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Abstract: This note is devoted to a fine study of the convergence of some weighted quadratic and cubic variations of a fractional Brownian motion B with Hurst index $H \in (0, 1/2)$. With the help of Malliavin calculus, we show that, correctly renormalized, the weighted quadratic variation of B that we consider converges in L^2 to an explicit limit when $H < 1/4$, while we conjecture that it converges in law when $H > 1/4$. In the same spirit, we also show that, correctly renormalized, the weighted cubic variation of B converges in L^2 to an explicit limit when $H < 1/6$.

Key words: Fractional Brownian motion - weighted quadratic variation - weighted cubic variation - exact rate of convergence.

1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, given a real $\kappa > 1$, the κ-power variation of a process X, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = 1\}$ of $[0, 1]$, is defined to be the sum

$$\sum_{k=0}^{n-1} |X_{t_{n,k+1}} - X_{t_{n,k}}|^\kappa.$$

For simplicity, consider from now on the case where $t_{n,k} = k/n$, for $n \in \mathbb{N}^*$ and $k \in \{0, \ldots, n\}$. In this paper, we shall point out some interesting phenomena when $X = B$ is a fractional Brownian motion with Hurst index $H \in (0, 1/2)$ and when the value of κ is 2 or 3. In fact, we will also drop the absolute value (when $\kappa = 3$) and we will introduce some weights. More precisely, we will consider:

$$\sum_{k=0}^{n-1} h(B_{k/n}) \Delta^\kappa B_{k/n}, \quad \kappa = 2, 3, \tag{1.1}$$

the function $h : \mathbb{R} \to \mathbb{R}$ being assumed smooth enough and where we note, for simplicity, $\Delta^\kappa B_{k/n}$ instead of $(B_{(k+1)/n} - B_{k/n})^\kappa$. Notice that, originally, the interest that we have in quantities of type (1.1) is motivated by the study of the exact rate of convergence for some approximation schemes of stochastic differential equations driven by B, see [5], [12] and [13].

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Now, let us recall some known results concerning κ-power variations which are today more or less classical. First, assume that the Hurst index is $H = 1/2$, that is B is a standard Brownian motion. Let μ_κ denote the κ-moment of a standard Gaussian random variable $G \sim \mathcal{N}(0,1)$. It is immediate to prove, by using central limit theorem that, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[n^{\kappa/2} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \mu_{2\kappa} - \mu_\kappa^2). \tag{1.2}$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.2). More precisely (see [1, 8] for a quite complete study of these phenomena): as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h(B_{k/n}) \left[n^{\kappa/2} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \overset{\text{Law}}{\longrightarrow} \sqrt{\mu_{2\kappa} - \mu_\kappa} \int_0^1 h(B_s) dW_s. \tag{1.3}$$

Here, W denotes another standard Brownian motion, independent of B.

Second, assume that $H \neq 1/2$, that is the case where the fractional Brownian motion B has not independent increments anymore. Then (1.2) has been extended by [2] (see also [15] for an elegant way to obtain (1.4)-(1.5) just below) and two cases are considered according to the evenness of κ:

- if κ is even and if $H \in (0, 3/4)$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left[n^{\kappa H} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_{H,\kappa}^2); \tag{1.4}$$

- if κ is odd and if $H \in (0, 1/2)$, as $n \to \infty$,

$$n^{\kappa H - 1/2} \sum_{k=0}^{n-1} \Delta^\kappa B_{k/n} \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_{H,\kappa}^2). \tag{1.5}$$

Here $\sigma_{H,\kappa} > 0$ is a constant depending only on H and κ, which can be computed explicitely. In fact, one can relax the restrictive conditions made on H in (1.4)-(1.5): in this case, the normalization is not the same anymore and one obtains limits which are not Gaussian but the value at time one of an Hermite process (see [4, 18]).

Now, let us proceed with the results concerning the weighted power variations in the case where $H \neq 1/2$. Following the ideas in [3][†] (see also [11]), one could prove that, when κ is even and when $H \in (1/2, 3/4)$, as $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h(B_{k/n}) \left[n^{\kappa H} \Delta^\kappa B_{k/n} - \mu_\kappa \right] \overset{\text{Law}}{\longrightarrow} \sigma_{H,\kappa} \int_0^1 h(B_s) dW_s, \tag{1.6}$$

where, once again, W is a standard Brownian motion independent of B. In other words, (1.6) shows in this case a similar behavior to that observed in the standard Brownian case, compare

[†]More precisely, in [3] one does not prove exactly (1.6) but a quite similar result.

with (1.3). In contradistinction, the asymptotic behavior of (1.1) is completely different of (1.3) or (1.6) when $H \in (0, 1/2)$ and κ is odd. The first result in this direction was discovered by Gradinaru, Russo and Vallois [6], when they showed that the following convergence holds when $H = 1/4$: as $\varepsilon \to 0$,

$$\int_0^t h(B_u) \frac{(B_{u+\varepsilon} - B_u)^3}{\varepsilon} du \ \xrightarrow{\mathrm{L}^2} \ -\frac{3}{2} \int_0^t h'(B_u) du. \tag{1.7}$$

In the same spirit, Gradinaru and myself [5] improved very recently (1.7), by working with sums instead of ε-integrals *à la* Russo-Vallois [16]. More precisely, we showed that we have, for any $H \in (0, 1/2)$ and any *odd* integer $\kappa \geq 3$: as $n \to \infty$,

$$n^{(\kappa+1)H-1} \sum_{k=0}^{n-1} h(B_{k/n}) \Delta^\kappa B_{k/n} \ \xrightarrow{\mathrm{L}^2} \ -\frac{\mu_{\kappa+1}}{2} \int_0^1 h'(B_s) ds. \tag{1.8}$$

At this level, we will make three comments. First, let us remark that the limits obtained in (1.7) and (1.8) do not involve an independent standard Brownian motion anymore, as it is the case in (1.3) or (1.6). Second, let us notice that (1.8) is not in contradiction with (1.5) since, when $H \in (0, 1/2)$, we have $(\kappa + 1)H - 1 < \kappa H - 1/2$ and (1.8) with $h \equiv 1$ is in fact a corollary of (1.5). Third, we want to add that exactly the same type of convergence than (1.7) had been already performed in [10], Theorem 4.1 (see also [9]), when, in (1.7), fractional Brownian motion B of Hurst index $H = 1/4$ is replaced by an iterated Brownian motion Z. It is not completely surprising, since this latter process is also centred, selfsimilar of index $1/4$ and has stationary increments. For the sake of completeness, let us finally mention that Swanson announced in [17] that, in a joint work with Burdzy, he will prove that the same also holds for the solution to a stochastic heat equation.

The aim of the present work is to prove the following result:

Theorem 1.1 *Let B be a fractional Brownian motion of Hurst index H. Then:*

1. *If $h : \mathbb{R} \to \mathbb{R} \in \mathscr{C}_b^2$ and if $H \in (0, 1/4)$, we have, as $n \to \infty$:*

$$n^{2H-1} \sum_{k=0}^{n-1} h(B_{k/n}) \left[n^{2H} \Delta^2 B_{k/n} - 1 \right] \ \xrightarrow{\mathrm{L}^2} \ \frac{1}{4} \int_0^1 f''(B_u) du. \tag{1.9}$$

2. *If $h : \mathbb{R} \to \mathbb{R} \in \mathscr{C}_b^3$ and if $H \in (0, 1/6)$, we have, as $n \to \infty$:*

$$n^{3H-1} \sum_{k=0}^{n-1} \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \ \xrightarrow{\mathrm{L}^2} \ -\frac{1}{8} \int_0^1 f'''(B_u) du. \tag{1.10}$$

Before giving the proof of Theorem 1.1, let us try to explain why (1.9) is *a priori* only available when $H < 1/4$ (of course, the same type of arguments could be also applied to understand why (1.10) is *a priori* only available when $H < 1/6$). For this purpose, let us first consider the

case where B is a standard Brownian motion (that is the case where $H = 1/2$). By using the independence of increments, we easily compute

$$E\left\{\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\} = 0,$$

and

$$E\left\{\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\}^2 = 2\,E\left\{\sum_{k=0}^{n-1} h^2(B_{k/n})\right\} \approx 2n\,E\left\{\int_0^1 h^2(B_u)du\right\}.$$

Although these two facts are of course not sufficient to guarantee that (1.3) holds when $\kappa = 2$, they however roughly explain why it is true. Now, let us go back to the general case, that is the case where B is a fractional Brownian motion of index $H \in (0, 1/2)$. In the sequel, we will show (see Lemmas 2.2 and 2.3 for precise statements) that

$$E\left\{\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\} \approx \frac{1}{4}n^{-2H}\sum_{k=0}^{n-1} E\left[h''(B_{k/n})\right],$$

and, when $H > 1/4$:

$$\begin{aligned} E\left\{\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\}^2 &\approx \sum_k E\left\{h^2(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]^2\right\} \\ &\approx 2\sum_k E\left[h^2(B_{k/n})\right] \approx 2n\,E\left\{\int_0^1 h^2(B_u)du\right\}, \end{aligned}$$

while, when $H < 1/4$:

$$\begin{aligned} E\left\{\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\}^2 &\approx \sum_{k\neq\ell} E\left\{h(B_{k/n})h(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]^2\left[n^{2H}\Delta^2 B_{\ell/n} - 1\right]^2\right\} \\ &\approx \frac{1}{16}n^{-4H}\sum_{k\neq\ell} E\left[h''(B_{k/n})h''(B_{\ell/n})\right] \\ &\approx \frac{1}{16}n^{2-4H}E\left\{\int_{[0,1]^2} h''(B_u)h''(B_v)dudv\right\} \end{aligned}$$

In other words, the quantity $\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]$, when B is a fractional Brownian motion of index $H \in (0, 1/4)$, behaves as in the case where B is a standard Brownian motion, at least for the first and second order moments. That is why, we conjecture that the following convergence certainly holds:

Conjecture: when $H \in (1/4, 1/2)$, $\dfrac{1}{\sqrt{n}}\displaystyle\sum_{k=0}^{n-1} h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right] \xrightarrow{\text{Law}} \sqrt{2}\int_0^1 h(B_s)dW_s$, as $n \to \infty$,

with W a standard Brownian motion independent of B. In order to prove this conjecture, a possible method would be to try to perform the classical moments method. For this, we should

analyze the moments at all orders of $\sum_{k=0}^{n-1} h(B_{k/n}) \left[n^{2H} \Delta^2 B_{k/n} - 1 \right]$ (and not only the first and second orders, as before). As we can suppose it, this work should be *a priori* quite long and technical. That is why we propose ourself to try to solve this conjecture in a forthcoming paper.

Finally, let us remark that (1.9) is of course not in contradiction with (1.4), since we have $2H-1 < -1/2$ if and only if $H < 1/4$ (it is an other reason which can explain the condition $H < 1/4$ in the first point of Theorem 1.1). Thus, (1.9) with $h \equiv 1$ is in fact a corollary of (1.4). Similarly, (1.10) is not in contradiction with (1.4), since we have $3H - 1 < -1/2$ if and only if $H < 1/6$ (this time, it can explain, in a sense, the condition $H < 1/6$ in the second point of Theorem 1.1).

Now, the sequel of this note is devoted to the proof of Theorem 1.1. Instead of the *pedestrian* technique performed in [5] or [6] (as their authors called it themselves), we stress on the fact that we choosed here to use a more elegant way via Malliavin calculus. It can be viewed as an other novelty of this paper.

2 Proof of Theorem 1.1

2.1 Notations and recalls

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [14] for further details. Let $B = (B_t)_{t \in [0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$ defined on a probability space $(\Omega, \mathscr{A}, P)$. We mean that B is a centered Gaussian process with the covariance function $\mathrm{E}(B_s B_t) = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right). \tag{2.11}$$

We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on [0,T]. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R_H(t,s).$$

We denote by $|\cdot|_{\mathfrak{H}}$ the associate norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1(B)$ associated with B. We denote this isometry by $\varphi \mapsto B(\varphi)$.

The covariance kernel $R_H(t,s)$ introduced in (2.11) can be written as

$$R_H(t,s) = \int_0^{s \wedge t} K_H(s,u) K_H(t,u) du,$$

where $K_H(t,s)$ is the square integrable kernel defined, for $s < t$, by

$$K_H(t,s) = \Gamma(H + \frac{1}{2})^{-1} (t-s)^{H-\frac{1}{2}} F\big(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\big), \tag{2.12}$$

where $F(a,b,c,z)$ is the classical Gauss hypergeometric function. By convention, we set $K_H(t,s) = 0$ if $s \geq t$.

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables, *i.e.* of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n))$$

where $n \geqslant 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to B is the element of $\mathrm{L}^2(\Omega, \mathfrak{H})$ defined by

$$D_s^B F \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i(s), \quad s \in [0,T].$$

In particular $D_s^B B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \;=\; \mathrm{E}\left[F^2\right] + \mathrm{E}\left[|D.F|_{\mathfrak{H}}^2\right].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{C}_b^1$ and if $(F_i)_{i=1,\ldots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1, \ldots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0,T]$:

$$D_s\, \varphi(F_1, \ldots, F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, \ldots, F_n) D_s F_i.$$

The divergence operator δ is the adjoint of the derivative operator D. If a random variable $u \in \mathrm{L}^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|\mathrm{E}\langle DF, u\rangle_{\mathfrak{H}}| \leq c_u \, \|F\|_{\mathrm{L}^2} \quad \text{for any } F \in \mathscr{S},$$

then $\delta(u)$ is defined by the duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\langle DF, u\rangle_{\mathfrak{H}},$$

for every $F \in \mathbb{D}^{1,2}$.

2.2 Proof of (1.9)

We will need several lemmas. The first one will be useful in order to control the remainder of the approximations we will make in the sequel.

Lemma 2.1 *Let us denote, for* $0 \leq \ell, k \leq n-1$*:*

$$\begin{aligned}
\alpha_{\ell,\delta\ell}^{(n)} &= \langle \mathbf{1}_{[0,\ell/n]}, n^H \mathbf{1}_{[\ell/n,(\ell+1)/n]}\rangle_{\mathfrak{H}} = \frac{1}{2}\, n^{-H}\left[(\ell+1)^{2H} - \ell^{2H} - 1\right] \\
\alpha_{k,\delta k}^{(n)} &= \langle \mathbf{1}_{[0,k/n]}, n^H \mathbf{1}_{[k/n,(k+1)/n]}\rangle_{\mathfrak{H}} = \frac{1}{2}\, n^{-H}\left[(\ell+1)^{2H} - \ell^{2H} - 1\right] \\
\alpha_{k,\delta\ell}^{(n)} &= \langle \mathbf{1}_{[0,k/n]}, n^H \mathbf{1}_{[\ell/n,(\ell+1)/n]}\rangle_{\mathfrak{H}} = \frac{1}{2}\, n^{-H}\left[(\ell+1)^{2H} - \ell^{2H} - |\ell+1-k|^{2H} + |\ell-k|^{2H}\right] \\
\alpha_{\ell,\delta k}^{(n)} &= \langle \mathbf{1}_{[0,\ell/n]}, n^H \mathbf{1}_{[k/n,(k+1)/n]}\rangle_{\mathfrak{H}} = \frac{1}{2}\, n^{-H}\left[(k+1)^{2H} - k^{2H} - |k+1-\ell|^{2H} + |k-\ell|^{2H}\right] \\
\alpha_{\delta k,\delta\ell}^{(n)} &= \langle n^H \mathbf{1}_{[k/n,(k+1)/n]} n^H \mathbf{1}_{[\ell/n,(\ell+1)/n]}\rangle_{\mathfrak{H}} = \frac{1}{2}\left[2|\ell-k|^{2H} - |\ell-k+1|^{2H} - |\ell-k-1|^{2H}\right].
\end{aligned}$$

Then, for any $0 \leq k < \ell \leq n-1$*:*

$$\begin{aligned}
\alpha^{(n)}_{\ell,\delta\ell} &= -\frac{1}{2}n^{-H} + O(\ell^{H-1}) \\
\alpha^{(n)}_{k,\delta k} &= -\frac{1}{2}n^{-H} + O(\ell^{H-1}) \\
\alpha^{(n)}_{k,\delta\ell} &= O\big((\ell-k)^{H-1}\big) \\
\alpha^{(n)}_{\ell,\delta k} &= O\big(\ell^{H-1} + (\ell-k)^{H-1}\big) \\
\alpha^{(n)}_{\delta k,\delta\ell} &= O\big((\ell-k)^{2H-2}\big),
\end{aligned}$$

where $O(\ell^\alpha)$ *– resp.* $O\big((\ell-k)^\alpha\big)$ *– means a quantity* $r_{k,\ell,n}$ *satisfying* $|r_{k,\ell,n}| \leq \mathrm{c}\,\ell^\alpha$ *for any* $n \geq 1$ *and any* $0 \leq k < \ell \leq n-1$*, for a certain constant* c *independent of* k*,* ℓ *and* n*.*

Proof. For $x \geq 0$, we can write:

$$|(x+1)^{2H} - x^{2H}| = 2H \int_0^1 \frac{du}{(x+u)^{1-2H}} \leq 2H\, x^{2H-1}.$$

We deduce, for $\ell \in \{0, \ldots, n-1\}$:

$$\alpha^{(n)}_{\ell,\delta\ell} = -\frac{1}{2}\, n^{-H} + O(n^{-H}\ell^{2H-1}) = -\frac{1}{2}\, n^{-H} + O(\ell^{H-1}).$$

For $\alpha^{(n)}_{k,\delta k}$, $\alpha^{(n)}_{k,\delta\ell}$ and $\alpha^{(n)}_{\ell,\delta k}$, the proofs are exactly the same.

For $x \geq 1$, we have

$$|(x+1)^{2H} + (x-1)^{2H} - 2x^{2H}| = 2H(2H-1) \int_{[0,1]^2} \frac{dudv}{(x+u-v)^{2-2H}} \leq 2H(2H-1)\, x^{2H-2}.$$

We finally deduce:

$$\alpha^{(n)}_{\delta k,\delta\ell} = (\ell-k+1)^{2H} + (\ell-k-1)^{2H} - 2(\ell-k)^{2H} = O((\ell-k)^{2H-2}).$$

□

Nota: In the remainder of the paper, for simplicity, we will note $\sum_k$ instead of $\sum_{k=0}^{n-1}$, and $\sum_{\ell\neq k}$ instead of $\sum_{0\leq k<\ell\leq n-1} + \sum_{0\leq \ell<k\leq n-1}$.

Lemma 2.2 *For* $h, g : \mathbb{R} \to \mathbb{R} \in \mathscr{C}^2_b$*, we have*

$$\sum_{\ell\neq k} E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\} = \frac{1}{4}n^{-2H}\sum_{\ell\neq k} E\left\{h''(B_{k/n})g(B_{\ell/n})\right\} + o(n^{2-2H}), \tag{2.13}$$

$$\sum_{k} E\left\{h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n} - 1\right]\right\} = \frac{1}{4}n^{-2H}\sum_{k} E\left\{h''(B_{k/n})\right\} + o(n^{1-H}), \tag{2.14}$$

$$\sum_{\ell\neq k} E\left\{h(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}\right\} = -\frac{1}{2}n^{-H}\sum_{\ell\neq k} E\left\{h'(B_{k/n})g(B_{\ell/n})\right\} + o(n^{1-H}) \tag{2.15}$$

and

$$\sum_{k} E\left\{h(B_{k/n})n^H\Delta B_{k/n}\right\} = -\frac{1}{2}n^{-H}\sum_{k} E\left\{h'(B_{k/n})\right\} + o(n^{1-H}). \tag{2.16}$$

Proof. Let us first prove (2.13). For $0 \leq \ell, k \leq n-1$, we can write:

$$\begin{aligned}
& E\left\{h(B_{k/n})g(B_{\ell/n})n^{2H}\Delta^2 B_{k/n}\right\} \\
= \;& E\left\{h(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}\,\delta(n^H\mathbf{1}_{[k/n,(k+1)/n]})\right\} \\
= \;& E\left\{h'(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}\right\}\alpha^{(n)}_{k,\delta k} + E\left\{h(B_{k/n})g'(B_{\ell/n})n^H\Delta B_{k/n}\right\}\alpha^{(n)}_{\ell,\delta k} \\
+ \;& E\left\{h(B_{k/n})g(B_{\ell/n})\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\} \\
= \;& E\left\{h'(B_{k/n})g(B_{\ell/n})\delta(n^H\mathbf{1}_{[k/n,(k+1)/n]})\right\}\alpha^{(n)}_{k,\delta k} + E\left\{h(B_{k/n})g'(B_{\ell/n})\delta(n^H\mathbf{1}_{[k/n,(k+1)/n]})\right\}\alpha^{(n)}_{\ell,\delta k} \\
= \;& E\left\{h''(B_{k/n})g(B_{\ell/n})\right\}(\alpha^{(n)}_{k,\delta k})^2 + 2\,E\left\{h'(B_{k/n})g'(B_{\ell/n})\right\}\alpha^{(n)}_{k,\delta k}\,\alpha^{(n)}_{\ell,\delta k} \\
+ \;& E\left\{h(B_{k/n})g''(B_{\ell/n})\right\}(\alpha^{(n)}_{\ell,\delta k})^2,
\end{aligned}$$

and, using Lemma 2.1, equality (2.13) follows. The proofs of (2.14), (2.15) and (2.16) are simpler and similar. □

Lemma 2.3 *For $h, g : \mathbb{R} \to \mathbb{R} \in \mathscr{C}^4_b$, we have*

$$\begin{aligned}
& \sum_{\ell\neq k} E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\left[n^{2H}\Delta^2 B_{\ell/n}-1\right]\right\} \\
= \;& \frac{1}{16}n^{-4H}\sum_{\ell\neq k} E\left\{h''(B_{k/n})g''(B_{\ell/n})\right\} + o(n^{2-4H})
\end{aligned} \tag{2.17}$$

and

$$\sum_{k} E\left\{h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]^2\right\} = 2\sum_{k} E\left\{h(B_{k/n})\right\} + o(n). \tag{2.18}$$

Proof. For $0 \leq \ell, k \leq n-1$, we can write:

$$\begin{aligned}
& E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]n^{2H}\Delta^2 B_{\ell/n}\right\} \\
= \;& E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]n^H\Delta B_{\ell/n}\,\delta(n^H\mathbf{1}_{[\ell/n,(\ell+1)/n]})\right\} \\
= \;& E\left\{h'(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]n^H\Delta B_{\ell/n}\right\}\alpha^{(n)}_{k,\delta\ell} \\
+ \;& E\left\{h(B_{k/n})g'(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]n^H\Delta B_{\ell/n}\right\}\alpha^{(n)}_{\ell,\delta\ell} \\
+ \;& 2\,E\left\{h(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}n^H\Delta B_{\ell/n}\right\}\alpha^{(n)}_{\delta k,\delta\ell} + E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& E\left\{h(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\left[n^{2H}\Delta^2 B_{\ell/n}-1\right]\right\} \\
= \;& E\left\{h'(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\delta(n^H\mathbf{1}_{[\ell/n,(\ell+1)/n]})\right\}\alpha^{(n)}_{k,\delta\ell} \\
+ \;& E\left\{h(B_{k/n})g'(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\delta(n^H\mathbf{1}_{[\ell/n,(\ell+1)/n]})\right\}\alpha^{(n)}_{\ell,\delta\ell} \\
+ \;& 2\,E\left\{h(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}\delta(n^H\mathbf{1}_{[\ell/n,(\ell+1)/n]})\right\}\alpha^{(n)}_{\delta k,\delta\ell} \\
= \;& E\left\{h''(B_{k/n})g(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}(\alpha^{(n)}_{k,\delta\ell})^2 + 2\,E\left\{h'(B_{k/n})g'(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}\alpha^{(n)}_{k,\delta\ell}\,\alpha^{(n)}_{\ell,\delta\ell} \\
+ \;& 2\,E\left\{h'(B_{k/n})g(B_{\ell/n})n^H\Delta B_{k/n}\right\}\alpha^{(n)}_{k,\delta\ell}\,\alpha^{(n)}_{\delta k,\delta\ell} + E\left\{h(B_{k/n})g''(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}(\alpha^{(n)}_{\ell,\delta\ell})^2 \\
+ \;& 2\,E\left\{h(B_{k/n})g'(B_{\ell/n})n^H\Delta B_{k/n}\right\}\alpha^{(n)}_{\ell,\delta\ell}\,\alpha^{(n)}_{\delta k,\delta\ell} + 2\,E\left\{h(B_{k/n})g(B_{\ell/n})\right\}(\alpha^{(n)}_{\delta k,\delta\ell})^2,
\end{aligned}
$$

and, using Lemmas 2.1 and 2.2, equality (2.17) holds. The proof of (2.18) is simpler and similar.

□

We are now in position to prove (1.9). Using Lemmas 2.2 and 2.3, we have on one hand:

$$
\begin{aligned}
& E\left\{n^{2H-1}\sum_k h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}^2 \qquad (2.19)\\
= \;& n^{4H-2}\sum_k E\left\{h^2(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]^2\right\} \\
+ \;& n^{4H-2}\sum_{\ell\neq k} E\left\{h(B_{k/n})h(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\left[n^{2H}\Delta^2 B_{\ell/n}-1\right]\right\} \\
= \;& \frac{1}{16}\,n^{-2}\sum_{\ell\neq k} E\left\{h''(B_{k/n})h''(B_{\ell/n})\right\}+O(n^{4H-1}).
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
& E\left\{n^{2H-1}\sum_k h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\times\frac{1}{4n}\sum_\ell h''(B_{\ell/n})\right\} \qquad (2.20)\\
= \;& \frac{n^{2H-2}}{4}\left(\sum_k E\left\{(hh'')(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}+\sum_{k\neq\ell} E\left\{h(B_{k/n})h''(B_{\ell/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]\right\}\right) \\
= \;& \frac{1}{16}\,n^{-2}\sum_{k\neq\ell} E\left\{h''(B_{k/n})h''(B_{\ell/n})\right\}+o(1).
\end{aligned}
$$

Now, we easily deduce (1.9). Indeed, thanks to (2.19)-(2.20), we obtain, by developing the square and by remembering that $H<1/4$, that

$$
E\left\{n^{2H-1}\sum_k h(B_{k/n})\left[n^{2H}\Delta^2 B_{k/n}-1\right]-\frac{1}{4n}\sum_k h''(B_{k/n})\right\}^2\longrightarrow 0,\quad \text{as } n\to\infty.
$$

Since $\frac{1}{4n}\sum_k h''(B_{k/n})\xrightarrow{\mathrm{L}^2}\frac{1}{4}\int_0^1 h''(B_u)du$ as $n\to\infty$, we have finally proved that (1.9) holds.

2.3 Proof of (1.10)

As in the previous section, we first need two technical lemmas.

Lemma 2.4 *For $h, g : \mathbb{R} \to \mathbb{R} \in \mathscr{C}_b^3$, we have*

$$\begin{aligned} & \sum_{\ell \neq k} E\left\{h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n}\right\} \\ = & -\frac{3}{2} n^{-H} \sum_{\ell \neq k} E\left\{h'(B_{k/n}) g(B_{\ell/n})\right\} - \frac{1}{8} n^{-3H} \sum_{\ell \neq k} E\left\{h'''(B_{k/n}) g(B_{\ell/n})\right\} + o(n^{2-3H}). \end{aligned} \tag{2.21}$$

Proof. For $0 \leq \ell, k \leq n-1$, we can write:

$$\begin{aligned} & E\left\{h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n}\right\} \\ = & E\left\{h(B_{k/n}) g(B_{\ell/n}) n^{2H} \Delta^2 B_{k/n}\, \delta(n^H \mathbf{1}_{[k/n,(k+1)/n]})\right\} \\ = & E\left\{h'(B_{k/n}) g(B_{\ell/n}) n^{2H} \Delta^2 B_{k/n}\right\} \alpha_{k,\delta k}^{(n)} + E\left\{h(B_{k/n}) g'(B_{\ell/n}) n^{2H} \Delta^2 B_{k/n}\right\} \alpha_{\ell,\delta k}^{(n)} \\ + & 2\, E\left\{h(B_{k/n}) g(B_{\ell/n}) n^{H} \Delta B_{k/n}\right\} \\ = & E\left\{h''(B_{k/n}) g(B_{\ell/n}) n^{H} \Delta B_{k/n}\right\} (\alpha_{k,\delta k}^{(n)})^2 + 2\, E\left\{h'(B_{k/n}) g'(B_{\ell/n}) n^{H} \Delta B_{k/n}\right\} \alpha_{k,\delta k}^{(n)} \alpha_{\ell,\delta k}^{(n)} \\ + & 3\, E\left\{h'(B_{k/n}) g(B_{\ell/n})\right\} \alpha_{k,\delta k}^{(n)} + E\left\{h(B_{k/n}) g''(B_{\ell/n}) n^{H} \Delta B_{k/n}\right\} (\alpha_{\ell,\delta k}^{(n)})^2 \\ + & 3\, E\left\{h(B_{k/n}) g'(B_{\ell/n})\right\} \alpha_{\ell,\delta k}^{(n)} \\ = & E\left\{h'''(B_{k/n}) g(B_{\ell/n})\right\} (\alpha_{k,\delta k}^{(n)})^3 + 3\, E\left\{h''(B_{k/n}) g'(B_{\ell/n})\right\} (\alpha_{k,\delta k}^{(n)})^2 \alpha_{\ell,\delta k}^{(n)} \\ + & 3\, E\left\{h'(B_{k/n}) g''(B_{\ell/n})\right\} \alpha_{k,\delta k}^{(n)} (\alpha_{\ell,\delta k}^{(n)})^2 + 3\, E\left\{h'(B_{k/n}) g(B_{\ell/n})\right\} \alpha_{k,\delta k}^{(n)} \\ + & E\left\{h(B_{k/n}) g'''(B_{\ell/n})\right\} (\alpha_{\ell,\delta k}^{(n)})^3 + 3\, E\left\{h(B_{k/n}) g'(B_{\ell/n})\right\} \alpha_{\ell,\delta k}^{(n)}. \end{aligned}$$

Using Lemma 2.1, we finally obtain (2.21). □

In the same spirit, we can prove:

Lemma 2.5 *For $h, g : \mathbb{R} \to \mathbb{R} \in \mathscr{C}_b^3$, we have*

$$\begin{aligned} & \sum_{\ell \neq k} E\left\{h(B_{k/n}) g(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} n^{3H} \Delta^3 B_{\ell/n}\right\} \\ = & \frac{9}{4} n^{-2H} \sum_{\ell \neq k} E\left\{h'(B_{k/n}) g'(B_{\ell/n})\right\} + \frac{3}{16} n^{-4H} \sum_{\ell \neq k} E\left\{h'(B_{k/n}) g'''(B_{\ell/n})\right\} \\ + & \frac{3}{16} n^{-4H} \sum_{\ell \neq k} + E\left\{h'''(B_{k/n}) g'(B_{\ell/n})\right\} + \frac{1}{64} n^{-6H} \sum_{\ell \neq k} E\left\{h'''(B_{k/n}) g'''(B_{\ell/n})\right\} + o(n^{2-6H}). \end{aligned} \tag{2.22}$$

Proof. Left to the reader: use the same technic than in the proof of Lemma 2.4. □

We are now in position to prove (1.10). Using Lemmas 2.4 and 2.5, we have on one hand

$$
\begin{aligned}
& E\left\{ n^{3H-1} \sum_k \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \right\}^2 \qquad (2.23) \\
= \;& n^{6H-2} \sum_k E \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right]^2 \\
+ \;& n^{6H-2} \sum_{\ell \neq k} E \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \left[h(B_{\ell/n}) n^{3H} \Delta^3 B_{\ell/n} + \frac{3}{2} h'(B_{\ell/n}) n^{-H} \right] \\
= \;& \frac{1}{64} n^{-2} \sum_{\ell \neq k} E \left\{ h'''(B_{k/n}) h'''(B_{\ell/n}) \right\} + O(n^{6H-1}).
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
& E\left\{ n^{3H-1} \sum_k \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] \times \frac{-1}{8n} \sum_\ell h'''(B_{\ell/n}) \right\} \qquad (2.24) \\
= \;& -\frac{n^{3H-2}}{8} \left(\sum_k E \left[(hh''')(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} (h'h''')(B_{k/n}) n^{-H} \right] \right. \\
& \left. + \sum_{k \neq \ell} E \left[h(B_{k/n}) h'''(B_{\ell/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) h'''(B_{\ell/n}) n^{-H} \right] \right) \\
= \;& \frac{1}{64} n^{-2} \sum_{k \neq \ell} E \left\{ h'''(B_{k/n}) h'''(B_{\ell/n}) \right\} + o(1).
\end{aligned}
$$

Now, we easily deduce (1.10). Indeed, thanks to (2.23)-(2.24), we obtain, by developing the square and by remembering that $H < 1/6$, that

$$
E\left\{ n^{3H-1} \sum_k \left[h(B_{k/n}) n^{3H} \Delta^3 B_{k/n} + \frac{3}{2} h'(B_{k/n}) n^{-H} \right] + \frac{1}{8n} \sum_k h'''(B_{k/n}) \right\}^2 \longrightarrow 0, \quad \text{as } n \to \infty.
$$

Since $-\frac{1}{8n} \sum_k h'''(B_{k/n}) \xrightarrow{\mathrm{L}^2} -\frac{1}{8} \int_0^1 h'''(B_u) du$ as $n \to \infty$, we have finally proved that (1.10) holds.

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