b(x, \varepsilon)\varepsilon y'$; the remaining entries of the corresponding T can

then be determined. Clearly, a , b resp. T have to satisfy restrictive properties, because (\tilde{f}, \tilde{g}) resp. (\tilde{R}) must be exactly of the same form as (f, g) resp. (R) . In the case of $f(x, \varepsilon) = \tilde{f}(x, \varepsilon) = 2x$, $g(0, 0) = \tilde{g}(0, 0) = 0$, this will be studied in details in subsection 4.2.

A priori for the weak equivalence, it is possible that $\det T(x, \varepsilon) = 0$ on some nonempty subset of a neighborhood of $(0, 0)$, but we will see in subsection 4.2 that $T(0, 0)$ is necessarily invertible in the case $f(x, \varepsilon) = \tilde{f}(x, \varepsilon) = 2x$, $g(0, 0) = \tilde{g}(0, 0) = 0$.

For any equivalence relation, it is a natural problem to look for invariants; we first focus on invariants for the relation \approx on the set \mathcal{R}_0 of 0-resonant equations. As a first invariant besides $f(x, 0)$, we choose a formal series in ε :

$$I(\varepsilon) = \hat{y}'(0, \varepsilon)$$

where \hat{y} is the only formal resonant solution of (f, g) which satisfies $\hat{y}(0, \varepsilon) = 1$. To simplify notation, we omit the hats on the invariants. As $\hat{y}(x, \varepsilon)$ is known to be Gevrey-1 uniformly with respect to x , Cauchy's formula implies that this invariant is also Gevrey of order 1.

Our second invariant, denoted by J , is the expansion in powers of $\varepsilon^{1/2}$ of the formal expression

$$J(\varepsilon) = \sum_{n \geq 0} \hat{\alpha}_n(\varepsilon) \int_0^{\delta i} e^{F(x, \varepsilon)/\varepsilon} x^n dx$$

where $F(x, \varepsilon) = \int_0^x f(t, \varepsilon) dt$, $\delta > 0$ is sufficiently small, \hat{y} is the above formal resonant solution and $\hat{\alpha}_n \in \mathbb{C}[[\varepsilon]]$ are given by

$$\hat{y}(x, \varepsilon)^{-2} = \sum_{n \geq 0} \hat{\alpha}_n(\varepsilon) x^n . \quad (5)$$

The choice of this second invariant is motivated by the solution of (f, g) , denoted by Zy , which satisfies $Zy(0, \varepsilon) = 0$, $(Zy)'(0, \varepsilon) = 1$; this solution can be expressed using any resonant solution y with $y(0, \varepsilon) = 1$ and J is the right hand side of the asymptotic expansion of the value of $Zy(x, \varepsilon)/y(x, \varepsilon)$ at $x = i\delta$ for δ sufficiently small. See subsection 4.1 for details. Furthermore, it will be shown in subsections 4.1, 7.1, 7.2 that $J = J_1 + \varepsilon^{1/2} J_2$ where J_1, J_2 are Gevrey-1 series. The series I, J_1, J_2 are the three Gevrey-1 series mentioned in the abstract and in the introduction. Of course, we have to prove that these are actually invariant with respect to \approx . This will be done in section 5 together with subsections 7.1, 7.2.

We denote by \mathcal{G}_1 the set of Gevrey-1 series in ε (this is sometimes written $\mathcal{G}_1 := \mathbb{C}[[\varepsilon]]_1$), by \mathcal{G} the set $\mathcal{G} := \mathbb{C}\{x\} \times \mathcal{G}_1 \times (\mathcal{G}_1 + \varepsilon^{1/2}\mathcal{G}_1)$ and by \mathcal{K} the set

$$\mathcal{K} := \left\{ (h, I, J) \in \mathcal{G} ; h'(0) = 2, J(\varepsilon) = \frac{i}{2}\sqrt{\pi\varepsilon} + O(\varepsilon) \right\} . \quad (6)$$

Theorem 2. — Denote by Φ_0 the mapping

$$\Phi_0 : \mathcal{R}_0 \rightarrow \mathcal{K}, (R) \mapsto (f(x, 0), I, J) , \quad (7)$$

which associates to each 0-resonant equation its invariants.

Then Φ_0 induces a bijection Φ_\approx between \mathcal{R}_0/\approx and \mathcal{K} ; in other words:

- (i) two 0-resonant equations are strongly equivalent if and only if they have the same invariants;
- (ii) for all (h, I, J) in \mathcal{K} there exists a 0-resonant equation with invariants h, I and J .

This theorem classifies the \approx -equivalent classes of 0-resonant equations by their invariants. For a general resonant equation (f, g) with $g(0, 0) = 2n$, $n \in \mathbb{N}$, we define its invariants to be the invariants of the 0-resonant equation satisfied by $y^{(n)}$; this equation is weakly equivalent to (f, g) (cf. subsection 7.3). We denote by $\Phi(f, g)$ the vector of invariants of a general resonant equation (cf. (7)). The weak equivalence \sim between equations yields an equivalence relation between invariants (h, I, J) , denoted by the same \sim sign; it turns out to be the following, see again section 5:

Definition 3. — Two elements (h, I, J) and $(\tilde{h}, \tilde{I}, \tilde{J})$ of \mathcal{K} are called *weakly equivalent* if $h = \tilde{h}$ and there are four functions A, B, C, D holomorphic in a neighborhood of $\varepsilon = 0 \in \mathbb{C}$ satisfying $AD - \varepsilon BC \equiv 1$, $A(0) = D(0) = 1$, such that

$$\tilde{I} = \frac{C + DI}{A + \varepsilon BI}, \quad \tilde{J} = (A + \varepsilon BI)^2 J - \varepsilon B(A + \varepsilon BI), \quad (8)$$

where ε has been again omitted (see remark above definition 1).

Remark: 1. As a straightforward calculation shows, this is indeed an equivalence relation.
 2. Observe that for two elements (h, I, J) and $(h, \tilde{I}, \tilde{J})$ of \mathcal{K} , there are uniquely determined formal series $A(\varepsilon), B(\varepsilon), C(\varepsilon), D(\varepsilon)$ satisfying the equalities of the definition; the elements are weakly equivalent if all these series are convergent. Indeed, separating $J = J_1 + \varepsilon^{1/2} J_2$ and $\tilde{J} = \tilde{J}_1 + \varepsilon^{1/2} \tilde{J}_2$ with $J_1, J_2, \tilde{J}_1, \tilde{J}_2 \in \mathbb{C}[[\varepsilon]]$, we obtain four equations for the four series A, B, C, D . Omitting again the argument ε , these are

$$\begin{aligned} (A + \varepsilon BI)\tilde{I} &= C + DI, & AD - \varepsilon BC &= 1, \\ \tilde{J}_1 &= (A + \varepsilon BI)^2 J_1 - \varepsilon B(A + \varepsilon BI), & \tilde{J}_2 &= (A + \varepsilon BI)^2 J_2. \end{aligned} \quad (9)$$

The last line uniquely determines A and B ; then the first one is a 2 by 2 system of linear equations with determinant $A + \varepsilon BI \neq 0$ for C and D .

Using this equivalence relation for the “invariants”, we present a result similar to theorem 2 concerning the weak equivalence of resonant equations. Furthermore, the method used to prove surjectivity yields a more precise result, which is stated below.

Theorem 4. — *The mapping Φ defined above definition 3 induces a bijection Φ_{\sim} between \mathcal{R}/\sim and \mathcal{K}/\sim . Moreover, each equivalence class of the relation \sim contains a 0-resonant equation (f, g) with f independent of ε and g linear in ε . More precisely, given $(h, I, J) \in \mathcal{K}$, there exist $\varphi = \varphi(\varepsilon)$ and $g_0 = g_0(x), g_1 = g_1(x)$ holomorphic in a neighborhood of $0 \in \mathbb{C}$ such that the equation $(h, g_0 + \varepsilon g_1)$ is 0-resonant and has invariants $(I + \varphi, J)$.*

- Remarks: 1 – By misuse of language, we call *invariants* the vectors (h, I, J) associated to an equation (f, g) , even if they are not properly invariant under \sim .
 2 – In each equivalence class of \mathcal{R}/\sim , there exist several resonant equations (h, g) for which g is linear in ε .

Sections 5 and 6 are devoted to the proofs of theorems 2 and 4 in the special case of 0-resonant equations with $f(x, \varepsilon) = 2x$; section 7 extends these results to general 0-resonant and resonant equations.

3 Auxiliary tools

In this section, we introduce several tools that will be useful in subsequent sections.

3.1 A right inverse of the variation operator

The first tool is a continuous right inverse of the Δ -operator described below (see (10)). It will be used in section 6. This inverse was already used in [4] in a simpler context.

We denote the sector of *direction* θ , *opening* $|S| := 2\delta$ and *radius* r by

$$S(\theta, \delta, r) = \{\varepsilon \in \mathbb{C} ; 0 < |\varepsilon| < r, |\arg \varepsilon - \theta| < \delta\}$$

Given $r, \mu > 0$ small enough, we consider, for $j \in \{0, \dots, 2h\}$, the sectors

$$S_j = S\left(j \frac{2\pi}{2h+1}, \frac{\pi}{2h} - \mu, r\right) .$$

where μ is chosen in such a way that these sectors form a *good covering* of 0, i.e. intersections $D_j := S_j \cap S_{j+1}$ (with $S_{2h+1} := S_0$) of consecutive sectors are non empty and triple intersections $D_j \cap D_{j+1}$ (with $D_{2h+1} := D_0$) are empty. We choose

$$\mu := \frac{\pi}{4h(2h+1)} .$$

In this way, $|S_j|/2 = \frac{\pi}{2h+1} + \mu$ and $|D_j|/2 = \mu$.

Let \mathcal{U} be the Banach space of $(2h+1)$ -tuples $\vec{y} = (y_0, \dots, y_{2h})$ of holomorphic and bounded functions $y_j : S_j \rightarrow \mathbb{C}$, endowed with the norm $\|\vec{y}\| = \max(\|y_0\|_\infty, \dots, \|y_{2h}\|_\infty)$.

Denote by \mathcal{D} the Banach space of $(2h+1)$ -tuples $\vec{d} = (d_0, \dots, d_{2h})$ of holomorphic functions $d_j : D_j \rightarrow \mathbb{C}$ such that $d_j(\varepsilon)/\varepsilon$ is bounded, with the norm

$$\|\vec{d}\| = \max_{j \in \{0, \dots, 2h\}} \sup_{\varepsilon \in D_j} \left| \frac{d_j(\varepsilon)}{\varepsilon} \right| .$$

The space \mathcal{D} can be identified to the space of holomorphic functions d on $D_0 \cup \dots \cup D_{2h}$ that are bounded with respect to the norm $\sup_\varepsilon |d(\varepsilon)/\varepsilon|$.

We consider the variation operator Δ defined by

$$\Delta \vec{y} = \vec{d} := (y_1 - y_0, \dots, y_{2h} - y_{2h-1}, y_0 - y_{2h}) . \quad (10)$$

This defines a mapping $\Delta : \mathcal{U}_0 \rightarrow \mathcal{D}$, where \mathcal{U}_0 is the subset of all $\vec{y} \in \mathcal{U}$ such that $\Delta \vec{y} \in \mathcal{D}$, i.e. $(y_{j+1} - y_j)(\varepsilon) = O(\varepsilon)$ as $\varepsilon \rightarrow 0$ for all $j \in \{0, \dots, 2h\}$. Given $\vec{d} \in \mathcal{D}$, to find $\vec{y} = (y_0, \dots, y_{2h}) \in \mathcal{U}_0$ such that $\Delta \vec{y} = \vec{d}$, it seems natural to use the classical formula

$$y_k : S_k \rightarrow \mathbb{C} , \quad \varepsilon \mapsto \frac{1}{2\pi i} \sum_{j=0}^{2h} \int_0^{\varepsilon_j} d_j(z) \frac{dz}{z - \varepsilon} \quad (11)$$

with

$$\varepsilon_j := r e^{i\alpha_j} \in \overline{D_j} \quad \text{and} \quad \alpha_j := \frac{(2j+1)\pi}{2h+1} ,$$

where the path of integration is the straight line from 0 to ε_j if $j \neq k$ and $j \neq k-1$, on the left of ε if $j = k$ and on the right of ε if $j = k-1$.

Actually it is well-known that the above \vec{y} satisfies $\Delta\vec{y} = \vec{d}$ with y_k analytic in S_k . However, this does not yield an operator from \mathcal{D} into \mathcal{U} , because y_k has logarithmic singularities at $\varepsilon = \varepsilon_k$ and $\varepsilon = \varepsilon_{k-1}$. We overcome this difficulty by considering some average of integrals similar to (11), namely, with $\delta := \mu/2$:

$$\Sigma\vec{d}(\varepsilon) = \frac{1}{2\delta} \int_{-\delta}^{\delta} \vec{y}_{\theta}(\varepsilon) d\theta \quad (12)$$

with $\vec{y}_{\theta} = (y_{\theta,0}, \dots, y_{\theta,2h})$, $y_{\theta,k} : S_k \rightarrow \mathbb{C}$, $\varepsilon \mapsto \frac{1}{2\pi i} \sum_{j=0}^{2h} \int_0^{\varepsilon_j e^{i\theta}} d_j(z) \frac{dz}{z - \varepsilon}$, ε_j and the paths of integration as above. Since $D_j = S(\alpha_j, \mu, r)$, $\varepsilon_j e^{i\theta}$ remains on $\overline{D_j}$ and sufficiently far away from the sides of $\overline{D_j}$.

Theorem 5. — 1 – The operator $\Sigma : \mathcal{D} \rightarrow \mathcal{U}_0$ given by (12) is bounded and satisfies

$$\Delta\Sigma = id : \mathcal{D} \rightarrow \mathcal{D} .$$

2 – We have $\ker \Delta = \mathcal{H}_{\varepsilon_0}$, the space of holomorphic and bounded functions on $D(0, \varepsilon_0)$.

3 – For all $\vec{g} \in \mathcal{U}_0$ we have $\Sigma\Delta\vec{g} - \vec{g} \in \mathcal{H}_{\varepsilon_0}$.

4 – Consider the operator L defined from \mathcal{D} into \mathcal{U}_0 by

$$L\vec{d} := \varepsilon\Sigma(\vec{d}) - \Sigma(\varepsilon\vec{d}) . \quad (13)$$

Then for all $\vec{d} \in \mathcal{D}$, the function $L\vec{d}$ is constant with respect to ε . Thus we use L as mapping \mathcal{D} into \mathbb{C} .

5 – More generally, for all $\vec{d} \in \mathcal{D}$ and all $n \in \mathbb{N}$, $\varepsilon^n \Sigma(\vec{d}) - \Sigma(\varepsilon^n \vec{d})$ is polynomial in ε of degree at most $n - 1$.

Remark: As above definition 1, ε denotes the identity function in the above theorem.

Proof. 1 – By construction, we have $(\Sigma\vec{d})_{j+1} - (\Sigma\vec{d})_j = d_j$, $j = 0, \dots, 2h$. The fact that $\Sigma\vec{d} \in \mathcal{U}$ will be shown below. It is then clear that $\Sigma\vec{d} \in \mathcal{U}_0$ since $\Delta\Sigma\vec{d} = \vec{d} \in \mathcal{D}$. In order to establish an upper bound of $\|\Sigma\|$, we first estimate \vec{y}_{θ} defined below (12) in a way similar to the proof of lemma 20 in [4]. We could refer to [4] for more details, but we prefer include a proof for completeness.

Let us look at the first coordinate $y_{\theta,0}$; the estimate will be the same for the other ones. We furthermore focus on $\varepsilon \in S_0$ with $\arg \varepsilon \in [0, \alpha_0 + 2\delta]$; the case $\arg \varepsilon < 0$ is similar.

If $\arg \varepsilon \leq \alpha_0 - 2\delta$ we simply use

$$|z - \varepsilon| \geq |z| \sin \delta \quad (14)$$

for any z on any segment $[0, \varepsilon_j e^{i\theta}]$ and obtain

$$|y_{\theta,0}(\varepsilon)| \leq \frac{2h+1}{2\pi} \frac{r}{\sin \delta} \|\vec{d}\| .$$

If $\arg \varepsilon > \alpha_0 - 2\delta$ (notice that only the integral involving d_0 poses a problem) we distinguish two cases:

In the case $\arg \varepsilon \leq \alpha_0 + \delta$ we use a path of integration γ_{θ} arbitrarily close to the segment

$[0, e^{i(\alpha_0+2\delta)}]$ and the arc of circle $C_\theta = (e^{i(\alpha_0+2\delta)}, e^{i(\alpha_0+\theta)})$. On the first piece, the same inequality (14) holds. On the remaining one, we have

$$\left| \int_{C_\theta} d_0(z) \frac{dz}{z - \varepsilon} \right| \leq \|d_0(z)\| \int_{C_\theta} \frac{|z|}{|z - \varepsilon|} |dz|. \quad (15)$$

In the case $\arg \varepsilon > \alpha_0 + \delta$ we first use Cauchy's formula which shows that

$$\int_{\gamma_\theta} d_0(z) \frac{dz}{z - \varepsilon} = 2\pi i d_0(\varepsilon) + \int_{\tilde{\gamma}_\theta} d_0(z) \frac{dz}{z - \varepsilon}$$

where now the path of integration is on the right of ε , arbitrarily close to $[0, e^{i(\alpha_0-2\delta)}]$ and the arc $(e^{i(\alpha_0-2\delta)}, e^{i(\alpha_0+\theta)})$. We still have (14) on the segment and (15) on the arc.

Therefore it remains to show that $I = \int_{-\delta}^{\delta} d\theta \int_{C_\theta} \frac{|z|}{|z - \varepsilon|} |dz|$ is bounded, where C_θ is either the arc $(e^{i(\alpha_0+2\delta)}, e^{i(\alpha_0+\theta)})$ if $\arg \varepsilon \leq \alpha_0 + \delta$, or the arc $(e^{i(\alpha_0-2\delta)}, e^{i(\alpha_0+\theta)})$ if $\arg \varepsilon > \alpha_0 + \delta$. This is done exactly as in [4]: by splitting the exterior integral in two parts at $\theta = \arg \varepsilon - \alpha_0$, we show that the double integral assumes its maximum for $\arg \varepsilon = \alpha_0$. Hence it suffices to estimate $2 \int_0^\delta d\theta \int_\theta^{2\delta} \frac{ds}{\left| e^{si} - \frac{|\varepsilon|}{r} \right|}$. As the denominator is at least $\sin s$, the theorem of Fubini-Tonelli yields $I \leq 2 \int_0^{2\delta} \frac{s}{\sin s} ds$.

2 – If \vec{y} satisfies $\Delta \vec{y} = 0$, then \vec{y} is a single valued function, hence holomorphic on $D(0, \varepsilon_0) \setminus \{0\}$. As this function is also bounded in a neighborhood of 0, the singularity at $\varepsilon = 0$ is removable and \vec{y} is holomorphic at $\varepsilon = 0$. Conversely, a function holomorphic on $D(0, \varepsilon_0)$ is clearly identified to a triple of \mathcal{U} with differences equal to zero.

3 – For $\vec{g} \in \mathcal{U}_0$, we have $\Delta(\Sigma \Delta \vec{g} - \vec{g}) = (\Delta \Sigma) \Delta \vec{g} - \Delta \vec{g} = 0$; then use item 2.

4 – Denote by $\mathcal{S} : \mathcal{D} \rightarrow \mathcal{U}_0$ the operator given by

$$\mathcal{S}(\vec{a}(\mu)) := \frac{1}{2\delta} \int_{-\delta}^{\delta} \sum_{j=0}^{2h} \left(\frac{1}{2\pi i} \int_0^{\varepsilon_j e^{i\theta}} a_j(\mu) d\mu \right) d\theta,$$

$\delta = \frac{\pi}{8h(2h+1)}$, $\varepsilon_j = r e^{i\alpha_j}$, $\alpha_j = \frac{(2j+1)\pi}{2h+1}$. We have $\Sigma(\vec{a})(\varepsilon) = \mathcal{S} \left(\frac{\vec{a}(\mu)}{\mu - \varepsilon} \right)$, therefore (13) reads

$$L\vec{d}(\varepsilon) = \mathcal{S} \left(\left(\varepsilon \vec{d}(\mu) - \mu \vec{d}(\mu) \right) \frac{1}{\mu - \varepsilon} \right) = -\mathcal{S}(\vec{d}(\mu)).$$

This shows that $L\vec{d}(\varepsilon)$ does not depend on ε .

5 – Let $L_n : \mathcal{D} \rightarrow \mathcal{D}$, $\vec{d} \mapsto L_n \vec{d} = \varepsilon^n \Sigma(\vec{d}) - \Sigma(\varepsilon^n \vec{d})$ (here the letter ε denotes multiplication by the variable). We have $L_n(\vec{d}) = \varepsilon L_{n-1}(\vec{d}) + L(\varepsilon^{n-1} \vec{d})$ with L given at item 4; hence the statement follows by induction. \square

3.2 Convergence of formal solutions

The second tool is due to Y. Sibuya [9]. It will be used in section 5 (proof of proposition 12). For completeness, a proof is included.

Let Ω be a domain (i.e. connected open subset) of \mathbb{C} and $\varepsilon_0 > 0$. We denote by $\overline{D} = \overline{D(0, \varepsilon_0)}$ the closed disk of radius ε_0 around the origin and by \mathcal{H} the Banach space of holomorphic bounded functions on $\Omega \times \overline{D}$ endowed with the usual norm. Let d and h be positive integers and $A \in \mathcal{M}(d, \mathcal{H})$ be a $d \times d$ -matrix with entries in \mathcal{H} .

Theorem 6. — *Suppose that the differential equation*

$$\varepsilon^h y'(x, \varepsilon) = A(x, \varepsilon)y(x, \varepsilon) \quad , \quad y \in \mathbb{C}^d \quad (16)$$

admits a non trivial formal solution $\hat{y} = \sum_{n \geq 0} z_n(x)\varepsilon^n$ whose coefficients z_n are analytic on Ω .

If there exists $x_0 \in \Omega$ such that $\hat{y}(x_0, \varepsilon)$ converges for $\varepsilon \in \overline{D}$, then \hat{y} converges for all $x \in \Omega$ and $\varepsilon \in \overline{D}$.

Proof. For \mathbb{C}^d , we can use any norm, for instance the maximum norm, for $\mathcal{M}(d, \mathbb{C})$ we use the associated matrix norm both denoted by $|\cdot|$. On $\mathcal{M}(d, \mathcal{H})$ we introduce the norm

$$\|A\| := \sup_{(x, \varepsilon) \in \Omega \times \overline{D}} |A(x, \varepsilon)| .$$

Let $R > 0$ be such that $D(x_0, R) \subset \Omega$. We first claim that it suffices to prove that

$$\hat{y} \text{ converges for } |x - x_0| < \rho := \min \left\{ \frac{\varepsilon_0^h}{\|A\|}, R \right\} \text{ and } |\varepsilon| \leq \varepsilon_0. \quad (17)$$

Indeed, for all x in Ω , there exists a finite sequence of points x_j , $j = 1, \dots, N$ such that $x_N = x$ and $x_{j+1} \in B(x_j, \rho_j)$ with $\rho_j = \min \left\{ \frac{\varepsilon_0^h}{\|A\|}, \text{dist}(x_j, \partial\Omega) \right\}$ for $j = 0, \dots, N - 1$. We conclude by induction that \hat{y} converges for x .

Without loss of generality, we assume that $x_0 = 0$. Here we prove even more.

Lemma 7 *Suppose that (16) admits a nontrivial formal solution $\hat{y} \in \mathbb{C}[[x, \varepsilon]]$ such that $\hat{y}(0, \varepsilon)$ converges for $\varepsilon \in \overline{D}$. Then $\hat{y}(x, \varepsilon)$ converges for $|x| < \rho$ and $\varepsilon \in \overline{D}$.*

The idea is to expand A and \hat{y} in powers of x :

$$A(x, \varepsilon) = \sum_{n \geq 0} A_n(\varepsilon)x^n, \quad A_n \in \mathcal{M}(d, \mathbb{C}\{\varepsilon\}) \quad \text{and} \quad \hat{y}(x, \varepsilon) = \sum_{n \geq 0} y_n(\varepsilon)x^n, \quad y_n \in \mathbb{C}[[\varepsilon]]^d .$$

First, Cauchy's inequalities yield that $A_n(\varepsilon)$ are holomorphic and bounded in \overline{D} and

$$\forall n \in \mathbb{N}, \forall \varepsilon \in \overline{D}, \quad \|A_n(\varepsilon)\| \leq \|A\| R^{-n} \leq \|A\| \rho^{-n} .$$

Identification of the coefficients of x^n in (16) yields

$$\varepsilon^h (n+1)y_{n+1}(\varepsilon) = \sum_{k=0}^n A_k(\varepsilon)y_{n-k}(\varepsilon) =: \varphi_n(\varepsilon) . \quad (18)$$

Lemma 8. — *For all $n \geq 0$, y_n converges for $|\varepsilon| \leq \varepsilon_0$.*

Proof. By induction on n . For $n = 0$, the series $y_0(\varepsilon) = \hat{y}(0, \varepsilon)$ converges for $|\varepsilon| \leq \varepsilon_0$, by assumption of the lemma.

If the statement of the lemma is valid for all $k \leq n$, then φ_n converges in $D(0, \varepsilon_0)$. Now we already know that y_{n+1} is a formal series in ε , hence the left hand side of (18), $\varepsilon^h(n+1)y_{n+1}$, is a series with valuation at least h , hence φ_n has a zero of order at least h at $\varepsilon = 0$. Therefore, $y_{n+1} = \frac{1}{n+1}S_0^h\varphi_n$, where S_0^h is the h -th iterate of the shift operator S_0 defined by $S_0\varphi(\varepsilon) = \frac{1}{\varepsilon}(\varphi(\varepsilon) - \varphi(0))$. This operator preserves the radius of convergence.

□

Lemma 9. — For all $n \geq 0$ and all $|\varepsilon| \leq \varepsilon_0$, we have $\|y_n(\varepsilon)\| \leq M\rho^{-n}$, with $M := \sup\{\|\hat{y}(0, \varepsilon)\| ; |\varepsilon| \leq \varepsilon_0\}$ and ρ given by (17).

Proof. Once again by induction on n . It is obvious for $n = 0$ since $y_0 = \hat{y}(0, \cdot)$. If it is true for $k \leq n$, then (18) gives

$$\|\varepsilon^h y_{n+1}(\varepsilon)\| \leq \frac{1}{n+1} \sum_{k=0}^n \|A\| \rho^{-k} M \rho^{k-n} = M \|A\| \rho^{-n} .$$

For $|\varepsilon| = \varepsilon_0$, taking into account that $\rho \leq \frac{\varepsilon_0^h}{\|A\|}$, we obtain $\|y_{n+1}(\varepsilon)\| \leq M\rho^{-(n+1)}$. Now the function y_{n+1} is analytic in the open disk $D = D(0, \varepsilon_0)$, therefore this estimate remains valid on D by the maximum modulus principle. □

Statement (17) now becomes evident: \hat{y} converges for $|x| < \rho$ and $|\varepsilon| \leq \varepsilon_0$. As shown above, the theorem follows □

4 0-Resonant equations in the special case : Preliminary results

In this section and the following two, we will only consider 0-resonant equations (f, g) with $f(x, \varepsilon) = 2x$; this simplifies many considerations and it seems easier to extend the results to general equations later. As f is fixed, we denote the equation (f, g) simply by (g) and the invariants simply by (I, J) . As will be shown in subsection 4.1, I and J have to satisfy the relation $J(\varepsilon) = \frac{i}{2}\sqrt{\pi\varepsilon} + I_0\varepsilon + O(\varepsilon^{3/2})$ in this case. Thus instead of the sets $\mathcal{R}, \mathcal{R}_0$ and \mathcal{K} introduced above theorem 2, we use $\mathcal{R}^o = \{g \in \mathbb{C}\{x, \varepsilon\} ; g(0, 0) = 0\}$ and $\mathcal{K}^o = \{(I, J) \in \mathcal{G}_1 \times (\mathcal{G}_1 + \varepsilon^{1/2}\mathcal{G}_1) ; J(\varepsilon) = \frac{i}{2}\sqrt{\pi\varepsilon} + I_0\varepsilon + O(\varepsilon^{3/2})\}$, instead of the mappings Φ_0, Φ of theorems 2 and 4, we use $\Phi^o : \mathcal{R}^o \rightarrow \mathcal{K}^o, (g) \mapsto (I, J)$.

4.1 The second invariant

Denote by Zy the solution of (g) that satisfies $Zy(0, \varepsilon) = 0, (Zy)'(0, \varepsilon) = 1$. The variation of constant formula gives:

$$Zy(x, \varepsilon) = y(x, \varepsilon) \int_0^x e^{\xi^2/\varepsilon} y(\xi, \varepsilon)^{-2} d\xi .$$

where y is any resonant solution of (g) with $y(0, \varepsilon) = 1$. Define

$$\tilde{J} := Zy(i\delta, \varepsilon)/y(i\delta, \varepsilon) = \int_0^{i\delta} e^{x^2/\varepsilon} y(x, \varepsilon)^{-2} dx$$

where $\delta > 0$ is an arbitrary constant. \tilde{J} is defined from a resonant solution and an arbitrary constant, but its expansion in powers of $\varepsilon^{1/2}$ (as $\varepsilon \rightarrow 0$) depends only on the formal resonant solution \hat{y} ; we define J as being this expansion. Namely, if we denote by $\hat{y}(x, \varepsilon)^{-2} = \sum_{n \geq 0} \hat{\alpha}_n(\varepsilon) x^n$ the expansion of \hat{y}^{-2} in powers of x , where each $\hat{\alpha}_n$ represents a formal series in ε , we put

$$J(\varepsilon) := \sum_{n \geq 0} \hat{\alpha}_n(\varepsilon) \frac{1}{2} i^{n+1} \Gamma\left(\frac{n+1}{2}\right) \varepsilon^{\frac{n+1}{2}} .$$

We will use in sections 5 and 6 the operator $K : \mathbb{C}[[x, \varepsilon]] \rightarrow \mathbb{C}[[\varepsilon^{1/2}]]$ which, to a formal series $u(x, \varepsilon) = \sum_{m, n \geq 0} a_{m, n} x^m \varepsilon^n$, associates the formal expansion

$$Ku(\varepsilon) := \sum_{m, n \geq 0} a_{m, n} \Gamma\left(\frac{m+1}{2}\right) \frac{1}{2} i^{m+1} \varepsilon^{\frac{m+1}{2} + n} .$$

With this notation, the second invariant of (g) is $J = K(\hat{y}^{-2})$.

Now we have $\hat{y}(x, \varepsilon) = 1 + x\hat{y}'(0, \varepsilon) + O(x^2)$, hence the first terms of \hat{y}^{-2} are given by $\hat{\alpha}_0(\varepsilon) = 1$ and $\hat{\alpha}_1(\varepsilon) = -2I(\varepsilon)$. Therefore $J(\varepsilon) = \frac{i}{2} \sqrt{\pi \varepsilon} + I_0 \varepsilon + O(\varepsilon^{3/2})$.

Furthermore, as \hat{y} is analytic in x and Gevrey-1 in ε then \hat{y}^{-2} is Gevrey-1, too, hence $\hat{\alpha}_n(\varepsilon) = \sum_{m \geq 0} \alpha_{m, n} \varepsilon^m$ satisfies

$$\exists A, C \geq 1 \forall m, n \in \mathbb{N}, \quad |\alpha_{m, n}| \leq A C^{m+n} m! .$$

Writing J in the form $J = J_1 + \sqrt{\varepsilon} J_2$:

$$J(\varepsilon) = \sum_{\nu \geq 1} \left(\sum_{m + \frac{n+1}{2} = \nu} \alpha_{m, n} \frac{1}{2} i^{n+1} \Gamma\left(\frac{n+1}{2}\right) \right) \varepsilon^\nu + \sqrt{\varepsilon} \sum_{\nu \geq 0} \left(\sum_{m + \frac{n}{2} = \nu} \alpha_{m, n} \frac{1}{2} i^{n+1} \Gamma\left(\frac{n+1}{2}\right) \right) \varepsilon^\nu$$

and using $\Gamma\left(\frac{n+1}{2}\right) \leq \left(\frac{n+1}{2}\right)!$ if n is odd, $\Gamma\left(\frac{n+1}{2}\right) \leq 2\left(\frac{n}{2}\right)!$ if n is even, and

$$\sum_{m+n'=\nu} m! n'! \leq 3\nu! ,$$

it follows that J_1 and J_2 are Gevrey-1 in ε .

4.2 Auxiliary statements

In terms of the scalar equations, the equivalence relations of definition 1 can be expressed as follows: We have $(g) \sim (\tilde{g})$ if and only if there exist two functions a and $b \in \mathcal{H}_0$ (i.e. holomorphic in a neighborhood of $(0, 0) \in \mathbb{C}^2$) such that the change of unknowns $\tilde{y} = a(x, \varepsilon)y + \varepsilon b(x, \varepsilon)y'$ transforms equation (g) to equation (\tilde{g}) . The corresponding matrix transformation is given by

$$T = \begin{pmatrix} a & b \\ \varepsilon c & d \end{pmatrix}, \quad c = a' - gb, \quad d = a + \varepsilon b' + 2xb . \quad (19)$$

We have $(g) \approx (\tilde{g})$ if moreover the functions a and b satisfy

$$a(0, \varepsilon) = 1, \quad a'(0, \varepsilon) = b(0, \varepsilon) = b'(0, \varepsilon) = 0$$

for all ε in a neighborhood of 0.

In addition to these equivalence relations, we will use the following intermediate relation. We write $(g) \sim_1 (\tilde{g})$ if $(g) \sim (\tilde{g})$ and if moreover the functions a and b which realize the equivalence satisfy $b(0, \varepsilon) = b'(0, \varepsilon) = 0$ and $a(0, \varepsilon) = 1$.

We write $(I, J) \sim_1 (\tilde{I}, \tilde{J})$ if there exists φ holomorphic in a neighborhood of $\varepsilon = 0$ such that $\tilde{I} = I + \varphi$ and $\tilde{J} = J$, in other words if $(I, J) \sim (\tilde{I}, \tilde{J})$ with $A = D \equiv 1$ and $B \equiv 0$ ($C = \varphi$).

We will show together with the analogues of theorems 2 and 4 for $\mathcal{R}^o, \mathcal{K}^o$ that Φ^o induces a bijection $\Phi_{\sim_1}^o : \mathcal{R}^o / \sim_1 \rightarrow \mathcal{K}^o / \sim_1$. Observe that the last sentence of theorem 4 now actually means that there exists an equation that is linear in ε in each equivalence class of \mathcal{K}^o / \sim_1 .

We now present two elementary results which will be very useful for the sequel. The first one expresses the equivalences between 0-resonant equations in terms of the functions a, b of (19). In the second one we construct functions a, b satisfying these conditions with prescribed initial values at $x = 0$.

Proposition 10. — *Let (g) and (\tilde{g}) be two 0-resonant equations. Recall that \mathcal{H}_0 is the space of functions holomorphic in a neighborhood of $(0, 0) \in \mathbb{C}^2$ and that we require $g(0, 0) = \tilde{g}(0, 0) = 0$.*

1 – *We have $(g) \sim (\tilde{g})$ if and only if there exist two functions a and b in \mathcal{H}_0 that satisfy both relations below at any point (x, ε) (in a neighborhood of $(0, 0)$) :*

$$(W) \quad a^2 + \varepsilon(ab' - a'b) + 2xab + \varepsilon gb^2 = 1, \quad (20)$$

$$(T) \quad \varepsilon a'' - \varepsilon g'b - 2\varepsilon gb' - ag + a\tilde{g} - 2xa' = 0, \quad (21)$$

as well as $a(0, 0) = 1$. As a consequence $a'(0, 0) = -b(0, 0)$.

2 – *We have $(g) \sim_1 (\tilde{g})$ if and only if there exist $a, b \in \mathcal{H}_0$ satisfying (W) and (T) at any (x, ε) small enough and such that $a(0, \varepsilon) = 1$, $b(0, \varepsilon) = b'(0, \varepsilon) = 0$ for all ε small enough. As a consequence $a'(0, 0) = 0$.*

3 – *We have $(g) \approx (\tilde{g})$ if and only if there exist $a, b \in \mathcal{H}_0$ satisfying (W) and (T) for all x, ε small enough and such that $a(0, \varepsilon) = 1$, $b(0, \varepsilon) = b'(0, \varepsilon) = a'(0, \varepsilon) = 0$ for all ε small enough.*

Remarks: 1 – The letters x and ε in formulae (W) and (T) denote the canonical projections as remarked above definition 1.

2 – The above statement has the advantage to avoid the use of the solutions y and \tilde{y} . Of course the equivalence between the equations is given by $\tilde{y} = ay + \varepsilon by'$.

3 – The transformation T of (19) satisfies the differential equation:

$$\varepsilon T' = \tilde{R}T - TR. \quad (22)$$

Proof of proposition 10. 1 – Assume that $(g) \sim (\tilde{g})$ and denote by a, b the functions that realize the equivalence. We can assume that $a(x, 0)$ and $b(x, 0)$ do not both vanish identically; otherwise one can realize the equivalence also by $\frac{1}{\varepsilon}a$ and $\frac{1}{\varepsilon}b$. Equation (22) yields

$$\begin{cases} a' &= c + gb \\ \varepsilon b' &= d - a - 2xb \\ \varepsilon c' &= gd - \tilde{g}a + 2xc \\ d' &= -\tilde{g}b - c \end{cases} \quad (23)$$

The first two equations give $c = a' - gb$ and $d = a + 2xb + \varepsilon b'$. With c and d replaced by their expressions, the third equation yields (T); the last one gives

$$\varepsilon b'' + 2xb' + (\tilde{g} - g + 2)b + 2a' = 0 . \quad (24)$$

Multiplying (24) by a and (T) by b , the difference of both equations is $2aa' + \varepsilon(ab'' - a''b) + 2ab + 2xa'b + 2xab' + \varepsilon g'b^2 + 2\varepsilon gbb' = 0$, which yields by integration

$$a^2 + \varepsilon(ab' - a'b) + 2xab + \varepsilon gb^2 = w(\varepsilon), \quad (25)$$

where w depends only upon ε . This is equation (W) except that 1 is replaced by $w(\varepsilon)$.

Actually, this equation can be directly obtained from the wronskians of (g) and (\tilde{g}) . Both wronskians are multiples of $\exp(x^2/\varepsilon)$, hence necessarily $\det T$ depends only on ε .

Now equation (T) with $\varepsilon = 0$ implies that $a(x, 0)$ satisfies some linear first order differential equation. As $g(0, 0) = \tilde{g}(0, 0) = 0$, we find that $a(x, 0) = C \exp\left(\int_0^x \frac{\tilde{g}(t, 0) - g(t, 0)}{2t} dt\right)$ with some constant C . We claim that $C \neq 0$. Otherwise, $a(x, 0) = 0$ for all x and (24) with $\varepsilon = 0$ implies as above that $b(x, 0) = \frac{D}{x} \exp\left(\int_0^x \frac{g(t, 0) - \tilde{g}(t, 0)}{2t} dt\right)$ with some constant D . Now $D = 0$ would imply that also $b(x, 0) = 0$ for all x which had been excluded at the beginning of the proof and $D \neq 0$ would imply that $b(x, 0)$ has a pole at $x = 0$ which contradicts our definition of equivalence. Thus we have shown that $C \neq 0$ and hence $a(0, 0) = C \neq 0$.

Putting $x = \varepsilon = 0$ in (25), we find that $w(0) = C^2 \neq 0$ and therefore there is a function $t(\varepsilon)$ analytic at $\varepsilon = 0$ with $t(0) = C$ such that $w(\varepsilon) = t(\varepsilon)^2$. Dividing a, b by $t(\varepsilon)$ finally yields one direction of statement 1.

Conversely, if a and b satisfy the assumptions of the theorem, then they satisfy (T) and (24), hence the elements a, b, c, d of the matrix defined by (19) satisfy (23) and T satisfies (22). Hence T realizes a transformation between the matrix equations associated to (g) and (\tilde{g}) , with $T(0, 0)$ invertible.

The statement $a'(0, 0) = -b(0, 0)$ follows by differentiation of (W) and evaluation at $x = \varepsilon = 0$ (and also directly from (24)).

The statements 2 and 3 follow from the first and the conditions imposed on T resp. a, b . □

The following proposition will be useful in the next section for finding 0-resonant equations equivalent to a given one.

Proposition 11. — *Suppose that A, B, C, D analytic in a neighborhood of $\varepsilon = 0$ are given and that $A(0) = D(0) = 1$, $C(0) = -B(0)$ and $AD - \varepsilon BC \equiv 1$. Let g be a function analytic near $x = \varepsilon = 0$ with $g(0, 0) = 0$. Then there exist a, b analytic in a neighborhood of $x = \varepsilon = 0$ satisfying condition (W) of proposition 10 and*

$$\begin{aligned} a(0, \varepsilon) &= A(\varepsilon), \quad a'(0, \varepsilon) = C(\varepsilon) + g(0, \varepsilon)B(\varepsilon) \\ b(0, \varepsilon) &= B(\varepsilon), \quad b'(0, \varepsilon) = \frac{1}{\varepsilon}(D(\varepsilon) - A(\varepsilon)) . \end{aligned} \quad (26)$$

Proof. Let us assume first that a, b exist that satisfy (W) and (26). Then the function $q = \frac{b}{a}$ has to satisfy $q(0, \cdot) = \frac{B}{A}$ and $q'(0, \cdot) = \frac{ab' - ba'}{a^2}(0, \cdot) = L$ with

$$L := \frac{1}{A^2} \left(\frac{1}{\varepsilon} A(D - A) - B(C + RB) \right) , \quad (27)$$

where $R(\varepsilon) = g(0, \varepsilon)$. Therefore we are led to put

$$q(x, \varepsilon) := \frac{B}{A}(\varepsilon) + xL(\varepsilon) + \frac{x^2}{2}Q(\varepsilon)$$

where the coefficient Q of the ‘‘quadratic’’ term will be determined later.

Division of equation (W) by a^2 (non zero in a neighborhood of 0), yields

$$1 + \varepsilon q' + 2xq + \varepsilon gq^2 = a^{-2}, \quad (28)$$

which leads to set $a := (1 + \varepsilon q' + 2xq + \varepsilon gq^2)^{-1/2}$ and $b := aq$.

Now let us check that a, b defined above satisfy the conditions we want for suitably chosen Q . (W) is obviously satisfied. Concerning (26) :

- Formula (28) yields $a(0, \cdot)^{-2} = 1 + \varepsilon L + \varepsilon R \left(\frac{B}{A}\right)^2 = \frac{1}{A^2} (A^2 + A(D - A) - \varepsilon B(C + RB) + \varepsilon RB^2) = \frac{1}{A^2} (AD - \varepsilon BC) = A^{-2}$, hence $a(0, \cdot) = A$ since both these holomorphic functions take the same value 1 at $\varepsilon = 0$.
- We have $b(0, \cdot) = a(0, \cdot)q(0, \cdot) = A \frac{B}{A} = B$.
- Differentiation of (28) yields

$$-2a^{-3}a' = \varepsilon q'' + 2xq' + 2q + \varepsilon g'q^2 + 2\varepsilon gqq' ,$$

therefore $-2A^{-3}a'(0, \cdot) = \varepsilon Q + 2\frac{B}{A} + \varepsilon g'(0, \cdot) \left(\frac{B}{A}\right)^2 + 2\varepsilon R\frac{B}{A}L$ which is of the form $\varepsilon Q + P$ with $P(0) = 2B(0)$. Thus $a'(0, \cdot) = C + RB$ is satisfied, if we choose

$$Q := \frac{1}{\varepsilon} \left(-2A^{-3}(C + RB) - P \right)$$

which is holomorphic, since $A(0) = 1$, $-2C(0) = 2B(0) = P(0)$ and $R(0) = 0$.

- Finally, $b = aq$ gives $b' = a'q + aq'$, hence $b'(0, \cdot) = (C + RB)\frac{B}{A} + AL$. Replacing L by its value (cf. (27)), this gives $b'(0, \cdot) = \frac{1}{A} \left((C + RB)B + \frac{1}{\varepsilon}A(D - A) - B(C + RB) \right) = \frac{1}{\varepsilon}(D - A)$. This completes the verification of (26). \square

5 Formal and analytic classification for the special case

In this section we will show that the quotient functions $\Phi_\varepsilon^o : \mathcal{R}^o / \sim \rightarrow \mathcal{K}^o / \sim$, $\Phi_{\varepsilon_1}^o : \mathcal{R}^o / \sim_1 \rightarrow \mathcal{K}^o / \sim_1$ and $\Phi_\varepsilon^\approx : \mathcal{R}^o / \approx \rightarrow \mathcal{K}^o$ analogous to those of theorems 2, 4 and just above proposition 10 in the case of $f(x, \varepsilon) = 2x$ are well defined and injective. This justifies the definitions of \sim and \sim_1 for the invariants. In the proof, we will use the formal equivalence \approx related to \approx .

For the formal equivalence, g and \tilde{g} are still holomorphic at $(0, 0)$ (and satisfy $g(0, 0) = \tilde{g}(0, 0) = 0$), but the functions a and b which realize the equivalence are only assumed to be in $\mathbb{C}[[x, \varepsilon]]$.

Indeed, thanks to lemma 7 we have the following.

Proposition 12. — *If (g) and (\tilde{g}) are two analytic 0-resonant equations with $(g) \approx (\tilde{g})$, then $(g) \approx (\tilde{g})$. In other words, if there exists $T \in \mathcal{GL}(2, \mathbb{C}[[x, \varepsilon]])$ with $T(0, \varepsilon) = \mathbf{1}$ such that $\tilde{\mathbf{y}} = T\mathbf{y}$ transforms (g) in (\tilde{g}) , then there exist $x_0, \varepsilon_0 > 0$ such that T converges for all $|x| < x_0$ and $|\varepsilon| < \varepsilon_0$.*

Proof. As already mentioned (see remark 3 below proposition 10), T is a solution of the linear singularly perturbed equation (22), considered as a system in \mathbb{C}^{n^2} and rewritten below:

$$\varepsilon T' = \tilde{R}T - TR$$

with the initial condition $T(0, \varepsilon) = \mathbf{1}$. Hence the statement follows from lemma 7. \square

This will be useful in order to prove the following result.

Theorem 13. — *Let (g) and (\tilde{g}) be two analytic 0-resonant equations with invariants respectively (I, J) and (\tilde{I}, \tilde{J}) .*

- 1 – *We have $(g) \approx (\tilde{g})$ if and only if $(I, J) = (\tilde{I}, \tilde{J})$.*
- 2 – *We have $(g) \sim (\tilde{g})$ if and only if $(I, J) \sim (\tilde{I}, \tilde{J})$.*
- 3 – *We have $(g) \sim_1 (\tilde{g})$ if and only if $(I, J) \sim_1 (\tilde{I}, \tilde{J})$.*

Before beginning the proof of this result, we need some further preliminaries.

As before, \hat{y} denotes the only formal resonant solution of (g) with $\hat{y}(0, \varepsilon) = 1$. Now we set

$$Z\hat{y}(x, \varepsilon) = e^{x^2/\varepsilon}\hat{y}(x, \varepsilon)\hat{v}(x, \varepsilon)$$

where $\hat{v}(x, \varepsilon) = \sum_{n \geq 0} v_n(x)\varepsilon^n$ is the only formal solution of

$$\varepsilon v' + 2xv = \varepsilon\hat{y}^{-2}.$$

This solution is determined recursively by $v_0 = 0$ and $2xv_{n+1} = z_n - v'_n$ for $n = 0, 1, \dots$, where z_n are given by $\hat{y}^{-2}(x, \varepsilon) = \sum_{n \geq 0} z_n(x)\varepsilon^n$. The function v_n has a pole of order at most $2n$ at $x = 0$.

This formal expression has the following analytic meaning. Given an analytic resonant solution y of (g) , a second (analytic) solution Zy of (g) is given by

$$Zy(x, \varepsilon) = y(x, \varepsilon) \int_0^x e^{\xi^2/\varepsilon} y(\xi, \varepsilon)^{-2} d\xi.$$

It is easily shown that \hat{v} is the asymptotic expansion of $e^{-x^2/\varepsilon} Zy(x, \varepsilon)/y(x, \varepsilon)$ as $\varepsilon \rightarrow 0$ in the domain $\operatorname{Re}(x^2) > 0$.

We recall the operator $K : \mathbb{C}[[x, \varepsilon]] \rightarrow \mathbb{C}[[\varepsilon^{1/2}]]$ which, to a formal series \hat{f} in x, ε , associates the formal expansion in $\varepsilon^{1/2}$ of $\int_0^{i\delta} e^{x^2/\varepsilon} \hat{f}(x, \varepsilon) dx$.

Precisely, if $\hat{f}(x, \varepsilon) = \sum_{m, n \geq 0} a_{m, n} x^m \varepsilon^n$, then $K\hat{f}$ is given by

$$K\hat{f}(\varepsilon) = \sum_{m, n \geq 0} a_{m, n} \Gamma\left(\frac{m+1}{2}\right) \frac{1}{2} i^{m+1} \varepsilon^{\frac{m+1}{2} + n}.$$

Recall that the second invariant of (g) is $J = K(\hat{y}^{-2})$. The following result will be useful in the proof of theorem 13.

Lemma 14. — 1 – Given $\hat{f} \in \mathbf{C}[[x, \varepsilon]]$, we have $K\hat{f} = 0$ if and only if there exists $\hat{g} \in \mathbf{C}[[x, \varepsilon]]$ such that $\hat{g}(0, \varepsilon) = 0$ and $\varepsilon\hat{g}' + 2x\hat{g} = \hat{f}$.

2 – More generally, for all $\hat{f} \in \mathbf{C}[[x, \varepsilon]]$ there exist unique formal series $\hat{g} \in \mathbf{C}[[x, \varepsilon]]$ and $\hat{a} \in \mathbf{C}[[\varepsilon]]$ such that

$$\varepsilon\hat{g}' + 2x\hat{g} = \hat{f} - \hat{a} .$$

3 – If $\hat{f}, \hat{a}, \hat{g}$ are as above then we have $K\hat{f} = \frac{i}{2}\sqrt{\pi\varepsilon} \hat{a} - \varepsilon\hat{g}(0, \cdot)$.

Proof. 2 – Let $\hat{f} \in \mathbf{C}[[x, \varepsilon]]$. Writing $\hat{f}(x, \varepsilon) = \sum_{n \geq 0} f_n(x)\varepsilon^n$, $\hat{g}(x, \varepsilon) = \sum_{n \geq 0} g_n(x)\varepsilon^n$, with $f_n, g_n \in \mathbf{C}[[x]]$, and $\hat{a}(\varepsilon) = \sum_{n \geq 0} a_n\varepsilon^n$, we obtain the equations

$$2xg_0(x) = f_0(x) - a_0$$

and for $n \geq 1$,

$$g'_{n-1}(x) + 2xg_n(x) = f_n(x) - a_n .$$

The condition $g_n \in \mathbf{C}[[x]]$ uniquely determines a_n , and we obtain a_n and g_n recursively:

$$a_0 = f_0(0) , \quad g_0(x) = \frac{1}{2x} (f_0(x) - a_0) ,$$

and for $n \geq 1$,

$$a_n = f_n(0) - g'_{n-1}(0) , \quad g_n(x) = \frac{1}{2x} (f_n(x) - a_n - g'_{n-1}(x)) .$$

3 – Straightforward computation shows that $K(\varepsilon\hat{g}' + 2x\hat{g}) = -\varepsilon\hat{g}(0, \varepsilon)$ if $\hat{g}(x, \varepsilon) = x^m$ for some $m \in \mathbb{N}$. As K apparently is $\mathbf{C}[[\varepsilon]]$ -linear and continuous w.r.t. the topology of formal series in x, ε , this proves the formula for all \hat{g} (one could also use the fact that $(K\hat{f})(\varepsilon)$ is the formal expansion of $\int_0^{i\delta} e^{x^2/\varepsilon} \hat{f}(x, \varepsilon) dx$). Now $K\hat{a}(\varepsilon) = \frac{i}{2}\sqrt{\pi\varepsilon} \hat{a}(\varepsilon)$ and hence $K\hat{f}(\varepsilon) = -\varepsilon\hat{g}(0, \varepsilon) + \frac{i}{2}\sqrt{\pi\varepsilon} \hat{a}(\varepsilon)$ follows from the preceding statement.

Statement 1 follows obviously from the 2 and 3. \square

Proof of theorem 13. Recall that \hat{y} is the unique formal solution of (g) with $\hat{y}(0, \varepsilon) = 1$ and that the equivalence relation $\tilde{y} = ay + \varepsilon by'$ between the equations implies (cf. also (22) and (23)) that also

$$\tilde{y}' = (a' - gb)y + (a + 2xb + \varepsilon b')y' . \quad (29)$$

Of course, $\hat{z} = a\hat{y} + \varepsilon b\hat{y}'$ is a non-trivial formal resonant solution of \tilde{g} , but it does not necessarily satisfy $\hat{z}(0, \varepsilon) = 1$ in the cases of \sim , \sim_1 .

We first prove that (I, J) is indeed an invariant for the relation \approx . If $(g) \approx (\tilde{g})$ then we have $a(0, \varepsilon) = 1, a'(0, \varepsilon) = b(0, \varepsilon) = b'(0, \varepsilon) = 0$, hence $\tilde{y} = a\hat{y} + \varepsilon b\hat{y}'$ is the formal resonant solution of (\tilde{g}) with $\tilde{y}(0, \varepsilon) = 1$. Thus $\tilde{I} = \tilde{y}'(0, \varepsilon) = \hat{y}'(0, \varepsilon) = I$. In order to show that $J = \tilde{J}$, it suffices to prove that the formal expression $K\hat{f}$ vanishes, where

$$\hat{f} := \tilde{y}^{-2} - \hat{y}^{-2} = (a\hat{y} + \varepsilon b\hat{y}')^{-2} - \hat{y}^{-2} .$$

In order to show this, put $\hat{g} := \frac{b}{\hat{y}\tilde{y}} = \frac{b}{\hat{y}(a\hat{y} + \varepsilon b\hat{y}')}$. A short calculation shows that $\varepsilon\hat{g}' + 2x\hat{g} = \hat{f}$, hence by lemma 14 we have $K\hat{f}(\varepsilon) = -\varepsilon\hat{g}(0, \varepsilon) = -\varepsilon b(0, \varepsilon)/\tilde{y}(0, \varepsilon) = 0$. Note that the formula

$$K(\tilde{y}^{-2} - \hat{y}^{-2})(\varepsilon) = -\varepsilon b(0, \varepsilon)/\tilde{y}(0, \varepsilon) \quad (30)$$

is still valid for relations \sim and \sim_1 .

Suppose now that $(g) \sim_1 (\tilde{g})$. Then we still have $a(0, \varepsilon) = 1, b(0, \varepsilon) = b'(0, \varepsilon) = 0$ but not necessarily $a'(0, \varepsilon) = 0$. Hence we have $\tilde{I}(\varepsilon) = \tilde{y}'(0, \varepsilon) = a'(0, \varepsilon) + I(\varepsilon)$. On the other hand, we still have $(\tilde{J} - J)(\varepsilon) = K(\tilde{y}^{-2} - y^{-2})(\varepsilon) = 0$. Thus we obtain $(\tilde{I}, \tilde{J}) \sim_1 (I, J)$ in this case. Observe that proposition 10 yields that $a'(0, 0) = 0$; this also follows from $(I, J) \sim_1 (\tilde{I}, \tilde{J})$ and $(I, J), (\tilde{I}, \tilde{J}) \in \mathcal{K}^o$.

In the case of $(g) \sim (\tilde{g})$, proposition 10 still allows us to assume that (W) holds and $a(0, 0) = 1$ and $a'(0, 0) = -b(0, 0)$. The formula $\tilde{y}(x, \varepsilon) = a(x, \varepsilon)\hat{y}(x, \varepsilon) + \varepsilon b(x, \varepsilon)\hat{y}'(x, \varepsilon)$ no longer gives the formal resonant solution of (\tilde{g}) reducing to 1 at $x = 0$; it is now given by $\tilde{z}(x, \varepsilon) = \frac{\tilde{y}(x, \varepsilon)}{\lambda(\varepsilon)}$ where $\lambda(\varepsilon) = \tilde{y}(0, \varepsilon) = a(0, \varepsilon) + b(0, \varepsilon)I(\varepsilon)$. Using (29), we find

$$\tilde{I}(\varepsilon) = \tilde{z}'(0, \varepsilon) = \frac{1}{\lambda(\varepsilon)} [(a'(0, \varepsilon) - g(0, \varepsilon)b(0, \varepsilon)) + (a(0, \varepsilon) + \varepsilon b'(0, \varepsilon))I(\varepsilon)].$$

With $A(\varepsilon) = a(0, \varepsilon), B(\varepsilon) = b(0, \varepsilon), C(\varepsilon) = a'(0, \varepsilon) - g(0, \varepsilon)b(0, \varepsilon)$ and $D(\varepsilon) = a(0, \varepsilon) + \varepsilon b'(0, \varepsilon)$, we obtain the first part of (8). Moreover by (30), $\tilde{J} = K(\lambda^2\tilde{y}^{-2}) = \lambda^2 K(\tilde{y}^{-2}) = \lambda^2 (K(\hat{y}^{-2}) - \varepsilon b(0, \varepsilon)/\tilde{y}(0, \varepsilon)) = \lambda^2 (J - \varepsilon B/\lambda)$, which corresponds to the second part of (8). The conditions for A, B, C, D required in (8) follow from the statements at the beginning of this paragraph.

Conversely, let 0-resonant equations (g) and (\tilde{g}) with $I = \tilde{I}$ and $J = \tilde{J}$ be given. We have to find a transformation T between the corresponding matrix equations (R) and (\tilde{R}) whose entries are formal series in x, ε that satisfies $T(0, \varepsilon) = 1$. We recall that \hat{v} denotes the only formal series solution of

$$\varepsilon v' + 2xv = \varepsilon \hat{y}(x, \varepsilon)^{-2}$$

(its coefficients may have poles at $x = 0$). A formal fundamental solution of (R) is given by:

$$Y = \begin{pmatrix} \hat{y} & \hat{z} \\ \varepsilon \hat{y}' & \varepsilon \hat{z}' \end{pmatrix}$$

where $\hat{z} := Z\hat{y} = e^{x^2/\varepsilon}\hat{y}\hat{v}$. A brief calculation shows that

$$Y = \begin{pmatrix} \hat{y} & 0 \\ \varepsilon \hat{y}' & \varepsilon \hat{y}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \hat{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{x^2/\varepsilon} \end{pmatrix}.$$

An analogous formula holds for a formal fundamental solution \tilde{Y} of (\tilde{R}) . Hence a transformation between both equations, given by $T := \tilde{Y}Y^{-1}$, can be written as

$$T = \begin{pmatrix} \tilde{y} & 0 \\ \varepsilon \tilde{y}' & \varepsilon \tilde{y}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \tilde{v} - v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ \varepsilon y' & \varepsilon y^{-1} \end{pmatrix}^{-1}.$$

It remains to show that the coefficients of ε^n of the entries of T have no poles at $x = 0$. Since $K(y^{-2}) = J = \tilde{J} = K(\tilde{y}^{-2})$, by lemma 14 there is $h \in \mathbb{C}[[x, \varepsilon]]$ such that $h(0, \varepsilon) = 0$

and $\varepsilon h' + 2xh = \tilde{y}^{-2} - y^{-2}$. Since $g := \frac{1}{\varepsilon}(\tilde{v} - v)$ satisfies $\varepsilon g' + 2xg = \tilde{y}^{-2} - y^{-2}$, too, we conclude that $u := g - h \in \mathbb{C}[[x, \varepsilon]]$ is a solution of $\varepsilon u' + 2xu = 0$, therefore $u = 0$. It follows that $g = h \in \mathbb{C}[[x, \varepsilon]]$ and $g(0, \varepsilon) = 0$. We obtain that

$$T = \begin{pmatrix} \tilde{y} & 0 \\ \varepsilon \tilde{y}' & \tilde{y}^{-1} \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ -\varepsilon y' & y \end{pmatrix}$$

has entries in $\mathbb{C}[[x, \varepsilon]]$. Moreover, $T(0, \varepsilon) = \begin{pmatrix} 1 & 0 \\ \varepsilon \tilde{I}(\varepsilon) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varepsilon I(\varepsilon) & 1 \end{pmatrix} = \mathbf{1}$ since $I = \tilde{I}$. This proves that $(g) \approx (\tilde{g})$ and thus by proposition 12, we obtain $g \approx (\tilde{g})$.

Now let (g) and (\tilde{g}) two 0-resonant equations such that their invariants satisfy $(I, J) \sim (\tilde{I}, \tilde{J})$ (resp. $(I, J) \sim_1 (\tilde{I}, \tilde{J})$). We show below

Lemma 15 *Under the above conditions, there exists a 0-resonant equation (\bar{g}) with invariants (\tilde{I}, \tilde{J}) satisfying $(\bar{g}) \sim (g)$ (resp. $(\bar{g}) \sim_1 (g)$).*

As this new equation (\bar{g}) and (\tilde{g}) have the same invariants, we have shown above that $(\tilde{g}) \approx (\bar{g})$. A fortiori $(\tilde{g}) \sim (\bar{g})$ and hence $(\tilde{g}) \sim (g)$ by transitivity. The proof for \sim_1 is the same.

It remains to prove lemma 15. The hypothesis of the lemma means that there exist A, B, C, D analytic in a neighborhood of $\varepsilon = 0$ satisfying $AD - \varepsilon BC \equiv 1$ and $A(0) = D(0) = 1$ such that (8) is satisfied. Observe that this and $(I, J), (\tilde{I}, \tilde{J}) \in \mathcal{K}^o$ imply that $C(0) = \tilde{I}_0 - I_0 = -B(0)$. As seen above, it is sufficient to find an analytic equation $(\bar{g}) \sim (g)$ such that the equivalence between them is realized by a, b satisfying $A(\varepsilon) = a(0, \varepsilon), B(\varepsilon) = b(0, \varepsilon), C(\varepsilon) = a'(0, \varepsilon) - g(0, \varepsilon)b(0, \varepsilon)$ and $D(\varepsilon) = a(0, \varepsilon) + \varepsilon b'(0, \varepsilon)$ as well as the conditions of proposition 10.

By proposition 11, we can find a, b with the above initial conditions satisfying (W). Condition (T) now uniquely determines a function \bar{g} analytic near $x = \varepsilon = 0$ with $\bar{g}(0, 0) = 0$. Thus (\bar{g}) is an equation equivalent to (g) such that the equivalence is realized by a, b , its invariants are related to the ones for (g) by (8) and hence they are equal to (\tilde{I}, \tilde{J}) .

The proof for \sim_1 is analogous. □

6 Construction of 0-resonant equations in the special case

In this section, we prove the surjectivity of the quotient functions $\Phi_\sim^o : \mathcal{R}^o / \sim \rightarrow \mathcal{K}^o / \sim$, $\Phi_{\sim_1}^o : \mathcal{R}^o / \sim_1 \rightarrow \mathcal{K}^o / \sim_1$ and $\Phi_{\approx}^o : \mathcal{R}^o / \approx \rightarrow \mathcal{K}^o$ analogous to those of theorem 2, 4 and below proposition 10 in the case $f(x, \varepsilon) = 2x$. In other words, we prove the existence of 0-resonant equations (g) with a prescribed pair of invariants.

Thus let $(I, J) \in \mathcal{K}^o$ be given, namely a Gevrey-1 series $I = I_0 + O(\varepsilon)$ and a series J in $\varepsilon^{1/2}$ of the form $J = J_1 + \varepsilon^{1/2} J_2$, with J_1, J_2 Gevrey-1, whose first terms are given by $J(\varepsilon) = \frac{i}{2} \sqrt{\pi \varepsilon} + I_0 \varepsilon + O(\varepsilon^{3/2})$.

We first prove that there exists a function φ , analytic (i.e. convergent) in a neighborhood of $\varepsilon = 0$ with $\varphi(0) = 0$, and an equation (g) with invariants $(I + \varphi, J)$. In other words, we first prove the surjectivity of Φ_\sim^o and $\Phi_{\sim_1}^o$. The surjectivity of $\Phi_\sim^o : \mathcal{R}^o / \approx \rightarrow \mathcal{K}^o$

will then result from lemma 15 of section 5.
Then we show that the equation is linear in ε .

As in section 3 (with $h = 1$) we consider, for $j = 0, 1, 2$, the three sectors

$$S_j = \left\{ \varepsilon \in \mathbb{C} ; 0 < |\varepsilon| < \varepsilon_0, \left| \arg \varepsilon - j \frac{2\pi}{3} \right| < \frac{\pi}{2} - \mu \right\}$$

where $\varepsilon_0, \mu > 0$ are small enough. \mathcal{U} denotes the Banach space of triples $\vec{y} = (y_0, y_1, y_2)$ of functions $y_j : S_j \rightarrow \mathbb{C}$ that are holomorphic and bounded, with the norm $\|\vec{y}\| = \max(\|y_0\|_{0,\infty}, \|y_1\|_{1,\infty}, \|y_2\|_{2,\infty}), \|y\|_{j,\infty} := \sup\{|y(\varepsilon)| ; \varepsilon \in S_j\}$.

With $D_j := S_j \cap S_{j+1}$ ($S_3 = S_0$), \mathcal{D} denotes the Banach space of triples $\vec{d} = (d_0, d_1, d_2)$ of holomorphic functions $d_j : D_j \rightarrow \mathbb{C}$ such that $d_j(\varepsilon)/\varepsilon$ is bounded, with the norm

$$\|\vec{d}\| = \max_{j=0,1,2} \sup_{\varepsilon \in D_j} \left| \frac{d_j(\varepsilon)}{\varepsilon} \right|$$

(identified to the corresponding space of holomorphic functions on $D_0 \cup D_1 \cup D_2$).

Δ denotes the difference operator $\Delta : \mathcal{U}_0 \rightarrow \mathcal{D}$, $\vec{y} \mapsto \vec{d}$ with $d_j = y_{j+1} - y_j$. As in section 3, \mathcal{U}_0 is the subset of \mathcal{U} of all \vec{y} such that $\Delta\vec{y} \in \mathcal{D}$. We recall the right inverse of Δ of section 3: $\Sigma : \mathcal{D} \rightarrow \mathcal{U}_0$, $\vec{d} \mapsto \Sigma\vec{d}$ defined by

$$\Sigma\vec{d}(\varepsilon) = \frac{12}{\pi} \sum_{j=0}^2 \int_{\frac{(2j+1)\pi - \pi}{24}}^{\frac{(2j+1)\pi + \pi}{24}} \left(\frac{1}{2\pi i} \int_0^{\varepsilon_0 e^{i\theta}} d_j(\mu) \frac{d\mu}{\mu - \varepsilon} \right) d\theta .$$

We recall the statement of theorem 5:

- 1 – Σ is bounded and satisfies $\Delta\Sigma = id : \mathcal{D} \rightarrow \mathcal{D}$.
- 2 – We have $\ker \Delta = \mathcal{H}_{\varepsilon_0}$, the space of holomorphic and bounded functions on $D(0, \varepsilon_0)$.
- 3 – For all $\vec{g} \in \mathcal{U}$ we have $\Sigma\Delta\vec{g} - \vec{g} \in \mathcal{H}_{\varepsilon_0}$.
- 4 – The operator L defined from \mathcal{D} into \mathcal{U} by

$$L\vec{d} := \varepsilon\Sigma(\vec{d}) - \Sigma(\varepsilon\vec{d}) \tag{31}$$

takes its values in \mathbb{C} . In other words, for all $\vec{d} \in \mathcal{D}$ the function $L\vec{d}$ is constant with respect to ε .

Now we set $B_\rho = \{x \in \mathbb{C} ; |x| < \rho\}$, where $\rho > 0$ will be determined later. \mathcal{U}_ρ and \mathcal{D}_ρ denote the spaces of functions of both variables x and ε analogous to \mathcal{U} and \mathcal{D} :

$$\mathcal{U}_\rho = \{\vec{y} = (y_0, y_1, y_2) ; y_j : B_\rho \times S_j \rightarrow \mathbb{C} \text{ holomorphic bounded, } j = 0, 1, 2\}$$

with the max-sup norm on j, x and ε ,

$$\mathcal{D}_\rho = \{\vec{d} = (d_0, d_1, d_2) ; d_j : B_\rho \times D_j \rightarrow \mathbb{C} \text{ holomorphic, } d_j(x, \varepsilon)/\varepsilon \text{ bounded, } j = 0, 1, 2\}$$

with the corresponding norm, and similarly $\mathcal{U}_{\rho,0}$ is the subset of \mathcal{U}_ρ of all \vec{y} such that $\Delta\vec{y} \in \mathcal{D}_\rho$. The operator Σ naturally induces a bounded linear operator also noted $\Sigma : \mathcal{D}_\rho \rightarrow \mathcal{U}_\rho$. The above statements naturally carry over to \mathcal{U}_ρ and \mathcal{D}_ρ . For instance, for all $\vec{d} \in \mathcal{D}_\rho$ the function $L\vec{d} = \varepsilon\Sigma(\vec{d}) - \Sigma(\varepsilon\vec{d})$ is constant with respect to ε .

The following Banach space will also be useful. Fix $\delta > 0$ and let $\mathcal{G}_{\rho,\delta}$ be the set of all $\vec{u} \in \mathcal{U}_\rho$ such that there exists $L > 0$ satisfying

$$|(u_{j+1} - u_j)(x, \varepsilon)| \leq L \exp(-(\rho^2 + \delta)/|\varepsilon|) \text{ for all } j, x \in B_\rho, \varepsilon \in D_j .$$

equipped with the norm $\|\vec{u}\|_{\mathcal{G}_{\rho,\delta}} = \max(\|\vec{u}\|_{\mathcal{U}_\rho}, G_{\rho,\delta}(\vec{u}))$ where

$$G_{\rho,\delta}(\vec{u}) = \sup\{|(u_{j+1} - u_j)(x, \varepsilon)|e^{(\rho^2 + \delta)/|\varepsilon|} ; j = 0, 1, 2, x \in B_\rho, \varepsilon \in D_j\} . \quad (32)$$

We recall the operator $K : C[[x, \varepsilon]] \rightarrow \mathbb{C}[[\varepsilon^{1/2}]]$ that associates, to a series $g = \sum_{n=0}^{\infty} g_n(x)\varepsilon^n$, the expansion in $\varepsilon^{1/2}$ of the expression

$$\sum_{n=0}^{\infty} \int_0^{i\infty} e^{x^2/\varepsilon} g_n(x) dx \varepsilon^n .$$

As indicated in subsection 4.1, Kg can be written $Kg = K_1g + \varepsilon^{1/2}K_2g$, where K_1g and K_2g are formal series in ε , K_1g without constant term. Recall also that, given a 0-resonant equation (g) , its second invariant is $J = K(\hat{y}^{-2})$, where \hat{y} is a formal 0-resonant solution of (g) satisfying $\hat{y}(0, \varepsilon) = 1$. By misuse of notation, we apply K also to functions of x, ε having an asymptotic expansion as $\varepsilon \rightarrow 0$ (and to triples of functions having a common asymptotic expansion, for example elements of $\mathcal{G}_{\rho,\delta}$).

Let k denote the restriction of K to functions of x only. The following properties of k and K will be useful:

Lemma 16. — *The operator k maps $\mathbb{C}\{x\}$ into $\varepsilon\mathbb{C}[[\varepsilon]]_1 + \sqrt{\varepsilon}\mathbb{C}[[\varepsilon]]_1$ and is bijective. The linear operator $k^{-1}K$ maps $\mathcal{G}_{\rho,\delta}$ into \mathcal{H}_ρ and satisfies*

$$\|k^{-1}K\vec{u}\| \leq \left(\frac{\rho^2}{\delta} + \frac{1}{2}\right) G_{\rho,\delta}(\vec{u}) + \left(1 + 2\frac{\rho^2}{\varepsilon_0}e^{\rho^2/\varepsilon_0}\right) \|\vec{u}\|_{\mathcal{U}_\rho} .$$

Proof. Since

$$\int_0^{i\infty} e^{x^2/\varepsilon} x^n dx = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) i^{n+1} \varepsilon^{\frac{n+1}{2}} ,$$

we have

$$k\left(\sum_{n \geq 0} g_n x^n\right) = \sum_{\nu \geq 0} \frac{g_{2\nu+1}}{2} \Gamma(\nu+1) (-1)^{\nu+1} \varepsilon^{\nu+1} + \sqrt{\varepsilon} \left(\sum_{\nu \geq 0} \frac{g_{2\nu}}{2} \Gamma\left(\nu + \frac{1}{2}\right) i (-1)^\nu \varepsilon^\nu\right)$$

which clearly shows that the correspondence between convergent power series in x and pairs of Gevrey-1 series in ε is one-to-one.

In order to prove the second statement, we first have to find a relation between a triple $\vec{u} \in \mathcal{G}_{\rho,\delta}$ and the right hand side $\hat{u}(x, \varepsilon) = \sum_{n=0}^{\infty} \tilde{u}_n(x)\varepsilon^n$ of its common asymptotic expansion. This is done by the well known Cauchy–Heine formula (cf. [10]); here we find

$$\tilde{u}_n(x) = \frac{1}{2\pi i} \sum_{j=1}^3 \int_{T_{j-1}}^{T_j} \frac{u_j(x, z)}{z^{n+1}} dz + \frac{1}{2\pi i} \sum_{j=1}^3 \int_0^{T_j} \frac{u_{j+1}(x, z) - u_j(x, z)}{z^{n+1}} dz \quad (33)$$

where $T_j \in \mathcal{S}_j \cap \mathcal{S}_{j+1}$ have modulus ε_0 , and the paths from 0 to T_j are line segments, those from T_{j-1} to T_j are close to the circular arcs between the two points.

A straightforward estimate implies for $n \geq 1$

$$|\tilde{u}_n(x)| \leq \|\vec{u}\|_{\mathcal{U}_\rho} \varepsilon_0^{-n} + \frac{3}{2\pi}(n-1)!(\rho^2 + \delta)^{-n} G_{\rho,\delta}(\vec{u}) \quad (34)$$

as well as $|\tilde{u}_0(x)| \leq \|\vec{u}\|_{\mathcal{U}_\rho} + \frac{3}{2\pi} G_{\rho,\delta}(\vec{u})$ for all $x \in B_\rho$ if $\varepsilon_0 \leq \rho^2 + \delta$.

Denote now by $S : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$ the operator given by

$$S(\varphi)(x) := -2x \int_0^x \varphi(\xi) d\xi$$

and by S^n its n -th iterate. With this notation, a short calculation shows that

$$k^{-1}K(\varepsilon^n x^m) = (-1)^n \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+1}{2} + n\right)} x^{2n+m} = S^n(x^m).$$

Hence $k^{-1}K(\hat{u}(x, \varepsilon)) = k^{-1}K(\sum_{n \geq 0} u_n(x) \varepsilon^n) = \sum_{n \geq 0} S^n(u_n)(x)$. Another short calculation shows that for all $v \in \mathcal{H}_\rho$, $n \geq 1$

$$|S^n(v)(x)| \leq \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} |x|^{2n} \|v\| \leq \frac{2}{(n-1)!} \rho^{2n} \|v\|. \quad (35)$$

Together with (34) we obtain

$$\begin{aligned} \|k^{-1}K(\hat{u})\| &\leq \|\vec{u}\|_{\mathcal{U}_\rho} + \frac{3}{2\pi} G_{\rho,\delta}(\vec{u}) + \sum_{n \geq 1} \frac{2}{(n-1)!} \rho^{2n} \left(\|\vec{u}\|_{\mathcal{U}_\rho} \varepsilon_0^{-n} + \frac{3}{2\pi} (n-1)!(\rho^2 + \delta)^{-n} G_{\rho,\delta}(\vec{u}) \right) \\ &\leq \left(1 + 2 \frac{\rho^2}{\varepsilon_0} e^{\rho^2/\varepsilon_0}\right) \|\vec{u}\|_{\mathcal{U}_\rho} + \left(\sum_{n \geq 1} \left(\frac{\rho^2}{\rho^2 + \delta}\right)^n + \frac{1}{2} \right) G_{\rho,\delta}(\vec{u}). \end{aligned}$$

This finally yields the wanted estimate. \square

Let us now consider both invariants I, J . From I we construct, by the classical truncated Borel-Laplace transform, a triple $\vec{I} \in \mathcal{U}$ (reducing ε_0 if necessary) of functions having I as common asymptotic expansion. Let $\vec{d} := \Delta \vec{I}$. It is known too that \vec{d} decreases exponentially as $\varepsilon \rightarrow 0$ (cf. [2] for details). Precisely there exists $\gamma > 0$ such that

$$\forall \varepsilon \in D_0 \cup D_1 \cup D_2, \quad |\vec{d}(\varepsilon)| \leq |\varepsilon| e^{-\gamma/|\varepsilon|} \quad (36)$$

(the factor $|\varepsilon|$ will be useful to fit with the norm of \mathcal{D}).

Assume for the moment that some $\vec{u} \in \mathcal{U}_\rho$ exists that satisfies

$$\vec{u} = f + \Sigma(\vec{d} \cdot Z\vec{u}) - x(\Sigma \vec{d})(0) \quad (37)$$

with $\vec{d} \cdot Z\vec{u} := (d_j Z u_j)_{j=0,1,2}$ and where $f \in \mathcal{H}_\rho$, the space of holomorphic functions on $B_\rho = D(0, \rho)$ (identified to the space of functions of \mathcal{U}_ρ constant with respect to ε), satisfies $f(0) = 1$ and $f'(0) = I_0$, the first term of I . Recall that Zu is defined by $Zu(x, \varepsilon) = u(x, \varepsilon) \int_0^x e^{\xi^2/\varepsilon} u(\xi, \varepsilon)^{-2} d\xi$.

In that case, we have $\Delta \vec{u} = \vec{d} \cdot Z\vec{u}$. We then deduce that the function

$$g(x, \varepsilon) := -\varepsilon \frac{\vec{u}''}{\vec{u}}(x, \varepsilon) + 2x \frac{\vec{u}'}{\vec{u}}(x, \varepsilon) \quad (38)$$

is holomorphic in a neighborhood of $(x, \varepsilon) = (0, 0)$, i.e. it is single valued with respect to ε and remains bounded as $\varepsilon \rightarrow 0$. Indeed, we can define functions $g_j : B_\rho \times S_j \rightarrow \mathbb{C}$ by $g_j(x, \varepsilon) := -\varepsilon \frac{u_j''}{u_j}(x, \varepsilon) + 2x \frac{u_j'}{u_j}(x, \varepsilon)$. Then u_j satisfy the differential equations $\varepsilon u_j'' - 2x u_j' + g_j(x, \varepsilon) u_j = 0$ for $\varepsilon \in S_j$. On the other hand, we know that Zu_j satisfies the same differential equation as u_j and thus, for $\varepsilon \in D_j$, u_j and $u_{j+1} = u_j + d_j(\varepsilon)Zu_j$ satisfy the same equation. Hence $g_j(x, \varepsilon) = g_{j+1}(x, \varepsilon)$ for $x \in B_\rho$, $\varepsilon \in D_j$; here $j = 0, 1, 2$. Hence the three functions g_j define one analytic function $g : B_\rho \times D(0, \varepsilon_0) \rightarrow \mathbb{C}$.

Since $\Delta(\vec{u}'(0, \varepsilon)) = (\Delta\vec{u})'(0, \varepsilon) = (\vec{d} \cdot Z\vec{u})'(0, \varepsilon) = \vec{d}'(\varepsilon) = \Delta\vec{I}'(\varepsilon)$, statement 3 of theorem 5 implies that $\varphi(\varepsilon) := \vec{u}'(0, \varepsilon) - \vec{I}'(\varepsilon) \in \mathbb{C}\{\varepsilon\}$; moreover the ‘‘correction’’ term in (37) yields $u'(0, 0) = f'(0) = I_0$.

This means, that for any analytic function f , if we find a solution \vec{u} of (37), then the equation (g), g defined by (38), has as first invariant the series expansion of $\vec{u}'(0, \varepsilon)$ which is equal to $I(\varepsilon) + \varphi(\varepsilon)$ where $\varphi \in \mathbb{C}\{\varepsilon\}$ with $\varphi(0) = 0$.

In order to solve (37), we consider it as a fixed point equation. Let us introduce the following subsets.

$$U := \{\vec{u} \in \mathcal{U}_\rho ; \|\vec{u} - 1\| \leq 1/2\} .$$

$$F := \{f \in \mathcal{H}_\rho ; f(0) = 1, f'(0) = I_0, \|f - 1\| \leq 1/4\} .$$

Of course, we only consider $\rho < \frac{1}{4|I_0|}$; otherwise F would be empty.

Lemma 17. — *Let $\gamma > 0$ satisfy (36) and $\rho^2 < \gamma$, then for $f \in F$ and ε_0 small enough the operator $\Psi : \vec{u} \mapsto f + \Sigma(\vec{d} \cdot Z\vec{u}) - x\Sigma(\vec{d})(0)$ is a contraction in U .*

Proof. For simplicity we omit the arrow $\vec{}$ on u and d in this proof.

Given $u \in U$, we have $\|u\| \leq \frac{3}{2}$ and $\|u^{-2}\| \leq 4$, hence

$$|Zu(x, \varepsilon)| \leq \|u\| \int_0^\rho e^{\xi^2/|\varepsilon|} \|u^{-2}\| d\xi \leq 6\rho e^{\rho^2/|\varepsilon|} . \quad (39)$$

We obtain using (36) that $\|d \cdot Zu\|_{\mathcal{D}} \leq 6\rho e^{(\rho^2 - \gamma)/\varepsilon_0}$ and $\|d\|_{\mathcal{D}} \leq e^{-\gamma/\varepsilon_0}$. If $M > 0$ denotes a bound for $\|\Sigma\|$ given by theorem 5, then this yields

$$\|\Psi(u) - 1\| = \|f - 1 + \Sigma(d \cdot Zu) - x\Sigma(d)(0)\| \leq \frac{1}{4} + 7M\rho e^{(\rho^2 - \gamma)/\varepsilon_0}$$

which is at most $\frac{1}{2}$ if $|\varepsilon| < \varepsilon_0 \leq \frac{\gamma - \rho^2}{\ln(28M\rho)}$ (in the case $28M\rho < 1$, there is no restriction on ε_0). So $\Phi^o(u) \in U$ for ε_0 sufficiently small.

Given $u_1, u_2 \in U$, we first write $Zu_1 - Zu_2$ in the form

$$\begin{aligned} (Zu_2 - Zu_1)(x, \varepsilon) &= (u_2(x, \varepsilon) - u_1(x, \varepsilon)) \int_0^x e^{\xi^2/\varepsilon} u_2^{-2}(\xi, \varepsilon) d\xi - \\ &u_1(x, \varepsilon) \int_0^x e^{\xi^2/\varepsilon} \left(\frac{1}{u_1(\xi, \varepsilon)u_2^2(\xi, \varepsilon)} + \frac{1}{u_1^2(\xi, \varepsilon)u_2(\xi, \varepsilon)} \right) (u_2(\xi, \varepsilon) - u_1(\xi, \varepsilon)) d\xi \end{aligned}$$

and find

$$|(Zu_2 - Zu_1)(x, \varepsilon)| \leq \int_0^x e^{|\xi|^2/|\varepsilon|} |d\xi| \left(4 + \frac{3}{2} \times (8 + 8) \right) \|u_2 - u_1\| \leq 28\rho e^{\rho^2/|\varepsilon|} \|u_2 - u_1\| .$$

This gives

$$\|\Psi(u_2) - \Psi(u_1)\| = \|\Sigma(d \cdot (Zu_2 - Zu_1))\| \leq 28M\rho e^{(\rho^2 - \gamma)/\varepsilon_0} \|u_2 - u_1\| .$$

Hence Ψ is a contraction provided that ε_0 is small enough, namely $\varepsilon_0 < \frac{\gamma - \rho^2}{\ln(28M\rho)}$ (in the case $28M\rho > 1$ only; otherwise no restriction for ε_0 is needed). \square

Thus for $f \in F$, equation (37) has a unique solution $\vec{u} \in U$ provided ε_0 is small enough. Thus we can consider now the operator $\vec{P} : F \rightarrow U$ which associates to any $f \in F$ the corresponding solution $\vec{u} \in U$ of (37).

In order to construct a 0-resonant equation with a pair of invariants of the form $(I + \varphi, J)$, it thus suffices to find $f \in F$ such that $K((\vec{P}f)^{-2}) = J$. Using the inverse k^{-1} introduced above, this amounts to finding f such that

$$\left((k^{-1}K)(\vec{P}f)^{-2} \right)^{-1/2} = g_0 \quad \text{with} \quad g_0 := (k^{-1}J)^{-1/2}. \quad (40)$$

We will show that N defined by $Nf := \left((k^{-1}K)(\vec{P}f)^{-2} \right)^{-1/2}$ is close to the identity, if ε_0 is sufficiently small and conclude that $Nf = g_0$ has a solution in F . Observe that the admissibility of (I, J) implies that $g_0(0) = 1$ and $g_0'(0) = I_0$; thus $g_0 \in F$ and even $\|g_0 - 1\| \leq \frac{1}{16}$ if ρ is chosen small enough.

From now on, we fix $\rho, \delta > 0$ satisfying additionally $2\rho^2 + \delta < \gamma$. First as in the proof of lemma 17, we show that $\|\vec{P}f - f\|_{\mathcal{U}_\rho} \leq 7M\rho e^{(\rho^2 - \gamma)/\varepsilon_0}$. Using that $\vec{u} = \vec{P}f$ satisfies (37), we find that (using $G_{\rho, \delta}$ defined in (32))

$$G_{\rho, \delta}(\vec{P}f - f) = G_{\rho, \delta}(\vec{P}f) = \sup\{|d_j(\varepsilon)(Zu_j)(x, \varepsilon)| e^{(\rho^2 + \delta)/|\varepsilon|} ; j = 0, 1, 2, x \in B_\rho, \varepsilon \in D_j\}$$

and hence by (36) and (39) that $G_{\rho, \delta}(\vec{P}f - f) \leq 6\rho\varepsilon_0 e^{(2\rho^2 + \delta - \gamma)/\varepsilon_0}$ for $f \in F$. As $\|(\vec{P}f)^{-1}\| \leq 2$ and $\|f^{-1}\| \leq 4/3$ (we consider the multiplicative inverses here), we find $\|(\vec{P}f)^{-2} - f^{-2}\|_{\mathcal{U}_\rho} \leq 63M\rho e^{(\rho^2 - \gamma)/\varepsilon_0}$ and $G_{\rho, \delta}((\vec{P}f)^{-2} - f^{-2}) = G_{\rho, \delta}((\vec{P}f)^{-2}) \leq 96\rho\varepsilon_0 e^{(2\rho^2 + \delta - \gamma)/\varepsilon_0}$ for $f \in F$. In both estimates, we used the formula $b^{-2} - a^{-2} = (a^{-2}b^{-1} + a^{-1}b^{-2})(a - b)$. Using lemma 16 and, of course, the fact that K reduces to k on $F \subset \mathcal{H}_\rho$, this yields

$$\|k^{-1}K(\vec{P}f)^{-2} - f^{-2}\|_{\mathcal{H}_\rho} \leq e^{(2\rho^2 + \delta - \gamma)/\varepsilon_0} \left(63M\rho \left(2\frac{\rho^2}{\varepsilon_0} + 1 \right) + 96\rho\varepsilon_0 \left(\frac{\rho^2}{\delta} + \frac{1}{2} \right) \right) =: L(\rho, \delta, \varepsilon_0)$$

for $f \in F$. Then, for sufficiently small $\varepsilon_0 > 0$, the right hand side $L(\rho, \delta, \varepsilon_0)$ of the last estimate is smaller than $1/9$. As $f \in F$ implies $\|f^{-2} - 1\| \leq 7/9$, the above estimate shows that $\|k^{-1}K(\vec{P}f)^{-2} - 1\| \leq 8/9$ and hence the square root of $k^{-1}K(\vec{P}f)^{-2}$ can be defined as the principal value. Furthermore, using the inequality $|b^{-1/2} - a^{-1/2}| \leq \frac{1}{2}(\min_{0 \leq t \leq 1} |a + t(b - a)|)^{-3/2} |b - a|$, we finally obtain that

$$\|Nf - f\|_{\mathcal{H}_\rho} \leq \frac{27}{2} L(\rho, \delta, \varepsilon_0) < \frac{1}{16} \quad (41)$$

for all $f \in F$, provided ε_0 is sufficiently small.

Consider now $F_2 = \{f \in F ; \|f - 1\|_{\mathcal{H}_\rho} \leq 1/8\}$ and the mapping $\psi : F_2 \rightarrow \mathcal{H}_\rho$ given by $\psi(f) = g_0 - (Nf - f)$ (cf. (40)). Recall that $\|g_0 - 1\| \leq \frac{1}{16}$ if ρ is sufficiently small. Thus ψ maps F_2 into itself because of (41) if ε_0 and ρ are sufficiently small. We will show below that ψ is a contraction and hence it has a unique fixed point in F_2 in this case.

In order to show that ψ is a contraction, fix $f \in F_2$ and $h \in \mathcal{H}_\rho$ with $\|h\|_{\mathcal{H}_\rho} = 1$ for a moment and consider the mapping $\Psi(t) = \psi(f + th)$, $t \in \mathbb{C}, |t| < 1/8$. Suppose that ε_0 is

small enough such that (41) holds for all $g \in F$. As all the elements $f + th$ considered are in F , (41) implies that $\|\Psi(t) - g_0\|_{\mathcal{H}_\rho} < 1/16$ for all t , $|t| < 1/8$. By Cauchy's inequality, we obtain $\|\Psi'(0)\| \leq 1/2$, i.e. $\|\psi'(f)h\|_{\mathcal{H}_\rho} \leq 1/2$ for all $f \in F_2$ and $\|h\|_{\mathcal{H}_\rho} = 1$. Hence $\|\psi'(f)\|_{\mathcal{L}(\mathcal{H}_\rho, \mathcal{H}_\rho)} \leq 1/2$ for all $f \in F_2$ and thus ψ is indeed a contraction if ε_0 is sufficiently small.

As we have shown that ψ has a fixed point $f \in F_2$, we conclude that the equation $Nf = g_0$ has a solution in $F_2 \subset F$ if ε_0 and ρ are sufficiently small. With this choice of f , the solution \vec{u} of the fixed point equation (37) has J as second invariant, and a first invariant of the form $I + \varphi$. This shows the surjectivity of the mappings $\Phi_\sim^o : \mathcal{R}^o / \sim \rightarrow \mathcal{K}^o / \sim$ and $\Phi_{\sim_1}^o : \mathcal{R}^o / \sim_1 \rightarrow \mathcal{K}^o / \sim_1$.

The surjectivity of $\Phi_\approx^o : \mathcal{R}^o / \approx \rightarrow \mathcal{K}^o$ follows easily: given $(I, J) \in \mathcal{K}^o$, the above proof shows that there exists a 0-resonant equation (\tilde{g}) with invariants $(\tilde{I}, \tilde{J}) \sim_1 (I, J)$. Then by lemma 15, there exists $(g) \sim_1 (\tilde{g})$ with invariants (I, J) . Thus the proof of the surjectivity of Φ_\approx^o is also complete.

We now prove

Proposition 18. — *Let $f \in \mathcal{H}_\rho$ and let \vec{u} be a solution of equation (37). Then the function g (holomorphic in a neighborhood of 0) defined by (38) is linear in ε , i.e. of the form $g(x, \varepsilon) = g_0(x) + \varepsilon g_1(x)$.*

Proof. As in the proof of theorem 6, the idea is to expand the ingredients in powers of x and to use induction. We have $f(x) = \sum_{n \geq 0} f_n x^n$, $g(x, \varepsilon) = \sum_{n \geq 0} g_n(\varepsilon) x^n$, $\vec{u}(x, \varepsilon) = \sum_{n \geq 0} \vec{u}_n(\varepsilon) x^n$, $Z\vec{u}(x, \varepsilon) = \sum_{n \geq 0} \vec{z}_n(\varepsilon) x^n$ with $f_n \in \mathbb{C}$, g_n convergent, $u_0 \equiv 1$, $z_0 \equiv 0$ and $z_1 \equiv 1$.

The triples \vec{u}_n are in \mathcal{U} (triples of holomorphic bounded functions from S_j into \mathbb{C}) but the triples \vec{z}_n are not necessarily bounded near 0. However the formulae for \vec{z}_n show that $\varepsilon^{[n/2]} \vec{z}_n$ is bounded as $\varepsilon \rightarrow 0$. It follows that the products $\vec{d} \vec{z}_n$ (with $\vec{d} = \Delta \vec{I}$) still decrease exponentially as $\varepsilon \rightarrow 0$.

Since $\vec{u}_0 = 1$, equation (g) yields for $n \geq 0$:

$$-g_n = \varepsilon(n+1)(n+2)\vec{u}_{n+2} - 2n\vec{u}_n + \sum_{j=0}^{n-1} g_j \vec{u}_{n-j} \quad (42)$$

and equation (37) gives

$$u_j = f_j + \Sigma(\vec{v}_j), \quad \vec{v}_j := \vec{d} \cdot \vec{z}_j \quad .$$

By induction, if for all $j \in \{0, \dots, n-1\}$ we have $g_j(\varepsilon) = g_{j,0} + \varepsilon g_{j,1}$, $g_{j,0}, g_{j,1} \in \mathbb{C}$, then

$$\begin{aligned} -g_n(\varepsilon) &= \varepsilon(n+1)(n+2)(f_{n+2} + \Sigma \vec{v}_{n+2}(\varepsilon)) \\ &\quad - 2n(f_n + \Sigma \vec{v}_n(\varepsilon)) + \sum_{j=0}^{n-1} (g_{j,0} + \varepsilon g_{j,1})(f_{n-j} + \Sigma \vec{v}_{n-j}(\varepsilon)) \quad . \end{aligned} \quad (43)$$

Replacing the terms of the form $\varepsilon \Sigma(\vec{v}_k(\mu))$ by $\Sigma(\mu \vec{v}_k(\mu)) - L(\vec{v}_k(\mu))$, where $L = \varepsilon \Sigma - \Sigma \varepsilon$ (cf theorem 5, part 4), and grouping together terms of same nature, (43) can be written in the form

$$-g_n(\varepsilon) = C_0 + \varepsilon C_1 + L\vec{V} + \Sigma \vec{W}(\varepsilon) \quad (44)$$

with $C_0, C_1 \in \mathbb{C}$ and $\vec{V}, \vec{W} \in \mathcal{U}$. By theorem 5, part 4, $L\vec{V}$ is constant with respect to ε . Applying Δ to (44) and using that $(\Delta f)(\varepsilon) = 0$ if $f(\varepsilon)$ converges at $\varepsilon = 0$, we obtain

$$0 = \Delta \Sigma \vec{W} = \vec{W} \quad .$$

Hence g_n has the form $g_n(\varepsilon) = g_{n,0} + \varepsilon g_{n,1}$ (with $g_{n,0} = -C_0 - L\vec{V}$ and $g_{n,1} = -C_1$). \square

Thus the analoga of theorems 2 and 4 in the case of $f(x, \varepsilon) = 2x$ have been completely proved.

7 Extension of the results to the general case

7.1 Extension to an intermediate class of 0-resonant equations

Suppose first that a 0-resonant equation (1) is given with $f(x, 0) = 2x$, i.e. $f(x, \varepsilon) = 2x + \varepsilon\varphi(x, \varepsilon)$. Then the transformation $y = a(x, \varepsilon)\tilde{y}$, $a(x, \varepsilon) = \exp(\frac{1}{2}\int_0^x \varphi(t, \varepsilon) dt)$ yields the equation $\varepsilon\tilde{y}'' - 2x\tilde{y}' + g_1(x, \varepsilon)\tilde{y} = 0$ with $g_1 = g - x\varphi - \frac{\varepsilon}{4}\varphi^2 + \frac{\varepsilon}{2}\varphi'$. This is also a 0-resonant equation with formal resonant solution $\tilde{y}(x, \varepsilon) = \hat{y}(x, \varepsilon)/a(x, \varepsilon)$. Hence its invariants are $\tilde{I} = \tilde{y}'(0, \varepsilon)/a(0, \varepsilon) = \tilde{y}'(0, \varepsilon) = I - \frac{1}{2}\varphi(0, \varepsilon)$ and \tilde{J} the formal series asymptotic of $\int_0^{\delta i} e^{x^2/\varepsilon}\tilde{y}(x, \varepsilon)^{-2} dx = \int_0^{\delta i} e^{x^2/\varepsilon + \int_0^x \varphi(t, \varepsilon) dt} \hat{y}(x, \varepsilon)^{-2} dx$, i.e. $\tilde{J} = J$. Thus $(f, g) \sim_1 (2x, g_1)$ and $(2x, I, J) \sim_1 (2x, \tilde{I}, \tilde{J})$.

This immediately shows that two equations $(2x + \varepsilon\varphi_1, g_1)$ and $(2x + \varepsilon\varphi_2, g_2)$ are weakly equivalent if and only if their invariants are weakly equivalent. The surjectivity statements of the analoga of theorem 2 and 4 proved in section 6 in the case $f(x, \varepsilon) = 2x$ immediately imply those of the present case $f(x, 0) = 2x$.

Two equations $(2x + \varepsilon\varphi_1, g_1)$ and $(2x + \varepsilon\varphi_2, g_2)$ are strongly equivalent if and only if the corresponding equations $(2x, \tilde{g}_1)$ and $(2x, \tilde{g}_2)$ are \sim_1 -equivalent and some transformation $\tilde{y}_2 = a\tilde{y}_1 + \varepsilon b\tilde{y}_1'$ realizing it satisfies $a(0, \varepsilon) = 1, b(0, \varepsilon) = b'(0, \varepsilon) = 0$ and $a'(0, \varepsilon) = \frac{1}{2}\varphi_1(0, \varepsilon) - \frac{1}{2}\varphi_2(0, \varepsilon)$; the most convenient way to see this is to go over to matrix notation in a way similar to the beginning of subsection 4.2. According to section 5, the last statement is equivalent to $\tilde{I}_2 = \tilde{I}_1 + \frac{1}{2}\varphi_1(0, \varepsilon) - \frac{1}{2}\varphi_2(0, \varepsilon)$ and $\tilde{J}_2 = \tilde{J}_1$. The above calculation shows that this is equivalent to $I_2 = I_1, J_2 = J_1$.

Thus theorems 2 and 4 are proved in the case of $f(x, 0) = h(x) = 2x$.

7.2 Extension to all 0-resonant equations

Given some analytic function $f_0(x)$ vanishing at $x = 0$ with $f_0'(0) = 2$, it is easy to find some analytic function $\varphi(t) = t + O(t^2)$ such that¹ $f_0(\varphi(t))\dot{\varphi}(t) = 2t$; it has to be the inverse function of $\Phi_0(x) = x + O(x^2)$ satisfying $\Phi_0(x)^2 = \int_0^x f_0(\xi) d\xi$.

Then any equation $(f_0(x) + \varepsilon h(x, \varepsilon), g(x, \varepsilon))$ can be transformed using $x = \varphi(t), v(t) = y(\varphi(t))$ into

$$\varepsilon\ddot{v} - (2t + \varepsilon f_1(t, \varepsilon))\dot{v} + g_1(t, \varepsilon)v = 0,$$

where $f_1(t, \varepsilon) = h(\varphi(t), \varepsilon)\dot{\varphi}(t) + \varepsilon\ddot{\varphi}(t)/\dot{\varphi}(t)$ and $g_1(t, \varepsilon) = g(\varphi(t), \varepsilon)\dot{\varphi}(t)^2$. The second equation is 0-resonant if and only if the first one is. Observe that the two equations are *not* equivalent according to any of our definitions. The change of variables permits, however, to carry over the results of the previous subsection to any 0-resonant equation.

First of all, two equations $(f_0 + \varepsilon h, g)$ and $(f_0 + \varepsilon\tilde{h}, \tilde{g})$ are equivalent via $y = a(x, \varepsilon)\tilde{y} + \varepsilon b(x, \varepsilon)\tilde{y}'$ if and only if the corresponding equations $(2t + \varepsilon f_1, g_1)$ and $(2t + \varepsilon\tilde{f}_1, \tilde{g}_1)$ are equivalent via $v = a(\varphi(t), \varepsilon)\tilde{v} + \varepsilon b(\varphi(t), \varepsilon)\dot{\varphi}(t)^{-1}\dot{\tilde{v}}$. This means that each of our three

¹In this subsection ' denotes differentiation with respect to x , $\dot{}$ with respect to t .

equivalence relations for $(f_0 + \varepsilon h, g)$ and $(f_0 + \varepsilon \tilde{h}, \tilde{g})$ corresponds to the same relation for $(2t + \varepsilon f_1, g_1)$ and $(2t + \varepsilon \tilde{f}_1, \tilde{g}_1)$.

Furthermore, a short calculation shows that both invariants I and J for an equation $(f_0 + \varepsilon h, g)$ and its corresponding equation $(2t + \varepsilon f_1, g_1)$ are the same. Thus theorem 2 and the first part of theorem 4 carry over to general 0-resonant equations immediately.

Finally, suppose that a vector $(f_0, I, J) \in \mathcal{K}$ is given. From the preceding section, we know that there is a 0-resonant equation² $(2t, h_0(t) + \varepsilon h_1(t))$ having the invariants $(I + \psi(\varepsilon), J)$ where $\psi(\varepsilon)$ is some convergent series. Using the transformation $v = \dot{\varphi}(t)^{-1/2} \tilde{v}$ with the above $\varphi(t)$, we find an equation of the form $(2t + \varepsilon \dot{\varphi}(t)/\dot{\varphi}(t), \tilde{h}_0(t) + \varepsilon \tilde{h}_1(t))$ having the invariants $I + \psi(\varepsilon) + \frac{1}{2}\ddot{\varphi}(0)$ and J . The above calculations show that the equation obtained using $x = \varphi(t)$, $y(\varphi(t)) = \tilde{v}(t)$ is of the form $(f_0(x), g_0(x) + \varepsilon g_1(x))$ with some functions g_0, g_1 analytic near $x = 0$ and that its invariants are still $I + \psi(\varepsilon) + \frac{1}{2}\ddot{\varphi}(0)$ and J . This proves the second part of theorem 4.

7.3 Extension to all resonant equations

Consider now a general resonant equation in its matrix form

$$(R) \quad \varepsilon \mathbf{y}' = R \mathbf{y} ,$$

with $\mathbf{y} = \begin{pmatrix} y \\ \varepsilon y' \end{pmatrix}$ and $R : (x, \varepsilon) \mapsto \begin{pmatrix} 0 & 1 \\ -\varepsilon g(x, \varepsilon) & f(x, \varepsilon) \end{pmatrix}$. If (R) is not 0-resonant, then by definition $g(0, 0) \neq 0$. Hence $\mathbf{y}_1 = R(x, \varepsilon)\mathbf{y}$ induces a weak equivalence between (R) and some equation (R_1) , $R_1 : (x, \varepsilon) \mapsto \begin{pmatrix} 0 & 1 \\ -\varepsilon g_1(x, \varepsilon) & f_1(x, \varepsilon) \end{pmatrix}$ with $f_1 = f + \varepsilon/g$, $g_1 = g - f' + f/g$. Clearly, this new equation is satisfied by $(y', \varepsilon y'')$ ^T if (R) is satisfied by $(y, \varepsilon y')$ ^T and the quantity $g_1(0, 0) = g(0, 0) - 2$.

This procedure can be repeated $n = g(0, 0)/2$ times and leads to a 0-resonant (matrix) equation (R_n) weakly equivalent to (R) which is satisfied by $(y^{(n)}, \varepsilon y^{(n+1)})$ ^T if (R) is satisfied by $(y, \varepsilon y')$; this means that the corresponding scalar equation has a solution $y^{(n)}$ if the scalar equation (f, g) has a solution y .

By our definition (see above definition 3), the invariants of (R) , i.e. (f, g) , are those of (R_n) . The first part of theorem 4, our only statement concerning general resonant equations, follows immediately.

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²Here the independent variable is denoted t

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