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Overstable analytic solutions for non-linear systems of difference equations with small step size containing an additional parameter

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Abstract: We study the existence of analytic solutions of the system of difference equations given by

$$y(\varepsilon, x + \varepsilon) = F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x)),$$

where F is an analytic function of ε, x, a and y on a certain neighborhood $V \times D_{r_1}(x_0) \times D_{r_2}(a_0) \times D_{r_3}(y_0) \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^n$ of the point $(0, x_0, a_0, y_0)$ with $y_0 = F(0, x_0, a_0, y_0)$. Under the assumption that the Jacobian $\frac{\partial F}{\partial y}(0, x, a_0, \phi_0(x)) - I$ is not invertible at some point $x = x_0$, we first show, under some transversality condition, the existence of a unique formal solution $(\hat{a}(\varepsilon), \hat{y}(\varepsilon, x))$ and establish its Gevrey-1 character. Then, we construct a quasi-solution, $(\tilde{a}(\varepsilon), \tilde{y}(\varepsilon, x))$, in the sense that, it solves our system equation up to an exponentially small remainder. The quasi-solution is obtained by taking the Borel-Laplace transforms, with some corrective terms, of the formal solution. Then, we show the existence of an analytic solution $(a(\varepsilon), y(\varepsilon, x))$, on some neighborhood of $(0, x_0)$ (to be constructed), that is exponentially close to the constructed quasi-solution. We also show the exponential closeness of any two quasi-solutions (and hence also of actual solutions) tending to the same slow curve. Finally, as an application of this theory, we give two numerical examples with some figures of canards solutions of some difference equation of second order.

Keywords: Difference equation; Difference operator; Inverse operator; Fixed point theorem; Gevrey asymptotic; Formal solution; Quasi-solution; Overstable solution; Nagumo norm; Weight norm; Borel-Laplace transform; Gronwall lemme

AMS Classification: 39A, 47B39

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1 Introduction

We consider a system of n difference equations of the form

$$y(\varepsilon, x + \varepsilon) = F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x)), \quad (1)$$

where x is a complex variable, ε is a small complex parameter, $a(\varepsilon)$ is a vector of additional parameters and F is an analytic function of ε, x, a and y on a certain domain. The precise hypotheses will be given later. We look for solutions $(a(\varepsilon), y(\varepsilon, x))$ of equation (1) that, as $\varepsilon \rightarrow 0$, tend to some given slow curve $(a_0, \phi_0(x))$ satisfying $F(0, x, a_0, \phi_0(x)) = \phi_0(x)$. Such solutions, if they exist, are so-called overstable solutions, in the sense that, as $\varepsilon \rightarrow 0$, they remain uniformly bounded with respect to x , in a full neighborhood of some point x_0 . It will also be shown that this is a strong restriction for $a(\varepsilon)$: Given two solutions $(a_1(\varepsilon), y_1(\varepsilon, x))$ and $(a_2(\varepsilon), y_2(\varepsilon, x))$ of equation (1) tending to $(a_0, \phi_0(x))$, the difference $(a_1(\varepsilon) - a_2(\varepsilon))$ has to be exponentially small.

A related system of *differential* equations of the form

$$\varepsilon^\sigma \frac{dy}{dx} = F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x))$$

was considered in [2, 11]. For the existence of solutions, it is important to know whether the Jacobian

$$A_0(x) = \frac{\partial F}{\partial y}(0, x, a_0, \phi_0(x))$$

is invertible at some point $x = x_0$ or not. In the case where the Jacobian is invertible, parameters $a(\varepsilon)$ are arbitrary and the existence of solutions is well-known (cf Sibuya [11]). In the case where the Jacobian is not invertible at some point $x = x_0$, a detailed discussion was given in [2]. There the authors first showed the existence of a formal solution $(\hat{a}(\varepsilon), \hat{y}(\varepsilon, x))$. Then its Gevrey character was studied and determined after introducing a certain “transversality” condition and the “Nagumo” norms. Then a quasi-solution was constructed and eventually the existence of an overstable solution $(a(\varepsilon), y(\varepsilon, x))$ was proved.

The framework of the present work will be more or less similar to that of [2]; globally, we follow the same strategy, but we have to modify several proofs *substantially* to be compatible with *difference* equations. The necessary modifications for the passage from differential to difference equations are not always obvious.

Also for our system (1), the existence of the solutions $y(\varepsilon, x)$ depends on the Jacobian $A_0(x)$, now however, defined by

$$A_0(x) = \frac{\partial F}{\partial y}(0, x, a_0, \phi_0(x)) - I, \quad (2)$$

Two cases must again be differentiated according to whether $A_0(x)$ is invertible at some point $x = x_0$ or not. In the case where $A_0(x)$ is invertible at some point $x = x_0$, no additional parameter $a(\varepsilon)$ is needed. This case was treated in some previous work [4]. The existence of a unique formal solution was shown and its Gevrey-1 character was established. The existence of an analytic solution was also shown separately.

Now, we discuss the case where $A_0(x)$ is not invertible at some point $x = x_0$. Here, we need to introduce a parameter $a(\varepsilon)$ in our system equation and assume a transversality condition analogous to [2]. First, we show the existence of a unique formal solution $(\hat{a}(\varepsilon), \hat{y}(\varepsilon, x))$ and we establish its Gevrey-1 character. Then, taking the Borel and the truncated Laplace transform of the formal solution, we show that we obtain a quasi-solution.

Finally, as a first **main result** of this work we show the existence of an overstable solution $(a(\varepsilon), y(\varepsilon, x))$ close to the constructed quasi-solution; actually their difference is exponentially small. We also show that any two quasi-solutions tending to the same slow curve are exponentially close to each other. This shows in particular that our “canards” are short-lived in the sense that the corresponding parameter values belong to an interval of exponentially small length. As an application of this theory, we give two numerical examples with some figures of canards solutions of some difference equation of second order. For the completeness of our work, we include, at the end, an appendix summarizing some previous results.

We mention some related works of: Baesens [1], Canalis-Durand, Ramis, Schäfke, Sibuya [2], El-Rabih [3], Fruchard [5], Fruchard, Schäfke [6, 8], Sibuya [11], and Wasow [12].

2 Assumptions

Consider the system (1) of difference equations; i.e.

$$y(\varepsilon, x + \varepsilon) = F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x)),$$

where ε is a small complex parameter.

We assume the following:

1. The function F is an analytic function of the variables x, a and y in some open neighborhood $D := D_{r_1}(x_0) \times D_{r_2}(a_0) \times D_{r_3}(y_0) \subset \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^n$ of (x_0, a_0, y_0) , where $D_r(z)$ denotes the open disk of radius $r > 0$ and of center z .
2. The function F is analytic in ε on some open small sector V of the complex plane with vertex at the origin. Moreover, F is asymptotic of Gevrey order 1 to the formal series $\sum_{k \geq 0} f_k(x, a, y) \varepsilon^k$ as $V \ni \varepsilon \rightarrow 0$ uniformly for $(x, a, y) \in D$, where the functions f_k are holomorphic on D . This means that there exist positive constants A, C such that for all $\varepsilon \in V$ and $(x, a, y) \in D$ and all $N \in \mathbb{N}^*$

$$\left| F(\varepsilon, x, a, y) - \sum_{k=0}^{N-1} f_k(x, a, y) \varepsilon^k \right| \leq C A^N \Gamma(N+1) |\varepsilon|^N.$$

The constant A is called the *type* of the Gevrey 1 expansion.

3. There exists a *slow curve* \mathcal{C}_0 of the *reduced equation* $F(0, x, a_0, y) = y$, i.e. there exists a holomorphic function $\phi_0 : D_{r_1}(x_0) \rightarrow \mathbb{C}^n$ satisfying $F(0, x, a_0, \phi_0(x)) = \phi_0(x)$ for $|x - x_0| < r_1$.
4. The Jacobian $A_0(x) = \frac{\partial F}{\partial y}(0, x, a_0, \phi_0(x)) - I$ is invertible except at $x = x_0$ and $\det A_0(x) = (x - x_0)^m k(x)$, where $m \in \mathbb{N}$ and $k(x)$ analytic near x_0 with $k(x_0) \neq 0$.
5. The mapping

$$\mathcal{H}^n(D_r) \times \mathbb{C}^m \rightarrow \mathcal{H}^n(D_r)$$

defined by

$$(y, a) \mapsto A_0 \cdot (y - \phi_0(x)) + B_0 \cdot (a - a_0),$$

is bijective for every $r < r_1$. Here $B_0(x) = \frac{\partial F}{\partial a}(0, x, a_0, \phi_0(x))$ and $\mathcal{H}(D_r)$ denotes the set of all functions holomorphic on D_r .

This assumption is the so-called *Hypothesis of Transversality* (H) introduced in [2].

3 Formal solutions

Without loss of generality, we can assume that $x_0 = 0$, $a_0 = 0$, and $\phi_0(x) = 0$, for otherwise, we replace x by $x_0 + x$, a by $a_0 + a$ and y by $\phi_0(x) + y$.

Hence we assume

1. F is analytic with respect to (x, a, y) on $D_{r_1}(0) \times D_{r_2}(0) \times D_{r_3}(0) \subset \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^n$.
2. F has a Gevrey-1 asymptotic expansion of type A in ε on V .
3. $F(0, x, 0, 0) = 0$ for $x \in D_{r_1}(0)$.

4. $A_0(x) = \frac{\partial F}{\partial y}(0, x, 0, 0) - I$ is invertible except at $x = 0$ and $\det A_0(x) = x^m k(x)$, where $m \in \mathbb{N}$ and $k(x)$ is analytic near 0 with $k(0) \neq 0$.

5. The mapping

$$\mathcal{H}^n(D_r) \times \mathbb{C}^m \rightarrow \mathcal{H}^n(D_r)$$

defined by

$$(y, a) \mapsto A_0 \cdot y + B_0 \cdot a$$

is bijective for every $r < r_1$; here $B_0(x)$ denotes $\frac{\partial F}{\partial a}(0, x, 0, 0)$.

Proposition 1 *Under the above assumptions, equation (1) has a unique formal solution $\hat{a}(\varepsilon) = \sum_{j=1}^{\infty} c_j \varepsilon^j$, $\hat{y}(\varepsilon, x) = \sum_{j=1}^{\infty} b_j(x) \varepsilon^j$, where $c_j \in \mathbb{C}^m$ and $b_j(x)$ are analytic in D_{r_1} . Moreover, \hat{a} is Gevrey-1 and \hat{y} is Gevrey-1 uniformly on $D_{\tilde{r}}(0)$ for all $\tilde{r} < r$, i.e. for each $\tilde{r} < r$ there exist numbers M, B such that $|c_j| \leq MB^j \Gamma(j+1)$ and $\sup_{|x| \leq \tilde{r}} |b_j(x)| \leq$*

$MB^j \Gamma(j+1)$ for all $j \in \mathbb{N}$.

Proof. For simplicity, we omit the “hats” in the following proof. We first show the existence of a formal solution. We rewrite equation (1) as an equation for formal series (except for the term $y(\varepsilon, x + \varepsilon) - y(\varepsilon, x)$ at the moment)

$$\begin{aligned} A_0(x)y(\varepsilon, x) + B_0(x)a(\varepsilon) &= y(\varepsilon, x + \varepsilon) - y(\varepsilon, x) - \varepsilon \phi_1(x) \\ &\quad - \sum_{|\vec{p}| + |\vec{q}| + k \geq 2} f_{\vec{p} \vec{q} k}(x) y^{\vec{p}}(\varepsilon, x) a^{\vec{q}}(\varepsilon) \varepsilon^k, \end{aligned} \tag{3}$$

where

- ϕ_1 and $f_{\vec{p} \vec{q} k}$ are n -dimensional vectors, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_m)$ and p_i, q_i, k are non-negative integers, $|\vec{p}| = p_1 + \dots + p_n$, $|\vec{q}| = q_1 + \dots + q_m$, $y^{\vec{p}} = y_1^{p_1} \dots y_n^{p_n}$, $a^{\vec{q}} = a_1^{q_1} \dots a_m^{q_m}$ where y_j (a_j) is the j th entry of y (a respectively).
- The entries of A_0, B_0, ϕ_1 and $f_{\vec{p} \vec{q} k}$ are holomorphic in x on $|x| < r_1$.

We proceed as in [2], but we also have to expand the term $y(\varepsilon, x + \varepsilon) - y(\varepsilon, x)$. We have

$$y(\varepsilon, x + \varepsilon) - y(\varepsilon, x) = \sum_{r=1}^{\infty} \frac{1}{r!} \left(\varepsilon \frac{d}{dx} \right)^r y(\varepsilon, x).$$

This gives

$$\begin{aligned} A_0(x)y(\varepsilon, x) + B_0(x)a(\varepsilon) &= \sum_{r=1}^{\infty} \frac{1}{r!} \left(\varepsilon \frac{d}{dx} \right)^r y(\varepsilon, x) - \varepsilon \phi_1(x) \\ &\quad - \sum_{|\vec{p}|+|\vec{q}|+k \geq 2} f_{\vec{p}\vec{q}k}(x) y^{\vec{p}}(\varepsilon, x) a^{\vec{q}}(\varepsilon) \varepsilon^k \end{aligned} \quad (4)$$

Putting $a(\varepsilon) = \sum_{j=1}^{\infty} c_j \varepsilon^j$, $y(\varepsilon, x) = \sum_{j=1}^{\infty} b_j(x) \varepsilon^j$ and looking at the coefficient of ε^n , $n = 0, 1, \dots$, we note that the right hand side of the above equation has coefficient 0 for ε^0 and that the coefficient of ε^1 is $-\phi_1(x)$. Furthermore, we easily see that the coefficients $\{c_j\}_{j=1}^l$, $\{b_j\}_{j=1}^l$ determine the coefficient of ε^{l+1} on the right hand side of equation (4). Thus, by the condition of transversality, the coefficients $\{c_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}}$ of the formal solution can be found recursively and are uniquely determined.

Next, we show that the formal solution is Gevrey-1. To do this, we construct a majorant equation of (4) having a unique series solution that is convergent. We rewrite equation (4) in the form:

$$\begin{aligned} A_0(x)y(\varepsilon, x) + B_0(x)a(\varepsilon) &= \sum_{r=1}^{\infty} \frac{1}{r!} \left(\varepsilon \frac{d}{dx} \right)^r y(\varepsilon, x) - \phi(\varepsilon, x) - B(\varepsilon, x)y(\varepsilon, x) - C(\varepsilon, x)a(\varepsilon) \\ &\quad - \sum_{|\vec{p}|+|\vec{q}| \geq 2} f_{\vec{p}\vec{q}}(\varepsilon, x) y^{\vec{p}}(\varepsilon, x) a^{\vec{q}}(\varepsilon), \end{aligned} \quad (5)$$

where

$$\begin{aligned} \phi(\varepsilon, x) &:= \sum_{k=1}^{\infty} \phi_k(x) \varepsilon^k, \quad B(\varepsilon, x) := \sum_{k=1}^{\infty} B_k(x) \varepsilon^k, \quad \text{and} \\ C(\varepsilon, x) &:= \sum_{k=1}^{\infty} C_k(x) \varepsilon^k, \quad f_{\vec{p}\vec{q}}(\varepsilon, x) := \sum_{k=0}^{\infty} f_{\vec{p}\vec{q}k}(x) \varepsilon^k; \end{aligned}$$

the entries of B_k, C_k are those of $f_{\vec{p}\vec{q}k}$ with $|\vec{p}| = 1$ or $|\vec{q}| = 1$, respectively. Note here that $B_k(x), k \geq 1$ are $n \times n$ matrices and $C_k(x), k \geq 1$ are $n \times m$ matrices.

Before continuing the proof, we recall the modified **Nagumo norms** introduced in [2]:

Definition 2 Given $\rho \in]0, r[$, we define a function d on the open circle $D_r(0)$ of center 0 and radius $r > 0$ by

$$d(x) = \begin{cases} r - |x| & \text{if } |x| \geq \rho \\ r - \rho & \text{if } |x| < \rho \end{cases}.$$

Then for nonnegative integers p and holomorphic functions f on $D_r(0)$, the Nagumo norm of f denoted by $\|f\|_p$ is defined by

$$\|f\|_p := \sup_{|x| < r} (|f(x)| d(x)^p).$$

The Nagumo norms are compatible with the elementary operations; their most important property is the following:

$$\text{For } f \in \mathcal{H}(D_r), p \in \mathbb{N}, \text{ we have } \|f'\|_{p+1} \leq e(p+1)\|f\|_p. \quad (6)$$

We refer the reader to appendix A.1 for more details.

Definition 3 [2] *We say that a series $g = \sum_{k=0}^{\infty} g_k(x)\varepsilon^k$ is majorized (" \ll ") by a series $h(z) = \sum_{l=0}^{\infty} h_l z^l$ if and only if*

$$\|g_j\|_j \leq h_j j! \text{ for } j = 0, 1, 2, \dots$$

We also recall the following two properties of \ll :

Given that $g \ll h(z)$ and $\tilde{g} \ll \tilde{h}(z)$, then

$$g\tilde{g} \ll h(z)\tilde{h}(z), \text{ and} \quad (7)$$

$$\varepsilon \frac{d}{dx} g \ll ezh(z). \quad (8)$$

The proof can be found in the appendix A.3.

Now, we denote by P_1 and P_2 the inverse of the mapping of the transversality condition 5, i.e.

$$P_1 : \mathcal{H}^n(D_r) \rightarrow \mathcal{H}^n(D_r), \quad P_2 : \mathcal{H}^n(D_r) \rightarrow \mathbb{C}^m$$

are the uniquely determined linear mappings satisfying

$$A_0(x)(P_1 f)(x) + B_0(x)(P_2 f)(x) = f(x), \text{ for } x \in D_r, f \in \mathcal{H}^n(D_r).$$

These P_1 and P_2 are bounded linear operators with respect to the Nagumo norms: There exists a constant $K > 0$ (independent of f) such that

$$\|P_1 f\|_p \leq K \|f\|_p, \text{ and} \quad (9)$$

$$\|P_2 f\|_p \leq K d(0)^{-p} \|f\|_p, \quad (10)$$

for nonnegative integers p and $f \in \mathcal{H}^n(D_r)$. We mention here that the Smith normal form of $A_0(x)$ and the so-called *shift operator* S (see appendix A.1) were involved in the proof of these two properties, see appendix A.2 for details.

Recall that we want to find a *majorant equation* related to equation (5). So, we rewrite it using the series $u(\varepsilon, x) := A_0(x)y(\varepsilon, x) + B_0(x)a(\varepsilon)$. We obtain

$$\begin{aligned} u(\varepsilon, x) = & \sum_{r=1}^{\infty} \frac{1}{r!} \left(\varepsilon \frac{d}{dx} \right)^r P_1 u(\varepsilon, x) - \phi(\varepsilon, x) - B(\varepsilon, x) P_1 u(\varepsilon, x) \\ & - C(\varepsilon, x) P_2 u(\varepsilon) - \sum_{|\vec{p}|+|\vec{q}|\geq 2} f_{\vec{p}\vec{q}}(\varepsilon, x) (P_1 u)^{\vec{p}}(\varepsilon, x) (P_2 u)^{\vec{q}}(\varepsilon). \end{aligned} \quad (11)$$

Once the formal series $u(\varepsilon, x)$ has been majorized, majorants of $y(\varepsilon, x)$ and $a(\varepsilon)$ can be obtained using the mappings P_1 and P_2 .

For the majorization of series of vectors, we use the maximum norm and for series of matrices, we use a compatible matrix norm. Then by applying the $\frac{\|\cdot\|_j}{j!}$ to the coefficients

of ε^j of equation (11), we can find majorant (scalar) series $\hat{\phi}(z)$, $\hat{B}(z)$, $\hat{C}(z)$ and $\hat{f}_{\vec{p}\vec{q}}(z)$ of $\phi(\varepsilon, x)$, $B(\varepsilon, x)$, $C(\varepsilon, x)$ and $f_{\vec{p}\vec{q}}(\varepsilon, x)$ respectively. Note here that we use the hypothesis that F is Gevrey-1.

It remains to treat the sum $\sum_{r=1}^{\infty} \frac{1}{r!} (\varepsilon \frac{d}{dx})^r P_1 u(\varepsilon, x)$.

Assume that $u \ll h(z)$, then $P_1 u \ll K h(z)$ and $P_2 u \ll K h(z)$, where K is the constant given in equations (9, 10).

By an idea of J.P.Ramis

$$\sum_{r=1}^{\infty} \frac{1}{r!} (\varepsilon \frac{d}{dx})^r P_1 u(\varepsilon, x) \ll \sum_{r=1}^{\infty} \frac{1}{r!} (ez)^r C_1 h(z) = C_1 (e^{ez} - 1) h(z). \quad (12)$$

By these properties and those given by equations (6, 7, 8, 9, 10) and Cauchy's estimate for the coefficients of a convergent power series, all these series have a common radius of convergence, and $\sum_{|\vec{p}|+|\vec{q}| \geq 2} \hat{f}_{\vec{p}\vec{q}}(z) g^{|\vec{p}|+|\vec{q}|}$ is convergent if $|z|$ and $|g|$ are small enough.

Then a majorant equation of (11) is given by:

$$\begin{aligned} h(z) &= K(e^{ez} - 1)h(z) + K\hat{\phi}(z) + K(\hat{B}(z) + \hat{C}(z))h(z) \\ &+ \sum_{n=2}^{\infty} \sum_{|\vec{p}|+|\vec{q}|=n} \hat{f}_{\vec{p}\vec{q}}(z) h(z)^n \end{aligned} \quad (13)$$

As $\hat{\phi}(0) = \hat{B}(0) = 0$, it is easy to see that this majorant equation has a unique formal solution $h(z) = \sum_{j=1}^{\infty} h_j z^j$, with h_j nonnegative. In the same way as in [2] (i.e. essentially by induction), it follows that our formal solution u of equation (11) satisfies $u \ll h(z)$.

By the implicit function theorem, the formal solution $h(z)$ converges and hence $h_j \leq MB^j$, $j = 1, 2, \dots$ where M, B are positive constants. Then $u \ll h(z)$ for the formal solution u of equation (11) and h of equation (13). This means that $\exists M, B$ such that $\|u_n(x)\|_n \leq MB^n n!$

By definition of $\|\cdot\|_n$, this implies that

$$|u_n(x)| \leq MB^n d(x)^{-n} n! \text{ for all } x \text{ with } |x| < r.$$

For all δ , $0 < \delta < r - \rho$, we also have

$$|u_n(x)| \leq MB^n \delta^{-n} n! \text{ for all } x \text{ with } |x| \leq r - \delta,$$

i.e. u is Gevrey-1 uniformly on $D_{\tilde{r}}(0)$ for all $\tilde{r} < r$.

Using P_1 and P_2 again, we obtain by equations (9, 10) that $y = P_1 u$, $a = P_2 u$ are Gevrey of order 1. \square

4 Quasi-solutions

So far, we have shown that equation (1) has a unique formal solution $\hat{a}(\varepsilon) = \sum_{j=1}^{\infty} c_j \varepsilon^j$, $\hat{y}(\varepsilon, x) = \sum_{j=1}^{\infty} b_j(x) \varepsilon^j$ with

$$\begin{aligned} |c_j| &\leq MB^j \Gamma(j+1), \\ \sup_{|x| \leq \tilde{r}} |b_j(x)| &\leq MB^j \Gamma(j+1), \end{aligned}$$

for some numbers M, B , i.e. they are Gevrey-1.

Recall that F is also Gevrey-1 asymptotic to $\sum_{j \geq 0} f_j(x, a, y) \varepsilon^j$.

Next, we take the Borel transforms $\tilde{c}(t)$, $\tilde{b}(t, x)$, and $\tilde{G}(t, x, a, y)$ of $\hat{a}(\varepsilon)$, $\hat{y}(\varepsilon, x)$ and $F(\varepsilon, x, a, y)$ respectively, i.e. we define:

$$\tilde{c}(t) := \sum_{j=1}^{\infty} c_j \frac{t^{j-1}}{(j-1)!}, \quad |t| < \frac{1}{B},$$

$$\tilde{b}(t, x) := \sum_{j=1}^{\infty} b_j(x) \frac{t^{j-1}}{(j-1)!}, \quad x \in D_{\tilde{r}}, \quad |t| < \frac{1}{B}, \text{ and}$$

$$\tilde{G}(t, x, a, y) := \sum_{j=1}^{\infty} f_j(x, a, y) \frac{t^{j-1}}{(j-1)!}, \quad (x, a, y) \in D, \quad |t| < \frac{1}{A}.$$

Recall that A is the constant introduced in section 3.

We choose T , $0 < T < \min(\frac{1}{B}, \frac{1}{A})$. We define $\tilde{a}(\varepsilon)$, $\tilde{y}(\varepsilon, x)$ and $\tilde{F}(\varepsilon, x, a, y)$ to be the Laplace transforms (with some corrective term) of $\tilde{c}(t)$, $\tilde{b}(t, x)$ and $\tilde{G}(t, x, a, y)$ respectively. We put

$$\tilde{a}(\varepsilon) := \int_0^T \tilde{c}(t) e^{-\frac{t}{\varepsilon}} dt + \varepsilon \tilde{c}(T) e^{-\frac{T}{\varepsilon}},$$

$$\tilde{y}(\varepsilon, x) := \int_0^T \tilde{b}(t, x) e^{-\frac{t}{\varepsilon}} dt + \varepsilon \tilde{b}(T, x) e^{-\frac{T}{\varepsilon}}, \text{ and}$$

$$\tilde{F}(\varepsilon, x, a, y) := f_0(x, a, y) + \int_0^T \tilde{G}(t, x, a, y) e^{-\frac{t}{\varepsilon}} dt + \varepsilon \tilde{G}(T, x, a, y) e^{-\frac{T}{\varepsilon}}.$$

Then, we know that

$$\tilde{a}(\varepsilon) \sim_1 \sum_{j=1}^{\infty} c_j \varepsilon^j,$$

$$\tilde{y}(\varepsilon, x) \sim_1 \sum_{j=1}^{\infty} b_j(x) \varepsilon^j, \text{ and}$$

$$\tilde{F}(\varepsilon, x, a, y) \sim_1 \sum_{j \geq 0} f_j(x, a, y) \varepsilon^j.$$

The notation \sim_1 means “Gevrey-1 asymptotic to” (see section 2).

Theorem 4 *As given above, $\tilde{a}(\varepsilon)$, $\tilde{y}(\varepsilon, x)$ are quasi-solutions of equation (1), i.e. there exists $K > 0$ such that*

$$\tilde{R}(\varepsilon, x) := \tilde{y}(\varepsilon, x + \varepsilon) - F(\varepsilon, x, \tilde{a}(\varepsilon), \tilde{y}(\varepsilon, x))$$

satisfies

$$\left| \tilde{R}(\varepsilon, x) \right| \leq K e^{-\frac{T}{|\varepsilon|}}, \quad \varepsilon \in V, \quad \varepsilon \rightarrow 0, \quad x \in D_{\tilde{r}}.$$

Proof. We use a theorem of Ramis-Sibuya [9] about the characterization of functions having Gevrey-1 asymptotics.

As $\tilde{a}(\varepsilon)$, $\tilde{y}(\varepsilon, x)$ and $F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x))$ are Gevrey-1 in ε , the theorem of Ramis-Sibuya implies that there exist a good covering $S_j = S(\alpha_j, \beta_j, \varepsilon_0)$, $1 \leq j \leq m$ of the punctured disk $D(0, \varepsilon_0)$ and functions $a_j(\varepsilon)$, $y_j(\varepsilon, x)$ and $F_j(\varepsilon, x, a(\varepsilon), y(\varepsilon, x))$ defined for $\varepsilon \in S_j$, $j = 1, \dots, m$ and satisfying

$$|a_{j+1}(\varepsilon) - a_j(\varepsilon)| = \mathcal{O}\left(e^{-\frac{\gamma}{|\varepsilon|}}\right),$$

$$|y_{j+1}(\varepsilon, x) - y_j(\varepsilon, x)| = \mathcal{O}\left(e^{-\frac{\gamma}{|\varepsilon|}}\right), \text{ and}$$

$$|F_{j+1}(\varepsilon, x, a(\varepsilon), y(\varepsilon, x)) - F_j(\varepsilon, x, a(\varepsilon), y(\varepsilon, x))| = \mathcal{O}\left(e^{-\frac{\gamma}{|\varepsilon|}}\right),$$

for $\varepsilon \in S_j \cap S_{j+1}$ with some positive constant γ and such that $\tilde{y} = y_1$, $\tilde{a} = a_1$ and $\tilde{F} = F_1$.

Trivially, the same estimates still hold for $y_j(\varepsilon, x + \varepsilon)$, i.e.

$$|y_{j+1}(\varepsilon, x + \varepsilon) - y_j(\varepsilon, x + \varepsilon)| = \mathcal{O}\left(e^{-\frac{\gamma}{|\varepsilon|}}\right), \quad \varepsilon \in S_j \cap S_{j+1}.$$

Also,

$$\tilde{y}(\varepsilon, x + \varepsilon) = y_1(\varepsilon, x + \varepsilon).$$

Note here that we have to reduce the x -domain $D_{\tilde{r}}(0)$ into $D_{\tilde{r}-\varepsilon}(0)$.

Let

$$R_j(\varepsilon, x) := y_j(\varepsilon, x + \varepsilon) - F_j(\varepsilon, x, a_j(\varepsilon), y_j(\varepsilon, x)).$$

Now, using standard estimates, it follows that the differences $R_{j+1} - R_j$ are also exponentially small:

$$|R_{j+1}(\varepsilon, x) - R_j(\varepsilon, x)| = \mathcal{O}\left(e^{-\frac{\gamma}{|\varepsilon|}}\right)$$

for $\varepsilon \in S_j \cap S_{j+1}$, $j = 1, \dots, n$ and $x \in D_{\tilde{r}-\varepsilon}(0)$. The theorem of Ramis-Sibuya implies that R_j , $j = 1, \dots, m$, have asymptotic expansions of Gevrey order 1 with a common right hand side. Since one of these R_j is \tilde{R} in question, this right hand side is $\hat{y}(\varepsilon, x + \varepsilon) - \hat{F}(\varepsilon, x, \hat{a}(\varepsilon), \hat{y}(\varepsilon, x)) = 0$, because \hat{a}, \hat{y} is a formal solution of equation (1). It immediately follows that all R_j are exponentially small. \square

5 Existence of analytic solutions

In this section, we want to prove our main theorem concerning the difference equation (1).

Theorem 5 *Consider the difference equation (1) and suppose that the hypotheses in section 4.2 are satisfied; moreover assume that $A_0(x_0) + I = \frac{\partial F}{\partial y}(0, x_0, a_0, \phi(x_0))$ is invertible.*

Then there exists a unique formal solution of equation (1)

$$\hat{y}(\varepsilon, x) = \sum_{j=0}^{\infty} b_j(x) \varepsilon^j, \quad \hat{a}(\varepsilon) = \sum_{j=0}^{\infty} c_j \varepsilon^j,$$

where $b_j(x)$ are analytic in $D_{r_1}(x_0)$ and $b_0 = \phi, c_0 = a_0$. Moreover these formal series are of Gevrey order 1.

For $\gamma \in \mathbb{R}$ and sufficiently small $\tilde{r}_1, \tilde{\delta} > 0$ such that $W = \{\varepsilon \mid |\varepsilon| < \tilde{\delta}, |\arg(\varepsilon) - \gamma| < \tilde{\delta}\}$ is contained in V , there exists an overstable solution (y^*, a^*) , $a^* : W \rightarrow \mathbb{C}^m$, $y^* : D_{\tilde{r}_1} \times W \rightarrow \mathbb{C}^n$ of equation (1) having (\hat{y}, \hat{a}) as asymptotic expansion of Gevrey order 1.

Proof. In order to find an actual solution of equation (1), we follow the strategy of [2].

As already remarked in section 4.3, we can assume without loss of generality that $x_0 = 0$, $a_0 = 0$, $\phi = 0$. For simplicity, we assume that $\gamma = 0$ and consider only real positive ε in the sequel; all considerations carry over for ε with small $|\arg(\varepsilon)|$.

We have already constructed the formal solution (\hat{y}, \hat{a}) in section 4.3 and, moreover, a quasi-solution (\tilde{y}, \tilde{a}) having it as an asymptotic expansion of Gevrey order 1 in the previous section. We insert $a = \tilde{a}$ in equation (1) and denote the resulting equation by

$$y(\varepsilon, x + \varepsilon) = G(\varepsilon, x, y(\varepsilon, x)). \quad (14)$$

We reduce equation (14) to an equation for a new unknown function $\tilde{\Delta}(\varepsilon, x)$ by

$$y(\varepsilon, x) = \tilde{y}(\varepsilon, x) + \tilde{\Delta}(\varepsilon, x).$$

We obtain

$$\tilde{\Delta}(\varepsilon, x + \varepsilon) = \tilde{G}(\varepsilon, x, \tilde{\Delta}(\varepsilon, x)) \quad (15)$$

where $\tilde{G}(\varepsilon, x, z) = G(\varepsilon, x, z + \tilde{y}(\varepsilon, x)) - \tilde{y}(\varepsilon, x + \varepsilon)$. Now $\tilde{\Delta}(\varepsilon, x) = 0$ is a quasi-solution of equation (15), i.e. $\tilde{G}(\varepsilon, x, 0)$ is exponentially small. Note that the domain of \tilde{G} is slightly smaller than that of G .

We will show that equation (15) has an analytic solution that is exponentially small. This will imply that equation (14) has an analytic solution $y(\varepsilon, x)$ given by $y(\varepsilon, x) = \tilde{y}(\varepsilon, x) + \tilde{\Delta}(\varepsilon, x)$, i.e. it differs from the quasi-solution only by an exponentially small function. Therefore this solution of equation (14) is Gevrey-1 since exponentially small functions are Gevrey-1.

In order to prove theorem 5, it thus suffices to solve equations of the form (14) satisfying

$$|G(\varepsilon, x, 0)| \leq K e^{-\frac{\alpha}{\varepsilon}} \text{ for } x \in D_{r_1} \quad (16)$$

with some positive constants α, K where $\frac{\partial G}{\partial y}(0, 0, 0)$ is invertible.

Before attacking this problem, we mention briefly the strategy followed in the work [2] for the analogous *differential* equation

$$\varepsilon \frac{d\tilde{\Delta}}{dx} = \tilde{G}(\varepsilon, x, \tilde{\Delta}(\varepsilon, x)) . \quad (17)$$

There, two different methods were used in order to show that equation (17) has an analytic solution (for small x and ε with small $|\arg \varepsilon|$) that is exponentially small. The first method uses a theorem on the growth of solutions of differential equations with parameter and the second one uses *Gronwall's* lemma.

Both of these methods can not be carried over to the theory of analytic *difference* equations because there is no local existence theorem for analytic solutions of difference equations.

Our method to solve an equation (14) satisfying (16) is the following: First split the linear part of G in several blocks, then construct a right inverse of this linear part block by block, rewrite equation (14) as a fixed point equation and finally apply the fixed point theorem on a certain Banach space with an exponential weight function.

We now have to prove

Proposition 6 *Consider the difference equation*

$$y(\varepsilon, x + \varepsilon) = G(\varepsilon, x, y(\varepsilon, x)), \quad (18)$$

with G analytic with respect to x, y and continuous with respect to ε, x, y , $0 \leq \varepsilon \leq \varepsilon_0$. We suppose that $\frac{\partial G}{\partial y}(0, 0, 0)$ is invertible and that there exist $K, \alpha > 0$ such that for $x \in D$, $0 \leq \varepsilon \leq \varepsilon_0$,

$$|G(\varepsilon, x, 0)| \leq K_1 e^{-\frac{2\alpha}{\varepsilon}}.$$

Then there exist a neighborhood Ω of 0, $\varepsilon_1 > 0$ and a solution $y : \Omega \times]0, \varepsilon_1[$ of equation (18) such that

$$|y(\varepsilon, x)| \leq e^{-\frac{\alpha}{\varepsilon}}, \forall x \in \Omega, 0 < \varepsilon < \varepsilon_1.$$

The proof of the proposition needs the two theorems stated below and both taken from [4] and lemma 7.

Theorem Given $\lambda_1, \dots, \lambda_n \in \mathbb{C} \setminus \{0, 1\}$, there exists a neighborhood Ω of 0 and bounded linear operators $T_{\varepsilon j} : \mathcal{H}_b(\Omega_\varepsilon) \rightarrow \mathcal{H}_b(\Omega_\varepsilon)$ such that $y_j := T_{\varepsilon j}(g)$, $g \in \mathcal{H}_b(\Omega_\varepsilon)$ implies that $y_j(\varepsilon, x + \varepsilon) = \lambda_j y_j(\varepsilon, x) + g(x)$ whenever $x \in \Omega_\varepsilon$ and $x + \varepsilon \in \Omega_\varepsilon$. Here $\Omega_\varepsilon = \Omega + [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $\mathcal{H}_b(\Omega_\varepsilon)$ denotes the Banach space of the holomorphic bounded functions on Ω_ε . Thus $T_\varepsilon : \mathcal{H}_b^n(\Omega_\varepsilon) \rightarrow \mathcal{H}_b^n(\Omega_\varepsilon)$ defined by $T_\varepsilon(g_j)_{j=1}^n = (T_{\varepsilon j}(g_j))_{j=1}^n$ is a bounded right inverse of the operator

$$\gamma_\varepsilon((y_j)_{j=1}^n)(\varepsilon, x) = (y_j(\varepsilon, x + \varepsilon) - \lambda_j y_j(\varepsilon, x))_{j=1}^n \quad (19)$$

Theorem Consider a matrix function $A(\varepsilon, x)$ of the form

$$A(\varepsilon, x) = \begin{pmatrix} A_1(\varepsilon, x) & A_2(\varepsilon, x) \\ A_3(\varepsilon, x) & A_4(\varepsilon, x) \end{pmatrix}$$

with the properties that $A_2(0, 0) = A_3(0, 0) = 0$ and that $A_1(0, 0)$ and $A_4(0, 0)$ are invertible and have no common eigenvalues. Then there exists a matrix function

$$P(\varepsilon, x) = \begin{pmatrix} I & P_2(\varepsilon, x) \\ P_3(\varepsilon, x) & I \end{pmatrix},$$

such that $P(\varepsilon, x + \varepsilon)^{-1} A(\varepsilon, x) P(\varepsilon, x) = B(\varepsilon, x)$ is block diagonal with

$$B(\varepsilon, x) = \begin{pmatrix} B_1(\varepsilon, x) & 0 \\ 0 & B_4(\varepsilon, x) \end{pmatrix},$$

and $B_1(0, 0) = A_1(0, 0)$, $B_4(0, 0) = A_4(0, 0)$.

The following lemma replaces Gronwall's lemma in the theory of difference equation with small step size.

Lemma 7 Let $\Omega \subset \mathbb{C}$ be a bounded, c -ascending domain with points x^\pm of maximal respectively minimal imaginary parts. We define

- $\Omega_\varepsilon = \Omega + [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ for each $\varepsilon > 0$,
- $D = \{(\varepsilon, x) | 0 < \varepsilon < \varepsilon_0, x \in \Omega_\varepsilon\}$ and
- $E = \{g : D \rightarrow \mathbb{C}^n | g \text{ continuous, holomorphic with respect to } x, \forall \varepsilon \in]0, \varepsilon_0[\text{ } g(\varepsilon, \cdot) \text{ bounded}\}.$

Let $x_0 \in \Omega$, $\varepsilon_0 > 0$, and $f : [0, \varepsilon_0] \times \Omega_{\varepsilon_0} \rightarrow M_{nn}(\mathbb{C})$ be bounded, continuous in ε , holomorphic in x such that

$$\sup_{\varepsilon, x} |f(\varepsilon, x) - id| \leq \left(\frac{1}{\beta} + \frac{1}{c^2(2\pi c - \beta)} \right)^{-1} \quad (20)$$

with some positive $\beta < 2\pi c$.

Then, there exists $T : E \rightarrow E$ with the properties:

1. T is linear.

2. Whenever $g \in E$ and $y = Tg$ then for all $\varepsilon \in]0, \varepsilon_0[$, $\forall x, x + \varepsilon \in \Omega_\varepsilon$,

$$y(\varepsilon, x + \varepsilon) = f(\varepsilon, x)y(\varepsilon, x) + g(\varepsilon, x). \quad (21)$$

3. There exists $L > 0$ such that for $g \in E$, $y = Tg$ and $0 < \varepsilon < \varepsilon_0$,

$$\sup_{x \in \Omega_\varepsilon} |y(\varepsilon, x)| \leq L e^{\frac{\beta}{\varepsilon}|x-x_0|} \sup_{x \in \Omega_\varepsilon} |g(\varepsilon, x)|.$$

Observe that contrary to the analogous statement for differential equations, it is needed that $f(\varepsilon, x)$ is not too far from the identity. Lemma 7 will be proved in section 6.

The rest of the proof of theorem 5 is analogous to [2]. Define

$$A(\varepsilon, x) := \frac{\partial G}{\partial y}(\varepsilon, x, 0).$$

Let $A_0 := A(0, 0)$. We can assume that A_0 is block diagonal, $A_0 = \text{bldiag}(A_{01}, \dots, A_{0\mu})$ where each block A_{0l} has exactly one eigenvalue λ_l and that $\lambda_1, \dots, \lambda_\mu$ are distinct. We can suppose that 1 is an eigenvalue of A_0 and that $\lambda_1 = 1$. In the case that $A_0 - id$ is nonsingular, the proof is easier than the subsequent one.

By the second of the theorems stated above, we can find a (rhomboidal) neighborhood Ω_0 of $x_0 = 0$, $\varepsilon_2 > 0$ and a matrix valued function $P :]0, \varepsilon_2[\times \Omega_0 \rightarrow M_{nn}(\mathbb{C})$ continuous, bounded, holomorphic with respect to x such that $P^{-1}(\varepsilon, x)$ is bounded,

$$(P(\varepsilon, x + \varepsilon))^{-1} A(\varepsilon, x) P(\varepsilon, x) = B(\varepsilon, x)$$

and

$$B(\varepsilon, x) = \text{bldiag}(B_1(\varepsilon, x), \dots, B_\mu(\varepsilon, x))$$

with blocks corresponding to $A_{01}, \dots, A_{0\mu}$, $B_l(0, 0) = A_{0l}$.

This suggest the transformation $y(\varepsilon, x) = P(\varepsilon, x)z(\varepsilon, x)$ which yields

$$z(\varepsilon, x + \varepsilon) = (P(\varepsilon, x + \varepsilon))^{-1} G(\varepsilon, x, P(\varepsilon, x)z(\varepsilon, x)) =: \tilde{G}(\varepsilon, x, z(\varepsilon, x)).$$

As obviously

$$\frac{\partial \tilde{G}}{\partial z}(\varepsilon, x, 0) = (P(\varepsilon, x + \varepsilon))^{-1} \frac{\partial G}{\partial y}(\varepsilon, x, 0) P(\varepsilon, x) = B(\varepsilon, x),$$

we can assume without loss of generality in the sequel that

$$A(\varepsilon, x) = \text{bldiag}(A_l(\varepsilon, x)), A_l(0, 0) = A_{0l}.$$

We can assume without loss of generality, furthermore, that A_{0l} , $l = 2, \dots, n$ are in Jordan canonical form and that their off diagonal elements are of absolute value smaller than δ , where δ is a certain positive constant determined later and only depending upon the eigenvalues $\lambda_2, \dots, \lambda_n$.

We rewrite equation (18) in the form

$$y(\varepsilon, x + \varepsilon) = R(\varepsilon, x) + \text{bldiag}(A_1(\varepsilon, x), \dots, A_\mu(\varepsilon, x))y(\varepsilon, x) + H(\varepsilon, x, y(\varepsilon, x)), \quad (22)$$

where $R(\varepsilon, x) = G(\varepsilon, x, 0) = \mathcal{O}(e^{-\frac{2\alpha}{\varepsilon}})$ and $H(\varepsilon, x, y(\varepsilon, x))$ contains the quadratic and higher order terms of G . As in [2] we rewrite equation (22) as a block system S

$$y_1(\varepsilon, x + \varepsilon) = A_1(\varepsilon, x)y_1(\varepsilon, x) + (S_1y)(\varepsilon, x) \quad (23)$$

$$y_j(\varepsilon, x + \varepsilon) = \lambda_j y_j(\varepsilon, x) + (S_jy)(\varepsilon, x), \quad j = 2, \dots, \mu \quad (24)$$

where

$$S_1y(\varepsilon, x) = R_1(\varepsilon, x) + H_1(\varepsilon, x, y(\varepsilon, x)) \quad (25)$$

$$S_jy(\varepsilon, x) = R_j(\varepsilon, x) + (A_j(\varepsilon, x) - \lambda_j id)y_j(\varepsilon, x) + H_j(\varepsilon, x, y(\varepsilon, x)), \quad j = 2, \dots, \mu \quad (26)$$

and

$$H(\varepsilon, x, y(\varepsilon, x)) = (H_1(\varepsilon, x, y(\varepsilon, x)), \dots, H_\mu(\varepsilon, x, y(\varepsilon, x))) \quad (27)$$

$$R(\varepsilon, x) = (R_1(\varepsilon, x), \dots, R_\mu(\varepsilon, x)) . \quad (28)$$

We choose a sufficiently small $\varepsilon_3 \in]0, \varepsilon_2]$ and a subdomain Ω of Ω_0 such that $\Omega_{\varepsilon_3} \subset \Omega_0$ and such that the first theorem recalled above holds. It can be chosen to be c -ascending with a $c > 1/2$. In the sequel, we might have to reduce its size so that $\sup\{|A_j(\varepsilon, x) - \lambda_j id| \mid 0 < \varepsilon < \varepsilon_3, x \in \Omega_\varepsilon\}$ is sufficiently small. Denote by E the vector space defined in lemma 7.

Then the operators

$$\begin{aligned} u \mapsto v, \quad v(\varepsilon, x) &= u(\varepsilon, x + \varepsilon) - A_1(\varepsilon, x)u(\varepsilon, x), \text{ respectively} \\ u \mapsto w, \quad w(\varepsilon, x) &= u(\varepsilon, x + \varepsilon) - \lambda_j u(\varepsilon, x), \quad j = 2, \dots, n \end{aligned} \quad (29)$$

have right inverses on E : a right inverse T_1 of the first operator exists according to lemma 7 if Ω is sufficiently small; the right inverses T_2, \dots, T_μ of the other operators exist according to the first theorem recalled above. It is thus sufficient to solve the fixed point equation

$$y = (T_1 S_1 y, \dots, T_\mu S_\mu y) =: \mathcal{T}(y) . \quad (30)$$

To do so, we show that the right hand side of equation (30) is a contraction on a closed subset of a certain Banach space. Choose $\beta \in]0, 2\pi c[$ such that $\beta \text{diam}(\Omega) \leq \alpha + \varepsilon_3 \log(\varepsilon_3)$ and let \mathcal{B} be the set of all functions $y(\varepsilon, x)$ with values in \mathbb{C}^n that are continuous on D , analytic for $x \in \Omega_\varepsilon$ for $\varepsilon \in]0, \varepsilon_3]$ and satisfy an estimate

$$|y(\varepsilon, x)| \leq L \exp(-\alpha/\varepsilon), \quad (31)$$

with some $L > 0$. We define the norm $\|y\|_\alpha$ of y as the smallest number L such that the above estimate of equation (31) holds. Then, \mathcal{B} becomes a Banach space with this norm and the usual addition and scalar multiplication of functions. There are constants K_2, K_3 such that

$$\sup_{x \in \Omega_\varepsilon} |T_1 v(\varepsilon, x)| \leq K_2 \varepsilon_3 \exp(\alpha/\varepsilon) \sup_{x \in \Omega_\varepsilon} |v(\varepsilon, x)| \quad (32)$$

for $0 < \varepsilon < \varepsilon_3$, $v \in \mathcal{B}^{n_1}$, where n_1 is the size of A_1 (cf lemma 7), and

$$\sup_{x \in \Omega_\varepsilon} |T_j w(\varepsilon, x)| \leq K_3 \sup_{x \in \Omega_\varepsilon} |w(\varepsilon, x)| \quad (33)$$

for $j = 2, \dots, \mu$, $0 < \varepsilon < \varepsilon_3$, $w \in \mathcal{B}^{n_j}$; here n_j is the size of A_j .

Next, let \mathcal{M} be the closed subset of \mathcal{B}^n of all y satisfying $|y(\varepsilon, x)| \leq \rho$ for all (ε, x) , where ρ is sufficiently small so that $H(\varepsilon, x, z)$ is defined for $\varepsilon \in]0, \varepsilon_3]$, $x \in \Omega_\varepsilon$, and $|z| \leq \rho$. Note that \mathcal{M} is non empty.

Now we show that the right hand side of equation (30) has a fixed point in \mathcal{M} if ρ and Ω are sufficiently small. We have

$$H(\varepsilon, x, z_2) - H(\varepsilon, x, z_1) = \left(\int_0^1 \frac{\partial H}{\partial z}(\varepsilon, x, z_1 + t(z_2 - z_1)) dt \right) (z_2 - z_1), \quad (34)$$

and $\frac{\partial H}{\partial z}(\varepsilon, x, 0) = 0$. Thus there exist $K_4 > 0$ such that

$$|H(\varepsilon, x, z_2) - H(\varepsilon, x, z_1)| \leq K_4 \max(|z_1|, |z_2|) |z_2 - z_1|$$

for $0 < \varepsilon < \varepsilon_3$, $x \in \Omega_\varepsilon$ and $|z_1|, |z_2| \leq \rho$. Hence for $y_1, y_2 \in \mathcal{M}$ we obtain

$$|(S_1(y_2))(\varepsilon, x) - (S_1(y_1))(\varepsilon, x)| \leq K_4 \max(\|y_1\|, \|y_2\|) \|y_2 - y_1\| \exp(-2\alpha/\varepsilon)$$

and hence by equation (32)

$$\|T_1 S_1(y_2) - T_1 S_1(y_1)\| \leq K_2 K_4 \varepsilon_3 \rho \|y_2 - y_1\|. \quad (35)$$

For $j = 2, \dots, \mu$, we find using equations (33, 34)

$$\|T_j S_j(y_2) - T_j S_j(y_1)\| \leq K_3 \left(L_j(\Omega) + K_4 e^{-\alpha/\varepsilon_3} \rho \right) \|y_2 - y_1\|, \quad (36)$$

where $L_j(\Omega) = \sup_{(\varepsilon, x) \in D} |A_j(\varepsilon, x) - \lambda_j id|$. Finally, because of the assumption of the theorem

$$\|\mathcal{T}(0)\| = \max(\|T_1 S_1(0)\|, \dots, \|T_\mu S_\mu(0)\|) \leq K_1 \max(K_2 \varepsilon_3, K_3 e^{-\alpha/\varepsilon_3}). \quad (37)$$

Now first choose¹ $\delta, \rho > 0$ and Ω sufficiently small so that the factors of $\|y_2 - y_1\|$ in equations (35, 36) are smaller than $1/2$, then choose $\varepsilon_3 > 0$ small enough so that the right hand side of equation (37) becomes smaller than $\rho/2$. Then we see that $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{B}^n$ satisfies the hypotheses of Banach's fixed point theorem and hence, \mathcal{T} has a unique fixed point y in \mathcal{M} . As discussed above equation (30), this yields the existence of an analytic solution y of (18) that is exponentially small: $|y(\varepsilon, x)| \leq \rho e^{-\alpha/\varepsilon}$. \square

6 Proof of lemma 7

If equation (21) is considered and only $f(0, x_0) = id$ is known, then by reducing Ω , as a c -ascending domain, one can achieve that f satisfies equation (20).

Equation (21) is somewhat analogous to $\varepsilon y' = (f(\varepsilon, x) - id)y + g$; only for the latter equation, Schäfke - Schäfke [10] or "Gronwall's lemma" can be used. In any case, the

¹see above equation (22).

local existence of a solution of the ordinary differential equation is not a problem. For the difference equation, the existence of a local solution was **not** known.

The idea of the proof is as follows: We use the well-known formula of Fruchard - Schäfke [7] providing a right inverse of the operator Δ_ε given by

$$\Delta_\varepsilon y(\varepsilon, x) = \frac{1}{\varepsilon} (y(\varepsilon, x + \varepsilon) - y(\varepsilon, x))$$

and apply the fixed point principle in a certain Banach space with *weight* function $e^{\beta|x-x_0|/\varepsilon}$.

Without loss of generality, we suppose that $x_0 = 0$. We define $b(\varepsilon, x) = f(\varepsilon, x) - id$, thus $b :]0, \varepsilon_0[\times \Omega_{\varepsilon_0} \rightarrow M_{nn}(\mathbb{C})$ is bounded by γ where $\gamma^{-1} = \beta^{-1} + c^{-2}(2\pi c - \beta)^{-1}$. Our equation (21) becomes

$$\varepsilon \Delta_\varepsilon y(\varepsilon, x) = b(\varepsilon, x)y(\varepsilon, x) + g(\varepsilon, x).$$

With a certain right inverse V_ε of Δ_ε to be introduced subsequently it suffices to solve

$$y(\varepsilon, x) = \frac{1}{\varepsilon} V_\varepsilon(b \cdot y) + \frac{1}{\varepsilon} V_\varepsilon(g), \quad (38)$$

where $b \cdot y$ denotes the function $(\varepsilon, x) \mapsto b(\varepsilon, x)y(\varepsilon, x)$. In the appropriate setting, $z \mapsto \frac{1}{\varepsilon} V_\varepsilon(b \cdot z)$ will be shown to define a contraction called θ_ε and thus

$$y(\varepsilon, x) = \frac{1}{\varepsilon} (id - \theta_\varepsilon)^{-1} V_\varepsilon(g)$$

will define a solution of equation (21) having, as we show, the wanted properties. This provides the operator T .

Now, we need to detail our construction. We put, as before, for $g \in D$.

$$Sg(\varepsilon, x) = \int_0^x g(\varepsilon, s) ds \quad (39)$$

$$U_\varepsilon g(\varepsilon, x) = \frac{1}{4} \int_{-\frac{1}{8}}^{\frac{1}{8}} U_{\varepsilon t} g(\varepsilon, x) dt \quad (40)$$

and for $-\frac{1}{8} \leq t \leq \frac{1}{8}$

$$\begin{aligned} U_{\varepsilon t} g(\varepsilon, x) &= \frac{1}{\varepsilon} \int_{\gamma_{x,t}^-} \frac{g(\varepsilon, \xi)}{1 - e^{\frac{2i\pi}{\varepsilon}(\xi-x)}} d\xi \\ &\quad - \frac{1}{\varepsilon} \int_{\gamma_{x,t}^+} \frac{g(\varepsilon, \xi)}{1 - e^{\frac{-2i\pi}{\varepsilon}(\xi-x)}} d\xi, \end{aligned} \quad (41)$$

where $\gamma_{x,t}^-$ is an ascending path joining $x^- + \varepsilon t$ and $x - \varepsilon/2$ (avoiding $x - \varepsilon$ and x) and such that $\text{Im } \xi$ is increasing as ξ varies on it, and $\gamma_{x,t}^+$ is an ascending path joining $x - \varepsilon/2$ and $x^+ + \varepsilon t$.

It was shown that $S - \varepsilon U_\varepsilon$ is a right inverse of Δ_ε (on Ω_ε) and that it defines a bounded linear operator on the set of **bounded** functions in E with respect to the maximum norm. Here, we need slightly different estimates. First, without loss of generality, we assume that g is bounded (otherwise, divide the linear equation by $\alpha(\varepsilon) := \sup_{x \in \Omega_\varepsilon} |g(\varepsilon, x)|$ which is continuous).

We introduce the Banach space

$$\mathcal{B} = \{h \in E \mid \exists k \forall (\varepsilon, x) \mid |h(\varepsilon, x)| \leq k e^{\beta|x|/\varepsilon}\},$$

with the norm $\|h\|_{\mathcal{B}} = \min\{k \mid k \text{ satisfies above estimate}\}$.

Lemma 8 For $h \in \mathcal{B}$ one has $\left\| \frac{1}{\varepsilon} Sh \right\|_{\mathcal{B}} \leq \frac{1}{\beta} \|h\|_{\mathcal{B}}$ and $\|U_{\varepsilon} h\|_{\mathcal{B}} \leq \frac{1}{c^3} \frac{1}{2\pi - \frac{\beta}{c}} \|h\|_{\mathcal{B}}$.

Proof. The first estimate follows from

$$\begin{aligned} \left| \frac{1}{\varepsilon} \int_0^x h(\varepsilon, s) ds \right| &\leq \frac{1}{\varepsilon} \int_0^x \|h\|_{\mathcal{B}} e^{\beta|s|/\varepsilon} |ds| \\ &\leq \frac{1}{\varepsilon} \|h\|_{\mathcal{B}} \int_0^{|x|} e^{\beta t/\varepsilon} dt \leq \frac{1}{\beta} \|h\|_{\mathcal{B}} e^{\beta|x|/\varepsilon}. \end{aligned}$$

The two integrals in the definition of $U_{\varepsilon t}$ will be called I^- and I^+ . We will only estimate I^- for some $t \in [-\frac{1}{8}, \frac{1}{8}]$ and x with $\text{Im } x > \text{Im } x^- + 2\varepsilon$; the estimates for I^+ are analogous and in the neighborhood of x^- , the estimates are as in [7]. As there, we find

$$\begin{aligned} |I^-| &= \left| \frac{1}{\varepsilon} \int_{\gamma_{x,t}^-} \frac{h(\varepsilon, \xi)}{1 - e^{\frac{2i\pi}{\varepsilon}(\xi-x)}} d\xi \right| \\ &\leq \frac{1}{\varepsilon c^2} \int_{x^-+t\varepsilon}^{x-\frac{\varepsilon}{2}} \exp\left(-\frac{2\pi}{\varepsilon} \text{Im}(x-\xi)\right) |h(\varepsilon, \xi)| |d\xi| \\ &\leq \frac{1}{\varepsilon c^2} \|h\|_{\mathcal{B}} e^{\beta|x|/\varepsilon} \int_{x^-+t\varepsilon}^{x-\frac{\varepsilon}{2}} \exp\left(-\frac{2\pi}{\varepsilon} \text{Im}(x-\xi) + \frac{\beta(|\xi|-|x|)}{\varepsilon}\right) |d\xi| \\ &\leq \frac{1}{\varepsilon c^2} \|h\|_{\mathcal{B}} e^{\beta|x|/\varepsilon} \int_{x^-+t\varepsilon}^{x-\frac{\varepsilon}{2}} \exp\left(-\frac{\rho}{\varepsilon} \text{Im}(x-\xi)\right) |d\xi|, \end{aligned}$$

where $\rho = 2\pi - \frac{\beta}{c}$, because the the argument of the exponential is $\frac{1}{\varepsilon}(-2\pi \text{Im}(x-\xi) + \beta|x-\xi|) \leq -\frac{1}{\varepsilon}(2\pi - \frac{\beta}{c}) \text{Im}(x-\xi)$. As we can parametrize by $t = \text{Im } \xi$, $|d\xi| \leq \frac{1}{c} dt$, we obtain that I^- is $\leq \frac{1}{c^3 \rho} \|h\|_{\mathcal{B}} e^{\beta|x|/\varepsilon}$. All this is only possible, of course, if $\beta < 2\pi c$ so that $\rho = 2\pi - \frac{\beta}{c} > 0$. \square

The remainder of the proof of lemma 7 is straightforward now. If

$$\sup_{\varepsilon, x} |b(\varepsilon, x)| \cdot \left(\frac{1}{\beta} + \frac{1}{c^2(2\pi c - \beta)} \right) := \gamma < 1,$$

then θ_{ε} is a contraction in \mathcal{B} and thus $(id - \theta_{\varepsilon})^{-1}$ is defined. We immediately obtain the lemma observing that $\left\| \frac{1}{\varepsilon} V_{\varepsilon} \right\|_{\mathcal{B}} \leq \frac{1}{\beta} + \frac{1}{c^2(2\pi c - \beta)}$.

7 Exponential closeness

Let two quasi-solutions $(a_1(\varepsilon), y_1(\varepsilon, x))$ and $(a_1(\varepsilon), y_1(\varepsilon, x))$ of equation (1) be given; we want to show that their difference is exponentially small.

More precisely, we consider the difference equation (1) and suppose that F, x_0, a_0, ϕ_0 satisfy the hypotheses in section 2, in particular, the hypothesis of transversality (H). We assume that $(a_j(\varepsilon), y_j(\varepsilon, x)), j = 1, 2$ are defined and bounded for $\varepsilon \in V$ and $x \in D_r(x_0)$, $r > 0$, holomorphic with respect x , that $a_j(\varepsilon) \rightarrow a_0$ and $y_j(\varepsilon, x) \rightarrow \phi_0(x)$ uniformly on $D_r(x_0)$ for $j = 1, 2$ and that

$$R_j(\varepsilon, x) := y_j(\varepsilon, x + \varepsilon) - F(\varepsilon, x, a_j(\varepsilon), y_j(\varepsilon, x)) = \mathcal{O}(\exp(-T/|\varepsilon|)) \quad (42)$$

for $j = 1, 2$, $V \ni \varepsilon \rightarrow 0$ uniformly for $x \in D_r(x_0)$ and T is a positive constant.

Theorem 9 *Under the above assumptions, there exist positive $K, \alpha > 0$ such that*

$$|a_2(\varepsilon) - a_1(\varepsilon)| \leq K \exp(-\alpha/|\varepsilon|)$$

for $\varepsilon \in V$ and for all positive $r_1 < r$, there exist positive M, β such that

$$|y_2(\varepsilon, x) - y_1(\varepsilon, x)| \leq M \exp(-\beta/|\varepsilon|)$$

for all x satisfying $|x - x_0| \leq r_1$ and $\varepsilon \in V$.

The theorem is in particular valid for actual *solutions* of equation (1) and shows that all overstable solutions reducing to the same slow curve are exponentially close to each other. In particular in the situation of our theorem 5, all parameter values yielding solutions of the difference equation have to be exponentially close to the one we constructed using Borel and Laplace transform.

Proof. Without loss in generality, we can assume that $a_0 = 0$, $x_0 = 0$ and $\phi_0 = 0$. We define $z(\varepsilon, x) = y_2(\varepsilon, x) - y_1(\varepsilon, x)$ and $b(\varepsilon) = a_2(\varepsilon) - a_1(\varepsilon)$. By subtraction of the difference equations (42), we obtain

$$z(\varepsilon, x + \varepsilon) = \tilde{A}(\varepsilon, x)z(\varepsilon, x) + \tilde{B}(\varepsilon, x)b(\varepsilon) + R(\varepsilon, x), \quad (43)$$

where $R = R_2 - R_1$ and

$$\begin{aligned} \tilde{A}(\varepsilon, x) &= Q_1(\varepsilon, x, y_2(\varepsilon, x), y_1(\varepsilon, x), a_2(\varepsilon), a_1(\varepsilon)), \\ \tilde{B}(\varepsilon, x) &= Q_2(\varepsilon, x, y_2(\varepsilon, x), y_1(\varepsilon, x), a_2(\varepsilon), a_1(\varepsilon)); \end{aligned}$$

here Q_1 and Q_2 are analytic functions satisfying

$$\begin{aligned} &F(\varepsilon, x, c_2, u_2) - F(\varepsilon, x, c_1, u_1) \\ &= Q_1(\varepsilon, x, u_2, u_1, c_2, c_1)(u_2 - u_1) + Q_2(\varepsilon, x, u_2, u_1, c_2, c_1)(c_2 - c_1) \end{aligned}$$

for all sufficiently small $u_j, c_j, j = 1, 2$; Q_j reduce to the partial derivatives if $u_2 = u_1, c_2 = c_1$. Using $A_0(x) = \frac{\partial F}{\partial y}(0, x, 0, 0) - I$ and $B_0(x) = \frac{\partial F}{\partial a}(0, x, 0, 0)$, we have that $\tilde{A}(\varepsilon, x) = \tilde{A}(\varepsilon, x) - A_0(x)$ and $\tilde{B}(\varepsilon, x) = \tilde{B}(\varepsilon, x) - B_0(x)$ tend to 0 uniformly on $D_r(0)$ as $V \ni \varepsilon \rightarrow 0$.

Hence we rewrite our equation in the following “prepared form”

$$\begin{aligned} &A_0(x)z(\varepsilon, x) + B_0(x)b(\varepsilon) \\ &= z(\varepsilon, x + \varepsilon) - z(\varepsilon, x) - \tilde{A}(\varepsilon, x)z(\varepsilon, x) - \tilde{B}(\varepsilon, x)b(\varepsilon) - R(\varepsilon, x). \end{aligned} \quad (44)$$

As we want to use our modified Nagumo norms (cf Definition 2), we need a lemma concerning the difference operator. In the present context, we need to reduce the radius of the domain of the functions because $z(\varepsilon, x + \varepsilon) - z(\varepsilon, x)$ is only defined on a subdomain. Thus we denote with fixed $0 < \rho < r$

$$\|f\|_{nt} = \sup_{|x| < t} \{|f(x)| d_t(x)^n\} \quad \text{where } d_t(x) = \min(t - |x|, t - \rho) \quad (45)$$

for $t \in]\rho, r[$. Note that $\|\cdot\|_{nt}$ is the modified Nagumo norm on D_t , so its properties (compatibility with elementary operations, differentiation, the shift operator S (cf appendix A.1) and thus also with the linear operators (P_1, P_2) ; cf equations (9, 10)) can be used. Observe that the constant for P_1 does not depend upon $t \in]\rho, r[$!

Lemma 10 Suppose that $\rho < t - |\varepsilon| < t \leq r$. If $f : D_t \rightarrow \mathbb{C}$ is holomorphic and $g : D_{t-|\varepsilon|} \rightarrow \mathbb{C}$ is defined by $g(x) = f(x + \varepsilon) - f(x)$, then

$$\|g\|_{n+1, t-|\varepsilon|} \leq e|\varepsilon| (n+1) \|f\|_{nt}.$$

Proof. Let $|x| < t - |\varepsilon|$. Then

$$|g(x)| \leq |\varepsilon| \max_{0 \leq s \leq 1} |f'(x + s\varepsilon)| \leq |\varepsilon| e(n+1) \|f\|_{nt} \max_{0 \leq s \leq 1} d_t(x + s\varepsilon)^{-n-1}$$

by property (6) (see p. 6).

Together with $d_t(x + s\varepsilon) \geq d_{t-|\varepsilon|}(x)$, this yields the lemma. \square

Now we apply the lemma to equation (44), using the fact that the inverse (P_1, P_2) of $(c, u) \mapsto A_0(\cdot)u + B_0(\cdot)c$ is a bounded linear operator with respect to $\|\cdot\|_{nt}$ and that its norm is independent of t . For sufficiently small ε , we find

$$\begin{aligned} \max(\|z(\varepsilon, \cdot)\|_{n+1, t-|\varepsilon|}, \|b(\varepsilon)\|_{n+1, t-|\varepsilon|}) &\leq M|\varepsilon|(n+1)\|z\|_{nt} + \\ &\frac{1}{2} \max(\|z(\varepsilon, \cdot)\|_{n+1, t-|\varepsilon|}, \|b(\varepsilon)\|_{n+1, t-|\varepsilon|}) + t^{n+1} \exp(-T/|\varepsilon|) \end{aligned}$$

for $\rho < t \leq r$ and $\varepsilon \in V$ sufficiently small and $n \in \mathbb{N}$. Now we define $L_n(\varepsilon) = \max(\|z(\varepsilon, \cdot)\|_{n, r-n|\varepsilon|}, \|b(\varepsilon)\|_{n, r-n|\varepsilon|})$ for such n, ε satisfying $n|\varepsilon| < r - \rho$. We obtain $L_{n+1}(\varepsilon) \leq M|\varepsilon|(n+1)L_n(\varepsilon) + \frac{1}{2}L_{n+1}(\varepsilon) + Mr^{n+1} \exp(-T/|\varepsilon|)$ provided $(n+1)|\varepsilon| < r - \rho$ and thus by induction

$$L_n(\varepsilon) \leq L_0(\varepsilon)(2M|\varepsilon|)^n n! + 2M \exp(-T/|\varepsilon|) \sum_{j=1}^n \frac{n!}{j!} (2M|\varepsilon|)^{n-j} r^j \quad (46)$$

if $n|\varepsilon| < r - \rho$.

Next, with some $\mu \in]0, r - \rho[$ to be determined and, for sufficiently small $\varepsilon \in V$, we choose n as the largest integer below $\mu/|\varepsilon|$. Then using Stirling's formula and observing that the last term of the sum is the largest provided $\mu < r/2M$, we find that there exists $\tilde{M} > 0$ such that

$$L_n(\varepsilon) \leq \tilde{M} n C^n, \quad C = \max(2M\mu/e, e^{-T/\mu r})$$

for sufficiently small ε . Using the relation between $\|g\|_{nt}$ and the values of $|g(x)|$ for $|x| < t - \delta$, we obtain for some $\delta > 0$ to be determined, that

$$\max_{|x| < r - \mu - \delta} |z(\varepsilon, x)|, |b(\varepsilon)| \leq \overline{M} |\varepsilon|^{-1} (C/\delta)^{\mu/|\varepsilon|}$$

with some constant $\overline{M} > 0$ and for all sufficiently small ε . For arbitrary $\delta < r - \rho$, we see that $C/\delta < 1$, if $\mu < r - \rho - \delta$ is chosen sufficiently small. This implies the statement of the theorem; the factor $|\varepsilon|^{-1}$ can be eliminated by slightly reducing β . \square

8 Examples

In this section, we consider two applications of our results to second order difference equations and deduce consequences for solutions on discrete sets.

The first example is

$$y(x + \varepsilon) = x^2 y(x - \varepsilon) + x - \varepsilon y(x)^2 + a. \quad (47)$$

It is equivalent to the following two-dimensional system

$$z(x + \varepsilon) = F(\varepsilon, x, z(x), a), \quad \text{where} \quad \begin{cases} F_1(\varepsilon, x, z, a) &= z_2 \\ F_2(\varepsilon, x, z, a) &= x^2 z_1 + x - \varepsilon z_2^2 + a \end{cases} \quad (48)$$

The slow curve $\phi(x) = (\phi_1(x), \phi_2(x))$ is given by $\phi_1(x) = \phi_2(x) = -\frac{1}{1+x}$, the Jacobian $\tilde{A}_0(x) = \frac{\partial F}{\partial z}(0, x, 1, \phi(x)) = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix}$ has eigenvalues x and $-x$ and hence $A_0(x) = \tilde{A}_0(x) - I$ is not invertible at $x = x_0 = 1$. It is easily seen that the hypothesis of transversality is satisfied with $B_0(x) = (0, 1)^T$. By our theorem 5, there exist $\delta > 0$ and an overstable solution $(a(\varepsilon), z(\varepsilon, x))$ of (48) on $|x - 1| < \delta$ and thus also an overstable solution $(a(\varepsilon), y(\varepsilon, x))$ of (47).

Consider now $x_1 < 1 < x_2$ and $\varepsilon > 0$ sufficiently small such that $N_1 = (1 - x_1)/\varepsilon$ and $N_2 = (x_2 - 1)/\varepsilon$ are integers (this is a condition on x_1, x_2 and ε); consider x_1, x_2 as fixed and ε as tending to 0 with the preceding restriction. Then put $t_l = 1 + l\varepsilon$, $l = -N_1, \dots, N_2$.

Our above result implies that for x_1, x_2 sufficiently close to 1 and ε sufficiently small, there exist $a(\varepsilon)$ and $y(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ such that (47) is satisfied and $a(\varepsilon) \rightarrow 1$, $y(\varepsilon, t_{l(\varepsilon)}) \rightarrow \phi_1(x)$ if $\varepsilon \rightarrow 0$ and $t_{l(\varepsilon)} \rightarrow x$.

In the numerical study of this problem, it is important that both eigenvalues of $\tilde{A}_0(x)$ are of absolute value less than 1 for $x < 1$ and both are larger than 1 for $x > 1$; thus the slow curve is attractive for $x < 1$ and repulsive for $x > 1$. Hence the solution $y^+(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ of (47) defined by $y^+(\varepsilon, x_2) = y^+(x_2 - \varepsilon) = 0$ and backward recursion will remain uniformly bounded on $[1, x_2]$ for any value of a , while its solution $y^-(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ defined by $y^-(\varepsilon, x_1) = y^-(x_1 + \varepsilon) = 0$ will be bounded on $[x_1, 1]$. For the “canard value” $a = a(\varepsilon)$, both are exponentially close to the overstable solution near $x = 1$ and hence $y^+(\varepsilon, 1) - y^-(\varepsilon, 1)$ is exponentially small for this value of a .

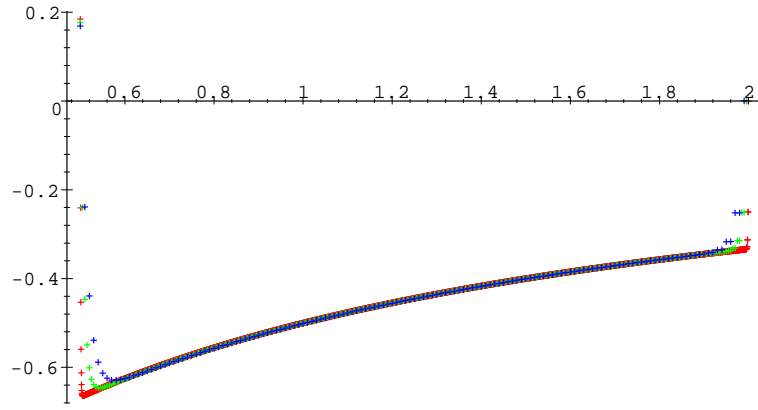
This gives a means to calculate some “canard value” (exponentially close to the one given by our overstable solution) numerically: Simply consider $y^+(\varepsilon, 1) - y^-(\varepsilon, 1) := g(a)$ as a function of a and solve $g(a) = 0$. Observe that it is unlikely to achieve $y^+(\varepsilon, t_l) = y^-(\varepsilon, t_l)$ for all l , as one parameter only allows to solve one equation and her equality for two subsequent values of l would be required.

Our numerical procedure worked for $x_1 = 1/2$, $x_2 = 2$ and $\varepsilon \in \{0.01, 0.005, 0.0005\}$; figure 1 shows the resulting $y^+(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$. Observe that our result does not allow to prove the existence of canard solutions on discrete sets for specific *given* x_1, x_2 ; here an approach similar to [3] might work. Our method, of course, yields boundary layers in x_1 and x_2 .

In order to illustrate the extreme sensitivity of the canard phenomenon, we replaced a canard value a by $a + 10^{-50}$ for $x_1 = 1/2, x_2 = 2, \varepsilon = 0.0005$. The resulting solution $y^+(\varepsilon, t_l)$ still follows the slow curve down to about $x \approx 0.68$ and then leaves it quickly (cf figure 2).

Our second example is a discrete version of the differential equation $\varepsilon y' = 1 - x^2 + (a - x)/y$ appearing in the study of the (continuous) canard solutions of the forced Van der Pol equation. Here, we replace $\varepsilon y'$ by $\frac{1}{2}(y(x + \varepsilon) - y(x - \varepsilon))$ and obtain the second order difference equation

$$y(x + \varepsilon) = y(x - \varepsilon) + 2 \left(1 - x^2 + \frac{a - x}{y(x)} \right). \quad (49)$$

Figure 1: Canard solutions for (47) and $\varepsilon \in \{0.01, 0.005, 0.0005\}$.

which is again equivalent to a two-dimensional system

$$z(x + \varepsilon) = F(\varepsilon, x, z(x), a), \quad \text{where} \quad \begin{cases} F_1(\varepsilon, x, z, a) &= z_2 \\ F_2(\varepsilon, x, z, a) &= z_1 + 2 \left(1 - x^2 + \frac{a-x}{z_2}\right) \end{cases} \quad (50)$$

As for the continuous canard equation, the slow curve $\phi(x)$ is given by $\phi_1(x) = \phi_2(x) = -\frac{1}{1+x}$. We find here

$$\tilde{A}_0(x) = \frac{\partial F}{\partial z}(0, x, 1, \phi(x)) = \begin{pmatrix} 0 & 1 \\ 1 & 2(x-1)(x+1)^2 \end{pmatrix}.$$

with eigenvalues $\lambda(x) = (x-1)(x+1)^2 + \sqrt{1 + (x-1)^2(x+1)^4}$ and $\tilde{\lambda}(x) = -1/\lambda(x)$; the matrix $B_0(x) = (0, -2(1+x))$.

At $x = x_0 = 1$, we find $\lambda(x) = 1$ and hence $A_0(x) = \tilde{A}_0(x) - I$ is not invertible at $x = 1$. It is again easily checked that the hypothesis of transversality is satisfied and hence, by our main result, there exist $\delta > 0$ and an overstable solution $(a(\varepsilon), z(\varepsilon, x))$ of (50) on $|x-1| < \delta$ and thus also an overstable solution $(a(\varepsilon), y(\varepsilon, x))$ of (49).

As in the first example, consider now $x_1 < 1 < x_2$ and $\varepsilon > 0$ sufficiently small such that $N_1 = (1-x_1)/\varepsilon$ and $N_2 = (x_2-1)/\varepsilon$ are integers; consider again x_1, x_2 as fixed and ε as tending to 0 with the preceding restriction. Then put $t_l = 1 + l\varepsilon$, $l = -N_1, \dots, N_2$.

Our above result implies that for x_1, x_2 sufficiently close to 1 and ε sufficiently small, there exist $a(\varepsilon)$ and $y(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ such that (49) is satisfied and $a(\varepsilon) \rightarrow 1$, $y(\varepsilon, t_{l(\varepsilon)}) \rightarrow \phi_1(x)$ if $\varepsilon \rightarrow 0$ and $t_{l(\varepsilon)} \rightarrow x$.

The numerical study of this example is however completely different from the previous one, because for $x < 1$, we have $|\lambda(x)| < 1 < |\tilde{\lambda}(x)|$ whereas for $x > 1$, we have $|\lambda(x)| > 1 > |\tilde{\lambda}(x)|$. Thus, the slow curve is neither attractive nor repulsive on any interval and the method applied to the previous example cannot work.

Our idea is to go over to $v(\varepsilon, t_l) = y(\varepsilon, t_l) + \frac{1}{1+t_l}$ and $b = a - 1$; then to the following system for v and $u(\varepsilon, t_l) := v(\varepsilon, t_l + \varepsilon) + \lambda(t_l)^{-1}v(\varepsilon, t_l)$ (here, the argument ε has been omitted)

$$\begin{aligned} u(t_l) - \lambda(t_l)u(t_l - \varepsilon) + 2(1+t_l)b &= h(\varepsilon, t_l, v(t_l - \varepsilon), v(t_l), b) \\ v(t_l + \varepsilon) + \lambda(t_l)^{-1}v(t_l) &= u(t_l) \end{aligned} \quad (51)$$

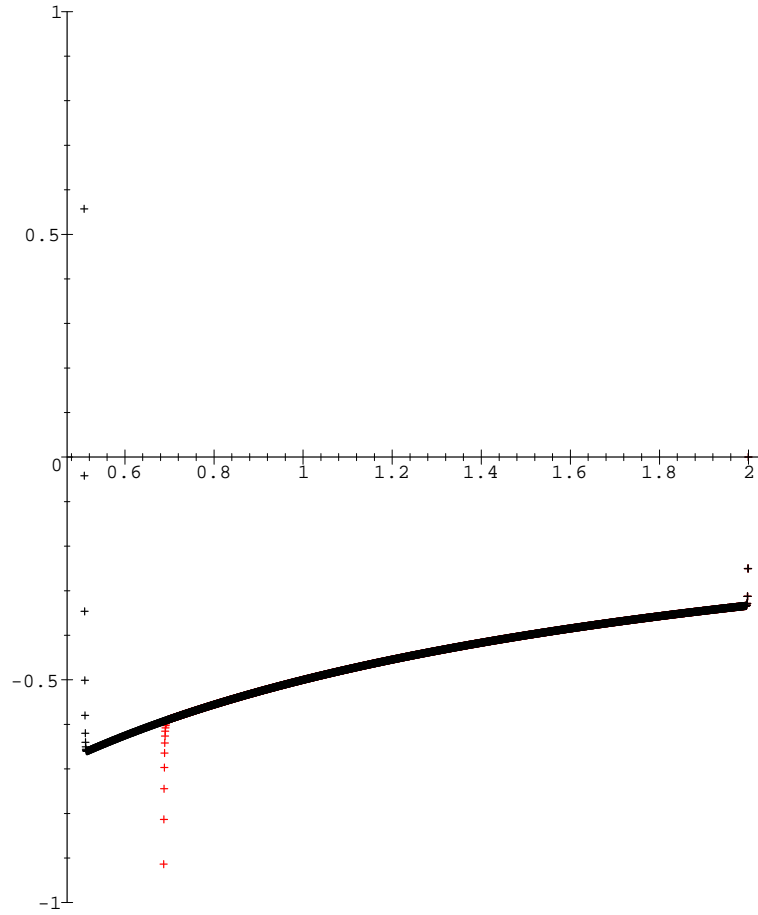


Figure 2: Sensitivity of the canard phenomenon for (47) .

where

$$h(\varepsilon, x, w, v, b) = (1 - \lambda(t)\lambda(t - \varepsilon)^{-1})w + (1 + t + \varepsilon)^{-1} - (1 + t - \varepsilon)^{-1} - 2[(1 - t^2)v + b](1 + t)^2v(1 - (1 + t)v)^{-1} .$$

We know that this system has a solution $(b(\varepsilon), (u(\varepsilon, t_l), v(\varepsilon, t_l), l = -N_1, \dots, N_2)$, provided x_1, x_2 are sufficiently close to 1 and $\varepsilon > 0$ is sufficiently small, that tends to a slow curve of this modified system as $\varepsilon \rightarrow 0$; it is the one obtained by evaluating our overstable solution at $x = t_l$, $l = -N_1, \dots, N_2$. Now (51) is a fixed point equation, hence given a sufficiently good approximation $b^{(0)}, u^{(0)}, v^{(0)}$ of this solution, one might be able to obtain a sequence $b^{(k)}, u^{(k)}, v^{(k)}$ converging to a solution by simply letting $b^{(k+1)}, u^{(k+1)}, v^{(k+1)}$ be a convenient solution of

$$\begin{aligned} u^{(k+1)}(t_l) - \lambda(t_l)u^{(k+1)}(t_l - \varepsilon) + 2(1 + t_l)b^{(k+1)} &= h(\varepsilon, t_l, v^{(k)}(t_l - \varepsilon), v^{(k)}(t_l), b^{(k)}) \\ v^{(k+1)}(t_l + \varepsilon) + \lambda(t_l)^{-1}v^{(k+1)}(t_l) &= u^{(k+1)}(t_l) . \end{aligned} \quad (52)$$

Thus we are lead to solve linear non-homogeneous equations

$$u(t_l) - \lambda(t_l)^{-1}u(t_{l-1}) + 2(1 + t_l)b = g(t_l) \quad (53)$$

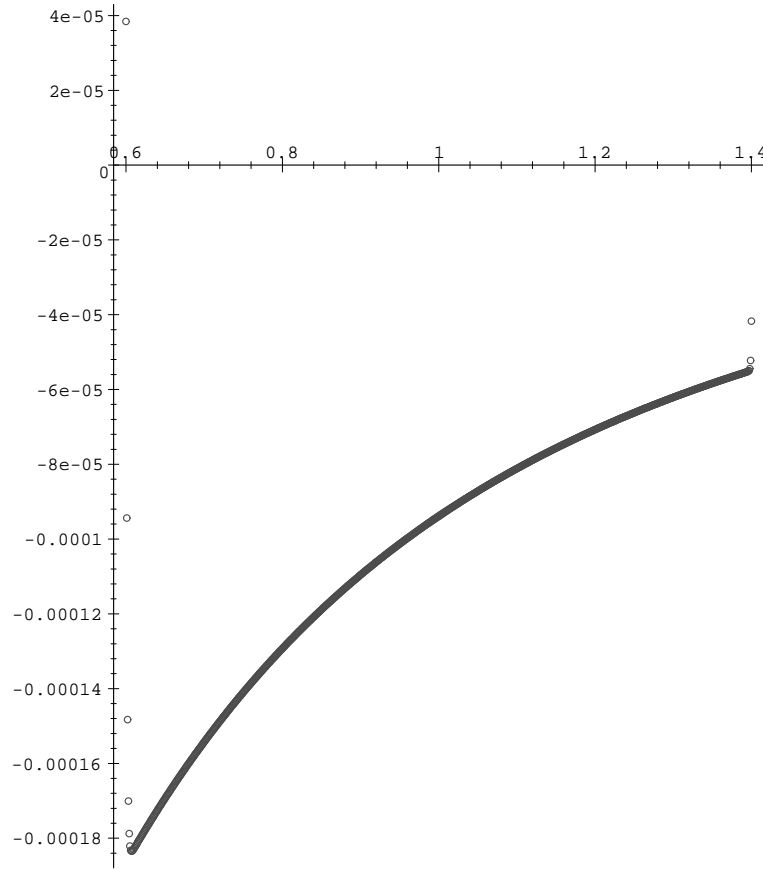


Figure 3: The canard solution for equation (49) .

and

$$v(t_{l+1}) + \lambda(t_l)v(t_l) = u(t_l) . \quad (54)$$

Equation (53) can be solved analogously to the previous example (see also [3]) by putting $u(x_1) = u(x_2) = 0$, calculating $u(t_l), l = -N_1, \dots, 0$ by forward recursion, $u(t_l), l = 0, \dots, N_2 - 1$ by backward recursion and adjusting b such that the two obtained values for $u(t_0)$ coincide.

For equation (54), one particular bounded solution can be found by putting $v(t_0) = u(t_0)/2$ – this is the “slow curve” of this linear problem – and calculating the other values $v(t_l)$ by forward and backward recursion. This works, because $\lambda(x) < 1$ for $x < 1$ and $\lambda(x) > 1$ for $x > 1$. Here, however, there are too many bounded solutions: We can add any (bounded) multiple of the homogeneous solution $v_h(t_l)$ satisfying $v_h(1) = 1$ to our particular solution and still have a bounded solution of (54). (One or both might have a *spike* at $x = 1$).

Here, the following observation helps. For a function $v(\varepsilon, x)$ analytic and bounded in a neighborhood of $[x_1, x_2]$, it can be shown that $\sum_{l=-N_1}^{N_2} v(t_l)v_h(t_l)$ is exponentially small; this will be the case for the overstable solution! Thus it seems reasonable to require that $\sum_{l=-N_1}^{N_2} v(t_l)v_h(t_l) = 0$ for our solution of (54). It can be shown that this defines a linear operator the norm of which is $\mathcal{O}(\varepsilon^{-1/2})$.

In this way, we computed b, u, v and thus a and $y(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ for $x_1 = 0.6$, $x_2 = 1.4$ and $\varepsilon = 0.001$. Figure 3 shows the values of $y(\varepsilon, t_l) + \frac{1}{1+t_l}$, $l = -N_1, \dots, N_2$

for our value of a ; the computation needed 200 digits! Figure 4 shows a solution with a small spike obtained by requiring $v(1) = \frac{3}{8}u(1)$. Also in this example the sensitivity is

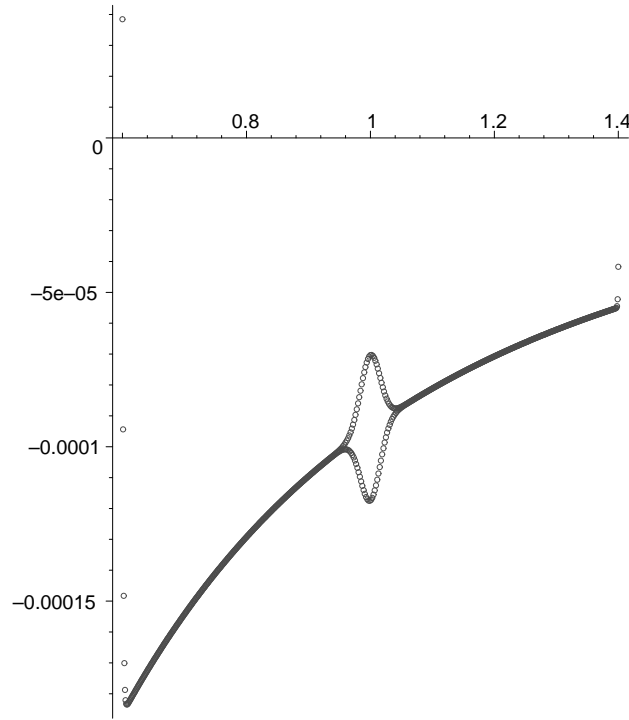


Figure 4: Canard solution with a spike for (54) .

extreme, now also with respect to initial values because the canard solution is neither attractive nor repulsive. This can be seen in figure 5 which presents the discrete solution $\tilde{y}(\varepsilon, t_l)$, $l = -N_1, \dots, N_2$ defined by $\tilde{y}(\varepsilon, x_2) = y(\varepsilon, x_2)$ and $\tilde{y}(\varepsilon, x_2 - \varepsilon) = y(\varepsilon, x_2 - \varepsilon) + 10^{-100}$ and the above values of x_1, x_2, ε and a .

A Auxiliary means

A.1 The Nagumo norms

The content of this appendix is taken from [2]. We included (and adapted) it here for completeness of the thesis.

Let numbers $0 < \rho < r$ be given. Consider the following function d on the open circle $D_r = D_r(0)$.

$$d(x) = \begin{cases} r - |x| & \text{if } |x| \geq \rho \\ r - \rho & \text{if } |x| < \rho \end{cases}$$

This is a modification of the function denoting the distance from x to the boundary of D_r and also depends upon ρ . We have the following property

Proposition 11 *If $x, y \in D_r$ then $|d(x) - d(y)| \leq |x - y|$.*

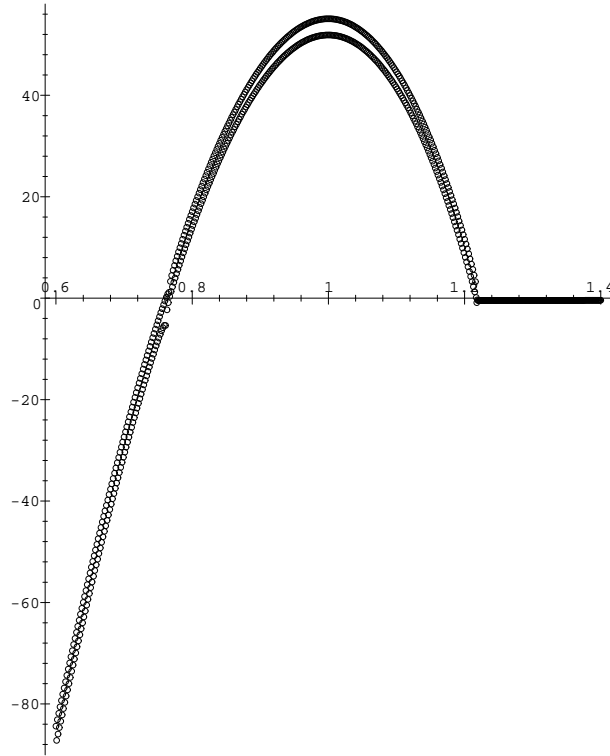


Figure 5: Sensitivity of the canard phenomenon for (54) .

Now we introduce (modified) Nagumo norms on $\mathcal{H}(D_r)$. For nonnegative integers p and for f holomorphic in D_r we put

$$\|f\|_p := \sup_{|x| < r} |f(x)| d(x)^p .$$

Note that the norms depend upon ρ but we do not indicate this. Of course $\|f\|_p$ is infinite for certain $f \in \mathcal{H}(D_r)$. If f is also continuous on the closure of D_r , then

$$\|f\|_p \leq (r - \rho)^p \sup_{|x| < r} |f(x)| .$$

In any case

$$\begin{aligned} |f(x)| &\leq \|f\|_p d(x)^{-p} \text{ for all } |x| < r , \\ |f(x)| &\leq \|f\|_p \delta^{-p} \text{ if } |x| \leq r - \delta, \ 0 < \delta < r - \rho . \end{aligned} \tag{55}$$

The larger p is, the larger is the set of functions f having finite norm $\|f\|_p$. Of course we have

$$\begin{aligned} \|f + g\|_p &\leq \|f\|_p + \|g\|_p \\ \|\alpha f\|_p &= |\alpha| \|f\|_p \end{aligned}$$

for $f, g \in \mathcal{H}(D_r)$ (except if $\alpha = 0, \|f\|_p = \infty$).

The norms are also compatible with multiplication

$$\|fg\|_{p+q} \leq \|f\|_p \|g\|_q \tag{56}$$

for $f, g \in \mathcal{H}(D_r)$ and nonnegative integers p, q (except for $f = 0, \|g\|_q = \infty$ or $g = 0, \|f\|_p = \infty$). The most important property is

Lemma 12 For $f \in \mathcal{H}(D_r)$ one has $\|f'\|_{p+1} \leq e(p+1) \|f\|_p$.

Here the norm of the derivative of a holomorphic function is estimated; it is important that it is estimated not only on a subset of D_r .

Proof. Let $K := \|f\|_p$. Then $|f(x)| \leq K d(x)^{-p}$ for $x \in D_r$. From Cauchy's theorem for $y \in D_r$

$$|f'(y)| = \left| \frac{1}{2\pi i} \oint_{|x-y|=R} \frac{f(x)}{(x-y)^2} dx \right| \leq \frac{1}{R} \max_{|x-y|=R} |f(x)| ,$$

where R can be chosen arbitrarily between 0 and $r - |y|$. By proposition 11 we have $d(x) \geq d(y) - R$ for $|x - y| = R$ and hence (if $R < d(y)$)

$$|f'(y)| \leq \frac{1}{R} K (d(y) - R)^{-p} .$$

We choose now $R = \frac{d(y)}{p+1}$ and obtain

$$|f'(y)| \leq K d(y)^{-p-1} (p+1) \left(1 - \frac{1}{p+1} \right)^{-p} \leq K d(y)^{-p-1} (p+1) e .$$

This yields the lemma. □

Besides differentiation we need the *shift operator* in our estimates. We recall its definition: For $f \in \mathcal{H}(D_r)$ denote by $Sf \in \mathcal{H}(D_r)$ the function

$$Sf(x) = \frac{f(x) - f(0)}{x} \text{ if } x \neq 0, \quad Sf(0) = f'(0) .$$

Here we have

Lemma 13 $\|Sf\|_p \leq \frac{2}{\rho} \|f\|_p$ for $f \in \mathcal{H}(D_r)$, $p \in \mathbb{N}$.

Proof. We have $|f(x)| \leq d(x)^{-p} \|f\|_p$ for $x \in D_r$ and $|f(0)| \leq \|f\|_p (r - \rho)^{-p} \leq d(x)^{-p} \|f\|_p$ for $x \in D_r$. Hence for $|x| \geq \rho$

$$\left| \frac{f(x) - f(0)}{x} \right| \leq \frac{2}{\rho} d(x)^{-p} \|f\|_p .$$

For $|x| < \rho$ we have $|Sf(x)| \leq \max_{|y|=\rho} |Sf(y)| \leq \frac{2}{\rho} (r - \rho)^{-p} \|f\|_p$ by the maximum principle and the above estimate. As $d(x) = r - \rho$ if $|x| < \rho$ we have $|Sf(x)| \leq \frac{2}{\rho} d(x)^{-p} \|f\|_p$ in all our cases. □

A.2 The ‘hypothesis of transversality’ (H)

We suppose that $A_0(x)$ is invertible except at $x = 0$ and hence

$$\det(A_0(x)) = x^m K(x)$$

where $m \in \mathbb{N}$ and $K(x)$ is analytic near 0 with $K(0) \neq 0$.

We consider equations of the form

$$A_0(x) y(x) + B_0(x) a = g(x) \quad (57)$$

for arbitrary functions g . Consider now the Smith normal form of $A_0(x)$ near 0

$$A_0(x) = C(x) x^M D(x),$$

where $C(0)$ and $D(0)$ are invertible matrices and $M = \text{diag}(m_1, \dots, m_n)$ with integers $0 \leq m_1 \leq \dots \leq m_n$, $m_1 + \dots + m_n = m$. Then the above equation is equivalent to

$$x^M D(x) y(x) + C(x)^{-1} B_0(x) a = C(x)^{-1} g(x) \quad (58)$$

For $l \in \{1, \dots, n\}$, we find

$$[D(x)y(x)]_l = \frac{[C(x)^{-1}g(x) - C(x)^{-1}B_0(x)a]_l}{x^{m_l}},$$

where $[]_l$ denotes the l^{th} component of a vector in \mathbb{C}^n . In order to obtain a function y holomorphic in a neighbourhood of 0, we must have

$$[S^k(C^{-1}B_0)(0)]_l a = [S^k(C^{-1}g)(0)]_l \quad (59)$$

for $l = 1, \dots, n$, $k = 0, \dots, m_l - 1$. Here $[]_l$ also denotes the l^{th} row of a $n \times m$ matrix and S is the *shift operator* defined above (lemma 13). If we want that this equation always has a unique solution, we need that the $m \times m$ matrix with the rows $[S^k(C^{-1}B_0)(0)]_l$, $l = 1, \dots, n$, $k = 0, \dots, m_l - 1$ is invertible. If this condition is satisfied then $a = (a_1, \dots, a_m)$ is uniquely determined by equation (59). Furthermore, we obtain a solution y of equation (58)

(and hence of equation (57)) corresponding to a holomorphic in a neighborhood of 0 by

$$y = D^{-1} S^M [C^{-1}(g - B_0 \cdot a)]$$

where S^M has the meaning of applying S to the first component m_1 times, to the second m_2 times, \dots .

As shown above, the condition that the above matrix is invertible is equivalent to the so-called ‘hypothesis of transversality’ **(H)**

(H) *The mapping $\mathcal{H}(D_r)^n \times \mathbb{C}^m \rightarrow \mathcal{H}(D_r)^n$ defined by $(y, a) \mapsto A_0 \cdot y + B_0 \cdot a$ is bijective for sufficiently small r , $0 < r < r_1$.*

Here $\mathcal{H}(D_r)$ denotes the set of function that are holomorphic on D_r . We say that ‘ B_0 is transversal to A_0 at $x = 0$ ’.

We now study the inverse of the mapping in the ‘hypothesis of transversality’ **(H)**. We denote by

$$\begin{aligned} P_1 : \mathcal{H}(D_r)^n &\longrightarrow \mathcal{H}(D_r)^n \\ P_2 : \mathcal{H}(D_r)^n &\longrightarrow \mathbb{C}^m \end{aligned}$$

the uniquely determined linear mappings satisfying

$$A_0(x)(P_1 f)(x) + B_0(x)(P_2 f)(x) = f(x) \quad \text{for } x \in D_r, f \in \mathcal{H}(D_r)^n. \quad (60)$$

Then we have

Lemma 14 *There is a constant $K > 0$ such that*

$$\|P_1 f\|_p \leq K \|f\|_p \text{ and } \|P_2 f\|_p \leq K \|f\|_p$$

(i.e. $\|P_2 f\| \leq K d(0)^{-p} \|f\|_p$) for nonnegative integers p and $f \in \mathcal{H}(D_r)^n$.

Proof. Let $f \in \mathcal{H}(D_r)$ with $\|f\|_p < \infty$ and $g := P_1 f$ and $a := P_2 f$. Then by a relation analogous to equation (58), g and a are determined by

$$x^M D(x) g(x) + C(x)^{-1} B_0(x) a = C(x)^{-1} f(x) \quad (61)$$

and a is determined (cf. equation (59)) by the equations

$$[S^k(C^{-1}B_0)(0)]_l a = [S^k(C^{-1}f)(0)]_l$$

for $l = 1, \dots, n$, $k = 0, \dots, m_{l-1}$, where S is the shift operator defined above. By the condition of transversality, the matrix of all $[S^k(C^{-1}B_0)(0)]_l$, $l = 1, \dots, n$, $k = 0, \dots, m_{l-1}$ is an invertible m by m matrix and hence $|a|$ is smaller than some K_1 times the vector norm of the vector of all $[S^k(C^{-1}f)(0)]_l$. Now, equations (55, 56) and lemma 13 yield that there is K_2 independent of f such that

$$\begin{aligned} |a| &\leq \left(1 + \left(\frac{2}{\rho}\right)^m\right) \|C^{-1}\|_0 d(0)^{-p} \|f\|_p \\ &\leq K_2 d(0)^{-p} \|f\|_p, \end{aligned}$$

and hence $\|a\|_p \leq K_2 \|f\|_p$.

Furthermore, we can write g in the form

$$g = D^{-1} S^M [C^{-1}(f - B_0 a)] ,$$

where S^M means applying S to the first component m_1 times, m_2 times to the second, and so on. Using equation (56) and lemma 13, we find K_3 independent of f such that

$$\begin{aligned} \|g\|_p &\leq \|D^{-1}\|_0 \left(1 + \left(\frac{2}{\rho}\right)^m\right) \|C^{-1}\|_0 \left(\|f\|_p + \|B_0\|_0 K_2 \|f\|_p\right) \\ &\leq K_3 \|f\|_p . \end{aligned}$$

This proves the lemma. □

A.3 The notion of the majorization

Definition 15 We say that a series $g = \sum_{k=0}^{\infty} g_k(x) \varepsilon^k$ is majorized (' \ll ') by a series $h(z) = \sum_{l=0}^{\infty} h_l z^l$ if and only if

$$\|g_j\|_j \leq h_j j! \text{ for } j = 0, 1, 2, \dots$$

We have the following relations for \ll :

Lemma 16 *Assume that $g \ll h(z)$ and $\tilde{g} \ll \tilde{h}(z)$. Then*

$$\begin{aligned} g \tilde{g} &\ll h(z) \tilde{h}(z) \\ \varepsilon \frac{d}{dx} g &\ll e z h(z) \end{aligned}$$

Proof. The first statement follows immediately from the definition of the product of two formal power series by using equation (56) and $k! (j - k)! \leq j!$ for $k \in [0, j]$. In order to show the second statement, we use lemma 12. If $g = \sum_{j=0}^{\infty} g_j(x) \varepsilon^j$ and $h(z) = \sum_{j=0}^{\infty} h_j z^j$ then $g \ll h(z)$ means $\|g_j\|_j \leq h_j j!$ for $j = 0, 1, \dots$. We calculate

$$\varepsilon \frac{d}{dx} g = \sum_{j=0}^{\infty} g'_j(x) \varepsilon^{j+1} = \sum_{j=1}^{\infty} g'_{j-1}(x) \varepsilon^j$$

$$e z h(z) = \sum_{j=0}^{\infty} e h_j z^{j+1} = \sum_{j=1}^{\infty} e h_{j-1} z^j$$

and hence we have to show that $\|g'_{j-1}\|_j \leq e h_{j-1} j!$ for $j = 1, 2, \dots$. This follows immediately from lemma 12 and the above ‘translation’ of the assumption $g \ll h(z)$. \square

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